Constant-normal sets and horizontally polynomial functions

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Summary

⋆ In Carnot groups the class of constant-normal sets, which in particular are monotone sets, is very rich.

⋆ In Carnot groups the class of horizontally affine functions, whose sublevel sets are monotone sets, is very rich.

⋆ We have a list of nice properties for constant-normal sets and horizontally affine functions.

collaborations with *G. Antonelli*, *C. Bellettini*, *S. Don*, *T. Moisala*, *D. Morbidelli*, *S. Rigot*, and *D. Vittone*.

Starting point

In **Euclidean**/Abelian geometry:

1. Sets with constant normal are half spaces;
2. Functions that are harmonic for all Laplacians are affine.

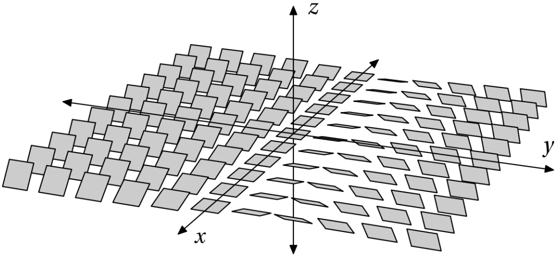
Remark. In non-commutative subRiemannian geometries (i.e., Carnot groups) these properties may FAIL.

Questions:

A. Can we still say something?!?
B. When these properties still hold?

Setting

$\mathbb{G}$ Carnot group, that is, simply connected Lie group whose Lie algebra $\mathfrak{g}$ is graded with Lie generating first layer V_1.



Distinguished set of curves: for $X \in V_1$ and $p \in \mathbb{G}$ the curve

$$t \mapsto p \exp(tX)$$

is a *horizontal line* in $\mathbb{G}$.

Definitions of special sets.

$\star$. Constant-normal sets.

For a half space $W \subset V_1$, a set $E \subset \mathbb{G}$ has *constant normal* if

$$X\mathbb{1}_E \geq 0, \qquad \text{for all } X \in W.$$

$\star$. Monotone sets.

A set $E \subset \mathbb{G}$ is *monotone* if for almost every horizontal line $L \subset \mathbb{G}$, the characteristic function $\mathbb{1}_E$ restricted to L agrees almost everywhere with a monotone function.

Remarks.

1. Every constant-normal set $E \subset \mathbb{G}$ is equivalent to a *precisely constant-normal* set: $\exists \widetilde{E} \subset \mathbb{G}$ with $\mathrm{vol}(\widetilde{E} \Delta E) = 0$ and

$$p \in \widetilde{E}, X \in W \implies p \exp(X) \in \widetilde{E}.$$

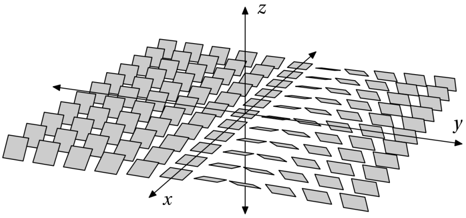
2. We don't know if every monotone set is equivalent to a *precisely monotone* set: $\exists ? \widetilde{E} \subset \mathbb{G}$ with $\mathrm{vol}(\widetilde{E} \Delta E) = 0$ and

$p \in \widetilde{E}, X \in V_1 \implies p \exp(\mathbb{R}X) \cap \widetilde{E}$ and its complement are convex.

Simple consequences

constant-normal
$\implies$ monotone

precisely constant-normal
$\implies$ precisely monotone



In the Heisenberg group, in exponential coordinates $\{z > 0\}$ is a precisely monotone set, without constant normal.

Constant-normal sets

Examples of constant-normal sets

Horizontal half spaces: $\exp(W \oplus V_2 \oplus \cdots \oplus V_s)$

In Engel group. $[X_1, X_2] = X_3\,,\ [X_1, X_3] = X_4$ In $\mathbb{R}^4$, a V_1 may be spanned by $X_1 = \partial_{x_1}\,,$ $X_2 = \partial_{x_2} + x_1\,\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4}$	$\{x \in \mathbb{R}^4 : x_2 > 0, 2x_4x_2 > x_3^2\}$
In rank-2 step-3 free Carnot group $[X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5$ In $\mathbb{R}^5$, a V_1 may be spanned by $X_1 = \partial_1,$ $X_2 = \partial_2 - x_1\partial_3 + \frac{x_1^2}{2}\partial_4 + x_1x_2\partial_5.$	$\{x \in \mathbb{R}^5 : x_2 > 0,$ $x_2^3x_4 - 2x_2^2x_3^2 - 6x_2x_3x_5 - 6x_5^2 > 0\}$

Properties of constant-normal sets

Theorem (with Bellettini)

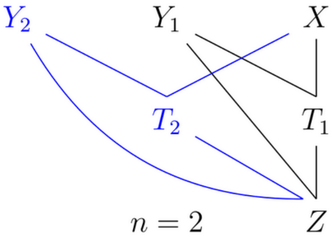
Every constant-normal set in a Carnot group admits a representative that
is a precisely constant-normal set that
is regularly open, is contractible, with contractible complement, and admits a cone property with respect to an open contractible cone.

When constant-normal sets are half spaces?

Theorem (with Moisala)

Let $\mathbb{G}$ be a Carnot group of step at most 3. Then some constant-normal set in $\mathbb{G}$ is not a half space if and only if its Lie algebra has one of the Engel-type algebras (see later) as a quotient.

The second Engel-type algebra $\mathbb{E}\mathbb{n}^2$ is the six-dimensional algebra whose diagram is presented below:



Rectifiability of finite-perimeter sets

When all constant-normal sets are half spaces, we have strong rectifiability of finite-perimeter sets:

Theorem (Franchi, Serapioni, and SerraCassano, after De Giorgi and Ambrosio)

Let $\mathbb{G}$ be a Carnot group on which the only constant-normal sets are the half spaces.
Then the reduced boundary of every finite-perimeter set in $\mathbb{G}$ is intrinsically C^1-rectifiable:
If $E \subseteq \mathbb{G}$ is a set of finite perimeter, then there exists a family $\{S_h : h \in \mathbb{N}\}$ of intrinsic C^1-regular hypersurfaces in $\mathbb{G}$ such that

$$\mathcal{F}E \setminus \bigcup_{h\in\mathbb{N}} S_h \text{ is negligible,}$$

with respect to the codimension-1 Hausdorff measure, where $\mathcal{F}E$ denotes the reduced boundary of E.

Rectifiability of finite-perimeter sets

Since constant-normal sets have a cone property, we always have some rectifiability of finite-perimeter sets:

Theorem (with Don, Moisala, and Vittone)

Let $\mathbb{G}$ be a Carnot group and let $E \subseteq \mathbb{G}$ be a set of finite perimeter. Then there exists a family $\{C_h : h \in \mathbb{N}\}$ of open cones in $\mathbb{G}$ and a family $\{\Gamma_h : h \in \mathbb{N}\}$ of subsets of $\mathbb{G}$ such that each Γ_h satisfies the C_h-cone property and

$$\mathcal{F}E = \bigcup_{h \in \mathbb{N}} \Gamma_h,$$

where $\mathcal{F}E$ denotes the reduced boundary of E.

When constant-normal sets are half spaces?

The property "all constant-normal sets are half spaces" can be reduced to an algebraic property:

Remark. Only half spaces are the constant-normal sets associated to a horizontal half space $W \subset V_1$ if and only if

$$\bigcup_{k=0}^{\infty} \exp(W)^k = \exp(W \oplus [\mathfrak{g}, \mathfrak{g}]).$$

Terminology. If this happen for all W, we say that the Carnot algebra $\mathfrak{g}$ is *semigenerated*.

Carnot algebra of type diamond

Definition (Type diamond)

Let $\mathfrak{g}$ be a Carnot algebra. We say that $\mathfrak{g}$ is *of type* $(\Diamond)$ if for each subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ for which $\mathfrak{h} \cap V_1$ has codimension 1 in V_1, there exists a basis $\{X_1, \ldots, X_m\}$ of V_1 such that

$$\mathrm{ad}^2_{X_i} X_j \in \mathfrak{h} \text{ and } \mathrm{ad}^2_{\mathrm{ad}^k_{X_i} X_j}(X_i) \in \mathfrak{h}, \qquad \forall i, j = 1, \ldots, m \forall k \geq 2.$$

Theorem (with Moisala)

Every Carnot algebra of type $(\Diamond)$ *is semigenerated.*

N.B. step 2 $\implies$ type $(\star)$ $\implies$ type $(\Diamond)$ $\implies$ semigenerated

A rephrasing

Corollary

If the Lie algebra of a Carnot group $\mathbb{G}$ is semigenerated (e.g., if it is of type $(\Diamond)$ or has step 3 and does not have any Engel-type algebra as a quotient), then the reduced boundary of every set of finite perimeter in $\mathbb{G}$ is intrinsically C^1-rectifiable.

Monotone sets

When monotone sets are half spaces?

When monotone sets are half spaces?
Very partial results:

for $\mathbb{H}^1$ by Cheeger-Kleiner,
for $\mathbb{H}^n$ by Naor-Young,
for $\mathbb{H}^1 \times \mathbb{R}$ by Morbidelli.

In the rank-2 step-2 free Carnot group there are monotone sets that are **not** half spaces.
It's not clear if there is an *algebraic* viewpoint for studying monotone sets.

Simplification:

study functions instead of sets

Definition (horizontally affine functions). On a Carnot group $\mathbb{G}$ a measurable function $f : \mathbb{G} \to \mathbb{R}$ is *horizontally affine* if one of the following equivalent conditions holds true:

(A.1) up to changing representative for f, one has that for every $X \in V_1$ and every $p \in \mathbb{G}$, the following map is affine:

$$t \in \mathbb{R} \mapsto f(p\exp(tX)) \in \mathbb{R} \, ;$$

(A.2) for every $X \in V_1$, one has $X^2 f = 0$;

(A.3) for every basis $X_1, \dots, X_m$ of V_1, the map f annihilates its subLaplacian, i.e., one has

$$X_1^2 f + \cdots + X_m^2 f = 0 \, .$$

(A.4) for every (resp. for some) basis $X_1, \dots, X_m$ of V_1 one has

$$X_i X_j f + X_j X_i f = 0, \qquad \forall i, j = 1, \dots, m \, .$$

Remarks

1. sublevel sets of horizontally affine functions are precisely monotone sets.

2. horizontally affine functions are harmonic (hence C^∞) and are the solutions of a *finite* system of PDEs.

3. level sets of horizontally affine functions are calibrated minimal surfaces. (In fact, precisely monotone sets are minimal surfaces.)

4. horizontal coordinates are horizontally affine functions and give vertical half spaces as sublevel sets.

5. If all sublevel sets of a horizontally affine function have constant normal with half space W, then they are the half spaces associated to W.

A result

Theorem (with Morbidelli and Rigot)

In step 2 Carnot groups, horizontally affine functions are completely determined (see later).

A broader viewpoint

Given a Lie group $\mathbb{G}$ and a subset $V \subseteq \mathrm{Lie}(\mathbb{G})$, a distribution T on $\mathbb{G}$ is *horizontally polynomial (with respect to V)* if

$$\forall X \in V \, \exists k \in \mathbb{N} : X^k T = 0, \text{ as distributions.}$$

Clearly, we will mostly assume that V is Lie-generating.
In this case, one may extract a linearly independent set $S \subseteq V$ that is Lie-generating and for which there is a uniform k.

Results

Theorem (with Antonelli)

If T is a distribution that is horizontally polynomial with respect to a Lie-generating subset, then T is represented by an analytic function.
Moreover, if the Lie group is nilpotent, then in exponential coordinates T is represented by a polynomial.

The polynomial degree of T is bounded by the degree of horizontal polynomiality of T and the step and rank of G.

In arbitrary Lie groups, the set of horizontally k-polynomial functions with respect to a Lie-generating subset form a finite-dimensional vector space.

Examples

In some Carnot group, there is a constant normal set that is not the sublevel set of a horizontally affine function.

E.g., in the previous model for the Engel group: $\{x_4 > 0\}$.

There is a constant normal set that is not the sublevel set of a horizontally polynomial function.

E.g., there are constant normal sets that are not of finite Riemannian perimeter [Bellettini-LeDonne2019], but level sets of polynomial functions are algebraic varieties.
N.B. horizont. polynomial = polynomial [Antonelli-LeDonne2020].

A sketch of a proof

1. Let T be a horizontally affine distribution in a Carnot group.

2. Approximate it with a horizontally affine smooth function (via convolution with mollifications).

3. Show that the jet of f at a point q determines f

$$
\begin{aligned}
(X_1 f)(qe^{tX}) &= \frac{d}{d\epsilon} f(qe^{tX} e^{t\epsilon X_1})|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} f(qe^{tX} e^{\epsilon X_1} e^{-tX} e^{tX})|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} f(qe^{tX} e^{\epsilon X_1} e^{-tX}) + tXf(qe^{tX} e^{\epsilon X_1} e^{-tX})|_{\epsilon=0} \\
&= \mathrm{Ad}_{\exp(tX)}(X_1) f_{|_q} + t\mathrm{Ad}_{\exp(tX)}(X_1) X f_{|_q} \\
&= e^{t\mathrm{ad}(X)}(X_1) f_{|_q} + t\, e^{t\mathrm{ad}(X)}(X_1) X f_{|_q}.
\end{aligned}
$$

A sketch of a proof (continuation)

4. If f is a horizontally k-polynomial smooth function then

$$(X_1 \dots X_r f)(q \exp(tX)) = \sum_{i=0}^{k-1} \frac{t^i}{i!} \mathrm{Ad}_{\exp(tX)}(X_1) \dots \mathrm{Ad}_{\exp(tX)}(X_r) X^i f_{|_q}.$$

5. $f(\exp(t_1 Y_1) \dots \exp(t_\ell Y_\ell))$ is a polynomial in $t_1, \dots, t_\ell$.

6. f has polynomial growth, with a growth exponent that only depends on the group (and on the degree of polinomiality).

A sketch of a proof (continuation)

7. Write f as sum/series of homogeneous affine polynomials. The bound on the growth exponent implies that f is a polinomial.

8. Since the space of horizontally affine smooth functions is a finite-dimensional vector space, then it is closed under convergence: every horizontally affine distribution is a polynomial.

□

Extra

Merci

Thanks

EXTRA

Theorem (with Morbidelli and Rigot)

Let $\mathbb{F}_{n,2}$ the free Carnot group of step 2 and rank n, seen as

$$\mathbb{F}_{n,2} = \Lambda^1(\mathbb{R}^n) \oplus \Lambda^2(\mathbb{R}^n).$$

Then $\mathrm{Aff}_{\mathrm{hor}}(\mathbb{F}_{n,2})$ and $\oplus_{k=0}^{n}\Lambda^k(\mathbb{R}^n)$ are isomorphic as real vector spaces.
Namely, a function $f : \mathbb{F}_{n,2} \to \mathbb{R}$ is in $\mathrm{Aff}_{\mathrm{hor}}(\mathbb{F}_{n,2})$ if and only if there is $(\eta_0, \ldots, \eta_n) \in \oplus_{k=0}^{n}\Lambda^k(\mathbb{R}^n)$ such that

$$f(\theta, \omega)\, dx_1 \wedge \cdots \wedge dx_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \omega^k \wedge \eta_{n-2k} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \theta \wedge \omega^k \wedge \eta_{n-2k-1}\ .$$

Theorem

Let $(\mathbb{G}, \cdot) = (V_1 \times V_2, [\cdot, \cdot])$ *be a step-two Carnot group. Then*

$$\mathrm{Aff}_{\mathrm{hor}}(\mathbb{G}) = \mathrm{Aff}(V_1 \times V_2)$$

if and only if the following holds true:

$$b : V_1 \times V_2 \to \mathbb{R} \text{ bilinear with } b(x, [x, x']) = 0 \; \forall\, x, x' \in V_1 \implies b = 0.$$