On the Sobolev quotient in sub-Riemannian geometry

Joint work with J.H.Cheng and P.Yang

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$$(Y) \qquad -c_n \Delta u + R_g = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

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Considering $\overline{R}$ as a Lagrange multiplier, one can try to find solutions by minimizing the *Sobolev-Yamabe quotient*

$$Q_{SY}(u) = \frac{\int_M \left(c_n |\nabla u|^2 + R_g u^2\right) dV}{\left(\int_M |u|^{2^*} dV\right)^{\frac{2}{2^*}}}; \qquad 2^* = \frac{2n}{n-2}.$$

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The *Sobolev-Yamabe constant* is defined as

$$Y(M, [g]) = \inf_{u \neq 0} Q_{SY}(u).$$

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• Since S^n is conformal to $\mathbb{R}^n$, one has that $Y(S^n,[g_{S^n}]) = S_n$.

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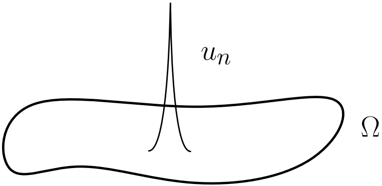
Minimizing sequences u_n tend to concentrate indefinitely inside Ω.



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- In 1984 Schoen proved that $Y(M,[g]) < S_n$ in all other cases, i.e. $n \leq 5$ or (M,g) locally conformally flat, unless (M,g) is *globally* conformal to the round sphere (S^n, g_{S^n}).

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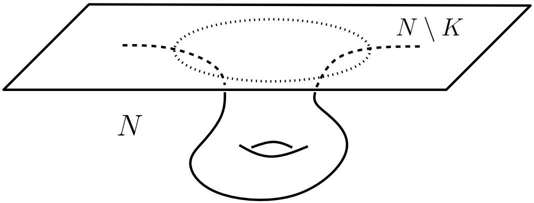

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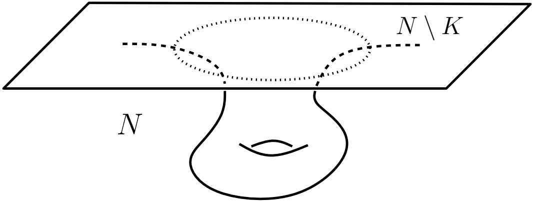

In general relativity these manifolds describe static gravitational systems.

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Then $(M \setminus \{p\}, \hat{g})$ is asymptotically flat, and

$$m(\hat{g}) = \lim_{x\to p} \left(G_p(x) - \frac{1}{d(x,p)} \right) = A.$$

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This condition is quite important for the study of biholomorphic mappings and the $\overline{\partial}$-Neumann problem ([Beals-Fefferman-Grossman, '83]).

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$$\mathring{Z}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}\right); \qquad \mathring{Z}_{\overline{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}\right),$$

ξ_0 is spanned by real and imaginary parts of $\mathring{Z}_1$.

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• Positivity of the mass implies that the Sobolev-Webster quotient of the manifold is lower than that of the sphere, and minimizers exist.

On the positivity condition for the Paneitz operator

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Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

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For small $s \neq 0$, the CR mass of S^3_s is negative ($m_s \simeq -18\pi s^2$).

Some ideas of the proof

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One then needs to verify that the two expansions match, obtaining then the asymptotic behaviour for $s \to 0$ of $A_{(s)}$, proportional to the mass. □

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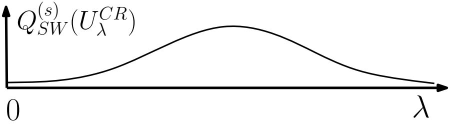
- For $|s| \neq 0$ small, the Webster quotient of the functions U^{CR}_λ has a profile of this kind (need to use Theorem 2 for λ large)



Theorem 3 ([Cheng-M.-Yang, '19])

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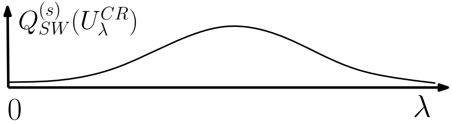
- If a function has low Sobolev-Webster quotient on a Rossi sphere S^3_s it has low Sobolev-Webster quotient also on the standard sphere $S^3 = S^3_0$. Minima for the Webster quotient on the standard S^3 were classifed in [Jerison-Lee, '88] as (CR counterparts of) Aubin-Talenti's functions.

- For $|s| \neq 0$ small, the Webster quotient of the functions U_λ^{CR} has a profile of this kind (need to use Theorem 2 for λ large)



One can then see that the minimum is not attained.

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In the (3D) CR case we obtain positive second variations also for special non-embeddable directions ([Bland, '94]). It would be interesting to observe this change of sign in general also for the mass and the Sobolev quotient.

Some open problems

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Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]).

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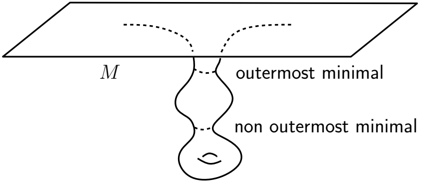
Penrose's inequality gives quantitative lower bounds on the ADM mass in terms of *outermost minimal surfaces*



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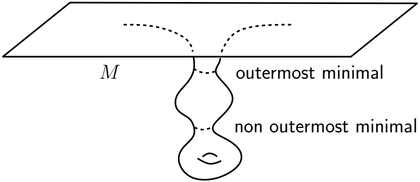

If A is the total area of the outermost minimal surfaces, Penrose's inequality asserts that $m \geq \sqrt{\frac{A}{16\pi}}$.

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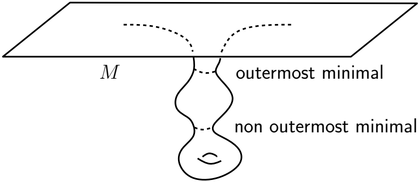

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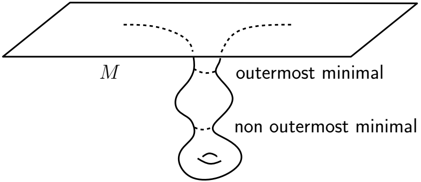

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The inequality was proved in [Huisken-Ilmanen, '01], [Bray, '01] using geometric flows. Are there any analogues in the CR case?

Thanks for your attention