Third order analysis of the end-point mapping

Roberto Monti (Padova)

joint work with

F. Boarotto (Padova) and F. Palmurella (ETH Zürich)

Sub-Riemannian Geometry and Interactions
Paris 6-11 September 2020

End-point mapping

$(M, \mathscr{D})$ sub-Riemannian manifold

- M smooth manifold
- $\mathscr{D} = \mathrm{span}\{f_1, \ldots, f_k\} \subset TM$ horiz. distribution (Hörmander condition)

End-point mapping

$(M, \mathscr{D})$ sub-Riemannian manifold

– M smooth manifold
– $\mathscr{D} = \mathrm{span}\{f_1, \dots, f_k\} \subset TM$ horiz. distribution (Hörmander condition)

Fix a point $q \in M$ and for $u \in X := L^2([0,1]; \mathbb{R}^k)$ let $\gamma = \gamma_{q,u}$ be the solution to

$$\dot{\gamma} = \sum_{i=1}^{k} u_i f_i(\gamma) \quad \text{on } [0,1], \qquad \gamma(0) = q.$$

End-point mapping

$(M, \mathscr{D})$ sub-Riemannian manifold

– M smooth manifold
– $\mathscr{D} = \mathrm{span}\{f_1, \ldots, f_k\} \subset TM$ horiz. distribution (Hörmander condition)

Fix a point $q \in M$ and for $u \in X := L^2([0,1]; \mathbb{R}^k)$ let $\gamma = \gamma_{q,u}$ be the solution to

$$\dot{\gamma} = \sum_{i=1}^{k} u_i f_i(\gamma) \quad \text{on } [0,1], \qquad \gamma(0) = q.$$

The end-point mapping is the smooth map $F = F_q : X \to M$

$$F(u) = \gamma_{q,u}(1).$$

End-point mapping

$(M, \mathscr{D})$ sub-Riemannian manifold

– M smooth manifold
– $\mathscr{D} = \mathrm{span}\{f_1, \ldots, f_k\} \subset TM$ horiz. distribution (Hörmander condition)

Fix a point $q \in M$ and for $u \in X := L^2([0,1]; \mathbb{R}^k)$ let $\gamma = \gamma_{q,u}$ be the solution to

$$\dot{\gamma} = \sum_{i=1}^{k} u_i f_i(\gamma) \quad \text{on } [0,1], \qquad \gamma(0) = q.$$

The end-point mapping is the smooth map $F = F_q : X \to M$

$$F(u) = \gamma_{q,u}(1).$$

The control $u \in X$ is *singular* if the differential $d_u F : X \to T_{F(u)} M$ is not surjective.

The end-point mapping is complicated

The main open problems in SR geometry:

–Regularity of singular length-minimizing curves
–Size of the image of singular extremals (Sard problem)

are related to our limited understanding of the end-point mapping at singular points.

Notation

X =Banach space
$\mathscr{F} : X \to M \times \mathbb{R}$ smooth mapping
$u \in X$ singular, say $u = 0$

Notation

X =Banach space
$\mathscr{F} : X \to M \times \mathbb{R}$ smooth mapping
$u \in X$ singular, say $u = 0$

In our application

$$\mathscr{F} = (F, J) : X \to M \times \mathbb{R} \equiv \mathbb{R}^n$$

is the extended end-point mapping, where $J : X \to \mathbb{R}$ is the length functional. We can assume $M \times \mathbb{R} = \mathbb{R}^n$ fixing a chart around $F(0) \in M$.

Notation

X =Banach space
$\mathscr{F} : X \to M \times \mathbb{R}$ smooth mapping
$u \in X$ singular, say $u = 0$

In our application

$$\mathscr{F} = (F, J) : X \to M \times \mathbb{R} \equiv \mathbb{R}^n$$

is the extended end-point mapping, where $J : X \to \mathbb{R}$ is the length functional. We can assume $M \times \mathbb{R} = \mathbb{R}^n$ fixing a chart around $F(0) \in M$.

The cokernel of the differential of $\mathscr{F}$ is

$$\mathrm{coker}(d_0\mathscr{F}) = T_{F(0)}M \times \mathbb{R}/\mathrm{Im}(d_0\mathscr{F}).$$

We denote by $\pi : T_{F(0)}M \times \mathbb{R} \to \mathrm{coker}(d_0\mathscr{F})$ the projection.

Strictly singular case

The singular point $u = 0$ is *strictly* singular when

$$\mathrm{coker}(d_0\mathscr{F}) = \mathrm{coker}(d_0 F) = T_{F(0)}M/\mathrm{Im}(d_0 F).$$

The J component is quotiented out by π.

Strictly singular case

The singular point $u = 0$ is *strictly* singular when

$$\mathrm{coker}(d_0 \mathscr{F}) = \mathrm{coker}(d_0 F) = T_{F(0)} M / \mathrm{Im}(d_0 F).$$

The J component is quotiented out by π.

In this case we may study the end-point map F and deduce properties for the extended map $\mathscr{F}$ (e.g., being an open mapping).

Second order analysis

Notation for the kth-order directional derivative at $u = 0$

$$d_0^k F(v) = \frac{d^k}{dt^k} F(tv)\Big|_{t=0}, \quad v \in X.$$

Second order analysis

Notation for the kth-order directional derivative at $u = 0$

$$d_0^k F(v) = \left. \frac{d^k}{dt^k} F(tv) \right|_{t=0}, \quad v \in X.$$

The intrinsic Hessian of F at $u = 0$ is the quadratic map $\mathscr{D}_0^2 F : \ker(d_0 F) \to \operatorname{coker}(d_0 F)$

$$\mathscr{D}_0^2 F(v) = \pi(d_0^2 F(v)), \quad v \in \ker(d_0 F).$$

Second order analysis

Notation for the kth-order directional derivative at $u = 0$

$$d_0^k F(v) = \frac{d^k}{dt^k} F(tv)\Big|_{t=0}, \quad v \in X.$$

The intrinsic Hessian of F at $u = 0$ is the quadratic map $\mathscr{D}_0^2 F : \ker(d_0 F) \to \operatorname{coker}(d_0 F)$

$$\mathscr{D}_0^2 F(v) = \pi(d_0^2 F(v)), \quad v \in \ker(d_0 F).$$

Definition. An element $v \in \ker(d_0 F)$ is a regular zero for $\mathscr{D}_0^2 F$ if:
– $\mathscr{D}_0^2 F(v) = 0$;
– the linear map $w \mapsto \mathscr{D}_0^2 F(v, w)$ is su from $\ker(d_0 F)$ to $\operatorname{coker}(d_0 F)$.

Second order analysis

Notation for the kth-order directional derivative at $u = 0$

$$d_0^k F(v) = \frac{d^k}{dt^k} F(tv)\Big|_{t=0}, \quad v \in X.$$

The intrinsic Hessian of F at $u = 0$ is the quadratic map $\mathscr{D}_0^2 F : \ker(d_0 F) \to \operatorname{coker}(d_0 F)$

$$\mathscr{D}_0^2 F(v) = \pi(d_0^2 F(v)), \quad v \in \ker(d_0 F).$$

Definition. An element $v \in \ker(d_0 F)$ is a regular zero for $\mathscr{D}_0^2 F$ if:
– $\mathscr{D}_0^2 F(v) = 0$;
– the linear map $w \mapsto \mathscr{D}_0^2 F(v, w)$ is su from $\ker(d_0 F)$ to $\operatorname{coker}(d_0 F)$.

Agracev-Sarychev theory

Theorem. Let $u \in X$ be strictly singular. If $\mathscr{D}_u^2 F$ has a regular zero then the extended map $\mathscr{F}$ is open at u.

Agracev-Sarychev theory

Theorem. Let $u \in X$ be strictly singular. If $\mathscr{D}_u^2 F$ has a regular zero then the extended map $\mathscr{F}$ is open at u.

Thus u is not J-optimal (it is not length optimal).

Agracev-Sarychev theory

Theorem. Let $u \in X$ be strictly singular. If $\mathscr{D}_u^2 F$ has a regular zero then the extended map $\mathscr{F}$ is open at u.

Thus u is not J-optimal (it is not length optimal).

Lemma. If for any $\lambda \neq 0$ orthogonal to $\mathrm{Im}(d_u F)$ the scalarization $\lambda \mathscr{D}_u^2 F$ has index

$$\mathrm{ind}(\lambda \mathscr{D}_u^2 F) \geq \dim(T_{F(u)}/\mathrm{Im}(d_u F)) =: \mathrm{corank}(u)$$

then $\mathscr{D}_u^2 F$ has a regular zero.

Agracev-Sarychev theory

Theorem. Let $u \in X$ be strictly singular. If $\mathscr{D}_u^2 F$ has a regular zero then the extended map $\mathscr{F}$ is open at u.

Thus u is not J-optimal (it is not length optimal).

Lemma. If for any $\lambda \neq 0$ orthogonal to $\mathrm{Im}(d_u F)$ the scalarization $\lambda \mathscr{D}_u^2 F$ has index

$$\mathrm{ind}(\lambda \mathscr{D}_u^2 F) \geq \dim(T_{F(u)}/\mathrm{Im}(d_u F)) =: \mathrm{corank}(u)$$

then $\mathscr{D}_u^2 F$ has a regular zero.

Corollary (Goh). Let (γ, u) be a strictly singular minimizing extremal. Then any adjoint curve $\lambda : [0,1] \to T^*M$ satisfies

$$\langle \lambda(t), [f_i, f_j](\gamma(t)) \rangle = 0 \quad t \in [0,1], \quad i,j = 1, \dots, k.$$

Agracev-Sarychev theory

Theorem. Let $u \in X$ be strictly singular. If $\mathscr{D}_u^2 F$ has a regular zero then the extended map $\mathscr{F}$ is open at u.

Thus u is not J-optimal (it is not length optimal).

Lemma. If for any $\lambda \neq 0$ orthogonal to $\mathrm{Im}(d_u F)$ the scalarization $\lambda \mathscr{D}_u^2 F$ has index

$$\mathrm{ind}(\lambda \mathscr{D}_u^2 F) \geq \dim(T_{F(u)}/\mathrm{Im}(d_u F)) =: \mathrm{corank}(u)$$

then $\mathscr{D}_u^2 F$ has a regular zero.

Corollary (Goh). Let (γ, u) be a strictly singular minimizing extremal. Then any adjoint curve $\lambda : [0,1] \to T^*M$ satisfies

$$\langle \lambda(t), [f_i, f_j](\gamma(t)) \rangle = 0 \quad t \in [0,1], \quad i,j = 1, \ldots, k.$$

Comments. 1) If Goh's condition is violated the index above is ∞.

Agracev-Sarychev theory

Theorem. Let $u \in X$ be strictly singular. If $\mathscr{D}_u^2 F$ has a regular zero then the extended map $\mathscr{F}$ is open at u.

Thus u is not J-optimal (it is not length optimal).

Lemma. If for any $\lambda \neq 0$ orthogonal to $\mathrm{Im}(d_u F)$ the scalarization $\lambda \mathscr{D}_u^2 F$ has index

$$\mathrm{ind}(\lambda \mathscr{D}_u^2 F) \geq \dim(T_{F(u)}/\mathrm{Im}(d_u F)) =: \mathrm{corank}(u)$$

then $\mathscr{D}_u^2 F$ has a regular zero.

Corollary (Goh). Let (γ, u) be a strictly singular minimizing extremal. Then any adjoint curve $\lambda : [0,1] \to T^*M$ satisfies

$$\langle \lambda(t), [f_i, f_j](\gamma(t)) \rangle = 0 \quad t \in [0,1], \quad i,j = 1, \ldots, k.$$

Comments. 1) If Goh's condition is violated the index above is ∞.
2) The bracket $[f_i, f_j]$ appears in the computation for $\mathscr{D}_u^2 F$ restricted to its correct domain, i.e., $\ker(d_u F)$.

Third intrinsic differential

Consider a smooth map $F : X \to M$. The subspace

$$\mathrm{dom}(\mathscr{D}_0^3 F) := \{v \in \ker(d_0 F) : \pi(d_0^2 F(v, x)) = 0 \text{ for all } x \in X\}$$

is chart-independent.

Third intrinsic differential

Consider a smooth map $F : X \to M$. The subspace

$$\mathrm{dom}(\mathscr{D}_0^3 F) := \{v \in \ker(d_0 F) : \pi(d_0^2 F(v, x)) = 0 \text{ for all } x \in X\}$$

is chart-independent.

We define $\mathscr{D}_0^3 F : \mathrm{dom}(\mathscr{D}_0^3 F) \to \mathrm{coker}(d_0 F)$ in the natural way:

$$\mathscr{D}_0^3 F(v) = \pi(d_0^3 F(v)).$$

Third intrinsic differential

Consider a smooth map $F : X \to M$. The subspace

$$\mathrm{dom}(\mathscr{D}_0^3 F) := \{v \in \ker(d_0 F) : \pi(d_0^2 F(v, x)) = 0 \text{ for all } x \in X\}$$

is chart-independent.

We define $\mathscr{D}_0^3 F : \mathrm{dom}(\mathscr{D}_0^3 F) \to \mathrm{coker}(d_0 F)$ in the natural way:

$$\mathscr{D}_0^3 F(v) = \pi(d_0^3 F(v)).$$

Definition. Let $w \in \ker(d_0 F)$ be such that $\mathscr{D}_0^2 F(w) = 0$.

Third intrinsic differential

Consider a smooth map $F : X \to M$. The subspace

$$\mathrm{dom}(\mathscr{D}_0^3 F) := \{v \in \ker(d_0 F) : \pi(d_0^2 F(v,x)) = 0 \text{ for all } x \in X\}$$

is chart-independent.

We define $\mathscr{D}_0^3 F : \mathrm{dom}(\mathscr{D}_0^3 F) \to \mathrm{coker}(d_0 F)$ in the natural way:

$$\mathscr{D}_0^3 F(v) = \pi(d_0^3 F(v)).$$

Definition. Let $w \in \ker(d_0 F)$ be such that $\mathscr{D}_0^2 F(w) = 0$.
An element $v \in \mathrm{dom}(\mathscr{D}_0^3 F)$ is a w-regular zero for the third differential if:
– $\mathscr{D}_0^3 F(v) = 0$;
– the linear map $u \mapsto \mathscr{D}_0^3 F(v,v,u)$ is surjective from $\mathrm{dom}(\mathscr{D}_0^3 F)$ to $\mathrm{coker}(d_0 F)/\mathrm{Im}(\mathscr{D}_0^2 F(w,\cdot))$.

Third order open mapping theorem

Theorem. Let $F : X \to M$ be a smooth mapping and let $u = 0$ be a singular point.

1) Case $\text{corank}(u) = 1$. If there exists $v \in \text{dom}(\mathscr{D}_0^3 F)$ such that $\mathscr{D}_0^3 F(v) \neq 0$ then F is open at $u = 0$.

Third order open mapping theorem

Theorem. Let $F : X \to M$ be a smooth mapping and let $u = 0$ be a singular point.

1) Case $\text{corank}(u) = 1$. If there exists $v \in \text{dom}(\mathscr{D}_0^3 F)$ such that $\mathscr{D}_0^3 F(v) \neq 0$ then F is open at $u = 0$.

2) Case $\text{corank}(u) \geq 1$. If there exist w and v such that v is a w-regular zero for $\mathscr{D}_0^3 F$ then F is open at $u = 0$.

Third order open mapping theorem

Theorem. Let $F : X \to M$ be a smooth mapping and let $u = 0$ be a singular point.

1) Case $\text{corank}(u) = 1$. If there exists $v \in \text{dom}(\mathscr{D}_0^3 F)$ such that $\mathscr{D}_0^3 F(v) \neq 0$ then F is open at $u = 0$.

2) Case $\text{corank}(u) \geq 1$. If there exist w and v such that v is a w-regular zero for $\mathscr{D}_0^3 F$ then F is open at $u = 0$.

Comments.

1) In our application the extended map $\mathscr{F}$ will be open at u.

Third order open mapping theorem

Theorem. Let $F : X \to M$ be a smooth mapping and let $u = 0$ be a singular point.

1) Case $\mathrm{corank}(u) = 1$. If there exists $v \in \mathrm{dom}(\mathscr{D}_0^3 F)$ such that $\mathscr{D}_0^3 F(v) \neq 0$ then F is open at $u = 0$.

2) Case $\mathrm{corank}(u) \geq 1$. If there exist w and v such that v is a w-regular zero for $\mathscr{D}_0^3 F$ then F is open at $u = 0$.

Comments.

1) In our application the extended map $\mathscr{F}$ will be open at u.

2) We could not find any condition of the algebraic type (extension of the notion of index for a quadratic form) ensuring the existence of a w-regular zero for a cubic map.

Third order open mapping theorem

Theorem. Let $F : X \to M$ be a smooth mapping and let $u = 0$ be a singular point.

1) Case $\mathrm{corank}(u) = 1$. If there exists $v \in \mathrm{dom}(\mathscr{D}_0^3 F)$ such that $\mathscr{D}_0^3 F(v) \neq 0$ then F is open at $u = 0$.

2) Case $\mathrm{corank}(u) \geq 1$. If there exist w and v such that v is a w-regular zero for $\mathscr{D}_0^3 F$ then F is open at $u = 0$.

Comments.

1) In our application the extended map $\mathscr{F}$ will be open at u.

2) We could not find any condition of the algebraic type (extension of the notion of index for a quadratic form) ensuring the existence of a w-regular zero for a cubic map.

Computation of $\mathscr{D}_u^3 F$

New notation $G(v) = F(u + v)$.

The end-point mapping is defined via a flow that can be expressed as a right (or left) chronological exponential and can be interpreted as an operator acting on $C^\infty(M)$.

Computation of $\mathscr{D}_u^3 F$

New notation $G(v) = F(u+v)$.

The end-point mapping is defined via a flow that can be expressed as a right (or left) chronological exponential and can be interpreted as an operator acting on $C^\infty(M)$.

In this sense we have the expansion

$$G(v) = \mathrm{Id} + d_0 G(v) + \frac{1}{2} d_0^2 G(v) + \frac{1}{6} d_0^3 G(v) + \mathrm{error}$$

Computation of $\mathscr{D}_u^3 F$

New notation $G(v) = F(u+v)$.

The end-point mapping is defined via a flow that can be expressed as a right (or left) chronological exponential and can be interpreted as an operator acting on $C^\infty(M)$.

In this sense we have the expansion

$$G(v) = \mathrm{Id} + d_0 G(v) + \frac{1}{2} d_0^2 G(v) + \frac{1}{6} d_0^3 G(v) + \mathrm{error}$$

The third order term is (we do not explain the notation)

$$\begin{aligned} d_0^3 G(v) = & \int_{\Sigma_3} [g^{\tau_3}_{v(\tau_3)}, [g^{\tau_2}_{v(\tau_2)}, g^{\tau_1}_{v(\tau_1)}]] d\tau_3 d\tau_2 d\tau_1 \\ & + \Big(\int_{\Sigma_2} [g^{\tau_2}_{v(\tau_2)}, g^{\tau_1}_{v(\tau_1)}] d\tau_2 d\tau_1 \Big) \circ \Big(\int_0^1 g^t_{v(t)} dt \Big) \\ & + \Big(\int_0^1 g^t_{v(t)} dt \Big) \circ \Big(\int_{\Sigma_2} [g^{\tau_2}_{v(\tau_2)}, g^{\tau_1}_{v(\tau_1)}] d\tau_2 d\tau_1 \Big) \\ & + \Big(\int_0^1 g^t_{v(t)} dt \Big) \circ \Big(\int_0^1 g^t_{v(t)} dt \Big) \circ \Big(\int_0^1 g^t_{v(t)} dt \Big) \end{aligned}$$

Comments

1. The above representation for the third differential is not unique.

Comments

1. The above representation for the third differential is not unique.

2. When $v \in \mathrm{dom}(\mathscr{D}_0^3 G)$ the only surviving term is the geometric one:

$$d_0^3 G(v) = \int_{\Sigma_3} [g_{v(\tau_3)}^{\tau_3}, [g_{v(\tau_2)}^{\tau_2}, g_{v(\tau_1)}^{\tau_1}]] d\tau_3 d\tau_2 d\tau_1$$

Comments

1. The above representation for the third differential is not unique.

2. When $v \in \mathrm{dom}(\mathscr{D}_0^3 G)$ the only surviving term is the geometric one:

$$d_0^3 G(v) = \int_{\Sigma_3} [g^{\tau_3}_{v(\tau_3)}, [g^{\tau_2}_{v(\tau_2)}, g^{\tau_1}_{v(\tau_1)}]] d\tau_3 d\tau_2 d\tau_1$$

3. After scalarization with λ orthogonal to $\mathrm{Im}(d_0 G)$ the representation becomes unique.

Third order necessary conditions

Theorem. Let (γ, u) be a strictly singular minimizing extremal.

Third order necessary conditions

Theorem. Let (γ, u) be a strictly singular minimizing extremal.

Assume that:

a) we have $\mathrm{corank}(u) = 1$;

b) $\mathrm{dom}(\mathscr{D}_u^3 F)$ is of finite codimension in $\ker(d_u F)$.

Third order necessary conditions

Theorem. Let (γ, u) be a strictly singular minimizing extremal.

Assume that:

a) we have $\mathrm{corank}(u) = 1$;

b) $\mathrm{dom}(\mathscr{D}_u^3 F)$ is of finite codimension in $\ker(d_u F)$.

Then any adjoint curve $\lambda : [0,1] \to T^*M$ satisfies for $i, j, \ell = 1, \dots, k$

$$\langle \lambda(t), [f_i, [f_j, f_\ell]](\gamma(t)) + [f_\ell, [f_j, f_i]](\gamma(t)) \rangle = 0 \quad t \in [0,1]. \qquad (*)$$

Comments

1. In the second order case, using the notion of index it is possible to cover the case of general corank.

Comments

1. In the second order case, using the notion of index it is possible to cover the case of general corank.

2. In the second order case, assumption b) reads

$$\mathrm{dom}(\mathscr{D}_u^2 F) = \ker(d_u F) \text{ is of finite codimension in } X$$

and it is trivially satisfied.

Comments

1. In the second order case, using the notion of index it is possible to cover the case of general corank.

2. In the second order case, assumption b) reads

$$\mathrm{dom}(\mathscr{D}_u^2 F) = \ker(d_u F) \text{ is of finite codimension in } X$$

and it is trivially satisfied.

3. Assumption b) does the following job. Contradicting $(*)$, there is enough room to construct a function $v \in L^2([0,1];\mathbb{R}^k)$ and in fact $v \in \mathrm{dom}(\mathscr{D}_0^3 G)$ such that

$$\mathscr{D}_0^3 G(v) \neq 0.$$

This contradicts the third order open mapping theorem.

That's all.

Thank you for your attention.

Hoping in better times.