Heat Content Asymptotics for sub-Riemannian Manifolds

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Let (M, g) a complete Riemannian manifold and $\Omega \subset M$ a relatively compact domain. Let $u(t, x)$ be the solution to

$$\begin{cases} (\partial_t - \Delta_g)u(t,x) & = & 0, & \text{for } t > 0, \ x \in \Omega \\ u(0,x) & = & 1, & \text{for } x \in \Omega \\ u(t,x) & = & 0, & \text{for } t > 0, \ x \in \partial\Omega. \end{cases}$$

Definition (Riemannian heat content)

The ***Riemannian heat content*** associated with Ω is

$$Q_\Omega(t) = \int_\Omega u(t,x) d\mu_g(x), \qquad \forall t > 0.$$

$d\mu_g$ denotes the Riemannian measure.

Theorem (van den Berg, Gilkey - 1994)

Assume that $\partial\Omega$ is smooth. Then the Riemannian heat content $Q_\Omega(t)$ has a complete asymptotic expansion in $\sqrt{t}$ as $t \to 0$. In particular, it holds

$$Q_\Omega(t) = \mathrm{Vol}(\Omega) - \sqrt{\frac{4t}{\pi}}\sigma_g(\partial\Omega) + \frac{t}{2}\int_{\partial\Omega} H d\sigma_g + o(t^{3/2}), \quad \textit{as } t \to 0.$$

- $\mathrm{Vol}(\Omega) = \int_\Omega d\mu_g$, and $\sigma_g(\partial\Omega) = \int_{\partial\Omega} d\sigma_g$, where σ_g is the Riemannian perimeter measure,
- $H = \mathrm{Tr}(II)$ is the Riemannian mean curvature of $\partial\Omega$.

Small time asymptotics of $Q_\Omega \Leftrightarrow$ geometry of $\partial\Omega$.

Sub-Riemannian manifolds I

Let M be a smooth, connected m-dimensional manifold. A sub-Riemannian structure on M is defined by a set of N global smooth vector fields $X_1, \dots, X_N$, called a ***generating frame***.

The ***distribution*** is defined, at each point $x \in M$, as:

$$\mathcal{D}_x = \text{span}\{X_1(x), \dots, X_N(x)\} \subseteq T_x M.$$

We assume that the distribution is ***bracket-generating***, i.e.

$$\text{span}\{X_i(x), [X_i, X_j](x), \dots\} = T_x M, \qquad \forall x \in M.$$

The generating frame induces a ***norm*** on the distribution at x, namely

$$\|v\|_x = \inf \left\{ \sum_{i=1}^{N} u_i^2 \mid \sum_{i=1}^{N} u_i X_i(x) = v \right\}, \qquad \forall\, v \in \mathcal{D}_x,$$

which induces a scalar product g_x on $\mathcal{D}_x$ by polarization.
Length minimization for admissible curves defines the ***sub-Riemannian distance***, d_{SR}.

The sub-Laplacian

Let ω be a smooth outer measure on M.
The ***divergence*** of $X \in \Gamma(TM)$ is the smooth function defined by:

$$\mathrm{div}_\omega(X)\omega = d(i_X\omega), \qquad \forall\, X \in \Gamma(TM).$$

The ***horizontal gradient*** of $f \in C^\infty(M)$, is the horizontal vector field, such that:

$$g_x(\nabla f(x), v) = d_x f(v), \qquad \forall\, v \in \mathcal{D}_x,\ x \in M.$$

The ***sub-Laplacian*** is the operator, acting on $C^\infty(M)$

$$\Delta = \mathrm{div}_\omega \circ \nabla.$$

$\Rightarrow$ Define the ***sub-Riemannian heat content***.

Let $\Omega \subset M$ be a relatively compact subset of M with smooth boundary.

Definition (Characteristic Points)

We say that $x \in \partial\Omega$ is a ***characteristic point***, and write $x \in \mathrm{Char}(\partial\Omega)$, if

$$\mathcal{D}_x \subseteq T_x(\partial\Omega).$$

If Ω is a non-characteristic domain, then

- $u(t,x)$, solution to the Dirichlet problem, is smooth up to the boundary of Ω (Kohn, Nirenberg - 1965),
- signed distance from the boundary δ is smooth in a neighborhood of $\partial\Omega$ (Franceschi, Prandi, Rizzi - 2017)

Main result I

Let M be any sub-Riemannian manifold and let Ω be a relatively compact domain in M. Denote by

$$\nu = -\nabla\delta, \qquad \sigma = |i_\nu \omega|_{\partial\Omega}, \qquad H = \mathrm{div}_\omega(\nu)_{|\partial\Omega},$$

here $\delta \colon \bar{\Omega} \to \mathbb{R}$ is the sub-Riemannian distance from the boundary.

Theorem 1 (Rizzi, R. - 2020)

Assume that $\partial\Omega$ has no characteristic points. Then, for any $N \geq 3$

$$Q_\Omega(t) = \omega(\Omega) - \sqrt{\frac{4t}{\pi}}\sigma(\partial\Omega) + \frac{t}{2}\int_{\partial\Omega} H d\sigma + \sum_{k=3}^{N} a_k t^{k/2} + o(t^N),$$

as $t \to 0$.

Main result II

Define the operator

$$N \colon \phi \mapsto 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta.$$

Theorem 2 (Rizzi, R. - 2020)

The coefficients a_k, for any $k \geq 1$, are of the form

$$a_k = \int_{\partial\Omega} D_k(\Delta\delta) d\sigma$$

where D_k is an operator, obtained as a homogeneous universal polynomials of degree $k-2$ evaluated in Δ, N. Moreover, there exists an inductive formula for the D_k's.

Higher-order coefficients

List of the first five coefficients:

- $a_2 = -\frac{1}{2}\int_{\partial\Omega} \Delta\delta d\sigma,$
- $a_3 = -\frac{1}{6\sqrt{\pi}}\int_{\partial\Omega} N\Delta\delta d\sigma,$
- $a_4 = -\frac{1}{16}\int_{\partial\Omega} \Delta^2\delta d\sigma,$
- $a_5 = \frac{1}{240\sqrt{\pi}}\int_{\partial\Omega} (N^3 - 8N\Delta)\Delta\delta d\sigma.$

The iterative procedure (due to Savo) has been implemented in Mathematica (the code is available online).

Euclidean setting

- Asymptotics up to order 1 in $\sqrt{t}$, ***M. van den Berg, E.B. Davies, 1989***;
- Asymptotics up to order 2 in $\sqrt{t}$, ***M. van den Berg, J.F. Le Gall, 1994***;

Riemannian setting

- Existence of a complete asymptotics, ***M. van den Berg, P.B. Gilkey, 1994***;
- Inductive formula for the coefficients, ***A. Savo, 1998***;

Sub-Riemannian setting

- Asymptotics up to order 2 in $\sqrt{t}$ in the Heisenberg group for a domain with no characteristic points, ***J. Tyson, J. Wang, 2018***.

Historical review

Euclidean setting

- Asymptotics up to order 1 in $\sqrt{t}$, ***M. van den Berg, E.B. Davies, 1989***; $\rightsquigarrow$ Formula for the Euclidean heat kernel
- Asymptotics up to order 2 in $\sqrt{t}$, ***M. van den Berg, J.F. Le Gall, 1994***; $\rightsquigarrow$ Probabilistic approach

Riemannian setting

- Existence of a complete asymptotics, ***M. van den Berg, P.B. Gilkey, 1994***; $\rightsquigarrow$ Invariance theory, Riemannian curvature
- Inductive formula for the coefficients, ***A. Savo, 1998***;

Sub-Riemannian setting

- Asymptotics up to order 2 in $\sqrt{t}$ in the Heisenberg group for a domain with no characteristic points, ***J. Tyson, J. Wang, 2018***. $\rightsquigarrow$ Relation with exit time of a rescaled Markov process

Proof of main results I

1. Localization principle;

2. Reduction to a 1-dimensional problem;

3. Duhamel's formula.

Step 1: Localization principle

Let $\Omega \subset M$ be open, relatively compact and smooth. Then, for any compact set $K \subset \mathring{\Omega}$

$$1 - u(t, x) = O(t^{\infty}), \qquad \text{as } t \to 0,$$

uniformly on K.

- Uniform off-diagonal estimates (Jerison, Sánchez-Calle - 1986);
- hypoelliptic version of the Kac's principle of not feeling the boundary (Jerison, Sánchez-Calle - 1986, Colin de Verdière, Hillairet, Trélat - 2020).

Proof of main results III

Recall that $Q_\Omega(t) = \int_\Omega u(t,x)d\omega(x)$ and write

$$Q_\Omega(t) = \omega(\Omega) - \int_\Omega (1 - u(t,x))\, d\omega(x)$$
$$= \omega(\Omega) - \underbrace{\int_\Omega (1 - u(t,x))\, \phi(x)d\omega(x)}_{I\phi(t)} + O(t^\infty),$$

where $\phi \equiv 1$ near the boundary and it is smooth.
Introduce a parameter $r \geq 0$:

$$I\phi(t) \rightsquigarrow I\phi(t,r) = \int_{\{x\in\Omega|\delta(x)>r\}} (1 - u(t,x))\, \phi(x)d\omega(x)$$

Proof of main results IV

Step 2: Reduction to a Neumann problem on $\mathbb{R}_+$

Assume that $\partial\Omega$ has no characteristic points. For a suitable choice of ϕ, then $I\phi(t,r)$ is smooth in $(0,+\infty)\times[0,+\infty)$ and

$$(\partial_t - \partial_r^2) I\phi(t,r) = source(t,r), \qquad \partial_r I\phi(t,0) = 0.$$

Finally, apply the Duhamel's principle to the 1D problem, to obtain the complete asymptotic expansion of $I\phi(t,0)$ as $t\to 0$, and thus of $Q_\Omega(t)$, as $t \to 0$.

Riemannian Approximation

Let $\{Z_1, \ldots, Z_L\}$ a global generating family for a Riemannian structure on M. Define g_ε as the scalar product induced by the family

$$\{X_1, \ldots, X_N, \varepsilon Z_1, \ldots, \varepsilon Z_L\}, \qquad \varepsilon < 1.$$

The ***Riemannian variation*** $\{(M, g_\varepsilon)\}$ of M approximates its sub-Riemannian structure:

$$d_\varepsilon \xrightarrow{\varepsilon \to 0} d_{\mathrm{SR}}, \qquad \text{unif. on the compact sets of } M.$$

Moreover, for the heat contents, it holds:

$$Q^\varepsilon_\Omega(t) \xrightarrow{\varepsilon \to 0} Q_\Omega(t) \qquad \text{uniformly on } [0, T].$$

Main result III

The Riemannian heat content has a complete asymptotic expansion by van den Berg, Gilkey:

$$Q_\Omega^\varepsilon(t) \sim \sum_{k=0}^{\infty} a_k^\varepsilon t^{k/2}, \qquad \text{as } t \to 0, \ \forall \varepsilon < 1.$$

Theorem 3 (Rizzi L., R.)

Assume that $\partial\Omega$ has no characteristic points. Then, for any $k \in \mathbb{N}$:

$$a_k^\varepsilon \xrightarrow{\varepsilon \to 0} a_k.$$

- There exist universal polynomials P_k such that

$$a_k = \int_{\partial\Omega} P_k(\delta, D\delta, \dots, D^k\delta)d\sigma,$$
$$a_k^\varepsilon = \int_{\partial\Omega} P_k(\delta_\varepsilon, D\delta_\varepsilon, \dots, D^k\delta_\varepsilon)d\sigma_\varepsilon.$$

- There exists a neighborhood of $\partial\Omega$ in Ω, such that

$$\delta_\varepsilon \xrightarrow{\varepsilon\to 0} \delta,$$

uniformly on U, with all the derivatives.

Even assuming $a_k^\varepsilon \to a_k$ as $\varepsilon \to 0$, we are not able to prove the sub-Riemannian result using Riemannian approximations. We would need:

- A priori existence of a complete asymptotics of $Q_\Omega(t)$;
- Exchange of limits:

$$\lim_{\varepsilon\to 0}\lim_{t\to 0}\frac{1}{t^{\frac{N}{2}}}\Big(Q_\Omega^\varepsilon(t)-\sum_{k=1}^{N} a_k^\varepsilon t^{\frac{k}{2}}\Big) \overset{?}{=} \lim_{t\to 0}\lim_{\varepsilon\to 0}\frac{1}{t^{\frac{N}{2}}}\Big(Q_\Omega^\varepsilon(t)-\sum_{k=1}^{N} a_k^\varepsilon t^{\frac{k}{2}}\Big).$$

Question

What happens to the asymptotics of $Q_\Omega(t)$ if $\mathrm{Char}(\partial\Omega) \neq \emptyset$?

Recall that, in the non-characteristic case, we have, as $t \to 0$:

$$Q_\Omega(t) = \omega(\Omega) - \sqrt{\frac{4t}{\pi}}\sigma(\partial\Omega) + \frac{t}{2}\int_{\partial\Omega} H d\sigma + \sum_{k=3}^{N} a_k t^{k/2} + o(t^N),$$

- Bracket-generating condition $\Rightarrow$ $\mathrm{Char}(\partial\Omega)$ has zero measure (Derridj - 1971, Balogh - 2003);
- The sub-Riemannian induced measure on $\partial\Omega$, σ, remains well-defined;
- $H \in L^1(\partial\Omega, d\sigma)$ (Danielli, Garofalo, Nhieu - 2012).

Recall the expression for $a_k \propto \int_{\partial\Omega} A_k d\sigma$, for $k = 3, 4, 5$ with

$$A_3 = N(\Delta\delta), \qquad A_4 = \Delta(\Delta\delta), \qquad A_5 = (N^3 - 8N\Delta)(\Delta\delta).$$

Theorem 4 (Rizzi, R. - 2020)

Consider the surface $\Sigma = \{z = 0\}$ in the Heisenberg group $\mathbb{H}^1$, equipped with the Lebesgue measure. Then

1. $A_3, A_4 \in L^1_{\mathrm{loc}}(\Sigma, d\sigma)$
2. $A_5 \notin L^1_{\mathrm{loc}}(\Sigma, d\sigma)$ *near* $\mathrm{Char}(\Sigma)$.

Open Problem: is it true that, for characteristic domains $\Omega \subset \mathbb{H}^1$, the asymptotic expansion of $Q_\Omega(t)$ holds up to some order $0 < k < 5$?

Thank you for your attention