On the regularity of abnormal minimizers for rank 2 sub-Riemannian structures

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in collaboration with

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Let us fix some notation

- M smooth (i.e., C^∞) manifold
- D is a smooth distribution on M of rank k (locally $D_q = \mathrm{span}\{X_1(q), \ldots, X_k(q)\}$ with $X_1, \ldots, X_k$ smooth vector fields)
- D is *bracket generating* and we denote by s its step
- g smooth metric on $D \longrightarrow \ell(\gamma)$ length of any horizontal curve $\gamma \longrightarrow$ sub-Riemannian distance d
- a horizontal curve $\gamma : [0, T] \to M$ is a length-minimizer if $d(\gamma(0), \gamma(T)) = \ell(\gamma)$
- a geodesic is a locally length-minimizing horizontal curve

Let us fix some notation and state the main problem

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- a geodesic is a locally length-minimizing horizontal curve
- a horizontal curve is said to be of class C^n if it is C^n when re-parameterized by arc-length

Long standing question

Are all sub-Riemannian geodesics of class C^∞? Or at least C^1?

In control terms

At least locally (we are anyhow going to study local properties):

- horizontal curves are solutions of $\dot{\gamma}(t) = u_1(t)X_1(\gamma(t)) + \cdots + u_k(t)X_k(\gamma(t))$ for $u \in L^1_{\mathrm{loc}}$
- length-minimizers $\longrightarrow$ solutions of the optimal control problem
$$\int_0^T \|u(t)\| dt \to \min$$
with fixed endpoints

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with fixed endpoints
- equivalently, we take $|u| \le 1$ and consider
$$T \to \min$$
(i.e., we restrict to arclength parameterized horizontal curves)
- C^1 regularity of length-minimizers $\longrightarrow$ C^0 continuity of u_{opt}

1st order conditions: Pontryagin maximum principle

$h_i : T^*M \to \mathbb{R}$ defined by

$$h_i(\lambda) = \langle \lambda, X_i(\pi(\lambda)) \rangle$$

where $\pi(\lambda) = q$ if $\lambda \in T_q^*M$

Theorem (PMP)

Let $\gamma : [0, T] \to M$ be an arclength parametrized length-minimizer. There exists a Lipschitz continuous curve $t \mapsto \lambda(t) \in T^*_{\gamma(t)}M \setminus \{0\}$ such that (at least) one of the following conditions is satisfied:

(N) $\dot{\lambda}(t) = \vec{H}(\lambda(t))$ *where* $H = \frac{1}{2}\sum_{i=1}^k h_i^2$

(A) $\dot{\lambda}(t) = \sum_{i=1}^k u_i(t)\vec{h}_i(\lambda(t))$ *and* $h_i(\lambda(t)) \equiv 0$ *for* $i = 1, \dots, k$

- case (N) implies that $\lambda(t)$ is C^∞: normal extremals are C^∞
- case (A) $\longrightarrow$ abnormal extremals may be just Lipschitz continuous
- notice that for abnormal extremals $\lambda(t) \in D^\perp_{\gamma(t)}$ for all t

Second order conditions: Goh

- The question of regularity of length-minimizers is reduced to length-minimizers that are strictly abnormal: there exist a lift satisfying (A) and no lift satisfying (N)
- Second order optimality condition for a strictly abnormal length-minimizer:

  $$\lambda(t) \in (D^2_{\gamma(t)})^\perp = \mathrm{span}\{X_i(\gamma(t)), [X_i, X_j](\gamma(t)) \mid i, j = 1, \ldots, k\}^\perp$$

  Goh condition

A selection of previous results

- If (M, D, g) has step 2, there are no strictly abnormal length-minimizers
  $\rightarrow$ every length-minimizer is C^∞
- If (M, D, g) has step 3, the situation is already more complicated. A positive answer is known at least for Carnot groups ($\rightarrow$ length-minimizers are C^∞) [Leonardi–Le Donne–Monti–Vittone, '13]
- If (M, D, g) is real-analytic, every length-minimizer is real-analytic on an open dense subset of its interval of definition [Sussmann, '14]

These results hold with no restriction on the rank of D

Theorem (Chitour–Jean–Trélat, '06)

For an open and dense set family of SR structures with rank ≥ 3 there are no strictly abnormal length-minimizers

Corners are not minimizers

Theorem (Hakavuori–Le Donne, '16)

Let M be a sub-Riemannian manifold. Let $T > 0$ and let $\gamma : [-T, T] \to M$ be a horizontal curve with left and right derivatives $\dot\gamma^-(0)$ and $\dot\gamma^+(0)$. If

$$\dot\gamma^+(0) \neq \dot\gamma^-(0)$$

then γ is not a length-minimizer.

$\to$ the absence of corners in abnormal length-minimizers cannot be deduced solely by PMP and Goh condition: more tricky ad hoc variations

- first proposed by in [Leonardi–Monti, '08] where the case rank$= 2$, step≤ 4 is handled
- systematic deployment in [Hakavuori–Le Donne, '16]

A result for rank 2 distributions

Proposition (Liu-Sussmann, '95, Agrachev-Sarychev, '95)

Let $D = \mathrm{span}\{X_1, X_2\}$ *and* γ *be a strictly abnormal length-minimizer. If* $|\langle \lambda(t), [X_1, [X_1, X_2]](\gamma(t))\rangle|$ $+|\langle \lambda(t), [X_2, [X_1, X_2]](\gamma(t))\rangle|$ *never vanishes, then* γ *is* C^∞

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Idea of the proof:

- $\dot\gamma(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t))$, $u \in L^\infty([0, T], \mathbb{S}^1)$
- set $h_{i_1\cdots i_m}(t) = \langle\lambda(t), [X_{i_1}, \ldots, [X_{i_m-1}, X_{i_m}]](\gamma(t))\rangle$
  then $h_1, h_2, h_{12} \equiv 0$ and

$$\dot h_{i_1\cdots i_m}(t) = u_1(t)h_{1i_1\cdots i_m}(t) + u_2(t)h_{2i_1\cdots i_m}(t)$$

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- in particular $0 = \dot h_{12} = u_1 h_{112} + u_2 h_{212}$

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- since $\zeta := (h_{212}, -h_{112})$ is not zero then $u = \pm\zeta/\|\zeta\|$

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- $\dot\zeta = \pm A\frac{\zeta}{\|\zeta\|}$ with $A = \begin{pmatrix} h_{1212} & h_{2212} \\ -h_{1112} & -h_{2112} \end{pmatrix}$
  optimality $\rightarrow$ $u = \zeta/\|\zeta\|$ or $u = -\zeta/\|\zeta\|$ on $[0, T]$

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  optimality $\rightarrow$ $u = \zeta/\|\zeta\|$ or $u = -\zeta/\|\zeta\|$ on $[0, T]$
- bootstrap argument $\rightarrow$ u and γ are C^∞

Resuming

The bracket generating distribution D defines a flag of subspaces

$$D_x = D_x^1 \subset D_x^2 \subset D_x^3 \subset \cdots \subset D_x^r = T_x M \qquad x \in M$$

This induces a dual decreasing sequence of subspaces of $T_x^* M$

$$\{0\} = (D_x^r)^\perp \subset \cdots \subset (D_x^4)^\perp \subset (D_x^3)^\perp \subset (D_x^2)^\perp \subset (D_x^1)^\perp \subset T_x^* M$$

- By construction, any abnormal lift satisfies $\lambda(t) \in (D^1)^\perp$

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When D has rank 2

- if $\lambda(t)$ does not cross $(D^3)^\perp$, then the length-minimizer is C^∞ [Liu-Sussmann, '95, Agrachev-Sarychev, '95]

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When D has rank 2

- if $\lambda(t)$ does not cross $(D^3)^\perp$, then the length-minimizer is C^∞ [Liu-Sussmann, '95, Agrachev-Sarychev, '95]

Question?

What can we say if $\lambda(t)$ crosses $(D^3)^\perp$?

Main result: one step further

Theorem (D. Barilari, Y. Chitour, F. Jean, D. Prandi, M. S., JMPA, 2020)

Let (D, g) be a rank 2 sub-Riemannian structure on M. Assume that $\gamma : [0, T] \to M$ is an arclength parameterized abnormal minimizer. If γ admits a lift satisfying $\lambda(t) \notin (D^4)^\perp$ for every $t \in [0, T]$, then γ is C^1.

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Corollary

Assume that the sub-Riemannian structure has rank 2 and step at most 4. Then all length-minimizers are of class C^1.

- The assumption in the corollary implies $\dim M \leq 8$
- if, in addition, all brackets of length 5 of X_1 and X_2 vanish, then length-minimizers can be proved to be C^∞
- [Chitour–Jean–Trélat, '06]: for generic rank 2 sub-Riemannian structures, $\lambda(t) \notin (D^4)^\perp$ for almost every $t \in [0, T]$

Basic idea of the proof

- abnormal length-minimizer $\gamma : [0, T] \to M$ parameterized by arclength:
  there exists $u \in L^\infty([0, T], \mathbb{S}^1)$ such that

$$\dot\gamma(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)), \quad \text{a.e. } t \in [0, T]$$

- we prove that the control $u = (u_1, u_2)$ admits left/right limits at every $t \in [0, T]$
- the two limits must coincide since corners are not minimizing
- then u is continuous, and the curve γ is C^1

Setting

We already noticed that, setting

$$h_{i_1\cdots i_m} = \langle \lambda, [X_{i_1}, \cdots, [X_{i_{m-1}}, X_{i_m}]]](\gamma)\rangle : [0, T] \to \mathbb{R}$$

$$\zeta = (h_{212}, -h_{112})$$

then $u = \frac{\zeta}{\|\zeta\|}$ or $u = -\frac{\zeta}{\|\zeta\|}$ on every interval where $\zeta \neq 0$ and

$$\dot{\zeta} = \pm A \frac{\zeta}{\|\zeta\|}, \qquad A = \begin{pmatrix} h_{1212} & h_{2212} \\ -h_{1112} & -h_{2112} \end{pmatrix}$$

u is C^∞ on the maximal open set where $\zeta \neq 0$

Problems arise when

- the vector ζ vanishes somewhere on $[0, T]$, i.e., the lift $\lambda(t)$ crosses $(D^3)^\perp$

Basic observations

- the non-autonomous matrix $A(t) = \begin{pmatrix} h_{1212}(t) & h_{2212}(t) \\ -h_{1112}(t) & -h_{2112}(t) \end{pmatrix}$ is Lipschitz continuous on the whole interval $[0, T]$
- A constant when all brackets of length 5 of X_1 and X_2 vanish
- A has zero trace since

  $$h_{1212} = \langle \lambda, [X_1, [X_2, [X_1, X_2]]] \rangle = \langle \lambda, [X_2, [X_1, [X_1, X_2]]] \rangle = h_{2112}$$

  by the Jacobi identity
- the condition "$\lambda(t)$ does not cross $(D^4)^\perp$" translates into a property for A:

Lemma

Assume that $\lambda(t) \notin (D^4_{\gamma(t)})^\perp$ for every $t \in [0, T]$. If $\zeta(t_0) = 0$ for some $t_0 \in [0, T]$, then $A(t_0) \neq 0$

Main technical result

The key point of the whole argument is the following statement

Theorem

Let (t_0, t_1) be a maximal interval on which $\zeta \neq 0$. Assume that $\zeta(t_1) = 0$ and $A(t_1) \neq 0$. Then $u(t) = \frac{\zeta(t)}{\|\zeta(t)\|}$ has a limit as $t \uparrow t_1$.

Recall $\dot{\zeta} = A(t)\frac{\zeta}{\|\zeta\|}$ or $\dot{\zeta} = -A(t)\frac{\zeta}{\|\zeta\|}$ on (t_0, t_1)
ζ time reparameterization of a trajectory of $\dot{z} = Az$

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ζ time reparameterization of a trajectory of $\dot{z} = Az$

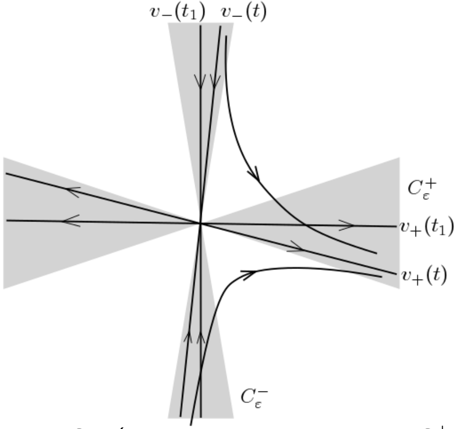
Since $A(t_1) \neq 0$ and $\mathrm{trace}(A(t_1)) = 0$ we have the following cases:

- $\det A(t_1) < 0 \rightarrow$ two real eigenvalues, limit occur along the stable or the unstable manifold for $A(t_1)$
- $\det A(t_1) > 0 \rightarrow$ excluded by general arguments
- $\det A(t_1) = 0 \rightarrow$ the difficult case!

For the case $\det A(t_1) \neq 0$ see also [Zelenko 1999]

Step 1: $u(t) = \frac{\zeta(t)}{\|\zeta(t)\|}$ has a limit when $\det A(t_1) < 0$

- $v_\pm(t)$ eigenvectors of $A(t)$ ($+ \longrightarrow$ eigenvalue of positive sign)
- $C_\varepsilon^\pm$ cone of width ε around $v_\pm(t_1)$



- a trajectory $\zeta \to 0$ stays in C_ε^- (otherwise it ends up in C_ε^+ and goes to infinity)
- take $\varepsilon \to 0$

Step 2: exclude the case $\det A(t_1) > 0$

Lemma

Let (t_0, t_1) be a maximal open interval of $[0, T]$ on which $\zeta \neq 0$ and assume that $\zeta(t_1) = 0$. Then $\det A(t_1) \leq 0$.

Assume by contradiction that $\det A(t_1) > 0$. Since $\mathrm{tr}\, A(t_1) = 0$, there exists $P \in \mathrm{GL}(2, \mathbb{R})$ such that

$$P^{-1}A(t_1)P = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

Then we can write

$$P^{-1}A(t)P = \begin{pmatrix} -\varepsilon_1(t) & -a + \varepsilon_2(t) \\ a + \varepsilon_3(t) & \varepsilon_1(t) \end{pmatrix}$$

- $\varepsilon_1, \varepsilon_2, \varepsilon_3$ Lipschitz continuous
- $\varepsilon_i(t) \to 0$ as $t \to t_1$, $i = 1, 2, 3$
- we should prove that ζ cannot spiral to zero in finite time

- Consider a time rescaling and polar coordinates

$$P^{-1}\zeta(t) = \rho(s(t))e^{i\theta(s(t))} \qquad s(t) := \int_{t_0}^{t} \frac{|P^{-1}\zeta(\tau)|}{|\zeta(\tau)|} d\tau$$

  and notice that $s(t_1) < +\infty$
- equation $\dot{\zeta} = A\frac{\zeta}{\|\zeta\|}$ can be rewritten with respect to the new time s

$$\begin{cases} \rho' = (-\alpha \cos 2\theta + \mu \sin 2\theta) \\ \theta' = \frac{1}{\rho}(\alpha \sin 2\theta + \mu \cos 2\theta + \eta) \end{cases}$$

- direct computations: $w := \alpha \sin 2\theta + \mu \cos 2\theta + \eta \rightarrow \exists M > 0$ such that

$$(\rho^2 w)' \geq -M\rho^2 w$$

  in a left-neighborhood of $s(t_1)$
- it is impossible for $\rho^2 w$ to tend to zero as $s \to s(t_1) < +\infty$

Recap

Recall that we want to prove the following result:

Theorem

Let (t_0, t_1) be a maximal interval on which $\zeta \neq 0$. Assume that $\zeta(t_1) = 0$ and $A(t_1) \neq 0$. Then $u(t)$ has a limit as $t \uparrow t_1$.

Since $A(t_1) \neq 0$ and $\mathrm{trace}(A(t_1)) = 0$ we have the following cases:

- $\det A(t_1) < 0 \rightarrow$ "distinct opposite eigenvalues", there exists a limit for u along the stable or unstable manifold of $A(t_1)$
- $\det A(t_1) > 0 \rightarrow$ excluded by general arguments
- $\det A(t_1) = 0$: to be done

Step 3.0: the case A constant and $\det A = 0$

- $\zeta = (h_{212}, -h_{112})$ satisfies

$$\dot{\zeta} = Au \qquad A = \begin{pmatrix} h_{2112} & h_{2212} \\ -h_{1112} & -h_{2112} \end{pmatrix}$$

- $u = \pm\frac{\zeta}{\|\zeta\|}$ when $\zeta \neq 0$
- A is a constant matrix with $\mathrm{tr}(A) = 0$, $\det(A) = 0$, $A \neq 0$ $\longrightarrow$ simiar to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
- $\zeta(\bar{t}) = 0$ at some $\bar{t} \in [0, T]$

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- $\zeta(\bar{t}) = 0$ at some $\bar{t} \in [0, T]$
- then necessarily $\zeta \equiv 0$ on $[0, T]$

Step 3.0: A constant, nilpotent, and $\zeta \equiv 0$

$$\dot{\zeta} = Au, \qquad A = \begin{pmatrix} h_{2112} & h_{2212} \\ -h_{1112} & -h_{2112} \end{pmatrix}$$

- $u(t)$ is in the kernel of A for a.e. $t \in [0, T]$
- A has one-dimensional kernel $\ker A = \mathrm{span}\{\bar{u}\}$, where $\bar{u}$ has norm one
- Then $u(t) = \sigma(t)\bar{u}$ for a.e. $t \in [0, T]$, with $\sigma(t) \in \{-1, 1\}$ and

  $$\dot{\gamma}(t) = \sigma(t) X_{\bar{u}}(\gamma(t)), \quad \text{a.e. } t \in [0, T],$$

  with $X_{\bar{u}}$ a constant vector field
- By length-minimality, σ is constant, and u is constant (hence smooth)

Step 3.0: A constant, nilpotent, and $\zeta \equiv 0$

$$\dot{\zeta} = Au, \qquad A = \begin{pmatrix} h_{2112} & h_{2212} \\ -h_{1112} & -h_{2112} \end{pmatrix}$$

- $u(t)$ is in the kernel of A for a.e. $t \in [0, T]$
- A has one-dimensional kernel $\ker A = \mathrm{span}\{\bar{u}\}$, where $\bar{u}$ has norm one
- Then $u(t) = \sigma(t)\bar{u}$ for a.e. $t \in [0, T]$, with $\sigma(t) \in \{-1, 1\}$ and

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  with $X_{\bar{u}}$ a constant vector field
- By length-minimality, σ is constant, and u is constant (hence smooth)
- The argument extends easily to the case where A is non-autonomous and $\zeta \equiv 0$ on an interval

Step 3.1: The case $\det A(t_1) = 0$

- The limit matrix $A(t_1)$ is similar to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, but we cannot exclude in this case that $\lim_{t \uparrow t_1} \zeta(t) = 0$
- we have to prove that $\exists \lim_{t \uparrow t_1} \frac{\zeta}{\|\zeta\|}$

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$$PA(t_1)P^{-1} = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)$$

Fix $t_* \in (t_0, t_1)$ and consider the new time

$$s = \int_{t_*}^{t} \frac{d\tau}{|\zeta(\tau)|}$$

With $\zeta \mapsto P^{-1}\zeta$, $A \mapsto P^{-1}AP$ we have
$A(s) = \left(\begin{array}{cc} -\varepsilon_1(s) & 1 + \varepsilon_2(s) \\ \varepsilon_3(s) & \varepsilon_1(s) \end{array} \right)$ where $\varepsilon_i \to 0$ as $s \to +\infty$, $i = 1, 2, 3$

Polar coordinates

ρ, θ polar coordinates for ζ

$$\frac{\rho'}{\rho} = \sin\theta\cos\theta + \varepsilon_\rho \qquad \theta' = -\sin^2\theta + \varepsilon_\theta$$

where $\varepsilon_\theta, \varepsilon_\rho \to 0$ as $s \to +\infty$

The dynamics of θ are a perturbation via ε_θ of

$$\theta' = -\sin^2\theta$$

$\to$ Two equilibria $\theta = 0, \pi$ joined by two clock-wise oriented heteroclinic trajectories

A dichotomy

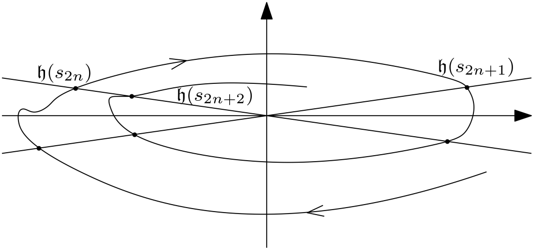
Lemma

We have the following dichotomy:

1. $\theta \to 0 \mod \pi$ *as* $s \to +\infty$
2. $\theta \to -\infty$ *as* $s \to +\infty$

Moreover, in case 2, for any $0 < \varepsilon < \pi/2$ there exist $(s_n)_{n\in\mathbb{N}}$ tending monotonically to infinity and such that

$$\theta(s_{2n}) = \pi - \varepsilon \mod 2\pi \qquad \theta(s_{2n+1}) = \varepsilon \mod 2\pi$$
$$\theta'(s) < 0 \quad \forall s \in [s_{2n}, s_{2n+1}]$$



The case of spirals

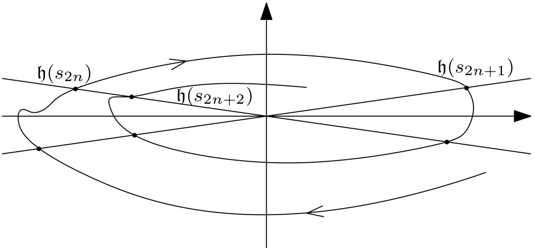
Out of a cone of width ε around the horizontal axis the dynamics are arbitrarily close to those of $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x$

By some computations one can estimate that, it t_n is s_n in the t time-scale, then

$$\frac{1}{t_{2n+1} - t_{2n}} \int_{t_{2n}}^{t_{2n+1}} \sin(\theta(t))dt \sim \varepsilon$$

for n large

Recall that $u(t) = \frac{\zeta(t)}{\|\zeta(t)\|} = (\cos\theta(t), \sin\theta(t))$



To the Limit!

- Fix $(\varepsilon^{(k)})_{k\in\mathbb{N}}$ decreasing to 0
- Associate with each $\varepsilon = \varepsilon^{(k)}$ a sequence $(s_n^{(k)})_{n\in\mathbb{N}}$ as in the previous slide
- Diagonal procedure $\longrightarrow$ sequence of times $(\varsigma_\ell)_{\ell\in\mathbb{N}}$ with $[\varsigma_{2\ell}, \varsigma_{2\ell+1}] = [s^{(\ell)}_{2n_\ell}, s^{(\ell)}_{2n_\ell+1}]$ and $\varsigma_\ell \to +\infty$
- Let τ_ℓ be ς_ℓ in the original time scale
- For every $\ell \geq 0$ consider the function $u_\ell \in L^\infty([0,1], \mathbb{S}^1)$ defined by

$$u_\ell(r) = u(\tau_{2\ell} + r(\tau_{2\ell+1} - \tau_{2\ell}))$$

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- By the weak-$\star$ compactness of bounded subsets of $L^\infty([0,1], \mathbb{R}^2)$, we can assume without loss of generality that $u_\ell \to u_\star$ in the weak-$\star$ topology
- Since $\frac{1}{\tau_{2\ell+1}-\tau_{2\ell}} \int_{\tau_{2\ell}}^{\tau_{2\ell+1}} u_2(r)dr \to 0$ we have that $(u_\star)_2 \equiv 0$. Moreover $(u_\star)_1$ has non-constant sign

Lemma (Blow up)

$u_\star$ optimal control for a curve in the nilpotent approximation to M at $\gamma(t_1)$

→ Contradiction with $(u_\star)_2 \equiv 0$ and $\mathrm{sign}[(u_\star)_1]$ non-constant

- Spirals towards a point at which $\det A(t_1) = 0$ are not length-minimizers
- There exists a left/right limit for the control u

This solves the cases where ζ vanishes at isolated points!

One still has to treat

- Segments where ζ identically vanishes (→ same as for A constant)
- Accumulation of points where ζ is not zero (→ adaptation)

It gets technical...

Conclusions and natural questions

Main result

C^1 regularity for curves that do not enter in $(D^4)^\perp$

- Can be extended to deeper singularity in rank 2? e.g. $(D^5)^\perp$
  - dynamical systems of higher order
- Can be extended to rank > 2?
  - maybe, but no more available tools from 2D ODEs
- Can one obtain (or disproof) further regularity than C^1?
  - not evident

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THANK YOU FOR THE ATTENTION