Sub-Laplacian Comparison Theorems on H-Type Foliations

Gianmarco Molino

SRGI, Université Sorbonne

9 September, 2020

Riemannian Geometry

- Riemannian manifolds allow for many notions of curvature

Riemannian Geometry

- Riemannian manifolds allow for many notions of curvature
- Metric, analytic, and even topological properties can be determined from a knowledge of curvature

Riemannian Geometry

- Riemannian manifolds allow for many notions of curvature
- Metric, analytic, and even topological properties can be determined from a knowledge of curvature
- How do these ideas fit in a subRiemannian setting?

Consequences of Laplacian Comparisons

Let $(\mathbb{M}, g)$ be a Riemannian manifold of dimension m and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n-1)\kappa g$.

Consequences of Laplacian Comparisons

Let $(\mathbb{M}, g)$ be a Riemannian manifold of dimension m and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n-1)\kappa g$.

Theorem (Bonnet-Meyers)

If $\kappa > 0$ then

- $\text{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$

Consequences of Laplacian Comparisons

Let $(\mathbb{M}, g)$ be a Riemannian manifold of dimension m and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n-1)\kappa g$.

Theorem (Bonnet-Meyers)

If $\kappa > 0$ then

- $\mathrm{diam}(\mathbb{M}) \leq \frac{\pi}{\sqrt{\kappa}}$
- *The fundamental group of $\mathbb{M}$ must be finite.*

Consequences of Laplacian Comparisons

Let $(\mathbb{M}, g)$ be a Riemannian manifold of dimension m and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n-1)\kappa g$.

Consequences of Laplacian Comparisons

Let $(\mathbb{M}, g)$ be a Riemannian manifold of dimension m and suppose there exists $\kappa \in \mathbb{R}$ such that $Ric \geq (n-1)\kappa g$.

Theorem (Bishop-Gromov)

Let $\mathbb{M}^m_\kappa$ be the Riemannian manifold of dimension m and constant sectional curvature κ. Denote by $B_{\mathbb{M}}(p,r)$ the Riemannian ball of radius r around $p \in \mathbb{M}$. Then

$$\phi(r) = \frac{B_{\mathbb{M}}(p,r)}{B_{\mathbb{M}^m_\kappa}(p_\kappa, r)}$$

is nonincreasing on $(0, \infty)$.

Connections

Denoting the space of vector fields $\Gamma(T\mathbb{M})$, an operator

$$\nabla\colon \Gamma(T\mathbb{M}) \times \Gamma(T\mathbb{M}) \to \Gamma(T\mathbb{M})$$

Connections

Denoting the space of vector fields $\Gamma(T\mathbb{M})$, an operator

$$\nabla : \Gamma(T\mathbb{M}) \times \Gamma(T\mathbb{M}) \to \Gamma(T\mathbb{M})$$

such that

1. $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$

Connections

Denoting the space of vector fields $\Gamma(T\mathbb{M})$, an operator

$$\nabla : \Gamma(T\mathbb{M}) \times \Gamma(T\mathbb{M}) \to \Gamma(T\mathbb{M})$$

such that

1. $\nabla_{fX+Y} Z = f \nabla_X Z + \nabla_Y Z$
2. $\nabla_X (fY) = df(X) Y + f \nabla_X Y$

is called a connection.

Levi-Civita Connection

Riemannian Geometry is characterized by the Levi-Civita connection,

Levi-Civita Connection

Riemannian Geometry is characterized by the Levi-Civita connection,

Theorem

Let $(\mathbb{M}, g)$ be a Riemannian manifold. There exists a unique connection ∇ on $\mathbb{M}$ such that

1. $\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. $\nabla_X Y - \nabla_Y X = [X, Y]$

Curvature

- Riemannian Curvature:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

Curvature

- Riemannian Curvature:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

- Sectional Curvature: (for orthonormal X, Y):

$$K(X,Y) = g(R(X,Y)Y, X)$$

Curvature

- Riemannian Curvature:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

- Sectional Curvature: (for orthonormal X, Y):

$$K(X,Y) = g(R(X,Y)Y, X)$$

- Ricci Curvature:

$$Ric(X,Y) = Tr(Z \mapsto g(R(Z,X)Y,Z))$$

Curvature

- Riemannian Curvature:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y - \nabla_Y X} Z$$

- Sectional Curvature: (for orthonormal X, Y):

$$K(X,Y) = g(R(X,Y)Y, X)$$

- Ricci Curvature:

$$Ric(X,Y) = Tr(Z \mapsto g(R(Z,X)Y, Z))$$

- Scalar Curvature:

$$s(X) = Tr(Y \mapsto Ric(X,Y))$$

Some Definitions

We set some notation. For a Riemannian manifold $(\mathbb{M}, g)$ and a point $p \in \mathbb{M}$, we define the distance function

$$d_p \colon \mathbb{M} \to \mathbb{R}, \;\; d_p(q) = d(p, q)$$

Some Definitions

We set some notation. For a Riemannian manifold $(\mathbb{M}, g)$ and a point $p \in \mathbb{M}$, we define the distance function

$$d_p \colon \mathbb{M} \to \mathbb{R}, \;\; d_p(q) = d(p, q)$$

Let $\gamma \colon [0, L] \to M$ be a minimizing geodesic. Then we define the curvatures

$$K^+(t) = \sup\{K(X_{\gamma(t)}, Y_{\gamma(t)}) \colon \gamma'(t) \in \operatorname{Span}(X_{\gamma(t)}, Y_{\gamma(t)})\}$$
$$K^-(t) = \inf\{K(X_{\gamma(t)}, Y_{\gamma(t)}) \colon \gamma'(t) \in \operatorname{Span}(X_{\gamma(t)}, Y_{\gamma(t)})\}$$

Some Definitions

We set some notation. For a Riemannian manifold $(\mathbb{M}, g)$ and a point $p \in \mathbb{M}$, we define the distance function

$$d_p \colon \mathbb{M} \to \mathbb{R}, \;\; d_p(q) = d(p, q)$$

Let $\gamma \colon [0, L] \to M$ be a minimizing geodesic. Then we define the curvatures

$$K^+(t) = \sup\{K(X_{\gamma(t)}, Y_{\gamma(t)}) \colon \gamma'(t) \in \mathrm{Span}(X_{\gamma(t)}, Y_{\gamma(t)})\}$$
$$K^-(t) = \inf\{K(X_{\gamma(t)}, Y_{\gamma(t)}) \colon \gamma'(t) \in \mathrm{Span}(X_{\gamma(t)}, Y_{\gamma(t)})\}$$

We define

$$\mathrm{Hess}\, f(X, Y) = \nabla^2 f(X, Y) = g(\nabla_X \nabla f, Y)$$
$$\Delta f = Tr(\mathrm{Hess}\, f)$$

Hessian Comparison Theorem

Theorem (Hessian Comparison)

Let $(\mathbb{M}_i, g_i), i \in \{1,2\}$ *be Riemannian manifolds,* $\gamma_i : [0,L] \to \mathbb{M}_i$ *be minimizing geodesics such that*

$$K^+_{\mathbb{M}_2}(t) \leq K^-_{\mathbb{M}_1}(t)$$

Hessian Comparison Theorem

Theorem (Hessian Comparison)

Let $(\mathbb{M}_i, g_i), i \in \{1,2\}$ *be Riemannian manifolds,* $\gamma_i : [0, L] \to \mathbb{M}_i$ *be minimizing geodesics such that*

$$K^+_{\mathbb{M}_2}(t) \leq K^-_{\mathbb{M}_1}(t)$$

Let $X_i \in \Gamma(T\mathbb{M}_i)$ *be such that for all* $t \in [0, L]$

- $\|X_1(\gamma_1(t))\|_{g_1} = \|X_2(\gamma_2(t))\|_{g_2}$
- $g_1(X_1(\gamma_1(t)), \gamma_1'(t)) = g_2(X_2(\gamma_2(t)), \gamma_2'(t))$

Hessian Comparison Theorem

Theorem (Hessian Comparison)

Let $(\mathbb{M}_i, g_i), i \in \{1,2\}$ *be Riemannian manifolds,* $\gamma_i : [0,L] \to \mathbb{M}_i$ *be minimizing geodesics such that*

$$K^{+}_{\mathbb{M}_2}(t) \leq K^{-}_{\mathbb{M}_1}(t)$$

Let $X_i \in \Gamma(T\mathbb{M}_i)$ *be such that for all* $t \in [0,L]$

- $\|X_1(\gamma_1(t))\|_{g_1} = \|X_2(\gamma_2(t))\|_{g_2}$
- $g_1(X_1(\gamma_1(t)), \gamma_1'(t)) = g_2(X_2(\gamma_2(t)), \gamma_2'(t))$

then denoting $p_i = \gamma_i(0), q_i = \gamma_i(t)$*,*

$$\text{Hess}\, d_{p_1}(X_1(q_1), X_1(q_1)) \leq \text{Hess}\, d_{p_2}(X_2(q_2), X_2(q_2))$$

Rough Sketch of proof:

Rough Sketch of proof:

- Consider variation of geodesics with fixed endpoints; Jacobi fields describe the infinitesimal variation

Rough Sketch of proof:

- Consider variation of geodesics with fixed endpoints; Jacobi fields describe the infinitesimal variation
- Define index $I(X,X) = \int_0^r \langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X \rangle - R(X, \dot\gamma, \dot\gamma, X)\, dt$

Rough Sketch of proof:

- Consider variation of geodesics with fixed endpoints; Jacobi fields describe the infinitesimal variation
- Define index $I(X,X) = \int_0^r \langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X \rangle - R(X, \dot\gamma, \dot\gamma, X)\, dt$
- $K_1^+ \leq K_2^- \implies I(X_1, X_1) \leq I(X_2, X_2)$

Rough Sketch of proof:

- Consider variation of geodesics with fixed endpoints; Jacobi fields describe the infinitesimal variation
- Define index $I(X,X) = \int_0^r \langle \nabla_{\dot\gamma} X, \nabla_{\dot\gamma} X\rangle - R(X,\dot\gamma,\dot\gamma,X)\; dt$
- $K_1^+ \leq K_2^- \implies I(X_1,X_1) \leq I(X_2,X_2)$
- Theorem follows from $\nabla^2 d = \alpha I(X,X)$

Rauch Comparison Theorem

Corollary (Rauch Comparison)

Take the same assumptions as in the previous theorem. Then

$$\Delta_1 d_{p_1}(q_2) \leq \Delta_2 d_{p_2}(q_2)$$

Rauch Comparison Theorem

Corollary (Rauch Comparison)

Take the same assumptions as in the previous theorem. Then

$$\Delta_1 d_{p_1}(q_2) \leq \Delta_2 d_{p_2}(q_2)$$

This presents a way to compare the behaviors of distance functions, but we still need to something to compare them to.

Model Spaces

Denote by $\mathbb{M}^m_\kappa$ the Riemannian manifold of constant sectional curvature κ and dimension m.

Model Spaces

Denote by $\mathbb{M}_\kappa^m$ the Riemannian manifold of constant sectional curvature κ and dimension m. Explicitly

$$\mathbb{M}_\kappa^m = \begin{cases} S^m(\kappa) & \kappa > 0 \\ \mathbb{R}^m & \kappa = 0 \\ \mathbb{H}^m(\kappa) & \kappa < 0 \end{cases}$$

Model Spaces

Denote by $\mathbb{M}^m_\kappa$ the Riemannian manifold of constant sectional curvature κ and dimension m. Explicitly

$$\mathbb{M}^m_\kappa = \begin{cases} S^m(\kappa) & \kappa > 0 \\ \mathbb{R}^m & \kappa = 0 \\ \mathbb{H}^m(\kappa) & \kappa < 0 \end{cases}$$

We refer to these as Model Spaces. We are able to compute Δd_p explicitly on these spaces, and use this as a basis for comparison.

Laplacian Comparison

Theorem (Laplacian Comparison)

Let $(\mathbb{M}, g)$ *be a Riemannian manifold with dimension n and suppose there exists* $\kappa \in \mathbb{R}$ *such that*

$$Ric \geq (n-1)\kappa g$$

Laplacian Comparison

Theorem (Laplacian Comparison)

Let $(\mathbb{M}, g)$ *be a Riemannian manifold with dimension* n *and suppose there exists* $\kappa \in \mathbb{R}$ *such that*

$$Ric \geq (n-1)\kappa g$$

Let $p, q \in \mathbb{M}$ *and denote* $r = d(p, q)$*.*

Laplacian Comparison

Theorem (Laplacian Comparison)

Let $(\mathbb{M}, g)$ *be a Riemannian manifold with dimension* n *and suppose there exists* $\kappa \in \mathbb{R}$ *such that*

$$Ric \geq (n-1)\kappa g$$

Let $p, q \in \mathbb{M}$ *and denote* $r = d(p,q)$. *Then*

$$\Delta d_p(q) \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r) & \kappa > 0 \\ \frac{n-1}{r} & \kappa = 0 \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \kappa < 0 \end{cases}$$

Comparison Function

On the model spaces, the Jacobi fields can be computed explicitly using the Jacobi equation

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}V + R(V,\dot{\gamma},\dot{\gamma}) = 0$$

Comparison Function

On the model spaces, the Jacobi fields can be computed explicitly using the Jacobi equation

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}V + R(V,\dot\gamma,\dot\gamma) = 0$$

then the upper bound on $\Delta d_p(q)$ is given by solving an ode.

Basic Definitions

Let $\mathbb{M}$ be a smooth manifold. We say that $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if

- $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
- and $g_{\mathcal{H}}$ is a fiberwise inner product on $\mathcal{H}$.

Basic Definitions

Let $\mathbb{M}$ be a smooth manifold. We say that $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold if

- $\mathcal{H}$ is a constant rank, bracket generating subbundle of $T\mathbb{M}$,
- and $g_{\mathcal{H}}$ is a fiberwise inner product on $\mathcal{H}$.

A main goal of sub-Riemannian geometry is to determine adequate notions of curvature that are able to support generalizations of the comparison theorems found in the Riemannian theory.

Some History

- Li-Zelenko 2011, Lee-Li 2013, Agrachev-Lee 2015, Lee-Li-Zelenko 2016: Comparison theorems on Sasakian manifolds
- Rizzi-Silveira 2015, 2017, Barilari-Rizzi 2016: Comparison theorems in 3 Sasakian case
- Baudoin-Bonnefont-Garofalo 2014, Baudoin-Grong-Kuwada-Thalmaier 2017: Eulerian approach to comparison theorems on Sasakian and 3 Sasakian manifolds

sR Manifolds with Metric Complement

Let $\mathbb{M}$ be a smooth manifold. We say that $(\mathbb{M}, \mathcal{H}, g)$ is a sub-Riemannian manifold with metric preserving complement or sRmc-manifold if

- $(\mathbb{M}, g)$ is a Riemannian manifold,
- the metric orthogonally splits as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$,
- and $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$ is a sub-Riemannian manifold.

We denote by $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ by g.

Motivating Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

Motivating Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

Define the vertical distribution as tangent to the leaves , $\mathbb{S}^1$,

Motivating Example: Hopf Fibration

Consider $\mathbb{S}^{2n+1}$ foliated as

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$$

Define the vertical distribution as tangent to the leaves , $\mathbb{S}^1$,

Then setting $\mathcal{H}$ to be orthogonal to $\mathcal{V}$ will make $(\mathbb{S}^{2n+1}, \mathcal{H}, g)$ a sRmc-manifold.

Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_\mathcal{H} + \frac{1}{\varepsilon} g_\mathcal{V}$$

Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_\mathcal{H} + \frac{1}{\varepsilon} g_\mathcal{V}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_\mathcal{H})$$

Gromov-Hausdorff Convergence

For a sRmc-manfiold $(\mathbb{M}, \mathcal{H}, g)$ we define the canonical variation of the metric

$$g_\varepsilon = g_\mathcal{H} + \frac{1}{\varepsilon} g_\mathcal{V}$$

which in the Gromov-Hausdorff sense

$$(\mathbb{M}, \mathcal{H}, g_\varepsilon) \xrightarrow{\varepsilon \to 0^+} (\mathbb{M}, \mathcal{H}, g_\mathcal{H})$$

The idea is to consider the convergence of Riemannian structures to the sub-Riemannian one.

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ *on* $(\mathbb{M}, \mathcal{H}, g)$ *such that*

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ *on* $(\mathbb{M}, \mathcal{H}, g)$ *such that*

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ *on* $(\mathbb{M}, \mathcal{H}, g)$ *such that*

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
    - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
    - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ *on* $(\mathbb{M}, \mathcal{H}, g)$ *such that*

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X, Z), Y \rangle_{\mathcal{H}} = \langle T(Y, Z), X \rangle_{\mathcal{H}}$

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ on $(\mathbb{M}, \mathcal{H}, g)$ such that

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X, Z), Y\rangle_{\mathcal{H}} = \langle T(Y, Z), X\rangle_{\mathcal{H}}$
   - $\langle T(Z, X), V\rangle_{\mathcal{V}} = \langle T(V, X), Z\rangle_{\mathcal{V}}$.

Hladky-Bott Connection

Theorem (Hladky '12 [5])

There exists a unique metric connection ∇ *on* $(\mathbb{M}, \mathcal{H}, g)$ *such that*

1. $\mathcal{H}$ *and* $\mathcal{V}$ *are* ∇*-parallel,*
2. *The torsion* T *of* ∇ *satisfies*
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. *For every* $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X, Z), Y \rangle_{\mathcal{H}} = \langle T(Y, Z), X \rangle_{\mathcal{H}}$
   - $\langle T(Z, X), V \rangle_{\mathcal{V}} = \langle T(V, X), Z \rangle_{\mathcal{V}}$.

This is called the Hladky-Bott connection.

Hladky-Bott Connection

We can explicitly write ∇ in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

Hladky-Bott Connection

We can explicitly write ∇ in terms of the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] + A_X Y & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

where the tensor A is defined by

$$\langle A_X Y, Z \rangle = \frac{1}{2} \left((\mathcal{L}_{X_{\mathcal{V}}} g)(Y_{\mathcal{H}}, Z_{\mathcal{H}}) + (\mathcal{L}_{X_{\mathcal{H}}} g)(Y_{\mathcal{V}}, Z_{\mathcal{V}}) \right)$$

Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- Bundle-like metric: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to

$$\mathcal{L}_{\mathcal{V}} g(\mathcal{H}, \mathcal{H}) = 0$$

Bundle-like Metrics and Totally Geodesic Foliations

There are two important properties we will require:

- Bundle-like metric: A foliation is said to have a bundle-like metric if the metric locally splits orthogonally. This is equivalent to

$$\mathcal{L}_{\mathcal{V}} g(\mathcal{H}, \mathcal{H}) = 0$$

- Totally geodesic foliation: A foliation is said to be totally geodesic if the geodesics of the fibers are embedded geodesics of the total space. This is equivalent to

$$\mathcal{L}_{\mathcal{H}} g(\mathcal{V}, \mathcal{V}) = 0$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

J Map

On $(\mathbb{M}, \mathcal{H}, g)$ we can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

If $\mathcal{V}$ is integrable,

$$\begin{cases} J_Z X \in \mathcal{H} & \text{if } Z \in \mathcal{V}, X \in \mathcal{H} \\ J_Z X = 0 & \text{otherwise} \end{cases}$$

We thus take the perspective

$$J \colon \mathcal{V} \to \mathrm{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an H-type foliation if

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an <u>H-type foliation</u> if

1. $(\mathbb{M}, \mathcal{V}, g)$ is a totally geodesic foliation with bundle-like metric, and

H-type Foliations

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be a sRmc-manfiold. We say that $(\mathbb{M}, \mathcal{H}, g, J)$ is an H-type foliation if

1. $(\mathbb{M}, \mathcal{V}, g)$ is a totally geodesic foliation with bundle-like metric, and
2. for all $X, Y \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$,

$$\langle J_Z X, J_Z Y\rangle_{\mathcal{H}} = \|Z\|^2 \langle X, Y\rangle_{\mathcal{H}}$$

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
- If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

Parallel Torsion

We also refine the definition of H-type foliations based on the behavior of derivatives of the Hladky-Bott torsion T.

- If $\delta_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ is of Yang-Mills type,
- If $\nabla_{\mathcal{H}} T = 0$ we say $\mathbb{M}$ has horizontally parallel torsion, and
- If $\nabla T = 0$ we say $\mathbb{M}$ has completely parallel torsion.

Lemma

All H-type foliations are Yang-Mills.

J^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

J^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the J^2 condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

J^2 condition

In many model cases, such as the Complex-, Quaternionic-, and Octonionic-Hopf fibrations we have a stronger property:

Definition

Let $(\mathbb{M}, \mathcal{H}, g)$ be an H-type foliation. We say that it satisfies the J^2 condition if for every $Z_1, Z_2 \in \mathcal{V}$ with $\langle Z_1, Z_2 \rangle = 0$ there exists $Z_3 \in \mathcal{V}$ such that

$$J_{Z_1} J_{Z_2} = J_{Z_3}$$

The H-type groups with this property were classified by (M. Cowling, A.H. Dooley, A. Korányi, and F.Ricci '91 [4])

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε,

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε,
- having horizontally parallel torsion,

Metric Connections, Jacobi Equation

For the remainder of the talk, let $(M, \mathcal{H}, g_\varepsilon)$

- be a sRmc-manifold,
- equipped with the canonical variation g_ε,
- having horizontally parallel torsion,
- and satisfying the J^2 condition.

Jacobi Equation

There's a subtlety that needs to be considered. The Levi-Civita connection ∇ is self-adjoint.

Jacobi Equation

There's a subtlety that needs to be considered. The Levi-Civita connection ∇ is self-adjoint.

As a consequence the Jacobi equation is simply

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}W + R(W,\dot{\gamma})\dot{\gamma} = 0$$

Jacobi Equation

There's a subtlety that needs to be considered. The Levi-Civita connection ∇ is self-adjoint.

As a consequence the Jacobi equation is simply

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}W + R(W,\dot{\gamma})\dot{\gamma} = 0$$

but this isn't true for general connections, or in particular the Bott connection.

Adjoint Connections and the Jacobi Equation

For an arbitrary connection ∇ with torsion

$$Tor(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

define its adjoint connection to be

$$\hat{\nabla}_X Y = \nabla_X Y - Tor(X, Y)$$

Adjoint Connections and the Jacobi Equation

For an arbitrary connection ∇ with torsion

$$Tor(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

define its adjoint connection to be

$$\hat{\nabla}_X Y = \nabla_X Y - Tor(X,Y)$$

notice, $\hat{\hat{\nabla}} = \nabla$.

Adjoint Connections and the Jacobi Equation

For an arbitrary connection ∇ with torsion

$$Tor(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

define its adjoint connection to be

$$\hat{\nabla}_X Y = \nabla_X Y - Tor(X,Y)$$

notice, $\hat{\hat{\nabla}} = \nabla$.

In general, the adjoint of a metric connection is not metric. As a consequence, terms involving the torsion of ∇ are introduced to the Jacobi equation.

Jacobi Equation for Metric Adjoint Connections

However, in the special case that both $\nabla, \hat{\nabla}$ are metric, the Jacobi equation along a geodesic γ is

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} W + \hat{R}(W, \dot{\gamma})\dot{\gamma} = 0$$

Jacobi Equation for Metric Adjoint Connections

However, in the special case that both $\nabla, \hat{\nabla}$ are metric, the Jacobi equation along a geodesic γ is

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} W + \hat{R}(W, \dot{\gamma})\dot{\gamma} = 0$$

This is a consequence of the commutation $\nabla_V \dot{\gamma} = \hat{\nabla}_{\dot{\gamma}} V$ for a Jacobi field V along a geodesic γ.

ε-invariant Metric Adjoint Connection

For any $\varepsilon > 0$ the connection

$$\hat{\nabla}^{\varepsilon}_X Y = \nabla_X Y + J^{\varepsilon}_X Y$$

will be metric with metric adjoint ∇^{ε}.

ε-invariant Metric Adjoint Connection

For any $\varepsilon > 0$ the connection

$$\hat{\nabla}^{\varepsilon}_X Y = \nabla_X Y + J^{\varepsilon}_X Y$$

will be metric with metric adjoint ∇^{ε}.

From this we recover a Jacobi equation for all $\epsilon > 0$.

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,
- $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,
- $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot{\gamma}} \nabla^\varepsilon_{\dot{\gamma}} W_i + \hat{R}^\varepsilon(W_i, \dot{\gamma})\dot{\gamma}, W_i \rangle_\varepsilon \geq 0$$

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,
- $\gamma \colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat{R}^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ *it holds that*

$$\sum_{i=0}^{k} \mathrm{Hess}^{\hat{\nabla}^\varepsilon}(d_p^\varepsilon)(W_i, W_i) \leq \sum_{i=0}^{k} \langle W_i, \hat{\nabla}^\varepsilon_{\dot\gamma} W_i \rangle_\varepsilon$$

The Comparison Principle

Theorem (Baudoin, Grong, Kuwada, & Thalmaier '17 [1])

- *Let* $x, y \in \mathbb{M}$,
- $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *a unit speed* g_ε*-geodesic connecting* x, y, *and*
- $W_1, \cdots, W_k$ *be a collection of vector fields along* γ *such that*

$$\sum_{i=0}^{k} \int_0^{r_\varepsilon} \langle \hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat{R}^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \geq 0$$

then at $y = \gamma(r_\varepsilon)$ *it holds that*

$$\sum_{i=0}^{k} \mathrm{Hess}^{\hat{\nabla}^\varepsilon}(d_p^\varepsilon)(W_i, W_i) \leq \sum_{i=0}^{k} \langle W_i, \hat{\nabla}^\varepsilon_{\dot\gamma} W_i \rangle_\varepsilon$$

with equality if and only if the W_i *are Jacobi fields.*

- Along a geodesic γ let V satisfy the Jacobi equation

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - \hat{R}(V, \dot{\gamma})\dot{\gamma} = 0$$

and initial conditions $V(0) = 0, V(r) = X$

- Along a geodesic γ let V satisfy the Jacobi equation

$$\hat{\nabla}_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - \hat{R}(V, \dot{\gamma})\dot{\gamma} = 0$$

and initial conditions $V(0) = 0, V(r) = X$

- Then it can be shown

$$\hat{\nabla}^2 d_p(q)(X, X) = I(V, V)$$

with (ε-invariant) index

$$I(V, V) = \int_0^r \langle \hat{\nabla}^{\varepsilon}_{\dot{\gamma}} V, \nabla^{\varepsilon}_{\dot{\gamma}} V \rangle - \hat{R}^{\varepsilon}(V, \dot{\gamma}, \dot{\gamma}, V)\, dt$$

- Along a geodesic γ let V satisfy the Jacobi equation

$$\hat{\nabla}_{\dot\gamma}\nabla_{\dot\gamma}V - \hat{R}(V,\dot\gamma)\dot\gamma = 0$$

and initial conditions $V(0)=0, V(r)=X$

- Then it can be shown

$$\hat{\nabla}^2 d_p(q)(X,X) = I(V,V)$$

with (ε-invariant) index

$$I(V,V) = \int_0^r \langle \hat{\nabla}^{\varepsilon}_{\dot\gamma}V, \nabla^{\varepsilon}_{\dot\gamma}V\rangle - \hat{R}^{\varepsilon}(V,\dot\gamma,\dot\gamma,V)\,dt$$

- This gives bounds on the behavior of $\mathrm{Hess}^{\hat{\nabla}^{\varepsilon}}(r_\varepsilon)$

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle.

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} | X \perp \mathcal{H}_{Sas} \oplus \mathrm{span}(Y)\}$$

Horizontal Splitting

We introduce an orthogonal splitting of the horizontal bundle. Fixing a vector field $Y \in \mathcal{H}$,

$$\mathcal{H} = \text{span}(Y) \oplus \mathcal{H}_{Riem}(Y) \oplus \mathcal{H}_{Sas}(Y)$$

where

$$\mathcal{H}_{Sas}(Y) = \{J_Z Y | Z \in \mathcal{V}\}$$

$$\mathcal{H}_{Riem}(Y) = \{X \in \mathcal{H} | X \perp \mathcal{H}_{Sas} \oplus \text{span}(Y)\}$$

Lemma

Denoting $n = \text{rk}(\mathcal{H}), m = \text{rk}(\mathcal{V})$, *we will have*

$$\dim(\mathcal{H}_{Sas}) = m, \quad \dim(\mathcal{H}_{Riem}) = n - m - 1$$

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0 \end{cases}$$

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0 \end{cases}$$

$$F_{Sas}(r,\kappa) = \begin{cases} \frac{\sqrt{\kappa}\left(\sin(\sqrt{\kappa}r) - \sqrt{\kappa}r\cos(\sqrt{\kappa}r)\right)}{2 - 2\cos(\sqrt{\kappa}r) - \sqrt{\kappa}r\sin(\sqrt{\kappa}r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}\left(\sqrt{\kappa}r\cosh(\sqrt{\kappa}r) - \sinh(\sqrt{\kappa}r)\right)}{2 - 2\cosh(\sqrt{\kappa}r) + \sqrt{\kappa}r\sinh(\sqrt{\kappa}r)} & \text{if } \kappa < 0 \end{cases}$$

Comparison Functions

Simliarly to the Riemannian case, we consider the comparision functions

$$F_{Riem}(r,\kappa) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}r) & \text{if } \kappa > 0 \\ \frac{1}{r} & \text{if } \kappa = 0 \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r) & \text{if } \kappa < 0 \end{cases}$$

$$F_{Sas}(r,\kappa) = \begin{cases} \frac{\sqrt{\kappa}\left(\sin(\sqrt{\kappa}r) - \sqrt{\kappa}r\cos(\sqrt{\kappa}r)\right)}{2-2\cos(\sqrt{\kappa}r) - \sqrt{\kappa}r\sin(\sqrt{\kappa}r)} & \text{if } \kappa > 0 \\ \frac{4}{r} & \text{if } \kappa = 0 \\ \frac{\sqrt{\kappa}\left(\sqrt{\kappa}r\cosh(\sqrt{\kappa}r) - \sinh(\sqrt{\kappa}r)\right)}{2-2\cosh(\sqrt{\kappa}r) + \sqrt{\kappa}r\sinh(\sqrt{\kappa}r)} & \text{if } \kappa < 0 \end{cases}$$

These comparison functions will correspond to the splitting of $\mathcal{H}$.

Hessian Comparisons

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

- *Let* $\gamma\colon [0, r_\varepsilon] \to \mathbb{M}$ *be a* g_ε*-geodesic. Then*

$$\mathrm{Hess}(r_\varepsilon)(\dot\gamma,\dot\gamma) \le \frac{\|\dot\gamma\|^2\left(1-\|\dot\gamma\|^2\right)}{r_\varepsilon}$$

- *If* $\mathrm{Sec}(X\wedge Y) \ge \rho > 0$ *for all unit* $X, Y \in \mathcal{H}_{Riem}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X,X) \le F_{Riem}(r_\varepsilon, K)$$

- *If* $\mathrm{Sec}(X\wedge J_Z X) \ge \rho > 0$ *for all unit* $X \in \mathcal{H}_{Sas}(\dot\gamma)$*, then*

$$\mathrm{Hess}(r_\varepsilon)(X,X) \le F_{Sas}(r_\varepsilon, K)$$

Where K *is a constant depending on* $\rho, \varepsilon, \|\nabla_{\mathcal{V}} r_\varepsilon\|$*, and* $\|\nabla_{\mathcal{H}} r_\varepsilon\|$*.*

Horizontal Ricci Curvature

We define the horizontal Ricci curvature as the horizontal trace of the Riemann tensor,

$$\begin{aligned}\mathrm{Ric}_{\mathcal{H}}(X,X) &= \sum_{i=0}^{n}\langle R^{\nabla}(W_i,X)X,W_i\rangle \\ &= \langle R^{\nabla}(Y,X)X,Y\rangle + \mathrm{Ric}_{Sas}(X,X) + \mathrm{Ric}_{Riem}(X,X)\end{aligned}$$

Horizontal Ricci Curvature

We define the horizontal Ricci curvature as the horizontal trace of the Riemann tensor,

$$\begin{aligned} \mathrm{Ric}_{\mathcal{H}}(X,X) &= \sum_{i=0}^{n} \langle R^{\nabla}(W_i, X)X, W_i \rangle \\ &= \langle R^{\nabla}(Y,X)X, Y \rangle + \mathrm{Ric}_{Sas}(X,X) + \mathrm{Ric}_{Riem}(X,X) \end{aligned}$$

where the splitting corresponds to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Bonnet-Meyers Estimates

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $$\frac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho}}$$

Bonnet-Meyers Estimates

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $\dfrac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{\pi}{\sqrt{\rho}}$
2. $\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{2\pi}{\sqrt{\rho}}$

Bonnet-Meyers Estimates

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $\dfrac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{\pi}{\sqrt{\rho}}$
2. $\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{2\pi}{\sqrt{\rho}}$
3. $\dfrac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \dfrac{2\pi\sqrt{3}}{\sqrt{\rho}}$

Bonnet-Meyers Estimates

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $$\frac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho}}$$
2. $$\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\rho}}$$
3. $$\frac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi\sqrt{3}}{\sqrt{\rho}}$$

and in each case the fundamental group of $\mathbb{M}$ must be finite.

Bonnet-Meyers Estimates

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $\rho > 0$. Then for unit $X \in \mathcal{H}$,

1. $$\frac{\mathrm{Ric}_{Riem}(X,X)}{n-m-1} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{\pi}{\sqrt{\rho}}$$
2. $$\mathrm{Sec}(X \wedge J_Z X) \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi}{\sqrt{\rho}}$$
3. $$\frac{\mathrm{Ric}_{Sas}(X,X)}{m} \geq \rho \implies \mathrm{diam}_0(\mathbb{M}) \leq \frac{2\pi\sqrt{3}}{\sqrt{\rho}}$$

and in each case the fundamental group of $\mathbb{M}$ must be finite.

The first two of these are sharp, as they are achieved in the complex, quaternionic, and octonionic Hopf fibrations.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\Delta_{\mathcal{H}} r_\varepsilon = \sum_{i=0}^{n} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

$$= \mathrm{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \mathrm{Hess}(r_\varepsilon)(J_{Z_i} Y, J_{Z_i} Y) + \sum_{i=0}^{n-m-1} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$.

sub-Laplacian

Similarly to the horizontal Ricci curvature, we can define the sub-Laplacian as the trace of the Hessian. For the distance function r_ε along a geodesic γ with $Y = \nabla_{\mathcal{H}} r_\varepsilon$,

$$\Delta_{\mathcal{H}} r_\varepsilon = \sum_{i=0}^{n} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

$$= \mathrm{Hess}(r_\varepsilon)(Y, Y) + \sum_{i=0}^{m} \mathrm{Hess}(r_\varepsilon)(J_{Z_i} Y, J_{Z_i} Y) + \sum_{i=0}^{n-m-1} \mathrm{Hess}(r_\varepsilon)(W_i, W_i)$$

for appropriate bases $\{W_i\}$ of $\mathcal{H}$ and $\{Z_i\}$ of $\mathcal{V}$. This splitting corresponds again to the decomposition

$$\mathcal{H} = \mathrm{span}(Y) \oplus \mathcal{H}_{Sas} \oplus \mathcal{H}_{Riem}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m-1)}{r_0}$$

Laplacian Comparisons

In each component of the horizontal decomposition we can use the previous comparisons on the Hessian to obtain

Theorem (Baudoin, Grong, M., & Rizzi '19 [3])

Let $(\mathbb{M}, g, \mathcal{H})$ be an H-type foliation with parallel horizontal Clifford structure and satisfying the J^2 condition, and with nonnegative horizontal Bott curvature. Then there exists a $C > 4$ such that

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n - m + 3 + C(m-1)}{r_0}$$

This is not sharp, but we can recover sharp estimates in each subspace.

References I

- F. Baudoin, E. Grong, K. Kuwada, and A. Thalmaier.
  Sub-Laplacian comparison theorems on totally geodesic Riemannian foliations.
  arXiv e-prints, Jun 2017, 1706.08489.
- F. Baudoin, E. Grong, G. Molino, and L. Rizzi.
  H-type Foliations.
  arXiv e-prints, Dec 2018, 1812.02563.
- F. Baudoin, E. Grong, G. Molino, and L. Rizzi.
  Comparison theorems on H-type sub-Riemannian manifolds.
  arXiv e-prints, Sept 2019, 1909.03532.
- M. Cowling, A. Dooley, A. Korányi, and F. Ricci.
  H-type groups and Iwasawa decompositions.
  Adv. Math., 81(1):1–41, 1991.

References II

R. K. Hladky.
Connections and curvature in sub-Riemannian geometry.
Houston J. Math., 38(4):1107–1134, 2012.

A. Moroianu and U. Semmelmann.
Clifford structures on Riemannian manifolds.
Adv. Math., 228(2):940–967, 2011.

Thank you for your attention!