Sub-Riemannian Brownian motion, functional inequalities on path space and horizontal Ricci curvature

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(Joint work with *Li-Juan Cheng* and *Erlend Grong*)

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Outline

1. Sub-Riemannian structures
2. Ricci curvature bounds and gradient estimates
3. Ricci curvature and analysis on path space
4. Analysis on path space over sub-Riemannian manifolds
5. Ricci curvature bounds in sub-Riemannian geometry

I. Sub-Riemannian structures

- (M, H, g_H) where
  - M smooth manifold, $\dim M = n$
  - $H \subsetneq TM$ subbundle ("horizontal directions"), $\operatorname{rank} H = m$
  - g_H fiberwise inner product on H

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- Let

$$d_H(x,y) = \inf\left\{ \int_0^1 |\dot{\gamma}(t)|\, dt : \ \gamma(0) = x,\ \gamma(1) = y,\ \dot{\gamma}(t) \in H_{\gamma(t)}\ \forall\ t \right\}$$

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- H bracket generating (i.e. $\operatorname{Lie}(H)(x) = T_xM$ for each $x \in M$)
  $\Longrightarrow (M, d_H)$ metric space

- Canonical sub-Riemannian Laplacian?

$$\Delta^H = \sum_{i=1}^{m} A_i^2 + Z \quad \text{(locally)}$$

$A_1, \ldots, A_m$ local orthonormal frame of H,
Z first order term (horizontal vector field)

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- Some notation:
  1. Consider

$$\sharp^H \colon T^*M \to H \subset TM, \quad \langle \sharp^H \alpha, v \rangle_{g_H} := \alpha(v),$$

for $\alpha \in T_x^*M,\ v \in H_x,\ x \in M$.
Note that $\ker \sharp^H = \mathrm{Ann}\, H$.

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Note that $\ker \sharp^H = \mathrm{Ann}\, H$.

  2. The map $\sharp^H$ induces a (degenerate) co-metric g_H^* on T^*M via

$$\langle \alpha, \beta \rangle_{g_H^*} = \langle \sharp^H \alpha, \sharp^H \beta \rangle_{g_H}.$$

- Let L be a second order partial differential operator on M. Its symbol $\sigma(L)$ is the symmetric, bilinear 2-tensor on T^*M determined by the relation

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- A second order PDO L (without constant term) is called sub-Laplacian with respect to (M, H, g_H) if

$$\sigma(L) = g_H^*.$$

We write $L = \Delta^H$.

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Define

$$\nabla^H f = \mathrm{pr}_H \nabla f \equiv \sharp^H df$$

and let Δ^H be the generator of the Dirichlet form

$$\mathcal{E}(f,h) := -\int_M \langle \nabla^H f, \nabla^H h \rangle_H \, d\mathrm{vol}_g.$$

Then $\Delta^H := -(\nabla^H)^* \nabla^H = \mathrm{trace}_H \nabla^2$ is a sub-Laplacian.

In the situation of the last example:

- Canonical variation of the metric

$$\varepsilon > 0 : \quad g_\varepsilon := g_H \oplus \frac{1}{\varepsilon} g_V$$

$\varepsilon \downarrow 0$: sub-Riemannian limit

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- Observation

$$\mathrm{Ric}^{g_\varepsilon}(u,u) \xrightarrow{\varepsilon \downarrow 0} -\infty \quad \text{for any horizontal unit vector } u$$

Natural connections on a sub-Riemannian manifold (M, H, g_H)

- Would like to have a connection ∇ on M which is horizontally compatible with (H, g_H) in the sense that the horizontal subbundle H is preserved under parallel transport, as well as its metric g_H

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- In terms of the corresponding *horizontal Hessian*,

$$\nabla^2 f \equiv \mathrm{Hess}\, f \in \Gamma(H^*\otimes H^*),\quad (\nabla^2 f)(A,B) = ABf - (\nabla_A B)f,$$

the associated sub-Laplacian Δ^H is given by

$$\Delta^H f = \mathrm{trace}_H \nabla^2 f,\quad f\in C^\infty(M).$$

- Note that horizontally compatible connections ∇ will always have torsion $\mathbf{T}$:

$$\nabla_A B - \nabla_B A - [A, B] = \mathbf{T}(A, B), \quad A, B \in \Gamma(H).$$

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- A horizontally compatible connection ∇ is uniquely determined by its torsion $\mathbf{T}$.
- Let V be a choice of complement to H. There exists a unique horizontally compatible partial connection ∇ with

$$\mathbf{T}(H, H) \subseteq V$$

Example Let again (M, g) and $g_H = g|H$. Then $TM = H \oplus_\perp V$ and

$$g = g_H \oplus g_V$$

Denote by ∇^g the Levi-Civita connection on M, g.

- (Bott connection) There is a canonical connection ∇ preserving the decomposition $TM = H \oplus V$:

$$\nabla_X Y = \begin{cases} \mathrm{pr}_H(\nabla^g_X Y), & X, Y \in \Gamma(H), \\ \mathrm{pr}_H([X, Y]), & X \in \Gamma(V),\ Y \in \Gamma(H), \\ \mathrm{pr}_V([X, Y]), & X \in \Gamma(H),\ Y \in \Gamma(V), \\ \mathrm{pr}_V(\nabla^g_X Y), & X, Y \in \Gamma(V), \end{cases}$$

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- $\nabla g = 0$
- its torsion $T^\nabla(X,Y)$ is vertical for X and Y horizontal, and zero if either X or Y is vertical

Standing assumptions

- Let V be a choice of a complement to H in (M, H, g_H).
  Let pr_H and pr_V be the corresponding projections. Write ∇ for the unique partial connection with $\mathbf{T}(H, H) \subseteq V$.

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$$\nabla_X Y, \quad X, Y \in \Gamma(H),$$

to an affine connection on M as follows:

$$\nabla_X Y = \begin{cases} \mathrm{pr}_H[X,Y] & \text{if } X \in \Gamma(V), Y \in \Gamma(H) \\ \mathrm{pr}_V[X,Y] & \text{if } X \in \Gamma(H), Y \in \Gamma(V) \end{cases}$$

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- Connections of this form satisfy the following properties:
  1. (i) both H and V are parallel with respect to ∇
  2. (ii) $\mathbf{T}(H, H) \subseteq V$
  3. (iii) $\mathbf{T}(H, V) = 0$.

  Conversely, any connection satisfying (i)-(iii) is of this form.

- (Metric preserving complement V) For simplicity, assume that

$$(L_Z \, \mathrm{pr}_H^* g_H)(X, X) = 0 \quad \text{for all } Z \in \Gamma(V) \text{ and } X \in \Gamma(H)$$

where L_Z denotes the Lie derivative with respect to Z.

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- Let $\mathrm{Ric}\colon TM \to TM$ be the Ricci tensor with respect to ∇:

$$\mathrm{Ric}(v) = \mathrm{trace}_H \, R^\nabla(v, \times)\times$$

  The object of our interest is

$$\mathrm{Ric}^H \in \Gamma(H^* \otimes H), \quad \mathrm{Ric}^H := \mathrm{Ric}|H \quad \text{(horizontal Ricci)}$$

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- We have

$$\operatorname{Ric}(v) = \operatorname{pr}_H \operatorname{Ric}^H \operatorname{pr}_H v, \quad v \in TM,$$

where $\operatorname{pr}_H\colon TM \to H$ is the projection with kernel V.

- **Example**

  Let (M,g) be a Riemannian manifold and $g_H = g|H$ such that $TM = H \oplus V$, and

$$g = g_H \oplus g_V \quad \text{and} \quad g_\varepsilon = g_H \oplus \frac{1}{\varepsilon} g_V, \quad \varepsilon > 0.$$

  Then

$$\mathrm{Ric}_{g_\varepsilon}(X,X) = \mathrm{Ric}^H(X,X) + \frac{1}{2\varepsilon}\langle J^2 X, X\rangle_H, \quad X \in \Gamma(H),$$

  where for $Z \in \Gamma(V)$, $J_Z \in \Gamma(\mathrm{End}\, TM)$ is defined by

$$\langle J_Z X, Y\rangle_{g_H} = \langle Z, T^\nabla(X,Y)\rangle_{g_V},$$

  and, for $Z_1, \ldots, Z_r$ any local vertical frame,

$$J^2 := \sum_{i=1}^{r} J_{Z_i} J_{Z_i}.$$

- *(Laplacian)* For a compatible connection ∇ as above let

$$\Delta^H = \mathrm{trace}_H \nabla^2_{\times,\times}$$

be the subelliptic Laplacian (the trace of the Hessian ∇^2 is taken over H with respect to the inner product g_H)

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- *(Sub-Riemannian Brownian)* A sub-Riemannian Brownian motion is a diffusion process X_t with generator Δ^H
- *(Stochastic development)* Let $X_0 = x$ then

$$dX_t = //_{0,t} \circ dB_t \quad \text{or} \quad dB_t = //_{0,t}^{-1} \circ dX_t$$

where B_t is a (classical) Brownian motion in H_x and

$$//_{0,t} := U_t \circ U_0^{-1} : H_x M \to H_{X_t} M$$

is *stochastic parallel transport along* of horizontal vectors along X (by construction isometries with respect to g_H).

Here U_t is the horizontal lift of X_t to the orthonormal frame bundle $\mathrm{O}(H)$ over M.

Functional inequalities

- Consider the semigroup generated by Δ^H:

$$P_t f = e^{t\Delta^H} f$$

We have

$$P_t f(x) = \mathbb{E}[f(X_t^x)\, \mathbb{1}_{\{t<\zeta(x)\}}], \quad x \in M.$$

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- Question: How is Ric^H related to functional inequalities for P_t?

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- (Bakry-Émery Ricci tensor)

$$\mathrm{Ric}^Z = \mathrm{Ric} - \nabla Z$$

where $\quad \mathrm{Ric}^Z(X, Y) := \mathrm{Ric}(X, Y) - \langle \nabla_X Z, Y \rangle$

Theorem (classical probabilistic representations)

Let $f \in \mathscr{B}_b(M)$ *and* $u(x,t) = P_t f(x)$ *be the (minimal) solution to*

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- *(Derivative formula) If $f \in C_b^1(M)$ and Ric^Z bounded below,*

$$(\nabla P_t f)(x) = \mathbb{E}\left[Q_t //_t^{-1} \nabla f(X_t^x)\right]$$

where the random transformations $Q_t \in \mathrm{End}(T_x M)$ are defined as solution to the pathwise ODE

$$dQ_t = -Q_t \,\mathrm{Ric}^Z_{//_t} \, dt, \quad Q_0 = \mathrm{id}_{T_x M}.$$

Here

$$\mathrm{Ric}^Z_{//_t} := //_t^{-1} \circ \mathrm{Ric}^Z_{X_t} \circ //_t \in End(T_x M)$$

is the equivariant representation of Ric^Z.

- In particular, if

$$\mathrm{CD}(K,\infty) \qquad \mathrm{Ric}^Z(v,v) \geq K|v|^2, \quad v \in TM,$$

for some constant K, then

$$|Q_t| \leq e^{-Kt}$$

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$$\text{(gradient estimate)} \qquad |\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad f \in C_b^1(M).$$

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- Actually, for $K \in \mathbb{R}$ the following two conditions are equivalent:

  - $\mathrm{CD}(K,\infty)$ $\qquad$ $\mathrm{Ric}(v,v) \geq K|v|^2, \quad v \in TM.$

  - (gradient estimate) $\qquad$ $|\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad f \in C_b^1(M).$

Well-known and classical: Let K be a real constant.
The following conditions are equivalent:

- **(Bakry-Émery lower curvature bound)**

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- **(Poincaré inequality)** for $p \in (1,2]$ and all $f \in C_c^\infty(M)$,

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- **(log-Sobolev inequality)** for all $f \in C_c^\infty(M)$,

$$P_t(f^2 \log f^2) - (P_t f^2) \log(P_t f^2) \leq \frac{2(1-e^{-2Kt})}{K} P_t |\nabla f|^2.$$

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Many other equivalent statements, e.g., transportation-cost inequalities; convexity properties of the entropy; Wang's dimension-free Harnack inequalities; Wang's log-Harnack inequalities, ...

Comparison with the sub-Riemannian case

- Example (Heisenberg group $\mathbb{H}^3$)

  $X, Y, Z \in \Gamma(\mathbb{H}^3), \quad [X, Y] = Z, \ [X, Z] = [Y, Z] = 0$

  $\mathbb{H} = \mathrm{span}(X, Y), \quad V = \mathbb{R} \cdot Z$

  Let

$$\Delta^H := X^2 + Y^2 \quad \text{and} \quad P_t f = (e^{t\Delta^H})f$$

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  Theorem (Hong-Quan Li, 2006)

  $$\exists\, C > 0, \quad |\nabla^H P_t f|_{g_H} \le C\, P_t |\nabla^H f|_{g_H}, \quad \forall f \in C_c^\infty(\mathbb{H}^3),$$

  where $\nabla^H f = \mathrm{pr}_H \nabla f$.

  The constant C must be strictly larger than 1!

Boundedness of Ric

The problem of characterizing boundedness of Ric in Riemannian geometry has been solved by A. Naber via analysis on path space:

$|\mathrm{Ric}| \le K$ (i.e. $-K \le \mathrm{Ric} \le K$ for some constant $K \ge 0$)
$\Longleftrightarrow$ certain functional inequalities on path space

III. Ricci curvature and analysis on path space

- For fixed $T > 0$, let $W^T = C([0,T];M)$ and

$$\mathscr{F}C_{0,T}^{\infty} = \Big\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : \\ 0 < t_1 < \ldots < t_n \le T,\ f \in C_c^{\infty}(M^n) \Big\}.$$

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- For $F \in \mathscr{F}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_{t_1},\dots,\gamma_{t_n})$, the intrinsic gradient is defined as

$$D_t^{//} F(X_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} //_{t,t_i}^{-1} \nabla^i f(X_{t_1},\dots,X_{t_n}), \quad t \in [0,T],$$

where ∇^i denotes the gradient with respect to the i-th component.

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- (L^2 gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left|\nabla_x \mathbb{E}[F(X^x_{[0,T]})]\right|^2 \leq e^{KT}\mathbb{E}^x\left[|D_0^{//}F|^2 + K\int_0^T e^{K(r-T)}|D_r^{//}F|^2\,dr\right].$$

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Important observation It is sufficient to check the estimates for very special $F \in \mathcal{F}C_0^\infty$. Namely:

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Important observation It is sufficient to check the estimates for very special $F \in \mathcal{F}C_0^\infty$. Namely:

(i) for $F(X^x_{[0,T]}) = f(X^x_t)$, and

(ii) for 2-point cylindrical functions of the form

$$F(X^x_{[0,T]}) = f(x) - \frac{1}{2}f(X^x_t)$$

From this observation, equivalence of the following two items follows:

(i) $|\mathrm{Ric}^Z| \le K$ for $K \ge 0$;

(ii) for $f \in C_c^\infty(M)$ and $t > 0$,

$$|\nabla P_t f|^2 \le e^{2Kt} P_t |\nabla f|^2 \quad \text{and}$$

$$\left|\nabla f - \frac{1}{2}\nabla P_t f\right|^2 \le e^{Kt}\,\mathbb{E}\left[\left|\nabla f - \frac{1}{2}//_{0,t}^{-1}\nabla f(X_t)\right|^2 + \frac{1}{4}\left(e^{Kt}-1\right)|\nabla f|^2(X_t)\right].$$

Path space characterization of pinched curvature

Let $F \in \mathscr{F}C^{\infty}_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$. Consider the gradients:

- *(intrinsic gradient)*

$$D^{//}_t F(X^x_{[0,T]}) = \sum_{i=1}^{n} \mathbb{1}_{\{t<t_i\}} //^{-1}_{t,t_i} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n});$$

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$$D_t F(X^x_{[0,T]}) = \sum_{i=1}^n 1\!\!1_{\{t<t_i\}} Q_{t,t_i} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n})$$

  where $Q_{t,r}$ takes values in the linear automorphisms of $T_{X^x_t} M$ satisfying for fixed $t \geq 0$:

$$\frac{dQ_{t,r}}{dr} = -Q_{t,r} \operatorname{Ric}^Z_{//_{t,r}}, \quad Q_{t,t} = \mathrm{id}; \quad r \geq t$$

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$$\frac{dQ_{t,r}}{dr} = -Q_{t,r} \, \mathrm{Ric}^Z_{//_{t,r}}, \quad Q_{t,t} = \mathrm{id}; \quad r \geq t$$

- *(balanced gradient)* For constants $k_1 \leq k_2$ let

$$\bar{D}_t^{//} F(X^x_{[0,T]}) = \sum_{i=1}^{n} \mathbb{1}_{\{t \leq t_i\}} e^{-\frac{k_1+k_2}{2}(t_i - t)} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n}).$$

Theorem (Path space characterization of pinched curvature)

The following conditions are equivalent:

1. $k_1 \le \mathrm{Ric}^Z \le k_2$;
2. (Gradient estimate) *for any* $F \in \mathscr{F}C_{0,T}^{\infty}$,

$$\left|\nabla_x \mathbb{E} F(X_{[0,T]}^x)\right| \le \mathbb{E}|\bar{D}_0^{//} F| + \frac{k_2-k_1}{2}\int_0^T e^{-k_1 s}\, \mathbb{E}|\bar{D}_s^{//} F|\, ds;$$

Theorem (Path space characterization of pinched curvature)

The following conditions are equivalent:

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2. (Gradient estimate) *for any* $F \in \mathscr{F}C^\infty_{0,T}$,
$$\left|\nabla_x \mathbb{E} F(X^x_{[0,T]})\right| \le \mathbb{E}|\bar{D}^{//}_0 F| + \frac{k_2-k_1}{2} \int_0^T e^{-k_1 s}\, \mathbb{E}|\bar{D}^{//}_s F|\, ds;$$
3. (Log-Sobolev inequality) *for any* $F \in \mathscr{F}C^\infty_{0,T}$ *and* $t_1 < t_2$ *in* $[0,T]$,
$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}] \log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\Big] \\
&\qquad - \mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}] \log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\Big] \\
&\le 2\int_{t_1}^{t_2} \left(1 + \frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)}\, ds\right) \\
&\qquad \times \left(\mathbb{E}|\bar{D}^{//}_t F|^2 + \frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)} \mathbb{E}|\bar{D}^{//}_s F|^2 ds\right) dt.
\end{aligned}$$

Theorem (continuation)

(iv) (Poincaré type inequality) *for* $F \in \mathscr{F}C^{\infty}_{0,T}$ *and* $t_1 < t_2$ *in* $[0, T]$,

$$\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_2}]^2\Big] - \mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_1}]^2\Big]$$
$$\leq \int_{t_1}^{t_2} \left(1 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)}\, ds\right)$$
$$\times \left(\mathbb{E}|\bar{D}_t^{//} F|^2 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)}\, \mathbb{E}|\bar{D}_s^{//} F|^2\, ds\right) dt.$$

- Let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

$$\mathcal{E}(F,F) = \mathbb{E}\left[\int_0^T |D_t^{//} F|^2 (X_{[0,T]})\, dt\right].$$

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- The log-Sobolev inequality or Poincaré inequality on path space can be used to derive spectral gap-lower bounds for the operator $\mathcal{L}$.

- It is well-known that a log-Sobolev inequality

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2\, H(T,k_1,k_2) \int_0^T |D_t^{//} F|^2(X_{[0,T]})\, dt$$

or a Poincaré inequality

$$\mathbb{E}\left[(F - \mathbb{E}[F])^2\right] \leq H(T,k_1,k_2) \int_0^T |D_t^{//} F|^2(X_{[0,T]})\, dt$$

for some explicit bound $H(T,k_1,k_2)$, give the spectral gap lower bound $H(T,k_1,k_2)^{-1}$ for the operator $\mathcal{L}$.

IV. Analysis on path space over sub-Riemannian manifolds

Let again ∇ be a partial connection on H, extended as above to a compatible connection on M.

Weitzenböck formula

- Consider the corresponding rough sub-Laplacian

$$L(\nabla) := \mathrm{trace}_H \nabla^2$$

  (on functions and 1-forms).

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- Would like to have a Weitzenböck type commutation formula of the form:

$$dLf = (L - \mathscr{R})df, \quad L = L(\nabla),$$

 where $\mathscr{R} \in \Gamma(\mathrm{End}(T^*M))$.

- Let $\hat{\nabla}$ be the adjoint connection to ∇, i.e.

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- **Proposition** Let L be a rough sub-Laplacian of a connection on M. There exists a vector bundle endomorphism

$$\mathscr{R} : T^*M \to T^*M$$

such that

$$(L - \mathscr{R})df = dLf, \quad f \in C^\infty(M),$$

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- In this case,

$$\mathscr{R} = \mathrm{Ric}^\nabla$$

where for $(\alpha, v) \in T^*M \oplus TM$,

$$\mathrm{Ric}^\nabla(\alpha)(v) = \mathrm{trace}_H R^\nabla(\cdot, v)\alpha(\cdot)$$

- **Proposition** (Weitzenböck formula)
  Then, for all $f \in C^\infty(M)$,

$$\left(L(\hat{\nabla}) - \mathscr{R}\right) df = dL(\hat{\nabla}) f = dL(\nabla) f = d\Delta^H f$$

Derivative formula

- Define $\hat{Q}_t = \hat{Q}_t(x) \in \mathrm{End}(T_xM)$ by

$$\frac{d}{dt}\hat{Q}_t = -\mathscr{R}_{\hat{//}_t}\hat{Q}_t\,, \quad \hat{Q}_0 = \mathrm{id}_{T_xM},$$

where $\mathscr{R} = \mathrm{Ric}^\nabla$ and $\mathscr{R}_{\hat{//}_t} = \hat{//}_t^{-1}\mathscr{R}\hat{//}_t$.

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- (Derivative formula) For $P_t = e^{t\Delta_H}$ and $f \in C^\infty(M)$, we have

$$dP_tf(x) = \mathbb{E}[\hat{Q}_t^* \hat{//}_t^{-1} df_{X_t(x)}]$$

Integration by parts on path space over a sub-Riemannian manifold

- Let (M, H, g_H) be a sub-Riemannian manifold equipped with a compatible connection ∇ and let

$$L = \mathrm{trace}_H \nabla^2_{\times,\times}$$

be defined as the trace of the Hessian ∇^2 over H with respect to the inner product g_H.

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- Assume that there is a decomposition $TM = H \oplus V$ such that
  - (a) both H and V are parallel with respect to ∇
  - (b) $\mathbf{T}(H, H) \subseteq V$
  - (c) $\mathbf{T}(H, V) = 0$.

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  a. both H and V are parallel with respect to ∇
  b. $\mathbf{T}(H, H) \subseteq V$
  c. $\mathbf{T}(H, V) = 0$.

  No choice of a Riemannian metric g on M satisfying $g|_H = g_H$ is required.

  Assume again that the complement V metric preserving.

- Let $X_t(x) \equiv X_t^x$ be the sub-Riemannian Brownian motion with generator L such that $X_0(x) = x$ and

$$dB_t^x = //_t^{-1} \circ dX_t(x), \quad B_0 = 0 \in H_x$$

Recall that B_t^x is a standard Brownian motion in H_x.

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- (Cameron-Martin space) Let

$$\mathbb{H} = \left\{ h \colon [0,T] \to H_x \text{ abs. cont.} \;\middle|\; \int_0^T |\dot{h}(t)|_{g_H}^2 dt < \infty \right\}$$

which becomes a Hilbert space with inner product

$$\langle h_1, h_2 \rangle_{\mathbb{H}} = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle_{g_H} dt.$$

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As usual, we write $\langle h, B^x \rangle_{\mathbb{H}} = \int_0^t \langle \dot{h}_s, dB_s^x \rangle_{g_H}$.

- For fixed $T > 0$, let $W^T = C([0,T]; M)$ and

$$\mathscr{F}C^\infty_{0,T} = \Big\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : \\ 0 < t_1 < \ldots < t_n \le T,\ f \in C^\infty_c(M^n) \Big\}$$

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be the class of smooth cylindrical functions on W^T.

- Let the operator $A_t : T_xM \to T_xM$ be given by

$$A_t = \int_0^t \mathbf{T}_{//_t}(\circ dB_t^x, \cdot)$$

(Note that $A_t(H_x) \subseteq V_x$ and $A_t(V_x) = 0$)

- For an adapted process h with paths in $\mathbb{H}$ let

$$S(h)_t = h_t + \int_0^t \mathbf{T}_{//_s}(\circ dB_s^x, h_s) \\ = h_t + \int_0^t dA_s h_s = \int_0^t (\mathrm{id} + A_t + A_s) dh_s.$$

- (Derivative operator on path space) For $F \in \mathscr{F}C_{0,T}^{\infty}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$ and $h \in \mathbb{H}$, let

$$D_h F(\gamma) = \sum_{i=1}^{n} \left\langle //_{t_i}^{-1} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n}), S(h)_{t_i} \right\rangle$$

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- Motivation For any horizontal curve γ on M (starting from x) with anti-development $u = \mathrm{Dev}^{-1}(\gamma)$ in $\mathbb{H}$, we have that

$$\left\{ \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Dev}(u + \varepsilon k) : \ k \in \mathbb{H} \right\} = \left\{ D_h|_{\gamma} : \ h \in \mathbb{H} \right\}$$

where the vector field D_h on path space is defined by

$$D_h|_{\gamma} = //_t^{-1} \left(h_t + \int_0^t \mathbf{T}_{//_s}(du_s, h_s) \right) = //_t^{-1} \left(h_t + \int_0^t dA_s h_s \right)$$

with $A_t = \int_0^t \mathbf{T}_{//_s}(du_s, h_s)$.

- Define $D_t F \in H_x$ such that

$$D_h F = \int_0^t \langle D_t F, \dot{h}_t \rangle_{g_H} dt.$$

It is straightforward to check that

$$D_t F := \sum_{i=1}^{n} \mathbb{1}_{\{t \le t_i\}} \sharp^H (\mathrm{id} + A_{t_i} - A_t)^* //_{t_i}^{-1} d_i f(\gamma_{t_1}, \dots, \gamma_{t_n}).$$

- The *gradient* DF is then given by the relation

$$\langle DF, h \rangle_{\mathbb{H}} = D_h F$$

Proposition (Integration by parts formula)

- For $F \in \mathscr{F}C^{\infty}_{0,T}$ and any adapted process h_t with paths in $\mathbb{H}$, we have

$$\mathbb{E}\left[\langle DF, h\rangle_{\mathbb{H}}\right] = \mathbb{E}\left[F \int_0^T \langle \dot{h}_t + \mathrm{Ric}_{//_t} h_t, dB_t\rangle_{g_H}\right].$$

- In particular, for $f \in C^{\infty}(M)$,

$$\mathbb{E}\left[\langle //_t^{-1} df_{X_t(x)}, S(h)_t\rangle\right] = \mathbb{E}\left[f(X_t(x)) \int_0^t \langle \dot{h}_s + \mathrm{Ric}_{//_s} h_s, dB_s\rangle_{g_H}\right].$$

Damped gradients and Quasi-invariance

- For $F \in \mathscr{F}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, define

$$\tilde{D}_t F(\gamma) := \sum_{i=1}^{n} 1_{\{t \le t_i\}} \sharp^H //_t^{-1} \hat{Q}^*_{t,t_i} \hat{//}^{-1}_{t,t_i} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n})$$

and

$$\tilde{D}_h F = \langle \tilde{D} F, h \rangle_{\mathbb{H}} = \int_0^T \tilde{D}_t F \, dh_t$$

Damped gradients and Quasi-invariance

- For $F \in \mathscr{F}C^{\infty}_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, define

$$\tilde{D}_t F(\gamma) := \sum_{i=1}^{n} 1_{\{t \le t_i\}} \sharp^H //_t^{-1} \hat{Q}^*_{t,t_i} \hat{//}^{-1}_{t,t_i} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n})$$

and

$$\tilde{D}_h F = \langle \tilde{D} F, h \rangle_{\mathbb{H}} = \int_0^T \tilde{D}_t F \, dh_t$$

- For adapted process h with paths in $\mathbb{H}$ one has

$$\mathbb{E}_x[\langle \tilde{D} F, h \rangle_{\mathbb{H}}] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{F(X^{\varepsilon}_{[0,T]}) - F(X_{[0,T]})}{\varepsilon} \right]$$

where

$$dX^{\varepsilon}_t = //^{\varepsilon}_t \circ dB_t + \varepsilon //^{\varepsilon}_t dh_t, \quad X^{\varepsilon}_0 = x$$

- Let $Q_t : T_xM \to T_xM$ be the solution of

$$Q_0 = \mathrm{id}_{T_xM}, \quad dQ_t = -\mathrm{Ric}_{//_t} Q_t dt$$

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- For any adapted process h_t with paths in $\mathbb{H}$, we then have

$$\langle \tilde{D} F, h \rangle_{\mathbb{H}} = \langle DF, k \rangle_{\mathbb{H}}, \quad k_t = Q_t \int_0^t Q_s^{-1} dh_s$$

and hence

$$\boxed{\mathbb{E}[\langle \tilde{D} F, h \rangle_{\mathbb{H}}] = \mathbb{E}\left[F \int_0^T \langle h, B^x \rangle_{\mathbb{H}} \right]}$$

V. Ricci curvature bounds in sub-Riemannian geometry

- (Derivative formula on path space)

  For $F \in \mathscr{F}C^{\infty}_{0,T}$ and $t > 0$, we have

$$D_t \mathbb{E}[F|\mathscr{F}_t] = \mathbb{E}[\tilde{D}_t F|\mathscr{F}_t]$$

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- (Semigroup derivative formula)

$$dP_t f(v) = \mathbb{E}\left[\left\langle //_t^{-1} df_{X_t(x)}, Q_t v + \int_0^t dA_r\, Q_r v \right\rangle\right], \quad v \in T_x M.$$

Theorem (Characterization of horizontal Ricci curvature)

Assume that V is metric preserving. For a non-negative constant K the following conditions are equivalent:

1. *(Bounded Ricci curvature) the horizontal Ricci curvature* $Ric^H = \mathrm{Ric}|_H \in \mathrm{End}(H)$ *is bounded by K, i.e.*

$$-K \le Ric^H \le K$$

2. *(Gradient estimate) for any* $F \in \mathscr{F}C_0^\infty$,

$$|D_0 \mathbb{E}_x[F]|_{g_H} \le \mathbb{E}_x\left[|D_0 F|_{g_H} + K \int_0^T e^{Ks} |D_s F|_{g_H}\, ds\right]$$

3. *(L^2 gradient estimate) for any* $F \in \mathscr{F}C_0^\infty$,

$$|D_0 \mathbb{E}_x[F]|_{g_H}^2 \le e^{-KT} \mathbb{E}_x\left[|D_0 F|_{g_H}^2 + K \int_0^T e^{Ks} |D_s F|_{g_H}^2\, ds\right]$$

Theorem (continuation)

iii. *(Log-Sobolev inequality)* *for any* $F \in \mathscr{F}C_0^\infty$ *and* $t > 0$ *in* $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F^2|\mathscr{F}_t]\log\mathbb{E}_x[F^2|\mathscr{F}_t]\Big] - \mathbb{E}_x[F^2]\log\mathbb{E}_x[F^2]$$
$$\leq 2\int_0^t e^{K(T-r)}\left(\mathbb{E}_x|D_rF|_{g_H}^2 + \frac{K}{2}\int_r^T e^{K(s-r)}\mathbb{E}_x|D_sF|_{g_H}^2\,ds\right)dr;$$

iv. *(Poincaré inequality)* *for any* $F \in \mathscr{F}C_0^\infty$ *and* $t > 0$ *in* $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F|\mathscr{F}_t]^2\Big] - \mathbb{E}_x[F]^2$$
$$\leq \int_0^t e^{K(T-r)}\left(\mathbb{E}_x|D_rF|_{g_H}^2 + \frac{K}{2}\int_r^T e^{K(s-r)}\mathbb{E}_x|D_sF|_{g_H}^2\,ds\right)dr.$$

- For non-symmetric bounds, i.e. $K_1 \leq \mathrm{Ric}^H \leq K_2$, one can give similar equivalent conditions redefining $\bar{D}_t F$ by

$$\bar{D}_t F = \sum_{i=1}^{n} \mathbb{1}_{\{t \leq t_i\}} e^{-\frac{K_1+K_2}{2}(t_i - t)} \sharp^H (\mathrm{id} + A_{t_i} - A_t)^* //_{t_i}^{-1} d_i F$$

- (Ornstein-Uhlenbeck operator)

  For $F, G \in \mathscr{F}C^{\infty}_{0,T}$ let

$$\mathscr{E}(F,G) = \mathbb{E}\langle DF, DG\rangle_{\mathbb{H}} = \mathbb{E}\left[\int_0^T \langle D_t F, D_t G\rangle_{g_H} \, dt\right].$$

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  Integration by parts formula implies the closability of the form.

- Let $\mathscr{L}$ be the generator of the the Dirichlet form

$$\mathscr{E}(F,F) = \mathbb{E}\left[\int_0^T |D_t F|_{g_H}^2\, dt\right].$$

  Let $\mathrm{gap}(\mathscr{L})$ denote its spectral gap.

Theorem Suppose there exists a constant $K \geq 0$ such that

$$\left|\mathrm{Ric}^H\right| \leq K.$$

Then

(i) (Poincaré inequality) for any $F \in \mathrm{dom}(\mathscr{E})$ with $\mathbb{E}[F] = 0$,

$$\mathbb{E}[F^2] \leq \frac{1}{2}(e^{KT} + 1)\mathscr{E}(F, F)$$

(ii) (Log-Sobolev inequality) for any $F \in \mathrm{dom}(\mathscr{E})$ with $\mathbb{E}[F^2] = 1$,

$$\mathbb{E}[F^2 \log F^2] \leq (e^{KT} + 1)\mathscr{E}(F, F)$$

(iii) (Spectral gap estimate) the following estimate holds:

$$\mathrm{gap}(\mathscr{L})^{-1} \leq \frac{1}{2}(e^{KT} + 1)$$

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