Localization and uniformity of asymptotics for sub-Riemannian heat kernels

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Joint work with Robert W. Neel

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Setting

Let M be a complete smooth sub-Riemannian manifold endowed with a smooth measure μ, allowing to define a Laplacian Δ and the sub-Riemannian heat equation

$$\partial_t f = \Delta f$$

We discuss small time asymptotics of the heat kernel (off the diagonal)

$$p_t(x, y) \qquad t \in \mathbb{R}^+, x, y \in M$$

In a series of papers, Barilari–Boscain–Neel and collaborators showed how to use Molchanov method to transform geometric information on sR manifolds into heat kernel asymptotics.

▶ We wish to reexplore these ideas, from foundations to applications, limitations and new extensions.

Molchanov method

Info on the uneven spatial distribution of heat $\longrightarrow$ Asymptotic expansions

The heat kernel doubles as a transition density, which implies the formula

$$p_t(x, y) = \int_M p_{t/2}(x, z) p_{t/2}(z, y) \, \mathrm{d}\mu(z)$$

If we are able to say that most of the heat is localized in $B \subset M$, this becomes

$$p_t(x, y) = \int_B p_{t/2}(x, z) p_{t/2}(z, y) \, \mathrm{d}\mu(z) + \text{remainder}$$

Question:
- Where is the heat located?

Léandre asymptotics

Theorem (Léandre asymptotics)

For any compact $K \subset M$, uniformly for $x, y \in K$,

$$\lim_{t \searrow 0} -4t \log p_t(x, y) = d^2(x, y)$$

Further, for any multi-index α and non-negative integer l, uniformly for $x, y \in K$,

$$\limsup_{t \searrow 0} 4t \log \left(\left| \partial_t^l Z_y^\alpha p_t(x, y) \right| \right) \leq -d^2(x, y)$$

(with Z_y^α denoting spatial derivative along a family of vector fields $(Z_1, \ldots, Z_m)$).

This theorem was shown by Léandre in 1987 for $M = \mathbb{R}^d$ and fields with bounded derivatives. See Bailleul–Norris 2018 for non-complete manifolds.

No assumptions need to be made on the pair (x, y).

Any compact can be covered with a finite number of charts on which Léandre asymptotics hold for pairs of points with distance less than some δ.

Thanks to the completeness assumption, we can set up $K \subset K' \subset K''$ compacts such that

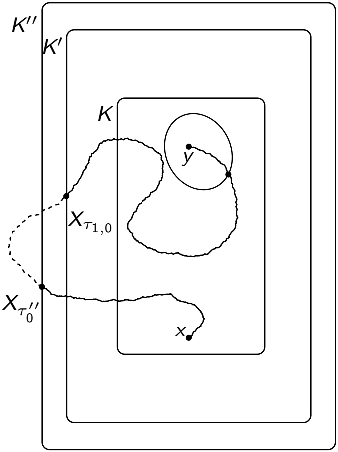
$$d(\partial K, \partial K') \geq \mathrm{diam}(K)/2$$
$$\text{and } d(\partial K', \partial K'') \geq \varepsilon$$

With X_t be the corresponding diffusion process starting from $x \in K$.

With $\rho \leq \delta$, we set the hitting times $\tau_{i,k} \leq \tau_i'' \leq \tau_{i+1,0}$ such that

$$X_{\tau_i''} \in \partial K'', X_{\tau_{i+1,0}} \in \partial K, d(X_{\tau_{i,k}}, X_{\tau_{i,k+1}}) = \rho,$$

we obtain coarse estimates on $p_t(x, y)$.



Localization

If the border of the larger compact is never hit, the diffusion might as well be zero, hence a splitting of the cases which leads to a localization theorem.

Theorem

For any compact $K \subset M$ and $\varepsilon > 0$, let K' be a compact such that $K \subset K'$ and $\mathrm{dist}(K, \partial K') > \mathrm{diam}(K)/2 + \varepsilon$.
If $p_t^{K'}$ is the heat kernel on K' with Dirichlet boundary conditions, then uniformly for $x, y \in K$,

$$p_t(x,y) = p_t^{K'}(x,y) + O\left(\exp\left(-\frac{(\mathrm{diam}(K)+\varepsilon)^2}{4t}\right)\right).$$

Where is the heat located for small time?

Since

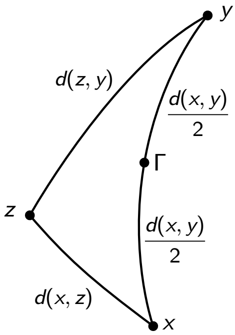
$$p_t(x,y) \leq C \exp\left(-\frac{d(x,y)^2}{4t}\right)$$

For all z in M,

$$p_{t/2}(x,z)p_{t/2}(z,y) \leq C \exp\left(-\frac{h_{x,y}(z)}{t}\right)$$

with $h_{x,y}$ the hinged energy functional

$$h_{x,y}(z) = \frac{1}{2}\left(d(x,z)^2 + d(z,y)^2\right).$$



- $h_{x,y}$ reaches its minimum when $d(x,z) = d(z,y) = d(x,y)/2$, the set Γ of midpoints of the geodesics between x and y

As soon as $d(x,z) \geq \frac{d(x,y)+\varepsilon}{2}$, then $d(z,y) \geq \frac{d(x,y)-\varepsilon}{2}$,

$$h_{x,y}(z) \geq \frac{(d(x,y)+\varepsilon)^2 + (d(x,y)-\varepsilon)^2}{8} \geq \frac{d(x,y)^2+\varepsilon^2}{4}$$

Thus the heat is located in Γ_ε a neighborhood of the midpoint set Γ:

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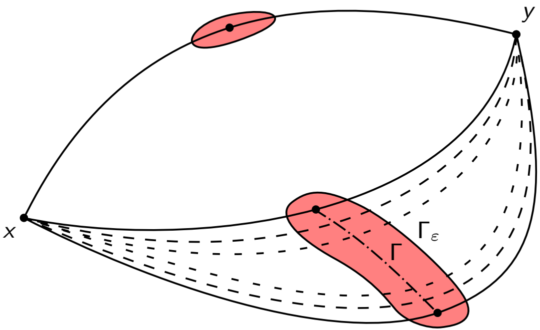

Molchanov method

Corollary (Molchanov method - V1)

Let $\mathcal{K}$ be a compact subset of M^2. Let l be a non-negative integer, α be a multi-index. For any $\varepsilon > 0$ small enough, we have uniformly for $(t, x, y) \in \mathbb{R}^+ \times \mathcal{K}$

$$\partial_t^l Z_y^\alpha p_t(x, y) = \int_{\Gamma_\varepsilon} \partial_t^l Z_y^\alpha \left(p_{t/2}(x, z) p_{t/2}(z, y)\right) \mathrm{d}\mu(z) + O\left(\mathrm{e}^{-\frac{d(x,y)^2+\varepsilon^2}{4t}}\right).$$

We know $p_t(x, y) \leq C\mathrm{e}^{-d(x,y)^2/4}$, and we also know that at $t/2$, the heat is located in Γ_ε with a proportional error bounded by the flat function $\mathrm{e}^{-\varepsilon^2/4t}$.

- Now we can build on this equation to get precise asymptotics.

Asymptotic expansions

Now we know that uniformly on compacts $\mathcal{K} \in M^2$, for t small enough,

$$p_t(x,y) \leq C \exp\left(-\frac{d(x,y)^2}{4t}\right)$$

The right-hand is completely flat, so what happens to $p_t(x,y) \times e^{\frac{d(x,y)^2}{4t}}$?

In fact, the answer depends heavily on the pair (x,y).

In 1988, Ben Arous gave an answer for pairs of points in $\mathbb{R}^d$, outside of a critical set $\mathcal{C} \in M^2$ such that $(x,y) \in \mathcal{C}$ if

- $x = y$ (points on the diagonal).
- There exists multiple length minimizing curves joining x and y.
- The unique geodesic joining x and y is conjugate.
- The unique geodesic joining x and y is not strongly normal (any subsection of a length minimizer is abnormal)

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Theorem (Uniform Ben Arous expansions)

There exist sequences of smooth functions $c_k : M^2 \setminus \mathcal{C} \to \mathbb{R}$, $r_k : \mathbb{R}^+ \times M^2 \setminus \mathcal{C} \to \mathbb{R}$, $k \in \mathbb{N}$, *such that for all* $n \in \mathbb{N}$, *for all* $(x, y) \in M^2 \setminus \mathcal{C}$, *for all* $t \in \mathbb{R}^+$

$$p_t(x, y) = t^{-d/2} e^{-\frac{d(x,y)^2}{4t}} \left(\sum_{k=0}^{N} c_k(x, y) t^k + t^{n+1} r_{n+1}(t, x, y) \right).$$

For any compact $\mathcal{K} \subset M^2 \setminus \mathcal{C}$, *any non-negative integer* l', *any multi-index* α', *there exists* t_0 *such that*

$$\sup_{0<t<t_0} \sup_{(x,y)\in\mathcal{K}} \left| \partial_t^{l'} Z_y^{\alpha'} r_{n+1}(t, x, y) \right| < \infty.$$

Additionally, $c_0(x, y) > 0$ *on* $M^2 \setminus \mathcal{C}$.

Again, Z_y^α denotes spatial derivative along a family of vector fields $(Z_1, \dots, Z_m)$.

- This results actually holds also for $\partial_t^l Z_y^\alpha p_t(x, y)$.

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$$\partial_t^l Z_y^\alpha p_t(x,y) = t^{-(|\alpha|+2l+d/2)} e^{-\frac{d(x,y)^2}{4t}} \left(\sum_{k=0}^{N} c_k(x,y) t^k + t^{n+1} r_{n+1}(t,x,y) \right).$$

For any compact $\mathcal{K} \subset M^2 \setminus \mathcal{C}$, any non-negative integer l', any multi-index α', there exists t_0 such that

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Additionally, if $\alpha = 0$, then $c_0(x,y) > 0$ on $M^2 \setminus \mathcal{C}$.

Applying the Molchanov method

We have to show that the result that holds on $\mathbb{R}^d$ also holds on sR-manifolds.

Cover a compact $\mathcal{K} \subset M^2 \setminus \mathcal{C}$ with a finite collection of charts: Ben Arous expansions hold uniformly on $\mathcal{K}$ for pairs of points (x, y) at distances less than δ.

Lemma

Let $\mathcal{K}$ be a compact subset of $M^2 \setminus \mathrm{Diag}$ such that all minimizers between pairs $(x, y) \in \mathcal{K}$ are strongly normal. Then $\{(x, z) \mid (x, y) \in \mathcal{K}, z \in \Gamma(x, y)\}$ is a compact set in $M^2 \setminus \mathcal{C}$.

If $d(x, y) \leq 2\delta - \varepsilon$ however, then

$$p_t(x, y) = \int_{\Gamma_\varepsilon} p_{t/2}(x, z) p_{t/2}(z, y) \, \mathrm{d}\mu(z) + O\left(\mathrm{e}^{-\frac{d(x,y)^2 + \varepsilon^2}{4t}}\right).$$

Ben Arous expansions hold for (x, z) and (z, y), where z is in Γ_ε.

If the expansions hold for (x, z) and (z, y) for $z \in \Gamma_\varepsilon$, then

$$p_t(x, y) = \int_{\Gamma_\varepsilon} \left[\left(\frac{2}{t}\right)^{d/2} e^{-\frac{d(x,z)^2}{t}} \phi_{t/2}(x, z) \right] \left[\left(\frac{2}{t}\right)^{d/2} e^{-\frac{d(z,y)^2}{t}} \phi_{t/2}(z, y) \right] d\mu(z) + O\left(e^{-\frac{d(x,y)^2+\varepsilon^2}{4t}} \right)$$

$$= \left(\frac{2}{t}\right)^{d} \int_{\Gamma_\varepsilon} e^{-\frac{h_{x,y}(z)}{t}} \Phi_{t/2}(x, y, z)\, d\mu(z) + O\left(e^{-\frac{d(x,y)^2+\varepsilon}{4t}} \right)$$

with $\phi_t(\cdot, \cdot) = \sum_{k=0}^{N} c_k(\cdot, \cdot) t^k + t^{n+1} r_{n+1}(t, \cdot, \cdot)$.

To prove the theorem, we have to show that

$$t^{-d/2} e^{\frac{d(x,y)}{4t}} \left(\frac{2}{t}\right)^{d} \int_{\Gamma_\varepsilon} e^{-\frac{h_{x,y}(z)}{t}} \Phi_{t/2}(x, y, z)\, d\mu(z) = \phi_t(x, y)$$

(In the sense that ϕ can be extended to pairs s.t. $d(x, y) \leq 2\delta - \varepsilon$.)

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$$+ O\left(\mathrm{e}^{-\frac{d(x,y)^2+\varepsilon^2}{4t}}\right)$$

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Laplace integrals asymptotics

A Laplace integral is an integral of the shape

$$I(t) = \int_{\mathbb{R}^d} \mathrm{e}^{-h(z)/t} \psi(z)\,\mathrm{d}z$$

with h the phase and ψ the amplitude, real smooth functions.

If h is well-behaved, we can give an expansion of $I(t)$ at $t = 0$ depending only on the jets of h and ψ at the minimas of h.

If $h(z) = z^{2k}$, (with $d = 1$), then for any N (see Estrada–Kanwal)

$$I(t) = \sum_{n=0}^{N} t^{\frac{2n+1}{2k}} \frac{\Gamma\left(\frac{2n+1}{2k}\right)}{k(2n)!} \partial_z^{2n}{}_{|z=0} \psi(z) + o\left(t^{\frac{2N+1}{2k}}\right)$$

If $h(z) = z_1^{2k_1} + \cdots + z_m^{2k_m}$, then we can get similar expressions by merging these expansions together.

Most importantly, the phase is responsible for the distribution of powers in t.

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Most importantly, the phase is responsible for the distribution of powers in t.

When you have a unique strongly normal and non-conjugate geodesic between x and y, then there exists a smooth change of coordinates on Γ_ε such that

$$h_{x,y}(z) = \frac{d(x,y)^2}{4} + z_1^2 + \cdots + z_n^2.$$

Hence a full expansion of

$$t^{-d/2}\left(\frac{2}{t}\right)^d \int_{\Gamma_\varepsilon} \mathrm{e}^{-\frac{h_{x,y}(z)-d(x,y)^2/4}{t}} \Phi_{t/2}(x,y,z)\,\mathrm{d}\mu(z)$$

by distributing the Laplace integral along the powers of t in

$$\Phi_{t/2}(x,y,z) = \sum_{k=0}^{n}\left(\frac{t}{2}\right)^k \left(\sum_{j=0}^{k} c_j(x,z)c_j(z,y)\right) + t^{n+1}R(t,x,y,z).$$

Taking into account uniformity of expansions, we extend the result to pairs of points in the compact $\mathcal{K}$ such that $d(x,y) \leq 2\delta - \varepsilon$.

- Repeating sufficiently many times, we get the expansions on the whole set $\mathcal{K}$ for any $\mathcal{K} \subset M^2 \setminus \mathcal{C}$.

Molchanov method: Laplace integral expression

Corollary (Molchanov method - V2)

Let $\mathcal{K}$ be a compact subset of $M^2 \setminus \mathrm{Diag}$ such that all minimizers between pairs $(x, y) \in \mathcal{K}$ are strongly normal.

Let $\phi : \mathbb{R}^+ \times M^2 \setminus \mathcal{C}$ the smooth function such that $\phi_t(x, y) = t^{d/2} e^{\frac{d(x,y)^2}{4t}} p_t(x, y)$.

For $\varepsilon > 0$ small enough, we have uniformly for all $(t, x, y) \in \mathbb{R}^+ \times \mathcal{K}$

$$p_t(x, y) = \left(\frac{2}{t}\right)^d \int_{\Gamma_\varepsilon} e^{-\frac{h_{x,y}(z)}{t}} \phi_{t/2}(x, z)\phi_{t/2}(z, y)\, d\mu(z) + O\left(e^{-\frac{d(x,y)^2+\varepsilon^2}{4t}}\right)$$

We are able to expand to $M^2 \setminus \mathrm{Diag}$ because when $\mathcal{K}$ is such a compact, the midpoint set still avoids $\mathcal{C}$.

Again similar results hold for derivatives of the heat kernel.

- Asymptotics on the cut locus can be given by following this logic.

Universal bounds of heat kernels

Barilari–Boscain-Neel 2012 showed that a priori comparisons of the hinged energy can lead to universal bounds.

For any pair $(x, y) \in M^2$ not on the diagonal, away from abnormals,

$$h_{x,y}(z) \geq \left(d(x,z) - \frac{d(x,y)}{2}\right)^2 + \frac{d(x,y)^2}{4}.$$

With $z_1 = d(x,z) - \frac{d(x,y)}{2}$, we then get

$$\int_{\Gamma_\varepsilon} \mathrm{e}^{-\frac{h_{x,y}(z)}{t}} \psi(z)\,\mathrm{d}\mu(z) \leq \mathrm{e}^{-\frac{d(x,y)^2}{4t}} \int_{\Gamma_\varepsilon} \mathrm{e}^{-\frac{z_1^2}{t}} \psi(z)\,\mathrm{d}\mu(z)$$

Likewise for any set of smooth coordinates $(z_1, \ldots, z_d)$ on Γ_ε, centered at a point of Γ, there exists $C > 0$ such that

$$h_{x,y}(z) \leq C\left(z_1^2 + \cdots + z_d^2\right) + \frac{d(x,y)^2}{4}$$

On compact sets, the bundle can be trivialized, allowing to uniformly control the change of variable $z_1 = d(x,z) - \frac{d(x,y)}{2}$.

(The same goes for families of coordinate charts centered at z)

Proposition (Universal bounds on heat kernel asymptotics)

Let $\mathcal{K}$ be a compact subset of $M^2 \setminus \mathrm{Diag}$ such that all minimizers between pairs $(x,y) \in \mathcal{K}$ are strongly normal.

For all non-negative integer l and multi-index α, there exists $C > 0$ such that for all $(x,y) \in \mathcal{K}$,

$$\partial_t^l Z_y^\alpha p_t(x,y) \leq \frac{C}{t^{|\alpha|+2l} t^{d-1/2}} e^{-\frac{d(x,y)^2}{4t}}.$$

In the case $\alpha = 0$ there also exists $C' > 0$ such that for all $(x,y) \in \mathcal{K}$,

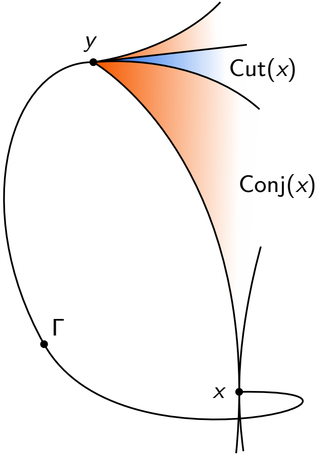
$$\frac{C'}{t^{2l} t^{d/2}} e^{-\frac{d(x,y)^2}{4t}} \leq \partial_t^l p_t(x,y).$$

Full expansions: the A-conjugate example

Assume there exists $x, y \in M$ such that the unique geodesic joining x and y is conjugate of type A_{2p-1}.

There exists a smooth change of coordinates on Γ_ε, centered at Γ, such that

$$h_{x,y}(z_1, \ldots, z_d) = \frac{d(x,y)^2}{4} + z_d^{2p} + \sum_{i=1}^{d-1} z_i^2.$$



Proposition (A-conjugate case)

Let $x, y \in M$ be such that the unique geodesic joining x and y is conjugate of type A_{2p-1}. Then for all non-negative integer l and multi-index α, there exists sequences $\nu_k \in \mathbb{R}$ and $\rho_k : \mathbb{R}^+ \to \mathbb{R}$, $k \in \mathbb{N}$, smooth, such that for all $n \in \mathbb{N}$,

$$\partial_t^l Z_y^\alpha p_t(x,y) = t^{-\left(|\alpha|+2l+\frac{d+1}{2}\right)+\frac{1}{2p}} e^{-\frac{d(x,y)^2}{4t}} \left(\sum_{k=0}^{n} \nu_k t^{k/p} + t^{\frac{n+1}{p}} \rho_{n+1}(t)\right),$$

and for $t_0 > 0$,

$$\sup_{(0,t_0)} |\rho_{n+1}(t)| < \infty.$$

Furthermore, if $\alpha = 0$, then $\nu_0 \neq 0$.

Conjugate cases are discussed in Boscain–Barilari–Charlot–Neel 2016

Other such examples are the Morse-bott case (Barilari–Boscain–Neel 2017, alternative methods by Inahama–Taniguchi 2017, Ludewig 2018).

Laplace asymptotics: diagonalizable vs non-diagonalizable phase

In all the presented cases, the hinged energy functional could be put in the diagonalized form

$$h_{x,y}(z) = z_1^{2k_1} + \cdots + z_m^{2k_m}$$

Then we can apply 1D full expansions results (of type Estrada–Kanwal) to get expansions that are sums of powers of t (with non-half integer powers only at the cut locus).

What type of expansions can be given for non-diagonalizable multivariate polynomials?

Discussions of Laplace integrals in Arnold's Singularities of differentiable maps points out that we should expect expansions of Laplace integrals of the form

$$\sum_{k,\alpha} c_{k,\alpha} t^{\alpha} \log^{k}(t)$$

with k integers and α belonging to a sequence of rationals.

Phase prescritption

Theorem

Let x, y be two points in $\mathbb{R}^d$, $d \geq 2$, and a and σ two positive real numbers. For any smooth $h : \mathbb{R}^{d-1} \to \mathbb{R}^+$ such that $h(0) = 0$ and h positive on $\partial B_0^{d-1}(a)$, there exists a Riemannian metric on $\mathbb{R}^d$ such that $\Gamma = \Gamma(x, y)$ is contained in a coordinate patch

$$(u_1, \dots, u_d) : U \to B_0^{d-1}(a) \times (-\delta, \delta)$$

and, for ε small enough,

$$h_{x,y|\Gamma_\varepsilon} = \frac{\sigma^2}{4} + h(u_1, \dots, u_{d-1}) + u_d^2.$$

In particular, Γ is given by the zero level set of h in $\{u_d = 0\}$.

Geometric construction: we deduce the shape of a front in Euclidean geometry that would produce such a $h_{x,y}$ near the midpoint, and complete it to full a Riemannian structure on $\mathbb{R}^d$.

Corollary

For any integers $d \geq 2$, $p \geq 1$, and $0 \leq k \leq d-2$, for any $\sigma > 0$, $(x, y) \in \mathbb{R}^d$, $x \neq y$, there exists a smooth Riemannian structure on $\mathbb{R}^d$ such that for some $C \neq 0$,

$$p_t(x, y) = e^{-\frac{\sigma^2}{4t}} t^{\frac{1}{2}+\frac{1}{2p}-d} \log(t)^k \big(C + o(1)\big).$$

This particular case is obtained by building on the map $x_1^{2p} \cdots x_{k+1}^{2p}$.

Example

Consider the map $h : \mathbb{R} \to \mathbb{R}^+$

$$h(u) = e^{-1/u^2} \sin^2(1/u^2) + \chi(u)$$

where χ is smooth, non-negative, 0 on $[-1, 1]$, 1 on $(-\infty, -2) \cup (2, \infty)$. This map has infinitely many zeros near 0. There is no hope of giving it an analytic normal form and thus obtaining info on potential expansions.

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Log-derivatives and cut locus detection

Outside the cut locus, Ben Arous expansions hold and imply

$$tZ_y^\alpha \log p_t(x,y) = -\frac{1}{4} Z_y^\alpha d^2(x,y) + O(t)$$

However, the squared distance is not twice differentiable at the cut locus.

- Classical idea: use derivatives of $\log p_t$ to detect the cut locus.

For that, we need a description of log derivatives that also holds on the cut.

We fold the remainder in the Molchanov method:

$$\left(\frac{t}{2}\right)^d p_t(x,y) = \int_{\Gamma_\varepsilon} \mathrm{e}^{-\frac{h_{x,y}(z)}{4t}} \phi_{t/2}(x,z) \underbrace{\psi^x_{t/2}(z,y)}_{\phi_{t/2}(z,y)+R_{t/2}(x,y)} \mathrm{d}\mu(z)$$

where $\partial_t^l Z_y^\alpha \left[R_{t/2}(x,y)\right] \leq C\mathrm{e}^{-\frac{\varepsilon^2}{8t}}$.

This is useful for computing $Z_y \log p_t(x,y) = \dfrac{Z_y p_t(x,y)}{p_t(x,y)}$:

$Z_y \log p_t(x,y) =$

$$\frac{\int_{\Gamma_\varepsilon} \left[-\frac{d(z,y)}{t}\psi^x_{t/2}(z,y) Z_y d(z,y) + Z_y \psi^x_{t/2}(z,y)\right] e^{-\frac{h_{x,y}(z)}{t}} \phi_{t/2}(x,z) d\mu(z)}{\int_{\Gamma_\varepsilon} e^{-\frac{h_{x,y}(z)}{t}} \phi_{t/2}(x,z)\psi^x_{t/2}(z,y)\, d\mu(z)}$$

In fact, for points in and out of the cut locus,

$$tZ_y \log p_t(x,y) = \mathbb{E}^{m_t}\left[-d(\cdot,y) Z_y d(\cdot,y)\right] + O(t)$$

$$\text{and} \quad tZ_yZ_y \log p_t(x,y) = \frac{1}{t}\operatorname{Var}^{m_t}\Big(d(\cdot,y)Z_yd(\cdot,y)\Big) + O(1).$$

To be compared with Ben Arous outside the cut

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$$\frac{\int_{\Gamma_\varepsilon} \left[-\frac{d(z,y)}{t} Z_y d(z,y) + \frac{Z_y \psi^x_{t/2}(z,y)}{\psi^x_{t/2}(z,y)}\right] e^{-\frac{h_{x,y}(z)}{t}} \phi_{t/2}(x,z)\psi^x_{t/2}(z,y) d\mu(z)}{\int_{\Gamma_\varepsilon} e^{-\frac{h_{x,y}(z)}{t}} \phi_{t/2}(x,z)\psi^x_{t/2}(z,y)\, d\mu(z)}$$

In fact, for points in and out of the cut locus,

$$t Z_y \log p_t(x,y) = \mathbb{E}^{m_t}\left[-d(\cdot,y) Z_y d(\cdot,y)\right] + O(t)$$

and $\quad t Z_y Z_y \log p_t(x,y) = \dfrac{1}{t} \operatorname{Var}^{m_t}\left(d(\cdot,y) Z_y d(\cdot,y)\right) + O(1).$

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$Z_y \log p_t(x,y) =$

$$Z_y \log p_t(x,y) = \mathbb{E}^{m_t}\left[-\frac{d(\cdot,y)}{t} Z_y d(\cdot,y) + Z_y\left(\log \psi^x_{t/2}(\cdot,y)\right)\right]$$

with m_t probability measure on Γ_ε of density $\dfrac{e^{-\frac{h_{x,y}(z)}{t}}\phi_{t/2}(x,z)\psi^x_{t/2}(z,y)}{\int_{\Gamma_\varepsilon} e^{-\frac{h_{x,y}(z)}{t}}\phi_{t/2}(x,z)\psi^x_{t/2}(z,y)\, d\mu(z)}$.

In fact, for points in and out of the cut locus,

$$\begin{aligned} tZ_y \log p_t(x,y) =& \mathbb{E}^{m_t}\left[-d(\cdot,y) Z_y d(\cdot,y)\right] + O(t) \\ \text{and} \quad tZ_y Z_y \log p_t(x,y) =& \frac{1}{t}\operatorname{Var}^{m_t}\left(d(\cdot,y) Z_y d(\cdot,y)\right) + O(1). \end{aligned}$$

To be compared with Ben Arous outside the cut

$$tZ_y Z_y \log p_t(x,y) = -\frac{1}{4} Z_y Z_y d^2(x,y) + O(t)$$

Theorem (Cut locus characterization)

Let x and y be such that all minimal geodesics from x to y are strongly normal, and $y \in \mathrm{Cut}(x)$. Then there exists a sequence $t_n \to 0$ and a vector field Z such that

$$\liminf_{n\to\infty} t_n^{1-\frac{1}{2d}} \left[t_n Z_y Z_y \log p_{t_n}(x,y)\right] > 0.$$

This holds when

$$\liminf_{n\to\infty} t_n^{-\frac{1}{2d}} \mathrm{Var}^{m_{t_n}} \left(d(\cdot,y) Z_y d(\cdot,y)\right) > 0$$

If $y \in \mathrm{Cut}(x)$ then for any sequence $t_n \to 0$, if m_{t_n} converges to a measure that is supported on one point only then x and y must be conjugate.

- If x and y are not conjugate $\liminf_{n\to\infty} \mathrm{Var}^{m_{t_n}} \left(d(\cdot,y) Z_y d(\cdot,y)\right) > 0$.
- Otherwise we have $\liminf_{n\to\infty} t_n^{-\frac{1}{2d}} \mathrm{Var}^{m_{t_n}} \left(d(\cdot,y) Z_y d(\cdot,y)\right) > 0$, where $t^{1-\frac{1}{2d}} = \left(t^{\frac{d-1}{2}+\frac{1}{4}}\right)^{d/2}$ indicates the comparison $h_{x,y}(z) \leq z_1^4 + z_2^2 + \cdots + z_d^2$.

Thank you for your attention