ON THE SET OF NON RADIATIVE SOLUTIONS FOR THE ENERGY CRITICAL WAVE EQUATION

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Abstract. Non radiative solutions of the energy critical non linear wave equation are global solutions u that furthermore have vanishing asymptotic energy outside the lightcone at both $t \to \pm\infty$:

$$\lim_{t\to\pm\infty} \|\nabla_{t,x} u(t)\|_{L^2(|x|\geqslant|t|+R)} = 0,$$

for some $R > 0$. They were shown to play an important role in the analysis of long time dynamics of solutions, in particular regarding the soliton resolution: we refer to the seminal works of Duyckaerts, Kenig and Merle, see [5] and the references therein.

We show that the set of non radiative solutions which are small in the energy space is a manifold whose tangent space at 0 is given by non radiative solutions to the linear equation (described in [2]). We also construct nonlinear solutions with an arbitrary prescribed radiation field.

1. Introduction

We consider solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ (I interval of $\mathbb{R}$) of the energy critical semilinear wave equation in dimension $3 \leqslant d \leqslant 6$:

$$\Box u = f(u), \tag{1}$$

with $f(x) = \pm|x|^{q-1}x$ or $f(x) = \pm x^q$ (if q is an integer), where $q = \frac{d+2}{d-2}$ is the $\dot{H}^1$-critical exponent. If u is a time dependent function, we denote $\vec{u} = (u, \partial_t u)$. Denote $\mathcal{H} := \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For a time interval $I \subset \mathbb{R}$, we define the spaces

$$W(I) = L^q(I, L^{2q}(\mathbb{R}^d)) \quad \text{and} \quad N(I) = L^1(I, L^2(\mathbb{R}^d))$$

together with

$$X(I) = \mathscr{C}(I, \dot{H}^1(\mathbb{R}^d)) \cap \mathscr{C}^1(I, L^2(\mathbb{R}^d)) \cap W(I),$$

with the natural norm

$$\|u\|_{X(I)} = \|u\|_{\mathscr{C}(I,\dot{H}^1(\mathbb{R}^d))} + \|\partial_t u\|_{\mathscr{C}(I,L^2(\mathbb{R}^d))} + \|u\|_{W(I)}.$$

We now define the linear and nonlinear flows: if $(u_0, u_1) \in \mathcal{H}$, then $\vec{u}_L(t) = S_L(t)(u_0, u_1)$ is the solution of the linear wave equation

$$\begin{cases} \Box u_L = 0, \\ \vec{u}_L(0) = (u_0, u_1). \end{cases} \tag{2}$$

Similarly, concerning the nonlinear equation, the problem is locally well posed for data (u_0, u_1) in $\mathcal{H}$ and furthermore, if they are small in that space, the non linear solution is global and scatters linearly as $t \to \pm\infty$: see for example Strauss

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[17], Rauch [16], Pecher [15], Ginibre-Velo [7] or Lindblad-Sogge [14] among others. In that case, we will denote $\vec{u}(t) = \mathcal{S}(t)(u_0, u_1)$ the solution to the nonlinear wave equation (1) with initial data $\vec{u}(0) = (u_0, u_1)$. We may write $S_L(u_0, u_1)$ and $\mathcal{S}(u_0, u_1)$ to denote the space time function $\vec{u}_L$ and $\vec{u}$ respectively.

For a space time function $\vec{v} \in X(\mathbb{R})$, we define its radiation energy outside a light cone (of base $R \geqslant 0$) by

$$E_{\mathrm{ext},R}(\vec{v}) := \frac{1}{2}\left(\lim_{t\to+\infty}(\|\nabla v\|^2_{L^2(|x|\geqslant t+R)} + \|\partial_t v\|^2_{L^2(|x|\geqslant t+R)}) \right.$$
$$\left. + \lim_{t\to-\infty}(\|\nabla u\|^2_{L^2(|x|\geqslant |t|+R)} + \|\partial_t u\|^2_{L^2(|x|\geqslant |t|+R)})\right),$$

provided that the limits exist.
If u is a global solution to the linear or non linear energy critical wave equation (2), or (1), due to finite speed of propagation, the energy outside a light cone

$$\|\nabla u\|^2_{L^2(|x|\geqslant t+R)} + \|\partial_t u\|^2_{L^2(|x|\geqslant t+R)}$$

is decreasing as a function of $t \geq 0$ and admits a limit as $t \to +\infty$, for any $R \geqslant 0$ (and also as $t \to -\infty$), and so its radiation energy is well defined for any $R \geqslant 0$.

We say that a space time function $\vec{v} \in X(\mathbb{R})$ is *non radiative* if $E_{\mathrm{ext},R}(\vec{v}) = 0$ for some $R \geqslant 0$. Non radiative solutions play a crucial role as the main obstruction in the energy channel method: this machinery was developed with great success, by Duyckaerts, Kenig, Merle and collaborators, to understand the long time behavior of solution to the *radial* energy critical non linear wave equation, in relation with the soliton resolution conjecture. We refer for example to [4, 5] and the references therein. We believe that a fine understanding of these particular solutions might constitute a useful step as well in the soliton resolution in the general case (without symmetry).

Our goal in this article is to give a description of an initial data which leads to non radiative solutions $\vec{u}$ to (1).

We described in [2], for odd dimensions, the linear space $P(R)$ of initial datum $(v_0, v_1) \in \mathcal{H}$ that give rise to a solution $\vec{v} = S_L(v_0, v_1)$ to the linear wave equation such that $E_{\mathrm{ext},R}(\vec{v}) = 0$, in terms of the the Radon transform of the initial data (v_0, v_1) and according to its decomposition in spherical harmonics: for the convenience of the reader, we give further details in the Appendix A, see in particular (20). This was first done for radial data in odd dimension by [9], and in even dimension in [11] (see also [1]), and it was extended to non radial data for odd dimensions in [2] and later in even dimension in [10].
Let us define the operator $\mathcal{T}$ as follows: for a function v defined on $\mathbb{R}^d$, $\mathcal{T}v$ is a function of two variables (s, ω), defined on $\mathbb{R} \times \mathbb{S}^{d-1}$ by its (partial) Fourier transform in the first variable s:

$$\mathcal{F}_{s\to\nu}(\mathcal{T}v)(\nu, \omega) = c_0|\nu|^{\frac{d-1}{2}}(e^{i\tau}\mathbb{1}_{\nu<0} + e^{-i\tau}\mathbb{1}_{\nu\geqslant 0})\hat{v}(\nu\omega), \tag{3}$$
$$\text{where} \quad \tau := \frac{d-1}{4}\pi \quad \text{and} \quad c_0 = \frac{1}{\sqrt{2(2\pi)^{d-1}}}.$$

The previous formula can also be expressed in term of the Radon transform $\mathcal{R}$: it is defined for $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathcal{R}f : \mathbb{R} \times \mathbb{S}^{d-1} \to \mathbb{R}, \quad (s, \omega) \mapsto \int_{\omega\cdot y=s} f(y)dy,$$

(with Lebesgue measure on the hyperplane $\{y : \omega \cdot y = s\}$) and it can be extended to $f \in L^2(\mathbb{R}^d)$. Then there holds

$$\mathcal{T} = m_d(D_s)\mathcal{R} \quad \text{where} \quad m_d(\nu) := c_0|\nu|^{\frac{d-1}{2}} \left(e^{i\tau}\mathbb{1}_{\nu<0} + e^{-i\tau}\mathbb{1}_{\nu\geqslant 0}\right).$$

In odd dimension, this relation simply writes:

$$\mathcal{T} = \frac{(-1)^{\frac{d-1}{2}}}{\sqrt{2(2\pi)^{d-1}}}\partial_s^{\frac{d-1}{2}}\mathcal{R}.$$

We refer to [2] for details.
Our statement regarding the radiation of linear wave solutions is as follows. It is closely related to the radiation field of Friedlander [6], see also Katayama [8] for a related result.

Proposition 1.1 (Radiation field and concentration of energy on the light cone, [2, Theorem 1.1])**.** *Let $(v_0, v_1) \in \mathcal{H}$, and $\vec{v} = S_L(v_0, v_1)$ be the linear solution to (2). Then as $t \to +\infty$, there holds the convergence in $L^2(\mathbb{R}^d)^{1+d}$*

$$\nabla_{t,x}v(t,x) - \frac{1}{\sqrt{2}|x|^{\frac{d-1}{2}}}(\partial_s\mathcal{T}v_0 - \mathcal{T}v_1)\left(|x| - t, \frac{x}{|x|}\right) \times \begin{pmatrix} -1 \\ x/|x| \end{pmatrix} \to 0. \tag{4}$$

Furthermore, one has

$$\lim_{t\to+\infty} \|\nabla v\|^2_{L^2(|x|\geqslant t+R)} = \lim_{t\to+\infty} \|\partial_t v\|^2_{L^2(|x|\geqslant t+R)} = \frac{1}{2}\|\partial_s\mathcal{T}v_0 - \mathcal{T}v_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}. \tag{5}$$

The function $\partial_s\mathcal{T}v_0 - \mathcal{T}v_1$ in (4) is called the radiation field (at $+\infty$) of $\vec{v}$. Note that changing v_1 to $-v_1$ and reversing time, we get the same result in negative time

$$\lim_{t\to-\infty} \|\nabla v\|^2_{L^2(|x|\geqslant t+R)} = \lim_{t\to-\infty} \|\partial_t v\|^2_{L^2(|x|\geqslant t+R)} = \frac{1}{2}\|\partial_s\mathcal{T}v_0 + \mathcal{T}v_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}, \tag{6}$$

so that

$$E_{\text{ext},R}(\vec{v}) = \|\partial_s\mathcal{T}v_0\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})} + \|\mathcal{T}v_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.$$

We want to define $\mathcal{P}(R)$ the (nonlinear) space of initial datum giving rise to nonlinear radiative solutions. More precisely, we denote

$$\mathcal{P}(R) = \{(u_0, u_1) : \mathcal{S}(u_0, u_1) \text{ is defined globally on } \mathbb{R} \text{ and } E_{\text{ext},R}(\mathcal{S}(u_0, u_1)) = 0\}.$$

Our first result states that around $0 \in \mathcal{H}$, $\mathcal{P}(R)$ is a submanifold of $\mathcal{H}$, whose tangent space at 0 is $P(R)$.

Theorem 1.2. *Let $d = 3$ or 5. Let $R > 0$, and denote π_R the orthogonal projection on $P(R)$ (in $\mathcal{H}$).*
There exists $\varepsilon > 0$ and a $\mathscr{C}^1$ map

$$\Phi : B_{\mathcal{H}}(0, \varepsilon) \to \mathcal{H}.$$

so that Φ is a diffeomorphism to its image $V = \Phi(B_{\mathcal{H}}(0, \varepsilon)) \subset B_{\mathcal{H}}(0, 2\varepsilon)$ whose differential at zero is the identity and satisfies

$$\forall (u_0, u_1) \in B_{\mathcal{H}}(0, \varepsilon), \quad \|(u_0, u_1) - \Phi(u_0, u_1)\|_{\mathcal{H}} \leqslant \|(u_0, u_1)\|^q_{\mathcal{H}}.$$

$$\forall (u_0, u_1) \in B_{\mathcal{H}}(0, \varepsilon), \quad \pi_R \circ \Phi(u_0, u_1) = \pi_R(u_0, u_1).$$

Moreover, when restricted to $P(R)$, we have $\Phi\left(P(R) \cap B_{\mathcal{H}}(0, \varepsilon)\right) = \mathcal{P}(R) \cap V$.
In particular, $\mathcal{P}(R) \cap V$ is a submanifold of $\mathcal{H}$ with tangent space at 0 equal to $P(R)$. Moreover, $(\pi_R)_{|\mathcal{P}(R)\cap V}$ is a chart from $\mathcal{P}(R) \cap V$ to $P(R) \cap B_{\mathcal{H}}(0, \varepsilon)$ with inverse Φ.

In particular, this result proves that there are a lot of nonlinear radiative solutions, at least as many as the linear set $P(R)$ which is actually a large space, see Appendix A.

Simple non radiative solutions can be constructed as follows: it suffices to consider a static solutions $u(t,x) = u(x)$ for $|x| \geq |t| + R$, with $-\Delta u = f(u)$. Such solutions outside of a ball have been precisely described in our recent work [3] for analytic nonlinearity (which is useful for (1) in dimension 3). The set $\mathcal{P}(R)^{stat}$ of such small solutions is also a manifold whose tangent set at 0 is the set $P(R)^{stat}$ of linear solutions of $\Delta u_L = 0$; but $\mathcal{P}(R)^{stat}$ is actually a strict subset of $\mathcal{P}(R)$, by a substantial margin: see Remark A.1 for more precisions.
$P(R)^{stat}$ is also a subset of $P(R)$, so we recover the inclusion $\mathcal{P}(R)^{stat}$ into $\mathcal{P}(R)$ at the tangent space level. Yet, in [3], we give a more precise statement: the nonlinear static solutions of $\mathcal{P}(R)^{stat}$ "look" like the linear one $P(R)^{stat}$ at infinity. In a suitable space Z_r of analytic functions on $\mathbb{S}^{d-1}$ adapted to the operator Δ, there exists a unique $u_L \in P(R)^{stat}$ so that

$$\|(u - u_L)(r\cdot)\|_{Z_r} \underset{r\to+\infty}{\longrightarrow} 0.$$

Moreover, the application $u \mapsto u_L$, that appears as a kind of scattering operator is a (local) bijection. It would be very interesting to obtain such precise description for the nonlinear non radiative solutions.

Our second result is related to wave operator: it says that given any radiation field F (as in (4)), there exists a unique nonlinear solution of (1) with this prescribed radiation field. The precise statement is as follows.

Theorem 1.3. *Let $3 \leqslant d \leqslant 6$ and $F \in L^2(\mathbb{R} \times \mathbb{S}^{d-1})$. Then, there exist $T \in \mathbb{R}$ and a unique $u \in X([T, +\infty)$ solution of the nonlinear equation (1) so that, as $t \to +\infty$,*

$$\nabla_{t,x} u(t,x) - \frac{1}{\sqrt{2}|x|^{\frac{d-1}{2}}} F\left(|x| - t, \frac{x}{|x|}\right) \times \begin{pmatrix} -1 \\ x/|x| \end{pmatrix} \to 0 \quad in \quad L^2(\mathbb{R}^d)^{1+d}.$$

Furthermore, if $\|F\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}$ is small enough, one can choose $T = 0$ and $u \in X(\mathbb{R})$ is defined globally.

This result is independent of Theorem 1.2, but relies on a linear scattering result in X and on Proposition 1.1, which ensures that the map giving the radiation of (from a linear solution) is onto.
We refer to [13, Theorem 1.1] for a result with a similar flavor, for wave type equations (with other nonlinearities) in dimension 3, but in different functional spaces; see also [12].

2. Proofs

The spaces $W(I)$, $X(I)$ and $W(I)$ were chosen to satisfy the following Strichartz and nonlinear estimates. For a constant independent of the interval I (or of $t \in \mathbb{R}$), we have

$$\begin{aligned} \|S_L(t)(u_0, u_1)\|_{X(\mathbb{R})} &\leqslant C \|(u_0, u_1)\|_{\mathcal{H}}, \\ \|(u(0), \partial_t u(0))\|_{\mathcal{H}} &\leqslant C \|u\|_{X(\mathbb{R})}, \\ \left\| \int_{-\infty}^{+\infty} \cos(\tau|D_x|) h(\tau) d\tau \right\|_{L^2(\mathbb{R}^d)} &\leqslant C \|h\|_{N(\mathbb{R})}, \\ \left\| \int_{-\infty}^{+\infty} \cos(\tau|D_x|) h(\tau) d\tau \right\|_{L^2(\mathbb{R}^d)} &\leqslant C \|h\|_{N(\mathbb{R})}, \end{aligned}$$

$$\left\| \int_{-\infty}^{+\infty} \sin(\tau|D_x|)h(\tau)d\tau \right\|_{L^2(\mathbb{R}^d)} \leqslant C \left\| h \right\|_{N(\mathbb{R})}, \tag{7}$$
$$\left\| \int_{.}^{+\infty} \frac{\sin((. - \tau)|D_x|)}{|D_x|} h(\tau)d\tau \right\|_{X(I)} \leqslant C \left\| h \right\|_{N(\mathbb{R})},$$
$$\left\| \int_{t}^{+\infty} \frac{\sin((t - \tau)|D_x|)}{|D_x|} h(\tau)d\tau \right\|_{\mathcal{H}} \leqslant C \left\| h \right\|_{N([t,+\infty))}.$$

The related Strichartz estimates can for example be found in [14, Theorem 3.1], see also [7]. Also notice that N is such that if $h \in N([A, +\infty))$ for some $A \in \mathbb{R}$, then

$$\|h\|_{N([t,+\infty))} \to 0 \text{ as } t \to +\infty, \tag{8}$$

(and similarly in a neighbourhood of $-\infty$). We will finally need the nonlinear estimate

$$\|f(u) - f(v)\|_{N(I)} \leqslant C \|u - v\|_{W(I)} (\|u\|_{W(I)}^{q-1} + \|v\|_{W(I)}^{q-1}). \tag{9}$$

It does hold in the cases considered for (1) since $|f'(s)| \leqslant C|s|^{q-1}$ and due to Hölder estimates. In fact, our proofs work in any functional setting that respects the above conditions (7)-(8)-(9).

Let us start by a a few observations related to the operator $\mathcal{T}$.

Definition 2.1. We denote:

$$L^2_{\text{odd}}(\mathbb{R} \times \mathbb{S}^{d-1}) := \left\{ F \in L^2(\mathbb{R} \times \mathbb{S}^{d-1});\ F(s, \omega) = -F(-s, -\omega), a.e. \right\},$$
$$L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1}) := \left\{ F \in L^2(\mathbb{R} \times \mathbb{S}^{d-1});\ F(s, \omega) = F(-s, -\omega), a.e. \right\}.$$

Lemma 2.2 ([2, Lemma 4.14]). *Let d be odd. $\mathcal{T}$ defines an isometry from $L^2(\mathbb{R}^d) \to L^2(\mathbb{R} \times \mathbb{S}^{d-1})$ and is therefore an isomorphism to its range defined by*

$$Range(\mathcal{T}) = \begin{cases} L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1}) & \text{if } d \equiv 1[4], \\ L^2_{\text{odd}}(\mathbb{R} \times \mathbb{S}^{d-1}) & \text{if } d \equiv 3[4]. \end{cases}$$

Similarly, $\partial_s \mathcal{T} : \dot{H}^1(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1} \times \mathbb{R})$ is isometric and

$$Range(\partial_s \mathcal{T}) = \begin{cases} L^2_{\text{odd}}(\mathbb{R} \times \mathbb{S}^{d-1}) & \text{if } d \equiv 1[4], \\ L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1}) & \text{if } d \equiv 3[4]. \end{cases}.$$

We obtain the following corollary.

Corollary 2.3. *Let d be odd and $R > 0$. There exists a continuous linear map $G_R^1 : L^2((R, +\infty) \times \mathbb{S}^{d-1}) \mapsto L^2(\mathbb{R}^d)$ so that for any $F \in L^2((R, +\infty) \times \mathbb{S}^{d-1})$, $\mathcal{T} G_R^1 F = F$ a.e. on $(R, +\infty)$.*
Similarly, there exists a continuous linear map $G_R^0 : L^2((R, +\infty) \times \mathbb{S}^{d-1}) \mapsto \dot{H}^1(\mathbb{R}^d)$ so that for any $F \in L^2((R, +\infty) \times \mathbb{S}^{d-1})$, $\partial_s \mathcal{T} G_R^0 F = F$ a.e. on $(R, +\infty)$.

Proof. We just prove the result for $\mathcal{T}$ and $d \equiv 1[4]$, the other cases being similar. Since $Range(\mathcal{T}) = L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1})$ is a closed subsets of the Banach space $L^2(\mathbb{R} \times \mathbb{S}^{d-1})$, we can apply the open mapping Theorem of Banach to define a continuous inverse $\mathcal{T}^{-1}$ from $L^2_{\text{even}}(\mathbb{R} \times \mathbb{S}^{d-1})$ to $L^2(\mathbb{R}^d)$. Let $\widetilde{F}$ be the even extension of $F \in L^2((R, +\infty) \times \mathbb{S}^{d-1})$ that is equal to zero on $s \in [-R, R]$. More precisely

$$\widetilde{F}(s, \omega) = F(s, \omega) \quad \text{for } s > R$$
$$\widetilde{F}(s, \omega) = F(-s, -\omega) \quad \text{for } s < -R$$
$$\widetilde{F}(s, \omega) = 0 \quad \text{for } s \in [-R, R].$$

It is clear that $\widetilde{F} \in L^2_{\text{even}}(\mathbb{R}\times\mathbb{S}^{d-1})$. Defining $G^1_R F = \mathcal{T}^{-1}\widetilde{F}$, we obtain $\mathcal{T}G^1_R F = \widetilde{F}$ which satisfies the expected result. □

Given a source term f, we can now construct a solution to the linear equation with this source term, which is non radiative.

Proposition 2.4. *Let d odd and X, N functional spaces satisfying* (7) *and* (8). *Let $h \in N(\mathbb{R})$. There exists a* continuous *linear map $T : N(\mathbb{R}) \to X(\mathbb{R})$, such that for any $h \in N(\mathbb{R})$, $u = Th$ is the unique element $u \in X(\mathbb{R})$ satisfying*

(1) u is solution of $\Box u = h$,
(2) $E_{\text{ext},R}(\vec{u}) = 0$,
(3) $\pi_R(\vec{u}(0)) = 0$.

Proof. Step 1. We first look for $\widetilde{u}$ satisfying the hypothesis 1) and 2), but not necessarily 3) We decompose $\tilde{u} = v + w$ where

$$v := \int_{-\infty}^{t} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau)d\tau,$$

so that $\Box v = h$ with morally 0 data at $-\infty$ and w solution of $\Box w = 0$ is to be chosen later on. Notice that changing t to $-t$ in (7), we get

$$\|\vec{v}(t)\|_{\mathcal{H}} \leqslant \left\| \int_{-\infty}^{t} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau)d\tau \right\|_{\mathcal{H}} \leqslant C \|h\|_{N((-\infty,t])}.$$

Using (8), this directly implies

$$\lim_{t\to-\infty} (\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \|\partial_t v\|^2_{L^2(\mathbb{R}^d)}) = 0. \tag{10}$$

Also, by (7), there hold

$$\|v\|_{X(\mathbb{R})} \leqslant C \|h\|_{N(\mathbb{R})}.$$

Let us now estimate the exterior energy (outside a truncated cone) of $\vec{v}$ as $t \to +\infty$. We write

$$\begin{aligned} v(t) &= \int_{-\infty}^{+\infty} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau)d\tau - \int_{t}^{+\infty} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau)d\tau \\ &= \frac{\sin(t|D_x|)}{|D_x|} \int_{-\infty}^{+\infty} \cos(\tau|D_x|)h(\tau)d\tau - \cos(t|D_x|) \int_{-\infty}^{+\infty} \frac{\sin(\tau|D_x|)}{|D_x|} h(\tau)d\tau \\ &\quad - \int_{t}^{+\infty} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau)d\tau \\ &=: \frac{\sin(t|D_x|)}{|D_x|} v_{1+} + \cos(t|D_x|)v_{0+} + r(t). \end{aligned} \tag{11}$$

In other words, $\vec{v} = S_L(v_{0+}, v_{1+}) + \vec{r}$. We estimate using (7)

$$\begin{aligned} \|v_{1+}\|_{L^2(\mathbb{R}^d)} &= \left\| \int_{-\infty}^{+\infty} \cos(\tau|D_x|)h(\tau)d\tau \right\|_{L^2(\mathbb{R}^d)} \leqslant C \|h\|_{N(\mathbb{R})}, \\ \|v_{0+}\|_{\dot{H}^1(\mathbb{R}^d)} &= \left\| \int_{-\infty}^{+\infty} \sin(\tau|D_x|)h(\tau)d\tau \right\|_{L^2(\mathbb{R}^d)} \leqslant C \|h\|_{N(\mathbb{R})}, \\ \|r\|_{X(\mathbb{R})} &= \left\| \int_{\cdot}^{+\infty} \frac{\sin((\cdot-\tau)|D_x|)}{|D_x|} h(\tau)d\tau \right\|_{X(\mathbb{R})} \leqslant C \|h\|_{N(\mathbb{R})}, \\ \|\vec{r}(t)\|_{\mathcal{H}} &= \left\| \int_{t}^{+\infty} \frac{\sin((t-\tau)|D_x|)}{|D_x|} h(\tau)d\tau \right\|_{\mathcal{H}} \leqslant C \|h\|_{N([t,+\infty))}. \end{aligned} \tag{12}$$

In particular, due to (8), we have

$$\lim_{t\to+\infty} (\|\nabla r\|^2_{L^2(\mathbb{R}^d)} + \|\partial_t r\|^2_{L^2(\mathbb{R}^d)}) = 0. \tag{13}$$

We will select $(w_0, w_1) = (w(0), \partial_t w(0))$ the data at initial time for w, so that $w(t) = S_L(t)(w_0, w_1)$. We can now compute the radiation of u in terms of (w_0, w_1) and (v_{0+}, v_{1+}). Indeed, for $t \to -\infty$, using (6) and (10), we have

$$\begin{aligned}
&\lim_{t\to-\infty}(\|\nabla\widetilde{u}\|^2_{L^2(|x|\geqslant|t|+R)} + \|\partial_t\widetilde{u}\|^2_{L^2(|x|\geqslant|t|+R)})\\
&\quad= \lim_{t\to-\infty}(\|\nabla w(t)\|^2_{L^2(|x|\geqslant|t|+R)} + \|\partial_t w(t)\|^2_{L^2(|x|\geqslant|t|+R)})\\
&\quad= \|\partial_s\mathcal{T}w_0 + \mathcal{T}w_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.
\end{aligned}$$

Similarly, for $t \to +\infty$, $\vec{\widetilde{u}}(t) = S_L(t)(w_0 + v_{0+}, w_1 + v_{1+}) + \vec{r}(t)$ so that using (13) and (5), we have

$$\begin{aligned}
&\lim_{t\to+\infty}(\|\nabla\widetilde{u}\|^2_{L^2(|x|\geqslant|t|+R)} + \|\partial_t\widetilde{u}\|^2_{L^2(|x|\geqslant|t|+R)})\\
&\quad= \lim_{t\to+\infty}(\|\nabla(v+w)(t)\|^2_{L^2(|x|\geqslant|t|+R)} + \|\partial_t(v+w)(t)\|^2_{L^2(|x|\geqslant|t|+R)})\\
&\quad= \|\partial_s\mathcal{T}(w_0+v_{0+}) - \mathcal{T}(w_1+v_{1+})\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.
\end{aligned}$$

Hence, summing up, we get:

$$\begin{aligned}
(14)\quad E_{\mathrm{ext},R}(\vec{\widetilde{u}}) &= \frac{1}{2}\|\partial_s\mathcal{T}w_0 + \mathcal{T}w_1\|_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}\\
&\quad+ \frac{1}{2}\|\partial_s\mathcal{T}(w_0+v_{0+}) - \mathcal{T}(w_1+v_{1+})\|_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}.
\end{aligned}$$

We therefore look for $(w_0, w_1) \in \dot{H}^1 \times L^2$ such that

$$(15)\qquad \begin{cases} \partial_s\mathcal{T}w_0 + \mathcal{T}w_1 = 0 \text{ on } [R,+\infty)\times\mathbb{S}^{d-1}, a.e.\\ \partial_s\mathcal{T}(w_0+v_{0+}) - \mathcal{T}(w_1+v_{1+}) = 0 \text{ on } [R,+\infty)\times\mathbb{S}^{d-1}, a.e. \end{cases}$$

Equivalently:

$$\begin{cases} 2\mathcal{T}w_1 = -\mathcal{T}v_{1+} + \partial_s\mathcal{T}v_{0+} \text{ on } [R,+\infty)\times\mathbb{S}^{d-1}, a.e.\\ 2\partial_s\mathcal{T}w_0 = -\partial_s\mathcal{T}v_{0+} + \mathcal{T}v_{1+} \text{ on } [R,+\infty)\times\mathbb{S}^{d-1}, a.e. \end{cases}$$

Due to Corollary 2.3, the previous equations can be solved with a continuous inverse. To summarize, we finally define

$$(16)\qquad w_1 = \frac{1}{2}G^1_R(-\mathcal{T}v_{1+} + \partial_s\mathcal{T}v_{0+}) \quad\text{and}\quad w_0 = \frac{1}{2}G^0_R(-\partial_s\mathcal{T}v_{0+} + \mathcal{T}v_{1+}).$$

Then (w_0, w_1) solve the system (15) and, thanks to Lemma 2.2 and (12), satisfy the estimates

$$\begin{aligned}
\|(w_0,w_1)\|_{\mathcal{H}} &\leqslant C\,\|\mathcal{T}v_{1+} - \partial_s\mathcal{T}v_{0+}\|_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}\\
&\leqslant C\,\|v_{1+}\|_{L^2(\mathbb{R}^d)} + C\,\|v_{0+}\|_{\dot{H}^1(\mathbb{R}^d)} \leqslant C\,\|h\|_{N(\mathbb{R})}.
\end{aligned}$$

Then we let $\vec{\widetilde{u}} = \vec{v} + S_L(w_0, w_1)$ where $\vec{v}$ is defined in (11) and (w_0, w_1) is defined in (16). Then $\Box\widetilde{u} = \Box v = h$ and, in view of (14), $E_{\mathrm{ext},R}(\vec{\widetilde{u}}) = 0$. Also, we have the bound

$$\|\widetilde{u}\|_{X(\mathbb{R})} \leqslant \|v\|_{X(\mathbb{R})} + \|w\|_{X(\mathbb{R})} \leqslant C\,\|h\|_{N(\mathbb{R})} + \|(w_0,w_1)\|_{\mathcal{H}} \leqslant C\,\|h\|_{N(\mathbb{R})}.$$

Step 2. Now that $\tilde{u}$ is defined, we simply let $u = \widetilde{u} - u_R$ where $\vec{u}_R = S_L(\pi_R(\widetilde{u}(0), \partial_t\widetilde{u}(0)))$: indeed, u_R is a non radiative solution, and solves $\Box u_R = 0$. Also, regarding continuity of the map, we just need to write

$$\|u_R\|_{X(\mathbb{R})} \leqslant C\,\|\pi_R(\widetilde{u}(0), \partial_t\widetilde{u}(0))\|_{\mathcal{H}} \leqslant \|(\widetilde{u}(0), \partial_t\widetilde{u}(0))\|_{\mathcal{H}} \leqslant C\,\|\widetilde{u}\|_{X(\mathbb{R})} \leqslant C\,\|h\|_{N(\mathbb{R})},$$

so that $\|u\|_{X(\mathbb{R})} \leqslant C\|h\|_{N(\mathbb{R})}$. This finishes the existence part.

Step 3. Concerning uniqueness: let u_1 and u_2 be two such solutions of the problem. In particular, $z = u_1 - u_2$ satisfy:

(1) z is solution of $\Box z = 0$,
(2) $E_{\text{ext},R}(\vec{z}) = 0$,
(3) $\pi_R(\vec{z}(0)) = 0$.

In particular, the first and second assumptions imply $(z(0), \partial_t z(0)) \in P(R)$ and therefore $\vec{z}(0) = \pi_R(\vec{z}(0))$. Together with the third assumption, we infer $(z(0), \partial_t z(0)) = 0$ and therefore $z = 0$, and $u_1 = u_2$. $\Box$

With Proposition 2.4 in hand, we can now prove the theorem.

Proof of Theorem 1.2. For $(u_0, u_1) \in \mathcal{H}$, let $\vec{u}_L = S_L(u_0, u_1)$. We are looking for a solution u of

$$u = u_L + T(f(u)). \tag{17}$$

Indeed, if $u \in X(\mathbb{R})$ solves (17), then

$$\Box u = \Box(Tf(u)) = f(u),$$

so that u solves (1). To solve (17), given $(u_0, u_1) \in P(R)$ with $\|(u_0, u_1)\|_{\mathcal{H}} \leqslant \varepsilon$, we use a fixed point argument on small closed balls $B(0, \varepsilon)$ of $X(\mathbb{R})$ for the map

$$G : r \mapsto T(f(u_L + r)).$$

Due to the continuity of $T : N(\mathbb{R}) \to X(\mathbb{R})$ (provided by Proposition 2.4), and using (7) and (9), we get for $r, \tilde{r} \in X(\mathbb{R})$,

$$\begin{aligned}
\|G(r)\|_{X(\mathbb{R})} &\leqslant C\,\|f(u_L + r)\|_{N(\mathbb{R})} \leqslant C\,\|u_L + r\|^q_{X(\mathbb{R})} \leqslant C(\varepsilon^q + \|r\|^q_X), \\
\|G(r) - G(r')\|_{X(\mathbb{R})} &\leqslant C\,\|f(u_L + r) - f(u_L + \widetilde{r})\|_{N(\mathbb{R})} \\
&\leqslant C\,\|r - \widetilde{r}\|_{W(\mathbb{R})}\,(\|u_L + r\|^{q-1}_{W(\mathbb{R})} + \|u_L + \widetilde{r}\|^{q-1}_{W(\mathbb{R})}) \\
&\leqslant C\,\|r - \widetilde{r}\|_X\,(\varepsilon^{q-1} + \|r\|^{q-1}_X + \|\widetilde{r}\|^{q-1}_X).
\end{aligned}$$

So, for ε small enough, G admits a unique fixed point v in $\overline{B}_{X(\mathbb{R})}(0, \varepsilon)$, the closed ball of radius ε in $X(\mathbb{R})$. Furthermore

$$\|v\|_{X(\mathbb{R})} = \|G(v)\|_{X(\mathbb{R})} \leqslant C\|(u_0, u_1)\|^q_{\mathcal{H}}. \tag{18}$$

Then $u := u_L + v$ solves (17). Also, by regularity of the Banach fixed point with parameter, the map $(u_0, u_1) \mapsto v$ is $\mathscr{C}^1$ from $B_{\mathcal{H}}(0, \varepsilon)$ to $X(\mathbb{R})$ (notice that the nonlinearity is $\mathscr{C}^1$), with differential 0 at $0 \in \mathcal{H}$, due to (18). Finally,

$$\pi_R(u(0), \partial_t u(0)) = \pi_R(u_0, u_1) + \pi_R(T(f(u))(0), \partial_t T(f(u))(0)) = \pi_R(u_0, u_1).$$

Therefore, the map

$$\Phi : (u_0, u_1) \mapsto (u, \partial_t u)(0),$$

(where u is as above) satisfies the first part of Theorem 1.2, up to possibly diminishing ε.

Now, assuming $(u_0, u_1) \in P(R)$, we define $\vec{u} = \mathcal{S}\Phi(u_0, u_1)$, the associated nonlinear solution. We have $f(u) \in N(\mathbb{R})$ due to (9) and as $E_{\text{ext},R}(S_L(u_0, u_1)) = 0$, the radiation of u is well defined and

$$E_{\text{ext},R}(\vec{u}) = E_{\text{ext},R}(T(f(u)), \partial_t T(f(u))) = 0,$$

so that $u \in \mathcal{P}(R)$. So, we have proved $\Phi(P(R) \cap B_{\mathcal{H}}(0, \varepsilon)) \subset \mathcal{P}(R) \cap V$.

Reciprocally, let $(v_0, v_1) \in \mathcal{P}(R) \cap V$. By definition of V, it can be written $(v_0, v_1) = \Phi(u_0, u_1)$ with $(u_0, u_1) \in B_{\mathcal{H}}(0, \varepsilon)$. Denoting $\vec{u} = \mathcal{S}\Phi(u_0, u_1) = \mathcal{S}(v_0, v_1)$, the associated nonlinear solution, we have, by definition of Φ, $\vec{u} = S_L(u_0, u_1) + T(f(u))$. In particular, as $E_{\text{ext},R}(T(f(u)), \partial_t T(f(u))) = 0$, we have

$$E_{\text{ext},R}(\vec{u}) = E_{\text{ext},R}(S_L(u_0, u_1)).$$

Now we assumed $(v_0, v_1) \in \mathcal{P}(R)$, so that $E_{\text{ext},R}(\vec{u}) = E_{\text{ext},R}(\mathcal{S}(v_0, v_1)) = 0$, and

$$E_{\text{ext},R}(S_L(u_0, u_1)) = 0.$$

Thus, $(u_0, u_1) \in P(R)$ and $(v_0, v_1) \in \Phi(P(R) \cap B_{\mathcal{H}}(0, \varepsilon))$.
The last statement of the theorem is only a rephrasing of the previous results in terms of submanifolds in Banach spaces. □

Now, we turn to the proof of Theorem 1.3 and begin by a Proposition stating that the radiation operator is onto.

Proposition 2.5 (Friedlander [6]). *The application*

$$\begin{aligned} \mathcal{H} &\longrightarrow L^2(\mathbb{R} \times \mathbb{S}^{d-1}) \\ (v_0, v_1) &\longmapsto \partial_s \mathcal{T} v_0 - \mathcal{T} v_1 \end{aligned}$$

is a bijective isometry.

Proof. For the convenience of the reader, we provide a proof with an explicit inversion formula in terms of Fourier transform. Formula (3) gives

$$\mathcal{F}_{s\to\nu}(\partial_s \mathcal{T} v_0 - \mathcal{T} v_1)(\nu, \omega) = c_0 |\nu|^{\frac{d-1}{2}} (e^{i\tau} \mathbb{1}_{\nu<0} + e^{-i\tau} \mathbb{1}_{\nu\geqslant 0})(i\nu \hat{v}_0(\nu\omega) - \hat{v}_1(\nu\omega)).$$

For the injectivity, we could compute directly that the application is an isometry, see for instance [2, Lemma 2.1.] for a closeby computation. Here we can directly check that $\partial_s \mathcal{T} v_0 - \mathcal{T} v_1 = 0$ implies $i\nu \hat{v}_0(\nu\omega) = \hat{v}_1(\nu\omega)$ almost everywhere in $\mathbb{R} \times \mathbb{S}^{d-1}$. Applying at (ν, ω) and $(-\nu, -\omega)$, it gives $(v_0, v_1) = (0, 0)$.
For the surjectivity, given $F \in L^2(\mathbb{R} \times \mathbb{S}^{d-1})$, denote for simplicity $\hat{F} = \mathcal{F}_{s\to\nu} F$, and define v_0 and v_1 by their Fourier transform as follows: for $\xi \in \mathbb{R}^d \setminus \{0\}$, with $\xi = \rho\omega$ where $\rho > 0$ and $\omega \in \mathbb{S}^{d-1}$, we set

$$\begin{aligned} \hat{v}_0(\xi) &= \frac{1}{2ic_0 \rho^{\frac{d+1}{2}}} \left(e^{i\tau} \hat{F}(\rho, \omega) - e^{-i\tau} \hat{F}(-\rho, -\omega) \right), \\ \hat{v}_1(\xi) &= -\frac{1}{2c_0 \rho^{\frac{d-1}{2}}} \left(e^{i\tau} \hat{F}(\rho, \omega) + e^{-i\tau} \hat{F}(-\rho, -\omega) \right). \end{aligned}$$

Then for $\omega \in \mathbb{S}^{d-1}$, we have for $\nu > 0$

$$\begin{aligned} \mathcal{F}_{s\to\nu}(\partial_s \mathcal{T} v_0 - \mathcal{T} v_1)(\nu, \omega) &= c_0 \nu^{\frac{d-1}{2}} e^{-i\tau} (i\nu \hat{v}_0(\nu\omega) - \hat{v}_1(\nu\omega)) \\ &= c_0 \nu^{\frac{d-1}{2}} e^{-i\tau} \Bigg(\frac{i\nu}{2ic_0 \nu^{\frac{d+1}{2}}} \left(e^{i\tau} \hat{F}(\nu, \omega) - e^{-i\tau} \hat{F}(-\nu, -\omega) \right) \\ &\qquad + \frac{1}{2c_0 \nu^{\frac{d-1}{2}}} \left(e^{i\tau} \hat{F}(\nu, \omega) + e^{-i\tau} \hat{F}(-\nu, -\omega) \right) \Bigg) \\ &= \hat{F}(\nu, \omega), \end{aligned}$$

and if $\nu < 0$,

$$\begin{aligned} \mathcal{F}_{s\to\nu}(\partial_s \mathcal{T} v_0 - \mathcal{T} v_1)(\nu, \omega) &= c_0 |\nu|^{\frac{d-1}{2}} e^{i\tau} \left(-i|\nu| \hat{v}_0(|\nu|(-\omega)) - \hat{v}_1(|\nu|(-\omega)) \right) \\ &= c_0 |\nu|^{\frac{d-1}{2}} e^{i\tau} \Bigg(-\frac{i|\nu|}{2ic_0 |\nu|^{\frac{d+1}{2}}} \left(e^{i\tau} \hat{F}(|\nu|, -\omega) - e^{-i\tau} \hat{F}(\nu, \omega) \right) \end{aligned}$$

$$+\frac{1}{2c_0|\nu|^{\frac{d-1}{2}}}\left(e^{i\tau}\hat F(|\nu|,-\omega)+e^{-i\tau}\hat F(\nu,\omega)\right)\Bigg)$$
$$=\hat F(\nu,\omega).$$

Hence there hold

$$(\partial_s\mathcal{T}v_0-\mathcal{T}v_1)=F.$$

We verify that (v_0,v_1) defined as above are indeed in $\mathcal{H}$.

$$\begin{aligned}\|v_0\|_{\dot H^1}^2&=\frac{1}{(2\pi)^d}\||\cdot|\hat v_0(\cdot)\|_{L^2}^2=\frac{1}{(2\pi)^d}\int_0^{+\infty}\rho^{d-1}\int_{\omega\in\mathbb{S}^{d-1}}\rho^2\left|\hat v_0(\rho\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{4c_0^2(2\pi)^d}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|e^{i\tau}\hat F(\rho,\omega)-e^{-i\tau}\hat F(-\rho,-\omega)\right|^2\,d\omega\,d\rho.\end{aligned}$$

$$\begin{aligned}\|v_1\|_{L^2}^2&=\frac{1}{(2\pi)^d}\|\hat v_1(\cdot)\|_{L^2}^2=\frac{1}{(2\pi)^d}\int_0^{+\infty}\rho^{d-1}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat v_1(\rho\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{4c_0^2(2\pi)^d}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|e^{i\tau}\hat F(\rho,\omega)+e^{-i\tau}\hat F(-\rho,-\omega)\right|^2\,d\omega\,d\rho.\end{aligned}$$

Finally, it is an isometry: indeed, $\dfrac{1}{4c_0^2(2\pi)^d}=\dfrac{1}{4\pi}$ and

$$\begin{aligned}&\left|e^{i\tau}\hat F(\rho,\omega)-e^{-i\tau}\hat F(-\rho,-\omega)\right|^2+\left|e^{i\tau}\hat F(\rho,\omega)+e^{-i\tau}\hat F(-\rho,-\omega)\right|^2\\&\qquad=2\left|\hat F(\rho,\omega)\right|^2+2\left|\hat F(-\rho,-\omega)\right|^2,\end{aligned}$$

so that

$$\begin{aligned}\|v_0\|_{\dot H^1}^2+\|v_1\|_{L^2}^2&=\frac{1}{2\pi}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat F(\rho,\omega)\right|^2+\left|\hat F(-\rho,-\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{2\pi}\int_0^{+\infty}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat F(\rho,\omega)\right|^2+\left|\hat F(-\rho,\omega)\right|^2\,d\omega\,d\rho\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\omega\in\mathbb{S}^{d-1}}\left|\hat F(\rho,\omega)\right|^2\,d\omega\,d\rho=\int_{\mathbb{R}}\int_{\omega\in\mathbb{S}^{d-1}}\left|F(s,\omega)\right|^2\,d\omega\,ds\\&=\|F\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}^2.\end{aligned}$$

□

Proof of Theorem 1.3. Step 1. We first construct the linear scattering state, that is find $(v_0,v_1)\in\mathcal{H}$ such that, denoting $\vec v_L=S_L(v_0,v_1)$, as $t\to+\infty$,

(19) $$\nabla_{t,x}v_L(t,x)-\frac{1}{\sqrt2|x|^{\frac{d-1}{2}}}F\left(|x|-t,\frac{x}{|x|}\right)\times\begin{pmatrix}-1\\x/|x|\end{pmatrix}\to0\quad\text{in}\quad L^2(\mathbb{R}^d)^{1+d}.$$

Due to Proposition 2.5, there exists $(v_0,v_1)\in\mathcal{H}$ so that

$$F=(\partial_s\mathcal{T}v_0-\mathcal{T}v_1).$$

In view of (4), we see that $\vec v_L=S_L(\cdot)(v_0,v_1)$ satisfies the expected asymptotic (19).

Step 2. We now construct $\vec u$, solution to (1) such that $\|\vec u-\vec v_L(t)\|_{\mathcal{H}}\to0$ as $t\to+\infty$: this is simply the wave operator, and is standard. We provide some elements of proof for the sake of completeness. We decompose $\vec u(t)=\vec v_L(t)+\vec w(t)$ and write $\vec w$ as solution of a fixed point problem. Let $T\in\mathbb{R}$ to be chosen later: the Duhamel formula on $[t,\tau]$ (for $\tau\geqslant t$) gives

$$\vec v_L(\tau)+\vec w(\tau)=S_L(\tau-t)(\vec v_L(t)+\vec w(t))+\int_t^\tau S_L(\tau-s)\begin{pmatrix}0\\f(v_L(s)+w(s))\end{pmatrix}ds.$$

Notice that $\vec{v}_L(t) = S_L(t-T)\vec{v}_L(T)$; compose by $S_L(t-\tau)$ and let $\tau \to +\infty$: as $\|S_L(t-\tau)\vec{w}(\tau)\|_{\mathcal{H}} = \|\vec{w}(\tau)\|_{\mathcal{H}}$ is meant to tend to 0, we arrive at the fixed point formulation:

$$\vec{w}(t) = \Psi\vec{w}(t), \quad \text{where} \quad \Psi\vec{v}(t) := -\int_t^{+\infty} S_L(t-s)\begin{pmatrix} 0 \\ f(v_L(s)+v(s)) \end{pmatrix} ds.$$

Let $T \in \mathbb{R}$ to be fixed later, we work in small closed balls $\overline{B}(0,\varepsilon)$ of $X([T,+\infty))$. By (7) and (9), we have for $\vec{v} \in X([T,+\infty))$,

$$\|\Psi\vec{v}\|_{X([T,+\infty))} \leqslant C\|f(v_L+v)\|_{N([T,+\infty))} \leqslant C\left(\|v_L\|^q_{W([T,+\infty))} + \|v\|^q_{W([T,+\infty))}\right).$$

Similarly,

$$\begin{aligned}\|\Psi\vec{v} - \Psi\vec{\tilde{v}}\|_{X([T,+\infty))} &\leqslant C\|f(v_L+v) - f(v_L+\tilde{v})\|_{N([T,+\infty)} \\ &\leqslant C\left(\|v_L\|^{q-1}_{W([T,+\infty))} + \|v\|^{q-1}_{W([T,+\infty))} + \|\tilde{v}\|^{q-1}_{W([T,+\infty))}\right)\|v-\tilde{v}\|_{W([T,+\infty))}.\end{aligned}$$

Let T be such that $\|v_L\|^{q-1}_{W([T,+\infty))} \leqslant \varepsilon$ be small enough, then Ψ admits a unique fixed point $\vec{w}$ in $B(0,\varepsilon)$, and $\vec{u} = \vec{v}_L + \vec{w}$ answers the question. □

Appendix A. Description of the set $P(R)$ of linear non radiative solutions

In this section, we gather some results of [2] where a precise description of the set $P(R)$ was performed for $R > 0$. This corresponds to classifying the linear solutions u that have vanishing asymptotic energy on the exterior light cone $|x| \geqslant t+R$ with $R > 0$, that is

$$E_{ext,R}(u) = 0.$$

By finite speed of propagation, initial data which are compactly supported in $|x| \leqslant R$ obviously satisfy this condition. We will call this space

$$\mathcal{K}_{R,comp} = \left\{(u_0,u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : (u_0,u_1)|_{\{|x|>R\}} = 0\right\}.$$

where the equality is in the distributional sense.
It turns out that these are not the only examples. We will now need some further notation.
We denote $(Y_\ell)_{\ell\in\mathbb{M}}$ a countable orthonormal basis of spherical harmonics of $\mathbb{S}^{d-1}$. Y_ℓ is the restriction to $\mathbb{S}^{d-1}$ of a harmonic (homogeneous) polynomial. For short, we will denote $l = l(\ell)$ the degree of this polynomial.
The non radiative functions will be the following. Denote for $k \in \mathbb{N}$,

$$\alpha_k := -l-d+2k+2.$$

α_k also depends on ℓ, but here and below, we silence this dependence to keep notations light. Then let

$$g_k(x) = \mathbb{1}_{\{|x|>R\}}|x|^{\alpha_k}Y_\ell\left(\frac{x}{|x|}\right).$$

Note that $g_k \in L^2 \iff \alpha_k < -d/2$. We introduce

$$\mathcal{N}^0_{R,\ell} = \operatorname{Span}\left(g_k;\ \text{for } k \in \mathbb{N} \text{ such that } \alpha_k < -d/2\right)$$

Similarly, let

$$f_k(x) = \begin{cases} \left(\frac{|x|}{R}\right)^{\alpha_k} Y_\ell\left(\frac{x}{|x|}\right) & \text{for } |x| > R \\ \left(\frac{|x|}{R}\right)^{l} Y_\ell\left(\frac{x}{|x|}\right) & \text{for } |x| \leqslant R. \end{cases}$$

Note that $f_k \in \dot{H}^1 \iff \alpha_k < -d/2+1$. Also, the value of f_k in $|x| \leqslant R$ is not very important; our choice allows to keep continuity and that the restriction $f_k|_{\{|x|<R\}}$ is a harmonic polynomial, so that f_k is orthogonal to (in $\dot{H}^1$) to functions with compact support in $B(0,R)$.
Let

$$\mathcal{N}^1_{R,\ell} = \operatorname{Span}\left(f_k;\ \text{for } k \in \mathbb{N} \text{ such that } \alpha_k < -d/2+1\right).$$

For any $\ell \in \mathbb{M}$, we note the space

$$P_\ell(R) = \mathcal{N}^0_{R,\ell} \times \mathcal{N}^1_{R,\ell}.$$

Remark A.1. For a fixed spherical harmonics Y_ℓ, only the value $k=0$ corresponding to $\alpha_0 = -l-d+2$ produces a solution of the stationary equation $\Delta u = 0$, and from [3] (in dimension 3), a nonlinear stationary solution defined outside a large ball: via time invariance, this yields a curve (manifold of dimension 1) of solutions stationary outside a light cone.
Theorem 1.2 constructs a non radiative solution for all elements in $P_\ell(R)$, which, except for those on the curve above, are *not* stationary outside a light cone.

One of the result of [2, Theorem 1.7] was the precise description of $P(R)$ in odd dimensions as follows.

$$P(R) = \mathcal{K}_{R,comp} \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\ell \in \mathbb{M}}} P_\ell(R). \tag{20}$$

(the orthogonality is related to the natural scalar product of $\dot{H}^1 \times L^2$).

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