

GLOBAL CONTROLLABILITY AND STABILIZATION FOR THE NONLINEAR SCHRÖDINGER EQUATION ON SOME COMPACT MANIFOLDS OF DIMENSION 3*

CAMILLE LAURENT†

Abstract. We prove global internal controllability in large time for the nonlinear Schrödinger equation on some compact manifolds of dimension 3. The result is proved under some geometrical assumptions: geometric control and unique continuation. We give some examples where they are fulfilled on \mathbb{T}^3 , S^3 , and $S^2 \times S^1$. We prove this by two different methods, both inherently interesting. The first one combines stabilization and local controllability near 0. The second one uses successive controls near some trajectories. We also get a regularity result about the control if the data are assumed smoother. If the H^1 norm is bounded, it gives a local control in H^1 with a smallness assumption only in L^2 . We use Bourgain spaces to solve the equation in H^1 .

Key words. controllability, stabilization, nonlinear Schrödinger equation, Bourgain spaces

AMS subject classifications. 93B05, 93D15, 35Q55, 35A21

DOI. 10.1137/090749086

Introduction. In this article, we study the internal stabilization and exact controllability for the defocusing nonlinear Schrödinger equation (NLS) on some compact manifolds of dimension 3:

$$(0.1) \quad \begin{cases} i\partial_t u + \Delta u &= |u|^2 u \text{ on } [0, +\infty[\times M, \\ u(0) &= u_0 \in H^1(M), \end{cases}$$

where Δ is the Laplace–Beltrami operator on M . The solution displays two conserved energies: the L^2 energy $\|u\|_{L^2}$ and the nonlinear energy, or H^1 energy,

$$E(t) = \frac{1}{2} \int_M |\nabla u|^2 + \frac{1}{4} \int_M |u|^4.$$

This equation arises in nonlinear optics, where it is obtained as an asymptotic regime of the Maxwell equations in a nonlinear medium (see, e.g., Sulem and Sulem [42]). In this context, the metric g can be interpreted as an inhomogeneity of the optical index. A more physically relevant situation could be to consider this equation on a domain. However, for the moment, this equation is not known to be globally well posed on an open set of dimension 3 (see [2], [6] for the two-dimensional case and [1] for the radial solutions on a ball). A compact manifold makes a good framework to understand the effect of geometry.

For the study of controllability, some similar results were obtained in dimension 2 in the article of Dehman, Gérard, and Lebeau [16], where exact controllability in H^1 is proved for the defocusing NLS on compact surfaces. Yet, the proof is based on Strichartz estimates which provide uniform well-posedness in dimension 3 only in H^s for $s > 1$. In [11], Burq, Gérard, and Tzvetkov managed to prove global existence and uniqueness in H^1 but failed to prove uniform well-posedness, which appears to

*Received by the editors February 9, 2009; accepted for publication (in revised form) December 28, 2009; published electronically April 9, 2010.

<http://www.siam.org/journals/sima/42-2/74908.html>

†Laboratoire de Mathématiques d’Orsay, UMR 8628 CNRS, Université Paris-Sud, Orsay Cedex F-91405, France (camille.laurent@math.u-psud.fr).

be of great importance in control problems. However, for certain specific manifolds, the strategy of $X^{s,b}$ spaces of Bourgain, extended to some other manifolds by Burq, Gérard, and Tzvetkov, succeeded in proving uniform well-posedness in H^s for some lower regularities. So, to our knowledge, this paper is the first one dealing with global controllability for the cubic NLS in three dimensions.

For control results, the $X^{s,b}$ spaces have already been used in dimension 1 at L^2 regularity: first Rosier and Zhang [41] obtained local results, and, independently, we proved global controllability in large time in [32]. We also quote the recent paper [40] about the control of the NLS on rectangles but still with local results. The $X^{s,b}$ spaces will also be our framework in this paper.

Under some specific assumptions that will be made precise later, we prove global controllability in large time two different ways, both inherently interesting: by stabilization and control near 0 or by some successive controls near some trajectories. This will provide global controllability towards 0, and the general result will follow by reversing time. The first strategy is very classical in control theory and has been used innumerable times (see, for example, Lee and Markus [34, p. 397] in finite dimension). The second strategy seems less classical, at least in this framework.

Our assumptions are fulfilled in the following cases ($\omega \subset M$ is the support of the control):

- \mathbb{T}^3 with $\omega = \{x \in \mathbb{R}^3 / (\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z}) \mid \exists i \in \{1, 2, 3\}, x_i \in]-\varepsilon, \varepsilon[+ \theta_i\mathbb{Z}\}$ (that is, a neighborhood of each face of the “cube,” fundamental volume of \mathbb{T}^3) with $\theta_i \in \mathbb{R}$. Moreover, we can easily extend this result to a cuboid with Dirichlet or Neumann boundary conditions; see [32] or [41].

- S^3 with ω a neighborhood of $\{x_4 = 0\}$ in $S^3 \subset \mathbb{R}^4$.

- $S^2 \times S^1$ with $\omega = (\omega_1 \times S^1) \cup (S^2 \times]0, \varepsilon[)$, where ω_1 is a neighborhood of the equator of S^2 .

THEOREM 0.1. *For any open set $\omega \subset M$ satisfying Assumptions 1, 2, and 3 (see below) and $R_0 > 0$, there exist $T > 0$ and $C > 0$ such that for every u_0 and u_1 in $H^1(M)$ with*

$$\|u_0\|_{H^1(M)} \leq R_0 \quad \text{and} \quad \|u_1\|_{H^1(M)} \leq R_0$$

there exists a control $g \in C([0, T], H^1)$ with $\|g\|_{L^\infty([0, T], H^1)} \leq C$ supported in $[0, T] \times \bar{\omega}$ such that the unique solution u in $X_T^{1,b}$ of the Cauchy problem

$$(0.2) \quad \begin{cases} i\partial_t u + \Delta u &= |u|^2 u + g \quad \text{on} \quad [0, T] \times M, \\ u(0) &= u_0 \in H^1(M) \end{cases}$$

satisfies $u(T) = u_1$.

In the rest of this article, ω will be related to a cut-off function $a = a(x) \in C^\infty(M)$ (whose existence is guaranteed by the Whitney theorem) taking real values and such that

$$(0.3) \quad \omega = \{x \in M : a(x) \neq 0\}.$$

The stabilization system we consider is

$$(0.4) \quad \begin{cases} i\partial_t u + \Delta u - a(x)(1 - \Delta)^{-1}a(x)\partial_t u &= (1 + |u|^2)u \quad \text{on} \quad [0, T] \times M, \\ u(0) &= u_0 \in H^1(M). \end{cases}$$

The link with the original equation can be made by the change of variable $w = e^{-it}u$. A more physically relevant damping term would be $ia(x)u$, as used in the

one-dimensional case in [32]. Yet, the damping term in (0.4) is especially fitted to the H^1 energy which is the regularity at which we solve the equation. The well-posedness of system (0.4) will be proved in section 2.1, and we can check that it satisfies the energy decay

$$(0.5) \quad E(u(t)) - E(u(0)) = - \int_0^t \left\| (1 - \Delta)^{-1/2} a(x) \partial_t u \right\|_{L^2}^2.$$

Our theorem states that, under some geometrical hypotheses, this yields an exponential decay.

THEOREM 0.2. *Let (M, ω) satisfy Assumptions 1, 2, and 3. Let $a \in C^\infty(M)$, as in (0.3). There exists $\gamma > 0$ such that for every $R_0 > 0$ there is a constant $C > 0$ such that inequality*

$$\|u(t)\|_{H^1} \leq C e^{-\gamma t} \|u_0\|_{H^1}, \quad t > 0,$$

holds for every solution u of system (0.4) with initial data u_0 such that $\|u_0\|_{H^1} \leq R_0$.

The independence of C and of the time of control T on the bound R_0 is an open problem. The fact that γ is independent of the size lies in the fact that it describes only the behavior near 0. However, it is unknown whether there is really a minimal time of controllability. This is in strong contrast with the linear case where exact controllability occurs in arbitrary small time and the conditions are geometric only for the open set ω . Moreover, some recent studies have analyzed the explosion of the control cost when T tends to 0: Phung [38] by reducing to the heat or wave equation, Miller [36] with resolvent estimates, and Tenenbaum and Tucsnak [43] with number theoretic arguments.

Let us now describe our assumptions. The first two deal with classical geometrical assumptions in control theory.

Assumption 1. Geometric control: There exists $T_0 > 0$ such that every geodesic of M , travelling with speed 1 and issued at $t = 0$, enters the set ω in a time $t < T_0$.

This condition is known to be sufficient for linear controllability; see Lebeau [33]. In section 9, we prove that it is necessary on S^3 for the nonlinear stabilization. However, there are some geometrical situations (especially when there are some unstable geodesics) in which it is not necessary. For example, we have linear controllability for any open set ω of \mathbb{T}^3 ; see Jaffard [26] and Komornik and Loreti [28] (see also [14]). This also holds for $M = S^2 \times S^1$ with $\omega = S^2 \times]0, \varepsilon[$ or $\omega = \omega_1 \times S^1$, where ω_1 is a neighborhood of the equator. In that case, our method fails to prove global results and we can prove only local controllability by perturbation (see Theorem 0.4).

Assumption 2. Unique continuation: For every $T > 0$, the only solution in $C^\infty([0, T] \times M)$ to the system

$$(0.6) \quad \begin{cases} i \partial_t u + \Delta u + b_1(t, x)u + b_2(t, x)\bar{u} = 0 & \text{on } [0, T] \times M, \\ u = 0 & \text{on } [0, T] \times \omega, \end{cases}$$

where $b_1(t, x)$ and $b_2(t, x) \in C^\infty([0, T] \times M)$, is the trivial one $u \equiv 0$.

We do not know if there exists a link between these two assumptions. In our three particular cases, this can be proved using Carleman estimates. There are some existing results about this, such as the one of Isakov [25] (for general anisotropic PDEs), Baudouin and Puel [4] (for global Carleman estimates), or Mercado, Osses, and Rosier [35] (in the special case of Schrödinger with flat metric but weaker geometrical assumptions). In the case of a Riemannian manifold with boundary, some

Carleman estimates were obtained by Triggiani and Xu [46] (see also an interesting discussion in section 10 about the existence of convex weights). Note also that these Carleman estimates can be used to treat some controllability problems directly; see, for instance, Lasiecka and Triggiani [29], Lasiecka, Triggiani, and Zhang [31], [30], and Triggiani [45]. For the convenience of the reader, we have chosen to give a proof of Carleman estimates on a compact manifold. It is given in Appendix B. They are quite similar to those of [46] but simpler because they are without boundary terms. We also believe that these estimates are of independent interest if the weight is weakly convex (as in [35] but with a metric).

The last assumption is a technical assumption that ensures that the Cauchy problem is well posed in H^1 . It yields a bilinear loss of $s_0 < 1$.

Assumption 3. There exist $C > 0$ and $0 \leq s_0 < 1$ such that for any $f_1, f_2 \in L^2(M)$ satisfying

$$f_j = \mathbf{1}_{\sqrt{1-\Delta} \in [N_j, 2N_j]}(f_j), \quad j = 1, 2,$$

one has the following bilinear estimates:

$$(0.7) \quad \begin{aligned} \|u_1 u_2\|_{L^2([0, T] \times M)} &\leq C \min(N_1, N_2)^{s_0} \|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)}, \\ u_j(t) &= e^{it\Delta} f_j, \quad j = 1, 2. \end{aligned}$$

It is known to be true in the following examples ($1/2+$ means any $s > 1/2$):

- \mathbb{T}^3 with $s_0 = 1/2+$; see [7].
- The irrational torus $\mathbb{R}^3/(\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z})$ with $\theta_i \in \mathbb{R}$ for which an estimate with $s_0 = 2/3+$ has recently been obtained in [8]. An easier proof for $s_0 = 3/4+$ can also be found in the beginning of [8] and in [15].
- S^3 with $s_0 = 1/2+$; see [13].
- $S^2 \times S^1$ with $s_0 = 3/4+$; see [13].

It yields some trilinear estimates in Bourgain spaces (see the definition below). For the control near a trajectory, we still have some particular assumptions that are again fulfilled in the particular geometries described above. Our result reads as follows.

THEOREM 0.3. *Let $T > 0$, and let (M, ω) be such that Assumptions 1, 3, 4, and 5 are fulfilled (see below). Let $1 \geq s > s_0$, and let $w \in X_T^{1,b}$ be a solution of*

$$(0.8) \quad \begin{cases} i\partial_t w + \Delta w \pm |w|^2 w &= g, \\ w(x, 0) &= w_0(x) \end{cases}$$

with $g \in C([0, T], H^1)$ supported in $[0, T] \times \bar{\omega}$.

Then, there exists $\varepsilon > 0$ such that for every $u_0 \in H^s$ with $\|u_0 - w_0\|_{H^s} < \varepsilon$ there exists $g_1 \in C([0, T], H^s)$ supported in $[0, T] \times \bar{\omega}$ such that the unique solution u in $X_T^{s,b}$ of (0.8) with $u(0) = u_0$ and g replaced by g_1 fulfills $u(T) = w(T)$.

Moreover, for any $u_0 \in H^1$ with $\|u_0 - w_0\|_{H^s} < \varepsilon$, the same conclusion holds with $g \in C([0, T], H^1)$.

An interesting fact is that the smallness assumption concerns only the H^s norm, even if we want a control in H^1 . For example, as in [17], if we assume $\|u_0\|_{H^1} \leq R_0$, we can find $N \in \mathbb{N}$ large enough such that the smallness assumption concerns only the N first frequencies (see Corollary 8.3). Of course, this result remains true in a lower dimension, where it was known only for the trajectory $w = 0$ (see [16]).

Let us describe the new hypothesis. Assumption 4 is a unique continuation result at weaker regularity.

Assumption 4. Unique continuation in H^1 : For every $T > 0$, the only solution in $C([0, T], H^1)$ to the system

$$(0.9) \quad \begin{cases} i\partial_t u + \Delta u + b_1(t, x)u + b_2(t, x)\bar{u} = 0 & \text{on } [0, T] \times M, \\ u = 0 & \text{on } [0, T] \times \omega, \end{cases}$$

where $b_1(t, x)$ and $b_2(t, x) \in L^\infty([0, T], L^3)$, is the trivial one $u \equiv 0$.

We do not know if it is really stronger than Assumption 2, but, for the moment, there are some examples where we are able to prove Assumption 2 but not Assumption 4 using some weak Carleman estimates (see Appendix B). For instance, on \mathbb{T}^3 , we are able to prove Assumption 2 for $\omega = \{x \in \mathbb{R}^3/\mathbb{Z}^3 \mid x_1 \in]0, \varepsilon[+ \mathbb{Z}\}$ but not Assumption 4. Yet, for the moment, we do not manage to deduce a controllability result from this statement.

The other new assumption is technical and yields quadrilinear estimates for a commutator.

Assumption 5. There exists $0 \leq s_0 < 1$ so that for any $\varepsilon \in [0, 1]$ we can find one constant $C > 0$ such that for any $f_1, f_2, f_3, f_4 \in L^2(M)$ satisfying

$$f_j = \mathbf{1}_{\sqrt{1-\Delta} \in [N_j, 2N_j[} (f_j), \quad j = 1, 2, 3, 4,$$

one has the following quadrilinear estimate:

$$(0.10) \quad \begin{aligned} & \sup_{\tau \in \mathbb{R}} \left| \int_{\mathbb{R}} \int_M \chi(t) e^{it\tau} u_1 u_2 \left((-\Delta)^{\varepsilon/2} u_3 u_4 - u_3 (-\Delta)^{\varepsilon/2} u_4 \right) dx dt \right| \\ & \leq C(N_1^\varepsilon + N_2^\varepsilon) (m(N_1, \dots, N_4))^{s_0} \|f_1\|_{L^2(M)} \|f_2\|_{L^2(M)} \|f_3\|_{L^2(M)} \|f_4\|_{L^2(M)}, \\ & \quad u_j(t) = e^{it\Delta} f_j, \quad j = 1, 2, 3, 4, \end{aligned}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is arbitrary and $m(N_1, \dots, N_4)$ is the product of the smallest two numbers among N_1, N_2, N_3, N_4 .

Moreover, the same result holds with u_i replaced by \bar{u}_i for i in a subset of $\{1, 2, 3, 4\}$.

For the three treated examples, we prove in Appendix A that Assumption 5 holds true with the same s_0 as in Assumption 3. We believe that it is the case for any manifold, but we did not manage to prove it.

As explained before, there are some examples for which we know that a geometric control assumption is not necessary. For instance, for any pair of manifolds M_1, M_2 and $\omega_1 \subset M_1$ such that ω_1 satisfies an observability estimate, $\omega_1 \times M_2$ satisfies the observability estimate for the linear Schrödinger equation. We can then use this remark and the work of Jaffard [26] and Komornik and Loreti [28] for the linear equation on \mathbb{T}^n to get some local nonlinear results. Since Theorem 0.3 is proved by a perturbative argument, we can also deduce controllability near 0 from these already known linear control results.

THEOREM 0.4. *If $w \equiv 0$ and (M, ω) is either*
 – $(\mathbb{T}^3, \text{any open set})$,
 – $(S^2 \times S^1, \omega_1 \times S^1)$, where ω_1 is a neighborhood of the equator of S^2 , or
 – $(S^2 \times S^1, S^2 \times]0, \varepsilon[)$,
then the same conclusion as in Theorem 0.3 is true.

Rosier and Zhang [40] simultaneously obtained the same result for \mathbb{T}^3 .

The proof of stabilization and of linear control with potential follows the same scheme as [16]. In a contradiction argument, we are led to prove the strong convergence to zero in $X_T^{s,b}$ of some weakly convergent sequence (u_n) solution to the damped NLS or Schrödinger with potential. Since the equation is subcritical, we use

some linearizability properties of the NLS in H^1 (see the work of Gérard [22] for the wave equation).

We first establish the strong convergence by some propagation of compactness. We adapt the argument of [16] inspired by Bardos and Masrour [3]. We use microlocal defect measures introduced by Gérard [21]. For a sequence (u_n) weakly convergent to 0 in $X_T^{s,b}$ satisfying

$$\begin{cases} i\partial_t u_n + \Delta u_n \rightarrow 0 & \text{in } X_T^{s-1+b,-b}, \\ a(x)u_n \rightarrow 0 & \text{in } L^2([0, T], H^s), \end{cases}$$

we prove that $u_n \rightarrow 0$ in $L^2_{loc}([0, T], H^s)$.

Once we know that the convergence is strong, we infer that the limit u is the solution to the NLS. We would like to use Assumption 2 or 4 of unique continuation to prove that it is 0. However, more regularity is needed to apply them. Again, we adapt the proof for $X^{s,b}$ spaces of propagation results of microlocal regularity coming from [16].

The rest of this article is organized as follows. The first section states and recalls some properties of the Bourgain spaces that will be used throughout this paper. The second section proves the well-posedness of the nonlinear equation with source and damping terms and its associated linearization near trajectories. In section 3, we prove that the equation is linearizable, namely, that at high frequency the nonlinear equation behaves as the linear one. Sections 4 and 5 are devoted to the propagation of regularity and compactness along the bicharacteristics which will be the essential tools for the proofs of stabilization and controllability. The main results of this article are proved in the last sections. The stabilization result is proved in section 6. In section 7, we prove the controllability of the linear equation that is obtained by linearization of the nonlinear one. This permits us to prove control near trajectories in section 8. In section 9, we prove that on S^3 our geometrical assumption is nearly optimal. In Appendix A, we prove some commutator estimates used in the proof of the regularity result of the control constructed in section 7. Appendix B is used to prove the assumption of unique continuation in our specific geometries thanks to some Carleman estimates.

In this article, b' will be a constant such that estimates of Lemma 1.1 hold. Actually, each of the trilinear estimates (with different s) that will be done will yield one $b' < 1/2$ but remains true if we choose a greater one. So we take $b' < 1/2$ as the largest of these constants. This allows us to choose one $b > 1/2$ with $1 > b + b'$.

In the rest of this paper, C will denote any constant whose value could change throughout this article.

1. Some properties of $X^{s,b}$ spaces. Since M is compact, Δ has a compact resolvent, and thus the spectrum of Δ is discrete. We choose $e_k \in L^2(M)$, $k \in M$, as an orthonormal basis of eigenfunctions of $-\Delta$ associated with eigenvalues λ_k . Denote P_k the orthogonal projector on e_k . We equip the Sobolev space $H^s(M)$ with the norm (with $\langle x \rangle = \sqrt{1 + |x|^2}$)

$$\|u\|_{H^s(M)}^2 = \sum_k \langle \lambda_k \rangle^s \|P_k u\|_{L^2(M)}^2.$$

The Bourgain space $X^{s,b}$ is equipped with the norm

$$\|u\|_{X^{s,b}}^2 = \sum_k \langle \lambda_k \rangle^s \left\| \langle \tau + \lambda_k \rangle^b \widehat{P}_k(\tau)u \right\|_{L^2(\mathbb{R}_\tau \times M)}^2 = \|u^\#\|_{H^b(\mathbb{R}, H^s(M))}^2,$$

where $u = u(t, x)$, $t \in \mathbb{R}$, $x \in M$, $u^\#(t) = e^{-it\Delta}u(t)$, and $\widehat{P_k u}(\tau)$ denotes the Fourier transform of $P_k u$ with respect to the time variable.

$X_T^{s,b}$ is the associated restriction space with the norm

$$\|u\|_{X_T^{s,b}} = \inf \{ \|\tilde{u}\|_{X^{s,b}} \mid \tilde{u} = u \text{ on }]0, T[\times M \}.$$

We also write $\|u\|_{X_T^{s,b}}$ if the infimum is taken on functions \tilde{u} equalling u on an interval

I. The following properties of $X_T^{s,b}$ spaces are easily verified:

1. $X^{s,b}$ and $X_T^{s,b}$ are Hilbert spaces.
2. If $s_1 \leq s_2$, $b_1 \leq b_2$, we have $X^{s_2,b_2} \subset X^{s_1,b_1}$ with continuous embedding.
3. For every $s_1 < s_2$, $b_1 < b_2$, and $T > 0$, we have $X_T^{s_2,b_2} \subset X_T^{s_1,b_1}$ with compact imbedding.
4. For $0 < \theta < 1$, the complex interpolation space $(X^{s_1,b_1}, X^{s_2,b_2})_{[\theta]}$ is as follows:
 $X^{(1-\theta)s_1+\theta s_2, (1-\theta)b_1+\theta b_2}$.

Property 4 can be proved with the interpolation theorem of Stein and Weiss for weighted L^p spaces (see [5, p. 114]).

Then, we list some additional trilinear estimates that will be used throughout this paper.

LEMMA 1.1. *If Assumption 3 is fulfilled, for every $r \geq s > s_0$, there exist $0 < b' < 1/2$ and $C > 0$ such that for any u and $\tilde{u} \in X^{r,b'}$*

$$(1.1) \quad \| |u|^2 u \|_{X^{r,-b'}} \leq C \|u\|_{X^{s,b'}}^2 \|u\|_{X^{r,b'}} ,$$

$$(1.2) \quad \| |u|^2 \tilde{u} \|_{X^{r,-b'}} \leq C \|u\|_{X^{s,b'}} \|u\|_{X^{r,b'}} \|\tilde{u}\|_{X^{r,b'}} ,$$

$$(1.3) \quad \| |u|^2 u - |\tilde{u}|^2 \tilde{u} \|_{X^{s,-b'}} \leq C \left(\|u\|_{X^{s,b'}}^2 + \|\tilde{u}\|_{X^{s,b'}}^2 \right) \|u - \tilde{u}\|_{X^{s,b'}} .$$

Moreover, the same estimates hold with $z_1 \overline{z_2} z_3$ replaced by any \mathbb{R} -trilinear form on \mathbb{C} .

The proof of the previous lemma can be found in [9], [12], or [23]. Yet, in Appendix A, we prove some slightly different estimates, but the proof gives an idea of how Lemma 1.1 is established. We also give some variants that will be used in the linearized version of our equations.

LEMMA 1.2. *If Assumption 3 is fulfilled, for every $-1 \leq s \leq 1$ and any $s_0 < r \leq 1$, there exist $0 < b' < 1/2$ and $C > 0$ such that for any $u \in X^{s,b'}$ and $a_1, a_2 \in X^{1,b'}$*

$$(1.4) \quad \| a_1 \overline{a_2} u \|_{X^{s,-b'}} \leq C \|a_1\|_{X^{1,b'}} \|a_2\|_{X^{1,b'}} \|u\|_{X^{s,b'}} ,$$

$$(1.5) \quad \| |a_1|^2 u \|_{X^{s,-b'}} \leq C \|a_1\|_{X^{1,b'}} \|a_1\|_{X^{r,b'}} \|u\|_{X^{s,b'}} .$$

Moreover, the same estimates hold with $z_1 \overline{z_2} z_3$ replaced by any \mathbb{R} -trilinear form on \mathbb{C} .

Proof. We first prove (1.5). Estimate (1.2) of Lemma 1.1 implies that the operator of multiplication by $|a_1|^2$ maps $X^{1,b'}$ into $X^{1,-b'}$ with norm $\|a_1\|_{X^{1,b'}} \|a_1\|_{X^{r,b'}}$. By duality, it maps $X^{-1,b'}$ into $X^{-1,-b'}$ with the same norm. We get the same result for $-1 \leq s \leq 1$ by interpolation, which yields (1.5). For (1.4), we observe that estimate

$$\| a_1 \overline{a_2} u \|_{X^{1,-b'}} \leq C \|a_1\|_{X^{1,b'}} \|a_2\|_{X^{1,b'}} \|u\|_{X^{1,b'}}$$

holds regardless of the position of the conjugate operator, and we get the result similarly. \square

Let us study the stability of the $X^{s,b}$ spaces with respect to some particular operations.

LEMMA 1.3. *Let $\varphi \in C_0^\infty(\mathbb{R})$ and $u \in X^{s,b}$; then $\varphi(t)u \in X^{s,b}$. If $u \in X_T^{s,b}$, then we have $\varphi(t)u \in X_T^{s,b}$.*

Proof. We write

$$\begin{aligned} \|\varphi u\|_{X^{s,b}} &= \|e^{-it\Delta}\varphi(t)u(t)\|_{H^b(\mathbb{R},H^s)} \\ &= \|(\varphi u)^\# \|_{H^b(\mathbb{R},H^s)} \leq C \|u^\# \|_{H^b(\mathbb{R},H^s)} \leq C \|u\|_{X^{s,b}}. \end{aligned}$$

We get the second result by applying the first one on any extension of u and taking the infimum. \square

In the case of pseudodifferential operators in the space variable, we have to deal with a loss in $X^{s,b}$ regularity compared to what we could expect. Some regularity in the index b is lost, due to the fact that a pseudodifferential operator does not keep the structure in time of the harmonics.

This loss is unavoidable, as we can see, for simplicity, on the torus \mathbb{T}^1 : we take $u_n = \psi(t)e^{inx}e^{i|n|^2t}$ (where $\psi \in C_0^\infty$ equal to 1 on $[-1, 1]$), which is uniformly bounded in $X^{0,b}$ for every $b \geq 0$. However, if we consider the operator B of order 0 of multiplication by e^{ix} , we get $\|e^{ix}u_n\|_{X^{0,b}} \approx n^b$. Yet, we do not have such a loss for the operator of the form $(-\Delta)^r$ which acts from any $X^{s,b}$ to $X^{s-2r,b}$. But if we do not make any further assumption on the pseudodifferential operator, we can show that our example is the worst one.

LEMMA 1.4. *Let $-1 \leq b \leq 1$, and let B be a pseudodifferential operator in the space variable of order ρ . For any $u \in X^{s,b}$ we have $Bu \in X^{s-\rho-|b|,b}$. Similarly, B maps $X_T^{s,b}$ into $X_T^{s-\rho-|b|,b}$.*

Proof. We first deal with the two cases $b = 0$ and $b = 1$, and we will conclude by interpolation and duality.

For $b = 0$, $X^{s,0} = L^2(\mathbb{R}, H^s)$, and the result is obvious.

For $b = 1$, we have $u \in X^{s,1}$ if and only if

$$u \in L^2(\mathbb{R}, H^s) \quad \text{and} \quad i\partial_t u + \Delta u \in L^2(\mathbb{R}, H^s)$$

with the norm

$$\|u\|_{X^{s,1}}^2 = \|u\|_{L^2(\mathbb{R},H^s)}^2 + \|i\partial_t u + \Delta u\|_{L^2(\mathbb{R},H^s)}^2.$$

Then, we have

$$\begin{aligned} \|Bu\|_{X^{s-\rho-1,1}}^2 &= \|Bu\|_{L^2(\mathbb{R},H^{s-\rho-1})}^2 + \|i\partial_t Bu + \Delta Bu\|_{L^2(\mathbb{R},H^{s-\rho-1})}^2 \\ &\leq C \left(\|u\|_{L^2(\mathbb{R},H^{s-1})}^2 + \|B(i\partial_t u + \Delta u)\|_{L^2(\mathbb{R},H^{s-\rho-1})}^2 \right. \\ &\quad \left. + \|[B, \Delta]u\|_{L^2(\mathbb{R},H^{s-\rho-1})}^2 \right) \\ &\leq C \left(\|u\|_{L^2(\mathbb{R},H^{s-1})}^2 + \|i\partial_t u + \Delta u\|_{L^2(\mathbb{R},H^{s-1})}^2 + \|u\|_{L^2(\mathbb{R},H^s)}^2 \right) \\ &\leq C \|u\|_{X^{s,1}}^2. \end{aligned}$$

Hence, B maps $X^{s,0}$ into $X^{s-\rho,0}$ and $X^{s,1}$ into $X^{s-\rho-1,1}$. Then, we conclude by interpolation that B maps $X^{s,b} = (X^{s,0}, X^{s,1})_{[b]}$ into $(X^{s-\rho,0}, X^{s-\rho-1,1})_{[b]} = X^{s-\rho-b,b}$, which yields the b loss of regularity as announced.

By duality, this also implies that for $0 \leq b \leq 1$, B^* maps $X^{-s+\rho+b,-b}$ into $X^{-s,-b}$. As there is no assumption on $s \in \mathbb{R}$, we also have the result for $-1 \leq b \leq 0$ with a loss $-b = |b|$.

To get the same result for the restriction spaces $X_T^{s,b}$, we write the inequality for an extension \tilde{u} of u , which yields

$$\|Bu\|_{X_T^{s-\rho-|b|,b}} \leq \|B\tilde{u}\|_{X^{s-\rho-|b|,b}} \leq C \|\tilde{u}\|_{X^{s,b}}.$$

Taking the infimum on all the \tilde{u} , we get the claimed result. \square

We will also use the following elementary estimate (see, e.g., [24] or [7]).

LEMMA 1.5. *Let (b, b') satisfy*

$$(1.6) \quad 0 < b' < \frac{1}{2} < b, \quad b + b' \leq 1.$$

If $f \in H^{-b'}(\mathbb{R})$ and we note that $F(t) = \Psi\left(\frac{t}{T}\right) \int_0^t f(t') dt'$, we have for $T \leq 1$

$$\|F\|_{H^b(\mathbb{R})} \leq CT^{1-b-b'} \|f\|_{H^{-b'}(\mathbb{R})}.$$

In the future aim of using a bootstrap argument, we will need some continuity in T of the $X_T^{s,b}$ norm of a fixed function.

LEMMA 1.6. *Let $0 < b < 1$ and $u \in X^{s,b}$; then the function*

$$\begin{cases} f & : &]0, T] & \longrightarrow & \mathbb{R}, \\ & & t & \longmapsto & \|u\|_{X_t^{s,b}} \end{cases}$$

is continuous. Moreover, if $b > 1/2$, there exists C_b such that

$$\lim_{t \rightarrow 0} f(t) \leq C_b \|u(0)\|_{H^s}.$$

Proof. By reasoning on each component on the basis, we are led to prove the result in $H^b(\mathbb{R})$. The most difficult case is the limit near 0. It suffices to prove that if $u \in H^b(\mathbb{R})$, with $b > 1/2$, satisfies $u(0) = 0$, and $\Psi \in C_0^\infty(\mathbb{R})$ with $\Psi(0) = 1$, then

$$\Psi\left(\frac{t}{T}\right) u \xrightarrow{T \rightarrow 0} 0 \quad \text{in } H^b.$$

Such a function u can be written $\int_0^t f$ with $f \in H^{b-1}$. Then, Lemma 1.5 gives the result we want if $u \in H^{b+\varepsilon}$. Nevertheless, if we have only $u \in H^b$, $\Psi\left(\frac{t}{T}\right)u$ is uniformly bounded. We conclude by a density argument. \square

The following lemma will be useful to control solutions on large intervals that will be obtained by piecing together solutions on smaller ones. We state it without proof.

LEMMA 1.7. *Let $0 < b < 1$. If $\bigcup]a_k, b_k[$ is a finite covering of $[0, 1]$, then there exists a constant C depending only on the covering such that for every $u \in X^{s,b}$*

$$\|u\|_{X_{[0,1]}^{s,b}} \leq C \sum_k \|u\|_{X_{[a_k, b_k]}^{s,b}}.$$

2. Existence of a solution to the NLS with source and damping terms.

2.1. Nonlinear equation. Let $a \in C^\infty(M)$ taking real values fixed. We will prove the existence of defocusing nonlinearities of degree 3: they will have the form $\alpha u + \beta|u|^2u$, with $\alpha, \beta \geq 0$.

PROPOSITION 2.1. *Let $T > 0$ and $s \geq 1$. Assume that M satisfies Assumption 3. Then, for every $g \in L^2([0, T], H^s)$ and $u_0 \in H^s$, there exists a unique solution u on $[0, T]$ in $X_T^{s,b}$ to the Cauchy problem*

$$(2.1) \begin{cases} i\partial_t u + \Delta u - \alpha u - \beta|u|^2 u &= a(x)(1 - \Delta)^{-1}a(x)\partial_t u + g \quad \text{on } [0, T] \times M, \\ u(0) &= u_0 \in H^s. \end{cases}$$

Moreover, the flow map

$$F : H^s(M) \times L^2([0, T], H^s(M)) \rightarrow X_T^{s,b}, \\ (u_0, g) \mapsto u$$

is Lipschitz on every bounded subset.

Proof. It is strongly inspired by the one of Bourgain [7] and Dehman, Gérard, and Lebeau [16] for the stabilization term. The proof is mainly based on estimates of Lemma 1.1.

First, we establish that the operator J defined by $Jv = (1 + ia(x)(1 - \Delta)^{-1}a(x))v$ is an isomorphism of H^s and $X^{s,b}$ ($s \in \mathbb{R}$ and $-1 \leq b \leq 1$).

J is an isomorphism of L^2 because of its decomposition in identity plus an antiself-adjoint part $J = 1 + A$. It is then an isomorphism of H^s with $s \geq 0$ by ellipticity and for every $s \in \mathbb{R}$ by duality. Using Lemma 1.4, we infer that if $-1 \leq b \leq 1$, A maps $X^{s,b}$ into itself. Moreover, J^{-1} (considered, for example, acting on $L^2([0, T] \times M)$) is a pseudodifferential operator of order 0 and satisfies $J^{-1} = 1 - AJ^{-1}$. Then, using again Lemma 1.4, we get that AJ^{-1} maps $X^{s,b}$ into $X^{s-|b|+2,b}$ and J is an isomorphism of $X^{s,b}$.

In the remainder of the proof, v will denote Ju . Hence, we can write system (2.1) as

$$(2.2) \begin{cases} \partial_t v - i\Delta v - R_0 v + i\beta|u|^2 u &= -ig \quad \text{on } [0, T] \times M, \\ v &= Ju, \\ v(0) &= v_0 = Ju_0 \in H^s, \end{cases}$$

where $R_0 = -i\Delta AJ^{-1} + i\alpha J^{-1}$ is a pseudodifferential operator of order 0.

First, we notice that if $g \in L^2([0, T], H^s)$, it also belongs to $X_T^{s,-b'}$ as $b' \geq 0$. We consider the functional

$$\Phi(v)(t) = e^{it\Delta}v_0 + \int_0^t e^{i(t-\tau)\Delta} [R_0 v - i\beta|u|^2 u - ig](\tau)d\tau.$$

We will apply a fixed point argument on the Banach space $X_T^{s,b}$. Let $\psi \in C_0^\infty(\mathbb{R})$ be equal to 1 on $[-1, 1]$. Then by construction (see [24])

$$\|\psi(t)e^{it\Delta}v_0\|_{X^{s,b}} = \|\psi\|_{H^b(\mathbb{R})} \|v_0\|_{H^s}.$$

Thus, for $T \leq 1$ we have

$$\|e^{it\Delta}v_0\|_{X_T^{s,b}} \leq C \|v_0\|_{H^s} \leq C \|u_0\|_{H^s}.$$

For $T \leq 1$, the one-dimensional estimate of Lemma 1.5 implies

$$\left\| \psi(t/T) \int_0^t e^{i(t-\tau)\Delta} F(\tau) \right\|_{X^{s,b}} \leq CT^{1-b-b'} \|F\|_{X^{s-b'}},$$

and then

$$\begin{aligned}
 (2.3) \quad & \left\| \int_0^t e^{i(t-\tau)\Delta} \left[R_0 v - i\beta |u|^2 u - ig \right] (\tau) d\tau \right\|_{X_T^{s,b}} \\
 & \leq CT^{1-b-b'} \left\| R_0 v - \beta i |u|^2 u - ig \right\|_{X_T^{s,-b'}} \\
 & \leq CT^{1-b-b'} \|R_0 v\|_{X_T^{s,0}} + \left\| |u|^2 u \right\|_{X_T^{s,-b'}} + \|g\|_{X_T^{s,-b'}} \\
 (2.4) \quad & \leq CT^{1-b-b'} \|v\|_{X_T^{s,b}} \left(1 + \|v\|_{X_T^{1,b}}^2 \right) + \|g\|_{X_T^{s,-b'}}.
 \end{aligned}$$

Thus

$$(2.5) \quad \|\Phi(v)\|_{X_T^{s,b}} \leq C \|u_0\|_{H^s} + \|g\|_{X_T^{s,-b'}} + CT^{1-b-b'} \|v\|_{X_T^{s,b}} \left(1 + \|v\|_{X_T^{1,b}}^2 \right)$$

and, similarly,

$$(2.6) \quad \|\Phi(v) - \Phi(\tilde{v})\|_{X_T^{s,b}} \leq CT^{1-b-b'} \|v - \tilde{v}\|_{X_T^{s,b}} \left(1 + \|v\|_{X_T^{s,b}}^2 + \|\tilde{v}\|_{X_T^{s,b}}^2 \right).$$

These estimates imply that if T is chosen small enough, Φ is a contraction on a suitable ball of $X_T^{s,b}$. Moreover, we have uniqueness in the class $X_T^{s,b}$ for the Duhamel equation and therefore for the Schrödinger equation.

We also prove propagation of regularity. If $u_0 \in H^s$, with $s > 1$, we have an existence time T for the solution in $X_T^{1,b}$ and another time \tilde{T} for the existence in $X_{\tilde{T}}^{s,b}$. By uniqueness in $X_T^{1,b}$, the two solutions are the same on $[0, \tilde{T}]$. Assume $\tilde{T} < T$. Then, $\|u(t, \cdot)\|_{H^s}$ explodes as t tends to \tilde{T} whereas $\|u(t, \cdot)\|_{H^1}$ remains bounded. Using local existence in H^1 and Lemma 1.7, we get that $\|u\|_{X_{\tilde{T}}^{1,b}}$ is finite. Applying tame estimate (2.5) on a subinterval $[T - \varepsilon, T]$, with ε small enough such that $C\varepsilon^{1-b-b'}(1 + \|v\|_{X_T^{1,b}}^2) < 1/2$, we obtain

$$\|v\|_{X_T^{s,b}} \leq C \|u(T - \varepsilon)\|_{H^s} + \|g\|_{X_T^{s,-b'}}.$$

Therefore, $u \in X_{\tilde{T}}^{s,b}$, and this contradicts the explosion of $\|u(t, \cdot)\|_{H^s}$ near \tilde{T} .

Next, we use energy estimates to get global existence. First, we will consider the energy

$$E(t) = \frac{1}{2} \int_M |\nabla u|^2 + \frac{1}{2} \alpha \int_M |u|^2 + \beta \frac{1}{4} \int_M |u|^4.$$

The energy is conserved if $g = 0$ and $a = 0$. It is nonincreasing if $g = 0$. In general, multiplying our equation by $\partial_t \bar{u}$, we have the relation

$$\begin{aligned}
 E(t) - E(0) &= - \int_0^t \left\| (1 - \Delta)^{-1/2} a(x) \partial_t u \right\|_{L^2}^2 - \Re \int_0^t \int_M g \overline{\partial_t u} \\
 &= - \int_0^t \left\| (1 - \Delta)^{-1/2} a(x) \partial_t u \right\|_{L^2}^2 - \Re \int_0^t \int_M (J^{-1*} g) \overline{\partial_t v} \\
 &= - \int_0^t \left\| (1 - \Delta)^{-1/2} a(x) \partial_t u \right\|_{L^2}^2 \\
 &\quad - \Re \int_0^t \int_M (J^{-1*} g) \overline{i\Delta v + R_0 v - i\beta |u|^2 u - ig}.
 \end{aligned}$$

If $0 \leq t \leq T$ (for this equation, there is not global existence in negative time) and $\beta > 0$, we get

$$\begin{aligned} E(t) &\leq E(0) + C \int_0^t \|\nabla(J^{-1*}g)\|_{L^2} \|\nabla u\|_{L^2} + \int_0^t \|g\|_{L^2} \|u\|_{L^2} \\ &\quad + \int_0^t \|g\|_{L^4} \|u\|_{L^4}^3 + \|g\|_{L^2([0,T] \times M)}^2 \\ &\leq E(0) + C \int_0^t \|g(\tau)\|_{H^1} \sqrt{E(\tau)} + C \int_0^t \|g(\tau)\|_{L^2} (E(\tau))^{1/4} \\ &\quad + C \int_0^t \|g(\tau)\|_{H^1} (E(\tau))^{3/4} + \|g\|_{L^2([0,T] \times M)}^2 \\ &\leq E(0) + C \int_0^t \|g(\tau)\|_{H^1} \left[1 + (E(\tau))^{3/4}\right] + \|g\|_{L^2([0,T] \times M)}^2. \end{aligned}$$

Therefore

$$\max_{0 \leq \tau \leq t} E(\tau) \leq E(0) + C \left[1 + \max_{0 \leq \tau \leq t} E(\tau)^{3/4}\right] \|g\|_{L^1([0,T],H^1)} + \|g\|_{L^2([0,T] \times M)}^2.$$

So we have finally

$$(2.7) \quad E(t) \leq C \left(1 + E(0)^4 + \|g\|_{L^2([0,T] \times M)}^8 + \|g\|_{L^1([0,T],H^1)}^4\right).$$

This implies that the energy is bounded if $g \in L^2([0, T], H^1)$ and yields global existence in $X_T^{1,b}$ for every $T > 0$. The fact that the flow is locally Lipschitz follows from estimate (2.6). \square

REMARK 2.1. *If $g = 0$, the solution of (2.1) satisfies the energy decay*

$$E(t) - E(0) = - \int_0^t \left\| (1 - \Delta)^{-1/2} a(x) \partial_t u \right\|_{L^2}^2.$$

This is obtained for initial data in H^2 by multiplying the equation by $\partial_t \bar{u}$ and can be extended to initial data in H^1 by approximation.

REMARK 2.2. *We have also proved that for any u_0, g with $\|u_0\|_{H^1} + \|g\|_{L^2([0,T],H^1)} \leq A$ the solution u of (2.1) satisfies*

$$\|u\|_{X_T^{1,b}} \leq C(T, A).$$

REMARK 2.3. *If we look carefully at inequality (2.3), we see that we have for $0 < \varepsilon < 1 - b - b'$*

$$\begin{aligned} &\left\| \int_0^t e^{(t-\tau)\Delta} \left[R_0 v - i |u|^2 u - i J g \right] (\tau) d\tau \right\|_{X^{1,b+\varepsilon}} \\ &\leq C T^{1-b-b'-\varepsilon} \left\| R_0 v - i |u|^2 u - i J g \right\|_{X_T^{1-b'}} \\ (2.8) \quad &\leq C T^{1-b-b'-\varepsilon} \|v\|_{X_T^{1,b}} \left(1 + \|v\|_{X_T^{1,b}}^2\right) + \|g\|_{L^2([0,T],H^1)} \end{aligned}$$

and we can then conclude that u is bounded in $X_T^{1,b+\varepsilon}$.

REMARK 2.4. We notice that for a solution of the equation the term of stabilization $a(x)(1 - \Delta)^{-1}a(x)\partial_t u$ belongs to $L^\infty([0, T], H^1(M))$ as expected. Actually, for a solution, this term acts as an operator of order 0. This is more visible using the equation fulfilled by $v = Ju$.

Then, in the aim of obtaining controllability near trajectories, we prove an appropriate existence theorem.

PROPOSITION 2.2. Suppose that Assumption 3 is fulfilled. Let $T > 0$, and let w be a solution in $X_T^{1,b}$ of

$$(2.9) \quad \begin{cases} i\partial_t w + \Delta w &= \pm |w|^2 w + g_1 \quad \text{on } [0, T] \times M, \\ w(0) &= w_0 \in H^1 \end{cases}$$

with $g_1 \in L^2([0, T], H^1)$. Then, for any $s \in]s_0, 1]$, there exists $\varepsilon > 0$ such that for any $u_0 \in H^s$ and $g \in L^2([0, T], H^s)$ with $\|u_0 - w_0\|_{H^s} + \|g_1 - g\|_{L^2([0, T], H^s)} \leq \varepsilon$ there exists a unique solution u in $X_T^{s,b}$ of (2.9). Moreover, for any $1 \geq r \geq s$ there exists $C = C(r, \|w\|_{X_T^{1,b}}, T) > 0$ such that, if $u_0 \in H^r$ and $g \in L^2([0, T], H^r)$, we have $u \in X_T^{r,b}$ and

$$(2.10) \quad \|u - w\|_{X_T^{r,b}} \leq C \left(\|u_0 - w_0\|_{H^r} + \|g_1 - g\|_{L^2([0, T], H^r)} \right).$$

REMARK 2.5. In the focusing case, the existence of w is not guaranteed for any w_0, g_1 , and T , and the result we prove assumes this existence.

REMARK 2.6. Here, we emphasize the fact that the assumption of smallness concerns only the H^s norm and not H^r . This is a consequence of the subcritical behavior.

Proof. We want to linearize the equation. If $u = w + r$ and $g = g_1 + g_r$, then

$$\begin{aligned} |w + r|^2 (w + r) &= |w|^2 w + 2|w|^2 r + w^2 \bar{r} + 2|r|^2 w + r^2 \bar{w} + |r|^2 r \\ &= |w|^2 w + 2|w|^2 r + w^2 \bar{r} + F(w, r). \end{aligned}$$

We are looking for the r solution of

$$(2.11) \quad \begin{cases} i\partial_t r + \Delta r &= 2|w|^2 r + w^2 \bar{r} + F(w, r) + g_r, \\ r(x, 0) &= r_0(x). \end{cases}$$

We make a proof similar to that of Proposition 2.1. We write only the necessary estimates. Inequalities (1.1) and (1.2) yield

$$\begin{aligned} \|r\|_{X_T^{r,b}} &\leq C \left(\|r_0\|_{H^r} + \|g_r\|_{L^2([0, T], H^r)} \right) + CT^{1-b-b'} \|w\|_{X_T^{1,b}}^2 \|r\|_{X_T^{r,b}} \\ &\quad + CT^{1-b-b'} \left(\|w\|_{X_T^{1,b}} \|r\|_{X_T^{r,b}} \|r\|_{X_T^{s,b}} + \|r\|_{X_T^{r,b}} \|r\|_{X_T^{s,b}}^2 \right). \end{aligned}$$

With T such that $CT^{1-b-b'} \|w\|_{X_T^{1,b}}^2 < 1/2$, it yields

$$(2.12) \quad \begin{aligned} \|r\|_{X_T^{r,b}} &\leq C \left(\|r_0\|_{H^r} + \|g_r\|_{L^2([0, T], H^r)} \right) \\ &\quad + CT^{1-b-b'} \left(\|w\|_{X_T^{1,b}} \|r\|_{X_T^{r,b}} \|r\|_{X_T^{s,b}} + \|r\|_{X_T^{r,b}} \|r\|_{X_T^{s,b}}^2 \right). \end{aligned}$$

First, we apply this with $r = s$. As we have proved in Lemma 1.6 the continuity with respect to T of $\|r\|_{X_T^{s,b}}$, we are in position to apply a bootstrap argument: for $\|r_0\|_{H^s} + \|g_r\|_{L^2([0, T], H^s)}$ small enough (depending only on $\|w\|_{X_T^{1,b}}$), we obtain

$$\|r\|_{X_T^{s,b}} \leq C \|r_0\|_{H^s} + \|g_r\|_{L^2([0, T], H^s)}.$$

Repeating the argument on every small interval, using that $\|r\|_{X_T^{s,b}}$ controls $L^\infty(H^s)$ and matching solutions with Lemma 1.7, we get the same result for every large interval, with a smaller constant ε , depending only on s, T , and $\|w\|_{X_T^{1,b}}$.

Then, we return to the general case $r \geq s$ and $CT^{1-b-b'}\|w\|_{X_T^{1,b}}^2 < 1/2$. For T small enough (depending only on r, ε , and $\|w\|_{X_T^{1,b}}$), estimate (2.12) becomes

$$\|r\|_{X_T^{r,b}} \leq C \left(\|r_0\|_{H^r} + \|g_r\|_{L^2([0,T],H^r)} \right).$$

Again, we obtain the desired result by piecing solutions together. \square

2.2. Linear equation with rough potential. The control near trajectories will be obtained by a perturbation of control of the linear Schrödinger equation with rough potential. The equations considered are the linearization of nonlinear equations and its dual version. We establish here the necessary estimates.

PROPOSITION 2.3. *Suppose Assumption 3. Let $T > 0, s \in [-1, 1], A > 0$, and $w \in X_T^{1,b}$ with $\|w\|_{X_T^{1,b}} \leq A$. For every $u_0 \in H^s$ and $g \in X_T^{s,-b'}$ there exists a unique solution u in $X_T^{s,b}$ of equation*

$$(2.13) \quad \begin{cases} i\partial_t u + \Delta u &= \pm 2|w|^2 u \pm w^2 \bar{u} + g \quad \text{on } [0, T] \times M, \\ u(0) &= u_0 \in H^s. \end{cases}$$

Moreover, there exists $C = C(s, A, T) > 0$ such that

$$(2.14) \quad \|u\|_{X_T^{s,b}} \leq C (\|u_0\|_{H^s} + \|g\|_{X^{s,-b'}}).$$

Proof. We make the same arguments as those above using estimates of Lemma 1.2. \square

3. Linearization in H^1 . The following result shows that any sequence of solutions with Cauchy data weakly convergent to 0 asymptotically behave as solutions of the linear equation. These types of results were first introduced by Gérard in [22] for the wave equation and are typical of subcritical situations.

PROPOSITION 3.1. *Suppose Assumption 3 is fulfilled. Let $(u_n) \in X_T^{1,b}$ be a sequence of solutions of*

$$(3.1) \quad \begin{cases} i\partial_t u_n + \Delta u_n - u_n - |u_n|^2 u_n &= a(x)(1 - \Delta)^{-1} a(x) \partial_t u_n \quad \text{on } [0, T] \times M, \\ u_n(0) &= u_{n,0} \in H^1(M) \end{cases}$$

such that

$$u_{n,0} \xrightarrow{H^1(M)} 0.$$

Then

$$|u_n|^2 u_n \xrightarrow{X_T^{1,-b'}} 0.$$

Proof. We prove that any subsequence (still denoted u_n) admits another subsequence converging to 0. The main point is the tame $X_T^{s,b}$ estimate of Lemma 1.1. For one $s_0 < s < 1$ we have

$$(3.2) \quad \| |u_n|^2 u_n \|_{X_T^{1,-b'}} \leq C \|u_n\|_{X_T^{s,b}}^2 \|u_n\|_{X_T^{1,b}}.$$

First, using Remark 2.2, we conclude that u_n is bounded in $X_T^{1,b}$, and actually by Remark 2.3, u_n is bounded in $X_T^{1,b+\varepsilon}$ for some $\varepsilon > 0$. By compact embedding of $X_T^{1,b+\varepsilon}$ into $X_T^{s,b}$ we obtain that u_n admits a subsequence converging weakly in $X_T^{1,b}$ and strongly in $X_T^{s,b}$ to a function $u \in X_T^{s,b}$ with $u(0) = 0$. $u_n(0)$ strongly converges to 0 in H^s and, by continuity of the nonlinear flow in H^s , u_n strongly converges to 0 in $X_T^{s,b}$. This yields the desired result thanks to (3.2). \square

4. Propagation of compactness. In this section, we adapt some theorems of Dehman, Gérard, and Lebeau [16] in the case of $X^{s,b}$ spaces. We recall that S^*M denotes the cosphere bundle of the Riemannian manifold M :

$$S^*M = \{(x, \xi) \in T^*M \quad : \quad |\xi|_x = 1\}.$$

PROPOSITION 4.1. *Let $r \in \mathbb{R}$. Let u_n be a sequence of solutions to*

$$i\partial_t u_n + \Delta u_n = f_n$$

such that for one $0 \leq b \leq 1$ we have

$$\|u_n\|_{X_T^{r,b}} \leq C, \quad \|u_n\|_{X_T^{r-1+b,-b}} \rightarrow 0, \quad \text{and} \quad \|f_n\|_{X_T^{r-1+b,-b}} \rightarrow 0.$$

*Then, there exists a subsequence $(u_{n'})$ of (u_n) and a positive measure μ on $]0, T[\times S^*M$ such that for every tangential (that is, without time derivative) pseudodifferential operator $A = A(t, x, D_x)$ of order $2r$ and of principal symbol $\sigma(A) = a_{2r}(t, x, \xi)$*

$$(A(t, x, D_x)u_{n'}, u_{n'})_{L^2(]0, T[\times M)} \rightarrow \int_{]0, T[\times S^*M} a_{2r}(t, x, \xi) \, d\mu(t, x, \xi).$$

*Moreover, if G_s denotes the geodesic flow on S^*M , one has for every $s \in \mathbb{R}$*

$$G_s(\mu) = \mu.$$

Proof. Existence of the measure: it is based on the Gårding inequality; see [21] for an introduction.

Propagation. Denote L the operator $L = i\partial_t + \Delta$. Let $\varphi = \varphi(t) \in C_0^\infty(]0, T[)$, and let $B(x, D_x)$ be a pseudodifferential operator of order 1, with principal symbol b_{2r-1} , $A(t, x, D_x) = \varphi(t)B(x, D_x)$. For $\varepsilon > 0$, we denote $A_\varepsilon = \varphi B_\varepsilon = A e^{\varepsilon \Delta}$ for the regularization.

As $A_\varepsilon u_n$ and $A_\varepsilon^* u_n$ are C^∞ , we can write

$$(Lu_n, A_\varepsilon^* u_n)_{L^2(]0, T[\times M)} = (f_n, A_\varepsilon^* u_n)_{L^2(]0, T[\times M)}$$

and

$$(A_\varepsilon u_n, Lu_n)_{L^2(]0, T[\times M)} = (A_\varepsilon u_n, f_n)_{L^2(]0, T[\times M)}.$$

We write by a classical way

$$\begin{aligned} \alpha_{n,\varepsilon} &= (Lu_n, A_\varepsilon^* u_n)_{L^2(]0, T[\times M)} - (A_\varepsilon u_n, Lu_n)_{L^2(]0, T[\times M)} \\ &= ([A_\varepsilon, \Delta]u_n, u_n) - i(\partial_t(A_\varepsilon)u_n, u_n). \end{aligned}$$

We will strongly use Lemmas 1.3 and 1.4 without citing them.

$\partial_t(A_\varepsilon)$ is of order $2r - 1$ uniformly in ε ; then,

$$\begin{aligned} \sup_\varepsilon (\partial_t(A_\varepsilon)u_n, u_n)_{L^2(]0,T[\times M)} &\leq C \|\partial_t(A_\varepsilon)u_n\|_{X_T^{-r+1-b,b}} \|u_n\|_{X_T^{r-1+b,-b}} \\ &\leq C \|u_n\|_{X_T^{r,b}} \|u_n\|_{X_T^{r-1+b,-b}}, \end{aligned}$$

which tends to 0 if $n \rightarrow \infty$.

But we also have

$$\begin{aligned} \alpha_{n,\varepsilon} &= (f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)} - (A_\varepsilon u_n, f_n)_{L^2(]0,T[\times M)}, \\ |(f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)}| &\leq \|f_n\|_{X_T^{r-1+b,-b}} \|A_\varepsilon^* u_n\|_{X_T^{-r+1-b,b}} \\ &\leq \|f_n\|_{X_T^{r-1+b,-b}} \|u_n\|_{X_T^{r,b}}. \end{aligned}$$

Then, $\sup_\varepsilon |(f_n, A_\varepsilon^* u_n)_{L^2(]0,T[\times M)}| \rightarrow 0$ when $n \rightarrow \infty$. The same estimate for the other terms gives $\sup_\varepsilon \alpha_{n,\varepsilon} \rightarrow 0$.

Finally, taking the supremum on ε tending to 0, we get

$$(\varphi[B, \Delta]u_n, u_n)_{L^2(]0,T[\times M)} \rightarrow 0 \quad \text{when } n \rightarrow \infty,$$

which means, in terms of measure,

$$\int_{]0,T[\times S^*M} \varphi(t) \{\sigma_2(\Delta), b_{2r-1}\} \, d\mu(t, x, \xi) = 0.$$

This is precisely the propagation along the geodesic flow. \square

COROLLARY 4.2. *Let $r \in \mathbb{R}$. Assume that $\omega \subset M$ satisfies Assumption 1 and $a \in C^\infty(M)$, as in (0.3). Let u_n be a sequence bounded in $X_T^{r,b'}$ with $0 < b' < 1/2$, weakly convergent to 0 and satisfying*

$$(4.1) \quad \begin{cases} i\partial_t u_n + \Delta u_n \rightarrow 0 & \text{in } X_T^{r,-b'}, \\ a(x)u_n \rightarrow 0 & \text{in } L^2(]0,T], H^r). \end{cases}$$

Then, we have $u_n \rightarrow 0$ in $X^{r,1-b'}$.

Proof. Let (u_{n_k}) be any subsequence of (u_n) . The assumption on b' and compact embedding allow us to apply Proposition 4.1. We can attach to (u_{n_k}) a microlocal defect measure in $L^2(]0,T], H^r)$ that propagates along the geodesics with infinite speed. The second assumption of (4.1) gives $a(x)\mu = 0$. By Assumption 1 and the fact that a is elliptic on ω , we have $\mu = 0$ on $]0,T[\times S^*M$; i.e., $(u_{n'}) \rightarrow 0$ in $L^2(]0,T], H^r)$, and $u_n \rightarrow u$ in $L^2(]0,T], H^r)$.

Then, we can pick t_0 such that $u_n(t_0) \rightarrow 0$ in H^r . Using Lemma 1.5 and assumptions on b' , we get for $T \leq 1$

$$\left\| \int_0^t e^{i(t-\tau)\Delta} f_n(\tau) d\tau \right\|_{X_T^{r,1-b'}} \leq C \|f_n\|_{X_T^{r,-b'}}.$$

Using the Duhamel formula, we conclude that $u_n \rightarrow 0$ in $X_T^{r,1-b'}$.

Then, the hypothesis $T \leq 1$ is easily removed by piecing solutions together as in Lemma 1.7. \square

5. Propagation of regularity. We write Proposition 13 of [16] with some $X^{s,b}$ assumptions on the second term of the equation.

PROPOSITION 5.1. *Let $T > 0$, let $0 \leq b < 1$, and let $u \in X_T^{r,b}$, $r \in \mathbb{R}$, be a solution of*

$$i\partial_t u + \Delta u = f \in X_T^{r,-b}.$$

*Given $\gamma_0 = (x_0, \xi_0) \in T^*M \setminus 0$, we assume that there exists a zeroth order pseudodifferential operator $\chi(x, D_x)$, elliptic in γ_0 such that*

$$\chi(x, D_x)u \in L^2_{loc}([0, T[, H^{r+\rho})$$

for some $\rho \leq \frac{1-b}{2}$. Then, for every $\gamma_1 \in \Gamma_{\gamma_0}$, the geodesic ray starting at γ_0 , there exists a pseudodifferential operator $\Psi(x, D_x)$, elliptic in γ_1 such that

$$\Psi(x, D_x)u \in L^2_{loc}([0, T[, H^{r+\rho}).$$

COROLLARY 5.2. *With the notations of the proposition, if an open set ω satisfies Assumption 1 and $a(x)u \in L^2_{loc}([0, T[, H^{r+\rho})$, with $a \in C^\infty(M)$, as in (0.3), then $u \in L^2_{loc}([0, T[, H^{r+\rho}(M)$.*

Proof of Proposition 5.1. We first regularize $u_n = e^{\frac{1}{n}\Delta}u$ with $\|u_n\|_{X_T^{r,b}} \leq C$. Set $s = r + \rho$. Let $B(x, D_x)$ be a pseudodifferential operator of order $2s - 1 = 2r + 2\rho - 1$ that will be chosen later, and let $A = A(t, x, D_x) = \varphi(t)B(x, D_x)$, where $\varphi \in C^\infty_0([0, T[)$.

If $L = i\partial_t + \Delta$, we write

$$\begin{aligned} & (Lu_n, A^*u_n)_{L^2([0, T[\times M)} - (Au_n, Lu_n)_{L^2([0, T[\times M)} \\ &= ([A, \Delta]u_n, u_n)_{L^2([0, T[\times M)} - (i\varphi' Bu_n, u_n)_{L^2([0, T[\times M)}, \\ & |(Au_n, f_n)_{L^2([0, T[\times M)}| \leq \|Au_n\|_{X_T^{-r,b}} \|f_n\|_{X_T^{r,-b}} \\ & \leq \|u_n\|_{X_T^{r+2\rho-1+b,b}} \|f_n\|_{X_T^{r,-b}}. \end{aligned}$$

As we have chosen $\rho \leq \frac{1-b}{2}$, we have $r + 2\rho - 1 + b \leq r$ and so

$$|(Au_n, f_n)_{L^2([0, T[\times M)}| \leq C \|u_n\|_{X_T^{r,b}} \|f_n\|_{X_T^{r,-b}} \leq C.$$

Similarly

$$|(\varphi' Bu_n, u_n)_{L^2([0, T[\times M)}| \leq C \|u_n\|_{X_T^{r,b}} \|u_n\|_{X_T^{r,-b}} \leq C.$$

Then,

$$([A, \Delta]u_n, u_n)_{L^2([0, T[\times M)} = \int_0^T \varphi(t) ([B, \Delta]u_n(t), u_n(t))_{L^2(M)} dt$$

is uniformly bounded. Then, we select B by means of symplectic geometry. Take $\gamma_1 \in \Gamma_{\gamma_0}$; U and V are two small conical neighborhoods, respectively, of γ_1 and γ_0 . For every symbol $c(x, \xi)$, of order s , supported in U , one can find a symbol $b(x, \xi)$ of

order $2s - 1$ such that

$$\frac{1}{i} \{ \sigma_2(\Delta), b(x, \xi) \} = |c(x, \xi)|^2 + r(x, \xi)$$

with $r(x, \xi)$ of order $2s$ and supported in V . We take B a pseudodifferential operator with principal symbol b so that $[B, \Delta]$ is a pseudodifferential operator of principal symbol $|c(x, \xi)|^2 + r(x, \xi)$. Then, if we choose $c(x, \xi)$ elliptic at γ_1 , we conclude that

$$\int_0^t \varphi(t) \|c(x, D_x)u_n(t, x)\|_{L^2}^2 dt \leq C.$$

This ends the proof of Proposition 5.1.

COROLLARY 5.3. *Here $\dim M \leq 3$ and $b > 1/2$. Let $u \in X_T^{1,b}$ be a solution of*

$$(5.1) \quad \begin{cases} i\partial_t u + \Delta u &= |u|^2 u + u \quad \text{on } [0, T] \times M, \\ \partial_t u &= 0 \quad \text{on }]0, T[\times \omega, \end{cases}$$

where ω satisfies Assumption 1. Then $u \in C^\infty(]0, T[\times M)$.

Proof. We have $u \in L^\infty([0, T], H^1)$ and so in $L^\infty([0, T], L^6)$ by Sobolev embedding. Then, we infer that $|u|^2 u \in L^\infty([0, T], L^2(M))$.

On $]0, T[\times \omega$, we have

$$\Delta u = |u|^2 u + u.$$

Therefore, $\Delta u \in L^2([0, T], L^2(\omega))$ and $u \in L^2(]0, T[, H^2(\omega))$. Since $H^2(\omega)$ is an algebra, we can go on the same reasoning to conclude that $u \in C^\infty(]0, T[\times \omega)$.

By applying once Corollary 5.2, we get $u \in L^2_{loc}([0, T], H^{1+\frac{1-b}{2}})$. Then we can pick t_0 such that $u(t_0) \in H^{1+\frac{1-b}{2}}$. We can then solve in $X^{1+\frac{1-b}{2}, b}$ our NLS equation with initial data $u(t_0)$. By uniqueness in $X_T^{1,b}$, we can conclude that $u \in X_T^{1+\frac{1-b}{2}, b}$.

By iteration, we get that $u \in L^2(]0, T[, H^r)$ for every $r \in \mathbb{R}$ and $u \in C^\infty([0, T] \times M)$. \square

COROLLARY 5.4. *If, in addition to Corollary 5.3, ω satisfies Assumption 2, then $u = 0$.*

Proof. Using Corollary 5.3, we infer that $u \in C^\infty(]0, T[\times M)$. Taking the time derivative of (5.1), $v = \partial_t u$ satisfies

$$(5.2) \quad \begin{cases} i\partial_t v + \Delta v + f_1 v + f_2 \bar{v} &= 0, \\ v &= 0 \quad \text{on }]0, T[\times \omega \end{cases}$$

for some $f_1, f_2 \in C^\infty(]0, T[\times M)$. Assumption 2 gives $v = \partial_t u = 0$. Multiplying (5.1) by \bar{u} and integrating, we get

$$\int_M |\nabla u|^2 + \int_M |u|^4 + \int_M |u|^2 = 0,$$

and so $u = 0$. \square

REMARK 5.1. *We have the same conclusion for the $u \in X_T^{1,b}$ solution of*

$$(5.3) \quad \begin{cases} i\partial_t u + \Delta u &= u \quad \text{on } [0, T] \times M, \\ \partial_t u &= 0 \quad \text{on }]0, T[\times \omega. \end{cases}$$

6. Stabilization. Theorem 0.2 is a consequence of the following proposition.

PROPOSITION 6.1. *Let $a \in C^\infty(M)$, as in (0.3). Under Assumptions 1, 2, and 3, for every $T > 0$ and every $R_0 > 0$, there exists a constant $C > 0$ such that inequality*

$$E(0) \leq C \int_0^T \left\| (1 - \Delta)^{-1/2} a(x) \partial_t u \right\|_{L^2}^2 dt$$

holds for every solution u of the damped equation

$$(6.1) \quad \begin{cases} i\partial_t u + \Delta u - (1 + |u|^2)u &= a(x)(1 - \Delta)^{-1} a(x) \partial_t u \quad \text{on } [0, T] \times M, \\ u(0) &= u_0 \in H^1 \end{cases}$$

and $\|u_0\|_{H^1} \leq R_0$.

Proof of Proposition 6.1 \Rightarrow Theorem 0.2. For any $f \in H^1(M)$, Sobolev embeddings yield

$$\begin{aligned} E(f) &\leq C \left(\|f\|_{H^1}^2 + \|f\|_{H^1}^4 \right), \\ \|f\|_{H^1} &\leq C (E(f))^{1/2}. \end{aligned}$$

As the energy is decreasing, if $\|u_0\|_{H^1} \leq R_0$, we can find another \tilde{R}_0 such that $\|u(t)\|_{H^1} \leq \tilde{R}_0$ for any $t > 0$. For this range of values, we have

$$(6.2) \quad C^{-1} (E(f))^{1/2} \leq \|f\|_{H^1} \leq C (E(f))^{1/2}$$

for one $C > 0$ depending on R_0 .

We apply Proposition 6.1 with this bound and obtain $E(t) \leq C e^{-\gamma(R_0)t} E(0)$. Then, for $t > t(R_0)$, we have $\|u(t)\|_{H^1} \leq 1$.

We take $\gamma(1)$ to be the decay rate corresponding to the bound 1. Therefore, for $t > t(R_0)$, we get that $\|u(t)\|_{H^1} \leq C e^{-\gamma(1)(t-t(R_0))} \|u(t(R_0))\|_{H^1}$. This yields a decay rate independent of R_0 as announced, while the coefficient C may strongly depend on R_0 . \square

REMARK 6.1. *If we make the change of unknown $w = e^{-it}u$, w is the solution of the new damped equation*

$$\begin{cases} i\partial_t w + \Delta w - |w|^2 w &= a(x)(1 - \Delta)^{-1} a(x) (\partial_t w - iw) \quad \text{on } [0, T] \times M, \\ w(0) &= u_0 \in H^1. \end{cases}$$

This modification is necessary because there is no exponential decay for the damped equation (6.1) with $|u|^2 u$ instead of $(1 + |u|^2)u$. We check, for example, that for $a = 1$ the solution $u(t)$ with constant Cauchy data u_0 is

$$|u(t)|^2 = \frac{|u_0|^2}{1 + |u_0|^2 t}.$$

This can be seen by working in polar coordinates $u(t) = \rho(t)e^{i\theta(t)}$ so that the solution satisfies $\dot{\rho} + i\rho\dot{\theta} = \frac{1}{i-1}\rho^3$ and $\frac{d}{dt}\left(\frac{1}{\rho^2}\right) = 1$ by taking the real part. Moreover, it also proves that the solution is global in time only on \mathbb{R}^+ (this restriction remains with the nonlinearity $(1 + |u|^2)u$).

Proof of Proposition 6.1. We argue by contradiction; we suppose the existence of a sequence (u_n) of solutions of (6.1) such that

$$\|u_n(0)\|_{H^1} \leq R_0$$

and

$$(6.3) \quad \int_0^T \left\| (1 - \Delta)^{-1/2} a(x) \partial_t u_n \right\|_{L^2}^2 dt \leq \frac{1}{n} E(u_n(0)).$$

We note that $\alpha_n = E(u_n(0))^{1/2}$. By the Sobolev embedding for the L^4 norm, we have $\alpha_n \leq C(R_0)$. So, up to extraction, we can suppose that $\alpha_n \rightarrow \alpha$. We will distinguish two cases: $\alpha > 0$ and $\alpha = 0$.

First case: $\alpha_n \rightarrow \alpha > 0$. By decreasing the energy, (u_n) is bounded in $L^\infty([0, T], H^1)$ and so in $X_T^{1,b}$. Then, as $X_T^{1,b}$ is a separable Hilbert we can extract a subsequence such that $u_n \rightharpoonup u$ weakly in $X_T^{1,b}$ and strongly in $X_T^{s,b'}$ for one $u \in X_T^{1,b}$ and $s > s_0$. Therefore, $|u_n|^2 u_n$ converges to $|u|^2 u$ in $X_T^{s,-b'}$.

Using (6.3) and passing to the limit in the equation verified by u_n , we get

$$\begin{cases} i\partial_t u + \Delta u &= |u|^2 u + u & \text{on } [0, T] \times M, \\ \partial_t u &= 0 & \text{on }]0, T[\times \omega. \end{cases}$$

Using Corollary 5.4, we infer that $u = 0$. Therefore, we have, up to new extraction, $u_n(0) \rightharpoonup 0$ in H^1 . Using Proposition 3.1 of linearization, we infer that $|u_n|^2 u_n \rightarrow 0$ in $X_T^{1,-b'}$. Moreover, by (6.3) we have

$$a(x)(1 - \Delta)^{-1} a(x) \partial_t u_n \xrightarrow{L^2([0, T], H^1)} 0$$

and the convergence is also in $X_T^{1,-b'}$.

Then, estimate (6.3) also implies $a(x) \partial_t u_n \xrightarrow{L^2([0, T], H^{-1})} 0$. Using (6.1), we obtain

$$a(x) [\Delta u_n - u_n - |u_n|^2 u_n - a(x)(1 - \Delta)^{-1} a(x) \partial_t u_n] \xrightarrow{L^2([0, T], H^{-1})} 0.$$

By Sobolev embedding, u_n tends to 0 in $L^\infty([0, T], L^p)$ for any $p < 6$. Therefore, $|u_n|^2 u_n$ converges to 0 in $L^\infty([0, T], L^q)$ for $q < 2$ and so in $L^2([0, T], H^{-1})$. Thus, we get

$$a(x)(\Delta - 1)u_n \xrightarrow{L^2([0, T], H^{-1})} 0.$$

Therefore, $(1 - \Delta)^{1/2} a(x) u_n = (1 - \Delta)^{-1/2} a(x)(1 - \Delta)u_n + (1 - \Delta)^{-1/2} [(1 - \Delta), a(x)]u_n$ converges to 0 in $L^2([0, T], L^2)$.

In conclusion, we have

$$\begin{cases} u_n \rightharpoonup 0 & \text{in } X_T^{1,b'}, \\ a(x)u_n \rightarrow 0 & \text{in } L^2([0, T], H^1), \\ i\partial_t u_n + \Delta u_n - u_n \rightarrow 0 & \text{in } X_T^{1,-b'}. \end{cases}$$

Thus, changing u_n into $e^{it} u_n$ and using that the multiplication by e^{it} is continuous on any $X_T^{s,b}$ (see Lemma 1.3), we are in position to apply Corollary 4.2. Hence, as we have $1 - b' > 1/2$, it yields

$$u_n(0) \xrightarrow{H^1} 0.$$

In particular, $E(u_n(0)) \rightarrow 0$, which is a contradiction to our hypothesis $\alpha > 0$.

Second case: $\alpha_n \rightarrow 0$. Let us make the change of unknown $v_n = u_n/\alpha_n$. v_n is the solution of the system

$$i\partial_t v_n + \Delta v_n - a(x)(1 - \Delta)^{-1}a(x)\partial_t v_n = v_n + \alpha_n^2 |v_n|^2 v_n$$

and

$$(6.4) \quad \int_0^T \left\| (1 - \Delta)^{-1/2} a(x) \partial_t v_n \right\|_{L^2}^2 dt \leq \frac{1}{n}.$$

For a constant depending on R_0 , we still have (6.2). Therefore, we write

$$\|v_n(t)\|_{H^1} = \frac{\|u_n(t)\|_{H^1}}{E(u_n(0))^{1/2}} \leq C \frac{E(u_n(t))^{1/2}}{E(u_n(0))^{1/2}} \leq C.$$

Therefore

$$(6.5) \quad \|v_n(0)\|_{H^1} = \frac{\|u_n(0)\|_{H^1}}{E(u_n(0))^{1/2}} \geq C > 0.$$

Thus, we have $\|v_n(0)\|_{H^1} \approx 1$ and v_n is bounded in $L^\infty([0, T], H^1)$.

By the same estimates we made in the proof of Proposition 2.1, we obtain

$$\|v_n\|_{X_T^{1,b}} \leq C \|v_n(0)\|_{H^1} + CT^{1-b-b'} \left(\|v_n\|_{X_T^{1,b}} + \alpha_n^2 \|v_n\|_{X_T^{1,b}}^3 \right).$$

Then, if we take $CT^{1-b-b'} < 1/2$, independent of v_n , we have

$$\|v_n\|_{X_T^{1,b}} \leq C(1 + \alpha_n^2 \|v_n\|_{X_T^{1,b}}^3).$$

By a bootstrap argument, we conclude that $\|v_n\|_{X_T^{1,b}}$ is uniformly bounded. Using Lemma 1.7, we conclude that it is bounded on $X_T^{1,b}$ for some large T , and then $\alpha_n^2 |v_n|^2 v_n$ tends to 0 in $X_T^{1,-b'}$.

Then, we can extract a subsequence such that $v_n \rightharpoonup v$ in $X_T^{1,b}$, where v is the solution of

$$\begin{cases} i\partial_t v + \Delta v &= v & \text{on } [0, T] \times M, \\ \partial_t v &= 0 & \text{on }]0, T[\times \omega. \end{cases}$$

It implies $v = 0$ by Remark 5.1. Estimate (6.4) yields that $a(x)(1 - \Delta)^{-1}a(x)\partial_t v_n$ converges to 0 in $L^2([0, T], H^1)$ and so in $X_T^{1,-b'}$.

We finish the proof as in the first case to conclude the convergence of v_n to 0 in $X_T^{1,b}$. This contradicts (6.5). \square

7. Controllability of the linear equation.

7.1. Observability estimate.

PROPOSITION 7.1. *Assume that (M, ω) satisfies Assumptions 1, 3, and 4. Let $a \in C^\infty(M)$, as in (0.3), taking real values. Then, for every $-1 \leq s \leq 1$, $T > 0$, and $A > 0$, there exists C such that estimate*

$$\|u_0\|_{H^s}^2 \leq C \int_0^T \|au(t)\|_{H^s}^2 dt$$

holds for every solution $u(t, x) \in X_T^{s,b}$ of the system

$$(7.1) \quad \begin{cases} i\partial_t u + \Delta u &= \pm 2|w|^2 u \pm w^2 \bar{u} \quad \text{on } [0, T] \times M, \\ u(0) &= u_0 \in H^s \end{cases}$$

with one w satisfying $\|w\|_{X_T^{1,b}} \leq A$.

Proof. We treat only the case with $2|w|^2 u + w^2 \bar{u}$. The others are similar. We argue by contradiction. Let $u_n \in X_T^{s,b}$ be a sequence of solutions to (7.1) with some associated w_n such that

$$(7.2) \quad \|u_n(0)\|_{H^s} = 1, \quad \int_0^T \|au_n\|_{H^s}^2 \rightarrow 0$$

and

$$\|w_n\|_{X_T^{1,b}} \leq A.$$

Proposition 2.3 of existence yields that u_n is bounded in $X_T^{s,b}$, and we can extract a subsequence such that u_n converges strongly in $X_T^{s-1+b,-b}$ to some $u \in X_T^{s,b}$ ($b < 1 - b' < 1$).

Then, using Lemma 1.2, we infer that $2|w_n|^2 u_n + w_n^2 \bar{u}_n$ is bounded in $X_T^{s,-b'}$. We can extract another subsequence such that it converges strongly in $X_T^{s-1+b,-b}$ (here we use $-b < -1/2 < -b'$) to some $\Psi \in X_T^{s,-b'}$. Denoting $r_n = u_n - u$ and $f_n = 2|w_n|^2 u_n + w_n^2 \bar{u}_n - \Psi$, we can apply Proposition 4.1 of propagation of compactness. As ω satisfies geometric control and $au_n \rightarrow 0$ in $L^2([0, T], H^s)$, we obtain that $r_n \rightarrow 0$ in $L^2_{loc}([0, T], H^s)$. r_n is also bounded in $X_T^{s,b}$, and we deduce, by interpolation, that r_n tends to 0 in $X_I^{s,b'}$ for every $I \subset\subset]0, T[$.

Now, we want to prove that $u \equiv 0$ using unique continuation. As w_n is bounded in $X_T^{1,b}$, we can extract a subsequence such that it converges weakly to some $w \in X_T^{1,b}$. We have to prove that u is the solution of a linear Schrödinger equation with potential. But the fact that $|w_n|^2 u_n$ converges weakly to $|w|^2 u$ is not guaranteed and actually uses the fact that the regularity H^1 is subcritical (see the article of Molinet [37], where the limit of the product is not the expected one).

We decompose

$$\begin{aligned} u_n |w_n|^2 - u |w|^2 &= (u_n - u) |w_n|^2 + u [|w_n - w|^2 - w(\overline{w - w_n}) - \overline{w}(w - w_n)] \\ &= \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Term I converges strongly to 0 in $X_T^{s,-b'}$ because $u_n - u$ tends to 0 in $X_T^{s,b'}$ and w_n is bounded in $X_T^{1,b}$. For term II, we use the tame estimate for ε such that $1 - \varepsilon > s_0$:

$$\|u |w_n - w|^2\|_{X_T^{s,-b'}} \leq \|u\|_{X_T^{s,b}} \|w_n - w\|_{X_T^{1-\varepsilon,b'}} \|w_n - w\|_{X_T^{1,b'}}.$$

By compact embedding, $w_n - w$ converges, up to extraction, strongly to 0 in $X_T^{1-\varepsilon,b'}$ and term II converges strongly in $X_T^{s,-b'}$. Terms III and IV converge weakly to 0 in $X_T^{-1,-b}$ and so in the distributional sense.

Finally, we conclude that the limit of $u_n |w_n|^2$ is $u |w|^2$. We obtain similarly that $w_n^2 \bar{u}_n$ converges in the distributional sense to $w^2 \bar{u}$. Therefore, u is the solution of

$$\begin{cases} i\partial_t u + \Delta u &= 2|w|^2 u + w^2 \bar{u}, \\ u &= 0 \quad \text{on } [0, T] \times \omega. \end{cases}$$

Using Corollary 5.2, we infer that $u \in L^2_{loc}([0, T], H^{s+\frac{1-b}{2}})$, and the existence proposition (Proposition 2.3) yields that it actually belongs to $X_T^{s+\frac{1-b}{2}, b}$. By iteration, we obtain that $u \in X_T^{1, b}$. Then, we can apply Assumption 4 and we in fact have $u = 0$.

We pick $t_0 \in [0, T]$ such that $u_n(t_0)$ converges strongly to 0 in H^s . Estimate (2.14) of the existence proposition (Proposition 2.3) yields strong convergence to 0 of u_n in $X_T^{s, b}$. Therefore, $\|u_n(0)\|_{H^s}$ tends to 0, which contradicts (7.2). \square

7.2. Linear control.

PROPOSITION 7.2. *Assume that (M, ω) satisfies Assumptions 1, 3, and 4. Let $-1 \leq s \leq 1$, $T > 0$, and $w \in X_T^{1, b}$. For every $u_0 \in H^s(M)$ there exists a control $g \in C([0, T], H^s)$ supported in $[0, T] \times \bar{\omega}$ such that the unique solution u in $X_T^{s, b}$ of the Cauchy problem*

$$(7.3) \quad \begin{cases} i\partial_t u + \Delta u &= \pm 2|w|^2 u \pm w^2 \bar{u} + g \quad \text{on } [0, T] \times M, \\ u(0) &= u_0 \in H^s(M) \end{cases}$$

satisfies $u(T) = 0$.

Proof. We treat only the case with $2|w|^2 u + w^2 \bar{u}$. Let $a(x) \in C^\infty(M)$ be real valued, as in (0.3). We apply the HUM method of Lions. We consider the system

$$\begin{cases} i\partial_t u + \Delta u &= 2|w|^2 u + w^2 \bar{u} + g, & g \in L^2([0, T], H^s), & u(T) = 0, \\ i\partial_t v + \Delta v &= 2|w|^2 v - w^2 \bar{v}, & v(0) = v_0 \in H^{-s}. \end{cases}$$

These equations are well posed in $X_T^{s, b}$ and $X_T^{-s, b}$ thanks to Proposition 2.3. The equation verified by v is the dual of the one of u for the real duality (the equation is not \mathbb{C} linear). Then, multiplying the first system by $i\bar{v}$, integrating, and taking the real part, we get (the computation is true for w, g , and v_0 smooth; we extend it by approximation)

$$\Re(u_0, v_0)_{L^2} = \Re \int_0^T (ig, v)_{L^2} dt,$$

where $(\cdot, \cdot)_{L^2}$ is the complex duality on $L^2(M)$. We define the continuous map $S : H^{-s} \rightarrow H^s$ by $Sv_0 = u_0$ with the choice

$$g = Av = -ia(x)(1 - \Delta)^{-s} a(x).$$

This yields

$$\begin{aligned} \Re(Sv_0, v_0)_{L^2} &= \Re \int_0^T (a(x)(1 - \Delta)^{-s} a(x)v, v) = \int_0^T \left\| (1 - \Delta)^{-s/2} a(x)v \right\|_{L^2}^2 \\ &= \int_0^T \|a(x)v\|_{H^{-s}}^2. \end{aligned}$$

Thus, S is self-adjoint and positive definite thanks to the observability estimate of Proposition 7.1. It therefore defines an isomorphism from H^{-s} into H^s . Moreover, we notice that the norms of S and S^{-1} are uniformly bounded as w is bounded in $X_T^{1, b}$. \square

PROPOSITION 7.3. *Assume $0 \leq s \leq 1$, $w = 0$, and (M, ω) is either*

- $(\mathbb{T}^3, \text{any open set})$,
- $(S^2 \times S^1)$ (a neighborhood of the equator) $\times S^1$,
- $(S^2 \times S^1, S^2 \times (\text{any open set of } S^1))$.

Then, the same conclusion as in Proposition 7.2 holds.

Proof. By following the proof of Proposition 7.2, we are reduced to proving an observability estimate:

$$\|u_0\|_{H^{-s}}^2 \leq C \int_0^T \|a(x)e^{it\Delta}u_0\|_{H^{-s}}^2 dt.$$

These results are already known for $s = 0$:

–For \mathbb{T}^3 , this was first proved by Jaffard [26] in dimension 2 and generalized to any dimension by Komornik and Loreti [28].

–The others example are of the form $(M_1 \times M_2, \omega_1 \times M_2)$, where ω_1 satisfies the observability estimate.

We can extend them to any s , with $0 \leq s \leq 1$, by writing

$$\|u_0\|_{H^{-s}} = \left\| (1 - \Delta)^{-s/2} u_0 \right\|_{L^2}$$

We conclude using the observability estimate in L^2 and commutator estimates.

Actually, Proposition 7.4 of the next section proves that controllability in L^2 implies controllability in H^s , $0 \leq s \leq 1$, with the HUM operator constructed on L^2 . This yields the observability estimate in H^{-s} , and, for that reason, we do not detail the previous argument. \square

7.3. Regularity of the control. This section is strongly inspired by the work of Dehman and Lebeau [17]. It expresses the fact that the HUM operator constructed on a space H^s propagates some better regularity. We extend this result to the Schrödinger equation with some rough potentials.

Let $T > 0$, $s \in [-1, 1]$, and $w \in X_T^{1,b}$. As in the the proof of Proposition 7.2, we denote $S = S_{s,T,w,a} : H^{-s} \rightarrow H^s$ the HUM operator of control associated with the trajectory w by $S\Phi_0 = u_0$, where

$$\begin{cases} i\partial_t \Phi + \Delta \Phi &= 2|w|^2 \Phi - w^2 \bar{\Phi}, \\ \Phi(x, 0) &= \Phi_0(x) \in H^{-s} \end{cases}$$

and u is the solution of

$$\begin{cases} i\partial_t u + \Delta u &= 2|w|^2 u + w^2 \bar{u} + A\Phi, \\ u(T) &= 0, \end{cases}$$

where $A = -ia(x)(1 - \Delta)^{-s}a(x)$.

PROPOSITION 7.4. *Suppose Assumptions 3 and 5 are fulfilled. Let $0 \leq s_0 < s, \leq 1$, $\varepsilon = 1 - s$, and $w \in X_T^{1,b}$. Denote $S = S_{s,T,w,a}$ the operator defined above. We assume that S is an isomorphism from H^{-s} into H^s . Then, S is also an isomorphism from $H^{-s+\varepsilon}$ into $H^{s+\varepsilon} = H^1$.*

Proof. First, we show that S maps $H^{-s+\varepsilon}$ into $H^{s+\varepsilon}$.

Let $\Phi_0 \in H^{-s+\varepsilon}$. By the existence proposition (Proposition 2.3), we have $\Phi \in X_T^{-s+\varepsilon,b}$; then $A\Phi \in L^2([0, T], H^{s+\varepsilon})$, and the existence proposition (Proposition 2.3) gives again $u \in X_T^{s+\varepsilon,b}$ and $u(0) = S\Phi_0 \in H^{s+\varepsilon}$.

To finish, we have to prove only that $S\Phi_0 = u_0 \in H^{s+\varepsilon}$ implies $\Phi_0 \in H^{-s+\varepsilon}$. As we already know that $\Phi_0 \in H^{-s}$, we need to prove that $(-\Delta)^{\varepsilon/2}\Phi_0 \in H^{-s}$. We use the fact that S is an isomorphism from H^{-s} into H^s . Denote $D^\varepsilon = (-\Delta)^{\varepsilon/2}$:

$$\begin{aligned} \|D^\varepsilon \Phi_0\|_{H^{-s}} &\leq C \|SD^\varepsilon \Phi_0\|_{H^s} \leq C \|SD^\varepsilon \Phi_0 - D^\varepsilon S\Phi_0\|_{H^s} + C \|D^\varepsilon S\Phi_0\|_{H^s} \\ &\leq C \|SD^\varepsilon \Phi_0 - D^\varepsilon u_0\|_{H^s} + C \|u_0\|_{H^{s+\varepsilon}}. \end{aligned}$$

Let φ be the solution of

$$\begin{cases} i\partial_t\varphi + \Delta\varphi &= 2|w|^2\varphi - w^2\bar{\varphi}, \\ \varphi(x, 0) &= D^\varepsilon\Phi_0(x) \end{cases}$$

and v the solution of

$$\begin{cases} i\partial_tv + \Delta v &= 2|w|^2v + w^2\bar{v} + A\varphi, \\ v(T) &= 0 \end{cases}$$

so that $v(0) = SD^\varepsilon\Phi_0$. We need to estimate $\|v(0) - D^\varepsilon u_0\|_{H^s}$. But $r = v - D^\varepsilon u$ is the solution of

$$\begin{cases} i\partial_tr + \Delta r &= 2|w|^2r + w^2\bar{r} - 2[D^\varepsilon, |w|^2]u - [D^\varepsilon, w^2]\bar{u} + A(\varphi - D^\varepsilon\Phi) - [D^\varepsilon, A]\Phi, \\ r(T) &= 0. \end{cases}$$

Then, using Proposition 2.3 we obtain

$$\begin{aligned} \|r_0\|_{H^s} \leq C \|r\|_{X_T^{s,b}} &\leq C \left(\| [D^\varepsilon, |w|^2]u \|_{X_T^{s,-b'}} + \| [D^\varepsilon, w^2]\bar{u} \|_{X_T^{s,-b'}} \right. \\ &\quad \left. + \| A(\varphi - D^\varepsilon\Phi) \|_{X_T^{s,-b'}} + \| [D^\varepsilon, A]\Phi \|_{X_T^{s,-b'}} \right). \end{aligned}$$

Lemma A.3 of Appendix A gives us some estimates about the commutators. For the last term, we notice that $[D^\varepsilon, A]$ is a pseudodifferential operator of order $\varepsilon - 2s - 1 \leq -2s$:

$$\|r_0\|_{H^s} \leq C \left(\|w\|_{X_T^{s+\varepsilon,b'}}^2 \|u\|_{X_T^{s,b'}} + \|A(\varphi - D^\varepsilon\Phi)\|_{X_T^{s,-b'}} + \|\Phi\|_{L^2([0,T],H^{-s})} \right).$$

We already know that $u \in X_T^{s,b'}$, $w \in X_T^{s+\varepsilon,b'}$, and $\Phi \in X_T^{-s,b}$. We have only to estimate $\|A(\varphi - D^\varepsilon\Phi)\|_{X_T^{s,-b'}} \leq C \|\varphi - D^\varepsilon\Phi\|_{L^2([0,T],H^{-s})}$. But $d = \varphi - D^\varepsilon\Phi$ is the solution of

$$\begin{cases} i\partial_td + \Delta d &= 2|w|^2d - w^2\bar{d} - 2[D^\varepsilon, |w|^2]\Phi + [D^\varepsilon, w^2]\bar{\Phi}, \\ d(x, 0) &= 0. \end{cases}$$

Thus, using Proposition 2.3, we get

$$\|\varphi - D^\varepsilon\Phi\|_{L^2(H^{-s})} \leq C \|d\|_{X_T^{-s,b'}} \leq C \left(\| [D^\varepsilon, |w|^2]\Phi \|_{X_T^{-s,-b'}} + \| [D^\varepsilon, w^2]\bar{\Phi} \|_{X_T^{-s,-b'}} \right).$$

The second part of Lemma A.3 allows us to conclude. \square

8. Control near a trajectory. Theorems 0.3 and 0.4 are consequences of the following proposition.

PROPOSITION 8.1. *Suppose Assumptions 3 and 5 are fulfilled. Let $T > 0$, and let $w \in X_T^{1,b}$ be a controlled trajectory, i.e., a solution of*

$$i\partial_tw + \Delta w = \pm|w|^2w + g_1 \quad \text{on } [0, T] \times M$$

with $g_1 \in L^2([0, T], H^1(M))$, supported in $\bar{\omega}$. Let $1 \geq s > s_0 \geq 0$. Assume that the HUM operator $S = S_{s,T,w,a}$, defined in section 7.3, is an isomorphism from H^{-s} into H^s .

There exists $\varepsilon > 0$ such that for every $u_0 \in H^s$ with $\|u_0 - w(0)\|_{H^s} < \varepsilon$ there exists $g \in C([0, T], H^s)$ supported in $[0, T] \times \bar{\omega}$ such that the unique solution u in $X_T^{s,b}$ of

$$(8.1) \quad \begin{cases} i\partial_t u + \Delta u &= \pm |u|^2 u + g, \\ u(x, 0) &= u_0(x) \end{cases}$$

fulfills $u(T) = w(T)$. Moreover, we can find another $\varepsilon > 0$ depending only on T, s, ω , and $\|w\|_{X_T^{1,b}}$ such that for any $u_0 \in H^1$ with $\|u_0 - w(0)\|_{H^s} < \varepsilon$ the same conclusion holds with $g \in C([0, T], H^1)$.

Proof. In the demonstration, we denote C some constants that could actually depend on $T, \|w\|_{X_T^{1,b}}$, and s . The final ε will have the same dependence. We make the proof for the defocusing case, but since there is no energy estimate, it is the same in the other situation.

We linearize the equation as in Proposition 2.2. If $u = w + r$, then r is the solution of

$$\begin{cases} i\partial_t r + \Delta r &= 2|w|^2 r + w^2 \bar{r} + F(w, r) + g - g_1, \\ r(x, 0) &= r_0(x) \end{cases}$$

with $F(w, r) = 2|r|^2 w + r^2 \bar{w} + |r|^2 r$. We seek g under the form $g_1 + A\Phi$, where Φ is the solution of the dual linear equation and $A = -ia(x)(1 - \Delta)^{-s}a(x)$, as in the linear control. The purpose is then to choose the adequate Φ_0 , and the system is completely determined.

With $\|r_0\|_{H^s}$ small enough, we are looking for a control such that $r(T) = 0$. More precisely, we consider the two systems

$$\begin{cases} i\partial_t \Phi + \Delta \Phi &= 2|w|^2 \Phi - w^2 \bar{\Phi}, \\ \Phi(x, 0) &= \Phi_0(x) \in H^{-s} \end{cases}$$

and

$$\begin{cases} i\partial_t r + \Delta r &= 2|w|^2 r + w^2 \bar{r} + F(w, r) + A\Phi, \\ r(x, T) &= 0. \end{cases}$$

Let us define the operator

$$\begin{aligned} L : H^{-s}(M) &\rightarrow H^s(M), \\ \Phi_0 &\mapsto L\Phi_0 = r(0). \end{aligned}$$

We split $r = v + \Psi$ with Ψ the solution of

$$\begin{cases} i\partial_t \Psi + \Delta \Psi &= 2|w|^2 \Psi + w^2 \bar{\Psi} + A\Phi, \\ \Psi(T) &= 0. \end{cases}$$

This corresponds to the linear control, and so $\Psi(0) = S\Phi_0$. v is the solution of

$$(8.2) \quad \begin{cases} i\partial_t v + \Delta v &= 2|w|^2 v + w^2 \bar{v} + F(w, r), \\ v(T) &= 0. \end{cases}$$

Then, r, v, Ψ belong to $X_T^{s,b}$ and $r(0) = v(0) + \Psi(0)$, which we can write as

$$L\Phi_0 = K\Phi_0 + S\Phi_0,$$

where $K\Phi_0 = v(0)$. $L\Phi_0 = r_0$ is equivalent to $\Phi_0 = -S^{-1}K\Phi_0 + S^{-1}r_0$. Defining the operator $B : H^{-s} \rightarrow H^{-s}$ by

$$B\Phi_0 = -S^{-1}K\Phi_0 + S^{-1}r_0,$$

the problem $L\Phi_0 = r_0$ is now to find a fixed point of B near the origin of H^{-s} . We will prove that B is contracting on a small ball $B_{H^{-s}}(0, \eta)$ provided that $\|r_0\|_{H^s}$ is small enough.

We may assume $T < 1$ and fix it for the rest of the proof (actually the norm of S^{-1} depends on T and even explodes when T tends to 0; see [36] and [43]). We have

$$\|B\Phi_0\|_{H^{-s}} \leq C(\|K\Phi_0\|_{H^s} + \|r_0\|_{H^s}).$$

So, we are led to estimate $\|K\Phi_0\|_{H^s} = \|v(0)\|_{H^s}$.

If we apply to (8.2) the estimate of Proposition 2.3, we get

$$\begin{aligned} \|v(0)\|_{H^s} &\leq \|v\|_{X_T^{s,b}} \\ &\leq C\|F(w, r)\|_{X_T^{s,-b'}} \\ &\leq C\|w\|_{X_T^{1,b}}\|r\|_{X_T^{s,b}}^2 + \|r\|_{X_T^{s,b}}^3. \end{aligned}$$

Then, we use the linear behavior near a trajectory of Proposition 2.2. We conclude that for $\|A\Phi\|_{L^2([0,T], H^s)} \leq \|\Phi\|_{X_T^{-s,b}} \leq C\|\Phi_0\|_{H^{-s}} < C\eta$ (see Proposition 2.3) small enough, we have

$$\|r\|_{X_T^{s,b}} \leq C\|\Phi_0\|_{H^{-s}}.$$

This yields

$$\|B\Phi_0\|_{H^{-s}} \leq C\left(\|\Phi_0\|_{H^{-s}}^2 + \|\Phi_0\|_{H^{-s}}^3 + \|r_0\|_{H^s}\right).$$

Choosing η small enough and $\|r_0\|_{H^s} \leq \eta/2C$, we obtain $\|B\Phi_0\|_{H^{-s}} \leq \eta$ and B reproduces the ball $B_{H^{-s}}(0, \eta)$.

If $u_0 \in H^1$, we want one g in $C([0, T], H^1)$, that is, $\Phi_0 \in H^{1-2s}$. We prove that B reproduces $B_{H^{-s}}(0, \eta) \cap B_{H^{1-2s}}(0, R)$ for R large enough.

Proposition 7.4 yields that S is an isomorphism from H^{1-2s} into H^1 . Then, we have by the same arguments as above:

$$\|B\Phi_0\|_{H^{1-2s}} \leq C(\|K\Phi_0\|_{H^1} + \|r_0\|_{H^1}),$$

$$\begin{aligned} \|v(0)\|_{H^1} &\leq C\|v\|_{X_T^{1,b}} \\ &\leq C\|F(w, r)\|_{X_T^{1,-b'}} \\ &\leq C\|w\|_{X_T^{1,b}}\|r\|_{X_T^{s,b}}\|r\|_{X_T^{1,b}} + \|r\|_{X_T^{s,b}}^2\|r\|_{X_T^{1,b}}, \end{aligned}$$

and

$$\|r\|_{X_T^{1,b}} \leq C\|\Phi_0\|_{H^{1-2s}}.$$

Then,

$$\|B\Phi_0\|_{H^{1-2s}} \leq C(R\eta + R\eta^2 + \|r_0\|_{H^1}).$$

Choosing η such that $C(\eta + \eta^2) < 1/2$ (it is important to notice here that this bound does not depend on the size of r_0 in H^1) and R large enough, we obtain that B reproduces $B_{H^{-s}}(0, \eta) \cap B_{H^{1-2s}}(0, R)$.

Let us prove that it is contracting for the H^{-s} norm. For that, we examine the systems

$$(8.3) \quad \begin{cases} i\partial_t(r - \tilde{r}) + \Delta(r - \tilde{r}) &= 2|w|^2(r - \tilde{r}) + w^2\overline{(r - \tilde{r})} \\ &+ F(w, r) - F(w, \tilde{r}) + A(\Phi - \tilde{\Phi}), \\ (r - \tilde{r})(T) &= 0, \end{cases}$$

$$\begin{cases} i\partial_t(v - \tilde{v}) + \Delta(v - \tilde{v}) &= 2|w|^2(v - \tilde{v}) + w^2\overline{(v - \tilde{v})} + F(w, r) - F(w, \tilde{r}), \\ (v - \tilde{v})(T) &= 0. \end{cases}$$

We obtain

$$(8.4) \quad \begin{aligned} \|B\Phi_0 - B\tilde{\Phi}_0\|_{H^{-s}} &\leq C\|(v - \tilde{v})(0)\|_{H^s} \leq C\|F(w, r) - F(w, \tilde{r})\|_{X_T^{s, -b'}} \\ &\leq C\left(\|r\|_{X_T^{s, b}} + \|\tilde{r}\|_{X_T^{s, b}} + \|r\|_{X_T^{s, b}}^2 + \|\tilde{r}\|_{X_T^{s, b}}^2\right)\|r - \tilde{r}\|_{X_T^{s, b}} \\ &\leq C(\eta + \eta^2)\|r - \tilde{r}\|_{X_T^{s, b}} \leq C\eta\|r - \tilde{r}\|_{X_T^{s, b}}. \end{aligned}$$

Considering (8.3), we deduce that

$$\begin{aligned} \|r - \tilde{r}\|_{X_T^{s, b}} &\leq C\|F(w, r) - F(w, \tilde{r})\|_{X_T^{s, -b'}} + C\|A(\Phi - \tilde{\Phi})\|_{L^2([0, T], H^s)} \\ &\leq C\eta\|r - \tilde{r}\|_{X_T^{s, b}} + C\|\Phi_0 - \tilde{\Phi}_0\|_{H^{-s}}. \end{aligned}$$

If η is taken small enough, it yields

$$(8.5) \quad \|r - \tilde{r}\|_{X_T^{s, b}} \leq C\|\Phi_0 - \tilde{\Phi}_0\|_{H^{-s}}.$$

Combining (8.5) with (8.4) we finally get

$$\|B\Phi_0 - B\tilde{\Phi}_0\|_{H^{-s}} \leq C\eta\|\Phi_0 - \tilde{\Phi}_0\|_{H^{-s}}.$$

This yields that B is a contraction on a small ball $B_{H^{-s}}(0, \eta)$, which completes the proof of Proposition 8.1. \square

COROLLARY 8.2. *Let $T > 0$, and let (M, ω) be such that Assumptions 1, 3, 4, and 5 are fulfilled. Then, the set of reachable states is open in H^s for $s_0 < s \leq 1$.*

In the next corollary, $\hat{f}(k)$ denotes the coordinates of a function f in the basis of eigenfunction of M .

COROLLARY 8.3. *Suppose the same assumptions as in Proposition 8.1. Let $E_0 > \|w_0\|_{H^1}$. Then, there exist N and ε such that for every u_0 and $u_1 \in H^1$ with*

$$(8.6) \quad \|u_0\|_{H^1} \leq E_0, \quad \|u_1\|_{H^1} \leq E_0,$$

$$(8.7) \quad \sum_{|k| \leq N} |\widehat{u}_0(k) - \widehat{w}_0(k)|^2 \leq \varepsilon, \quad \sum_{|k| \leq N} |\widehat{u}_1(k) - \widehat{w}_T(k)|^2 \leq \varepsilon$$

we can find a control $g \in L^\infty([0, T], H^1)$ supported in $[0, T] \times \omega$ such that the unique solution of (8.1) with control g and $u(0) = u_0$ satisfies $u(T) = u_1$.

Proof. We build the control in two steps: the first brings the system from u_0 to $w(T/2)$ and the second from $w(T/2)$ to u_1 . Actually, the second step is the same by reversing time, and we describe only the first one.

Let $s_0 < s < 1$. We first check that the first part of the conclusion of Proposition 8.1 is true without Assumption 5. It gives one $\tilde{\varepsilon} > 0$ such that if $\|u_0 - w_0\|_{H^s} \leq \tilde{\varepsilon}$, we have a control to $w(T/2)$ in time $T/2$ with $g \in C([0, T/2], H^1)$. We check only that once E_0 is chosen we can find N and ε such that assumptions (8.6) and (8.7) imply $\|u_0 - w_0\|_{H^s} \leq \tilde{\varepsilon}$. \square

We also obtain a first proof of global controllability. The assumptions we make are stronger than those in Theorem 0.1, which will be proved using stabilization. However, in the examples we treat, the assumptions are fulfilled.

COROLLARY 8.4. *Theorem 0.1 is true under the stronger assumptions (Assumptions 1, 3, and 4).*

Proof. We will make successive controls near some free nonlinear trajectory so that the energy decreases. The main argument is that the ε of Theorem 0.3 depends only on $\|w\|_{X_T^{1,b}}$ and if the trajectory is a free nonlinear trajectory, then the ε depends only on $\|w_0\|_{H^1}$. We just have to be careful that each new free trajectory fulfills $\|w\|_{X_T^{1,b}} \leq A$ for one fixed constant A .

Fix $T > 0$. There exists C_1 such that

$$\|f\|_{H^1} \leq C_1 \left(E(f) + \sqrt{E(f)} \right)^{1/2} \quad \forall f \in H^1(M).$$

Denote $A = C_1(E(w_0) + \sqrt{E(w_0)})$. There exists a constant such that $\|w_0\|_{H^1} \leq A$ implies $\|w\|_{X_T^{1,b}} \leq B$ for w the solution of

$$\begin{cases} i\partial_t w + \Delta w &= |w|^2 w \quad \text{on } [0, T] \times M, \\ w(0) &= w_0. \end{cases}$$

Let ε be the constant so that Theorem 0.3 is true for any w with $\|w\|_{X_T^{1,b}} \leq B$. We choose the arrival point $u_T = (1 - \varepsilon/A)w_T$ such that

$$\|u_T - w_T\|_{H^1} = \varepsilon/A \|w_T\|_{H^1} \leq C_1 \left(E(w_T) + \sqrt{E(w_T)} \right) \varepsilon/A = \varepsilon.$$

We have a control g supported in $[0, T] \times \bar{\omega}$ such that the solution u of

$$\begin{cases} i\partial_t u + \Delta u &= |u|^2 u + g \quad \text{on } [0, T] \times M, \\ u(0) &= w_0 \end{cases}$$

satisfies $u(T) = u_T$. If $1 - \varepsilon/A \in [0, 1]$, we have

$$E(u_T) = \frac{1}{2} \int_M \left| \left(1 - \frac{\varepsilon}{A}\right) \nabla w_T \right|^2 + \frac{1}{4} \int_M \left| \left(1 - \frac{\varepsilon}{A}\right) w_T \right|^4 \leq \left(1 - \frac{\varepsilon}{A}\right)^2 E(w_T).$$

Moreover, we still have

$$\|u_T\|_{H^1} \leq C_1 \left(E(u_T) + \sqrt{E(u_T)} \right)^{1/2} \leq A.$$

Then, we can reiterate this process with the same ε . We construct a sequence of solutions $u_n \in X_{[nT, (n+1)T]}^{1,b}$ and of controls $g_n \in C([nT, (n+1)T], H^1)$ such that

$$\begin{cases} i\partial_t u_n + \Delta u_n &= |u_n|^2 u_n + g_n \quad \text{on } [nT, (n+1)T] \times M, \\ u_n(nT) &= u_{n-1}(nT) \end{cases}$$

and

$$E(u_n(nT)) \leq (1 - \varepsilon/A)^{2n} E(w_0) \leq C(1 - \varepsilon/A)^{2n} \left(\|w_0\|_{H^1}^2 + \|w_0\|_{H^1}^4 \right).$$

But, we have

$$\|u_n(nT)\|_{H^1}^2 \leq C_1 \left(E(u_n(nT)) + \sqrt{E(u_n(nT))} \right)^{1/2}.$$

Therefore, it can be made arbitrarily small for large n . This allows us to use local controllability near the trajectory 0. We obtain global controllability making the same proof in negative time. \square

9. Necessity of geometric control assumption on S^3 . In this section, we prove that on S^3 the geometric control is necessary for stabilization to occur. The argument uses some concentration of eigenfunctions. This concentration was also used by Burq, Gérard, and Tzvetkov [10] to prove some ill-posedness results.

PROPOSITION 9.1. *Let Γ be a geodesic of S^3 , and let $a \in C^\infty(S^3)$ such that $\text{Supp}(a) \cap \Gamma = \emptyset$. Then, for every $R_0 > 0$, C , and $\gamma > 0$ there exist $T > 0$ and $u_0 \in H^1(S^3)$ with $\|u_0\|_{H^1} \leq R_0$ such that*

$$\|u(T)\|_{H^1} > Ce^{-\gamma T} \|u\|_{H^1}$$

for u the solution of equation

$$(9.1) \begin{cases} i\partial_t u + \Delta u - (1 + |u|^2)u &= a(x)(1 - \Delta)^{-1}a(x)\partial_t u \quad \text{on } [0, T] \times S^3, \\ u(0) &= u_0 \in H^1. \end{cases}$$

Proof. Let T be such that $Ce^{-\gamma T} \leq 1/2$. By changes of coordinates, we can assume that $\Gamma = \{x_3 = x_4 = 0\}$. We will use the eigenfunctions $\Phi_n = c_n(x_1 + ix_2)^n$ that concentrate on the subset $\{x_3 = x_4 = 0\}$. c_n is chosen such that $\|\Phi_n\|_{H^1} = R_0$, and so $c_n \approx n^{1/2-1}$. We have $-\Delta\Phi_n = \lambda_n\Phi_n$ with $\lambda_n = n(n+2)$. Let u_n be the solution of (9.1) with $u_n(0) = \Phi_n$. Let $v_n = e^{i(\lambda_n-1)t}\Phi_n$ be the solution of the linear equation

$$\begin{cases} i\partial_t v_n + \Delta v_n - v_n &= 0 \quad \text{on } [0, T] \times S^3, \\ v_n(0) &= \Phi_n. \end{cases}$$

Then, $r_n = u_n - v_n$ is the solution of

$$\begin{cases} i\partial_t r_n + \Delta r_n - r_n &= a(x)(1 - \Delta)^{-1}a(x)\partial_t r_n + R_n \quad \text{on } [0, T] \times S^3, \\ r_n(0) &= 0 \end{cases}$$

with $R_n = |u_n|^2 u_n + a(x)(1 - \Delta)^{-1}a(x)\partial_t v_n$.

Proposition 3.1 about linearization yields that $|u_n|^2 u_n \rightarrow 0$ in $X_T^{1,-b'}$. For the other term in R_n , we use the concentration of the Φ_n :

$$\begin{aligned} \|a(x)(1 - \Delta)^{-1}a(x)\partial_t v_n\|_{X_T^{1,-b'}} &\leq \|a(x)(1 - \Delta)^{-1}a(x)\partial_t v_n\|_{L^2([0,T],H^1)} \\ &\leq \|a(x)\partial_t v_n\|_{L^2([0,T],H^{-1})} \leq (\lambda_n + 1) \|a(x)\Phi_n\|_{L^\infty(S^3)}. \end{aligned}$$

Let $\delta > 0$ such that we have $x_3^2 + x_4^2 > \delta$ on $\text{Supp } a$. Hence, we have $|(x_1 + ix_2)|^2 = x_1^2 + x_2^2 = 1 - x_3^2 - x_4^2 < 1 - \delta$:

$$(\lambda_n + 1) \|a(x)\Phi_n\|_{L^\infty(S^3)} \leq C(\lambda_n + 1)c_n(1 - \delta)^{n/2}.$$

Since λ_n and c_n are at most polynomial in n , we deduce that R_n tends to 0 in $X_T^{1,-b'}$. By some arguments similar to the proof of the continuity of the flow map of Proposition 2.1, we infer that r_n tends to 0 in $X_T^{1,b}$. Then, $\|u_n(T)\|_{H^1}$ tends to R_0 and, for n large enough, we have $\|u_n(T)\|_{H^1} > R_0/2$. \square

With a similar proof, we could show the same result on $S^2 \times S^1$ if $\text{Supp}(a) \cap (\Gamma \times S^1) = \emptyset$ for some geodesic Γ of S^2 . Yet, it does not imply geometric control.

The construction of Ralston [39] proves that, actually, a necessary condition for stabilization is that the support of $a(x)$ intersects any stable closed geodesic (see also the work of Thomann [44], where this concentration is used to prove ill-posedness). In the case of S^3 , we use the geometric fact that every closed geodesic is stable.

Appendix A. Some commutator estimates. This section is devoted to the proof of some commutator estimates used in Proposition 7.4. More precisely, we study the action of $[(-\Delta)^{\varepsilon/2}, a_1 a_2]$ on $X^{s,b}$ where a_i are rough. We first give a simple proof for \mathbb{T}^3 (rational or not) and then a general one under Assumption 5. Then, we show that this assumption is fulfilled for S^3 and $S^2 \times S^1$. We will need an elementary lemma.

LEMMA A.1. *If $0 \leq \varepsilon \leq 1$, we have for any norm $\|k\|^\varepsilon - \|k_3\|^\varepsilon \leq \|k - k_3\|^\varepsilon$.*

Proof. Using the triangular inequality, we get $\|k\|^\varepsilon - \|k_3\|^\varepsilon \leq \|k - k_3\|^\varepsilon$. Then, we are reduced to the case of \mathbb{R}^{+*} : we prove that for $z, t \in \mathbb{R}^{+*}$ we have $(z+t)^\varepsilon - z^\varepsilon \leq t^\varepsilon$, which is an easy consequence of the Minkowski inequality for $1 \leq 1/\varepsilon \leq +\infty$. \square

A.1. An easier proof for \mathbb{T}^3 .

LEMMA A.2. *Let $M = \mathbb{R}^3/(\theta_x \mathbb{Z} \times \theta_y \mathbb{Z} \times \theta_z \mathbb{Z})$ with $(\theta_x, \theta_y, \theta_z) \in \mathbb{R}^3$. Denote s_0 the constant taken from Assumption 3. Let $s > s_0$ and $0 \leq \varepsilon \leq 1$. Then, there exists $b' < 1/2$ such that $u_3 \mapsto [\Delta^{\varepsilon/2}, u_1 u_2] u_3$ maps any $X^{s,b'}$ into $X^{s,-b'}$, where $u_1 u_2$ denotes the operator of multiplication by $u_1 u_2$ with $u_i \in X^{s+\varepsilon, b'}$ for $i \in \{1, 2\}$. This function $[\Delta^{\varepsilon/2}, u_1 u_2]$ also maps $X^{-s, b'}$ into $X^{-s, -b'}$. Moreover, the same result holds with u_i replaced by \bar{u}_i for i in a subset of $\{1, 2, 3\}$.*

Proof. We choose the norm $|k| = \sqrt{(\theta_x k_x)^2 + (\theta_y k_y)^2 + (\theta_z k_z)^2}$ so that

$$-\widehat{\Delta} u(k) = |k|^2 \widehat{u}(k).$$

By duality, it is equivalent to prove

$$\int_{\mathbb{R} \times M} [(-\Delta)^{\varepsilon/2}, u_1 u_2] u \bar{v} \leq C \|u_1\|_{X^{s+\varepsilon, b'}} \|u_2\|_{X^{s+\varepsilon, b'}} \|u\|_{X^{s, b'}} \|v\|_{X^{-s, b'}}.$$

Using the Parseval theorem and denoting $k = k_1 + k_2 + k_3$, $\tau = \tau_1 + \tau_2 + \tau_3$,

$$\begin{aligned} & \int_{\mathbb{R} \times M} [(-\Delta)^{\varepsilon/2}, u_1 u_2] u \bar{v} \\ &= \int_{\tau_1, \tau_2, \tau_3} \sum_{k_1, k_2, k_3} \widehat{u}_1(k_1, \tau_1) \widehat{u}_2(k_2, \tau_2) (|k|^\varepsilon - |k_3|^\varepsilon) \widehat{u}(k_3, \tau_3) \bar{\widehat{v}}(k, \tau) \\ &\leq \int_{\tau_1, \tau_2, \tau_3} \sum_{k_1, k_2, k_3} \|k|^\varepsilon - |k_3|^\varepsilon| \left| \widehat{u}_1(k_1, \tau_1) \widehat{u}_2(k_2, \tau_2) \widehat{u}(k_3, \tau_3) \bar{\widehat{v}}(k, \tau) \right|. \end{aligned}$$

Lemma A.1 and $k - k_3 = k_1 + k_2$ yield

$$\begin{aligned} & \left| \int_{\mathbb{R} \times M} [(-\Delta)^{\varepsilon/2}, u_1 u_2] u \bar{v} \right| \\ &\leq C \int_{\tau_1, \tau_2, \tau_3} \sum_{k_1, k_2, k_3} (|k_1|^\varepsilon + |k_2|^\varepsilon) |\widehat{u}_1(k_1, \tau_1)| |\widehat{u}_2(k_2, \tau_2)| |\widehat{u}(k_3, \tau_3)| |\widehat{v}(k, \tau)|. \end{aligned}$$

Denoting u_1 the function with Fourier transform $|\widehat{u}_1(k_1, \tau_1)|$ we obtain

$$\begin{aligned} \left| \int_{\mathbb{R} \times M} [(-\Delta)^{\varepsilon/2}, u_1 u_2] u \bar{v} \right| &\leq C \int_{\mathbb{R} \times M} (\Delta^{\varepsilon/2} u_1) u_2 u \bar{v} + \int_{\mathbb{R} \times M} u_1 (\Delta^{\varepsilon/2} u_2) u \bar{v} \\ &\leq C \|u_1\|_{X^{s+\varepsilon, b'}} \|u_2\|_{X^{s+\varepsilon, b'}} \|u\|_{X^{s, b'}} \|v\|_{X^{-s, b'}}. \end{aligned}$$

Here, we have finished the proof using the trilinear Bourgain estimate because $s > s_0$. If we estimate this integral using the trilinear estimate at the negative level H^{-s} , we obtain the second result. \square

A.2. General proof under Assumption 5.

LEMMA A.3. *Denote s_0 the constant taken from Assumption 5. Let $s > s_0$ and $0 \leq \varepsilon \leq 1$. Then, there exists $b' < 1/2$ such that $u_3 \mapsto [(-\Delta)^{\varepsilon/2}, u_1 u_2] u_3$ maps any $X^{s, b'}$ into $X^{s, -b'}$, where $u_1 u_2$ denotes the operator of multiplication by $u_1 u_2$ with $u_i \in X^{s+\varepsilon, b'}$ for $i \in \{1, 2\}$. This function $[\Delta^{\varepsilon/2}, u_1 u_2]$ also maps $X^{-s, b'}$ into $X^{-s, -b'}$. Moreover, the same result holds with u_i replaced by $\overline{u_i}$ for i in a subset of $\{1, 2, 3\}$.*

Proof. The proof follows the techniques of Bourgain and Burq, Gérard, and Tzvetkov. Here, we were inspired more precisely by [23]. We recall the notations $u^\# = e^{-it\Delta} u(t)$, $u^N = \mathbf{1}_{\sqrt{1-\Delta} \in [N, 2N]} u$, where N is a dyadic number and $\widehat{u}(\tau)$ is the Fourier transform of u with respect to the time variable. First, with some dyadic integers N_i fixed, we estimate the integral

$$\begin{aligned} I(N_1, \dots, N_4) &= \int_{\mathbb{R} \times M} u_1^{N_1} u_2^{N_2} \left[((-\Delta)^{\varepsilon/2} u_3^{N_3}) \overline{u_4^N} - u_3^{N_3} (-\Delta)^{\varepsilon/2} \overline{u_4^N} \right] dt dx \\ &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}_t \times M_x} \iiint \int_{\mathbb{R}_{\tau_1, \tau_2, \tau_3, \tau_4}} e^{it(\tau_1 + \tau_2 + \tau_3 - \tau)} e^{it\Delta} \widehat{u_1^{N_1\#}}(\tau_1) e^{it\Delta} \widehat{u_2^{N_2\#}}(\tau_2) \\ &\quad \times \left[((-\Delta)^{\varepsilon/2} e^{it\Delta} \widehat{u_3^{N_3\#}}(\tau_3)) e^{it\Delta} \widehat{u_4^N}(\tau) - e^{it\Delta} \widehat{u_3^{N_3\#}}(\tau_3) (-\Delta)^{\varepsilon/2} e^{it\Delta} \widehat{u_4^N}(\tau) \right]. \end{aligned}$$

By near orthogonality in H^b and partition of unity, $u_j = \sum_{n \in \mathbb{Z}} \varphi(t - n/2) u_j(t)$, we are led to the special case where the u_j are supported in time in the interval $]0, 1[$. Select $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi = 1$ on $[0, 1]$. Thus, estimates (0.10), applied with τ_j fixed, and the Cauchy–Schwarz inequality in $(\tau_1, \tau_2, \tau_3, \tau_4)$ give for any $b > 1/2$

$$\begin{aligned} |I(N_1, \dots, N_4)| &\leq C(N_1^\varepsilon + N_2^\varepsilon) (m(N_1, \dots, N_4))^{s_0} \prod_{j=1}^4 \int_{\tau_j} \left\| \widehat{u_j^{N_j\#}}(\tau_j) \right\|_{L^2(M)} \\ \text{(A.1)} \quad &\leq C(N_1^\varepsilon + N_2^\varepsilon) (m(N_1, \dots, N_4))^{s_0} \prod_{j=1}^4 \left\| u_j^{N_j} \right\|_{X^{0, b}(\mathbb{R} \times M)}. \end{aligned}$$

This estimate is very satisfactory for the space regularity. Yet, for the regularity in time, it requires $b > 1/2$, which is too much for our purpose. We will interpolate with some crude estimates in space but better in time.

For the case where N_1 is large, we estimate $|I(N_1, \dots, N)|$ using Sobolev embeddings $H^{1/4}(\mathbb{R}) \subset L^4(\mathbb{R})$:

$$\text{(A.2)} \quad |I(N_1, \dots, N_4)| \leq C(N_3^\varepsilon + N_4^\varepsilon) (m(N_1, \dots, N_4))^{3/2} \prod_{j=1}^4 \left\| u_j^{N_j} \right\|_{X^{0, 1/4}(\mathbb{R} \times M)}.$$

In another case where the frequency N_3 is large, we will use an argument close to the one in [18]. In that case, we cannot afford a loss in the frequency N_3 . We use the fact

that $[u_1^{N_1}u_2^{N_2}, \Delta^{\varepsilon/2}]$ is a pseudodifferential operator of order less than 0 (if $\varepsilon \leq 1$). Then,

$$\begin{aligned}
 |I(N_1, \dots, N_4)| &= \left| \int_{\mathbb{R} \times M} [u_1^{N_1}u_2^{N_2}, \Delta^{\varepsilon/2}] u_3^{N_3} \overline{u_4^{N_4}} \right| \\
 &\leq C \int_{\mathbb{R}} \left\| [u_1(t)^{N_1}u_2(t)^{N_2}, \Delta^{\varepsilon/2}] \right\|_{L^2 \rightarrow L^2} \|u_3(t)\|_{L^2(M)} \|u_4(t)\|_{L^2(M)} dt \\
 &\leq \int_{\mathbb{R}} \sum_{\alpha=0}^m \|\partial^\alpha u_1 u_2(t)\|_{L^\infty(M)} \|u_3(t)\|_{L^2(M)} \|u_4(t)\|_{L^2(M)} dt \\
 \text{(A.3)} \quad &\leq C \max(N_1, N_2)^\mu \prod_{j=1}^4 \|u_j^{N_j}\|_{X^{0,1/4}(\mathbb{R} \times M)},
 \end{aligned}$$

where μ depends on the dimension and on ε .

Let us now begin the summation of the harmonics. As in [23], we decompose each function

$$u = \sum_K u_K, \quad u_K = \mathbf{1}_{K \leq (i\partial_t + \Delta) < 2K}(u),$$

where K denotes the sequence of dyadic integers. Notice that

$$\|u\|_{X^{0,b}}^2 \approx \sum_K K^{2b} \|u_K\|_{L^2(\mathbb{R} \times M)}^2 \approx \sum_K \|u_K\|_{X^{0,b}}^2.$$

Then, we decompose the integral in the sum of the following elementary integrals:

$$\begin{aligned}
 &I(N_1, \dots, N_4, K_1, \dots, K_4) \\
 &= \int_{\mathbb{R} \times M} a_1^{N_1, K_1} a_2^{N_2, K_2} \left[((-\Delta^{\varepsilon/2}) u^{N_3, K_3}) \overline{v}^{N, K} - u^{N_3, K_3} (-\Delta^{\varepsilon/2}) \overline{v}^{N, K} \right] dt dx.
 \end{aligned}$$

Estimate (A.1) leads to (for every $b > 1/2$)

$$\begin{aligned}
 &|I(N_1, \dots, N_4, K_1, \dots, K_4)| \\
 &\leq (N_1^\varepsilon + N_2^\varepsilon) m(N_1, \dots, N_4)^{s_0} (K_1 K_2 K_3 K_4)^b \prod_{j=1}^4 \|u_j^{N_j, K_j}\|_{L^2}.
 \end{aligned}$$

We will interpolate this estimate with different inequalities. We distinguish three cases: $N_4 \leq C(N_1 + N_2 + N_3)$ with $N_3 < \max(N_1, N_2)$ or $\max(N_1, N_2) \leq N_3$, and $N_4 > C(N_1 + N_2 + N_3)$ with C large enough. Without loss of generality, we can assume $N_1 \geq N_2$.

First case: $N_3 < \max(N_1, N_2) = N_1$ and $N_4 \leq C(N_1 + N_2 + N_3)$. Estimate (A.2) gives

$$\begin{aligned}
 |I(N_1, \dots, N_4, K_1, \dots, K_4)| &\leq (N_3^\varepsilon + N_4^\varepsilon) m(N_1, \dots, N_4)^{3/2} \\
 &\quad \times (K_1 K_2 K_3 K_4)^{1/4} \prod_{j=1}^4 \|u_j^{N_j, K_j}\|_{L^2}.
 \end{aligned}$$

Then, for every $\theta \in [0, 1]$

$$\begin{aligned}
 |I(N_1, \dots, N_4, K_1, \dots, K_4)| &\leq C(N_1^\varepsilon + N_2^\varepsilon)^{1-\theta} (N_3^\varepsilon + N_4^\varepsilon)^\theta m(N_1, \dots, N_4)^{(1-\theta)s_0 + 3\theta/2} \\
 &\quad \times (K_1 K_2 K_3 K_4)^{b(1-\theta) + \theta/4} \prod_{j=1}^4 \|u_j^{N_j, K_j}\|_{L^2}.
 \end{aligned}$$

We denote $s(\theta) = (1 - \theta)s_0 + 3\theta/2$ and $b(\theta) = b(1 - \theta) + \theta/4$:

$$|I(N_1, \dots, N, K_1, \dots)| \leq C(N_1^\varepsilon + N_2^\varepsilon)^{1-\theta} (N_3^\varepsilon + N_4^\varepsilon)^\theta m(N_1, \dots, N_4)^{s(\theta)} \times (K_1 K_2 K_3 K_4)^{b(\theta)-b'} \prod_{j=1}^4 \|u_j^{N_j}\|_{X^{0,b'}}.$$

By choosing some appropriate θ and $b' < 1/2 < b$, we can make the series in K convergent if $b(\theta) - b' < 0$. This yields

$$\begin{aligned} |I(N_1, \dots, N_4)| &\leq C(N_1^\varepsilon + N_2^\varepsilon)^{1-\theta} (N_3^\varepsilon + N_4^\varepsilon)^\theta m(N_1, \dots, N_4)^{s(\theta)} \prod_{j=1}^4 \|u_j^{N_j}\|_{X^{0,b'}} \\ &\leq CN_1^{(1-\theta)\varepsilon-s-\varepsilon} N_4^{s+\theta\varepsilon} N_2^{s(\theta)-s-\varepsilon} N_3^{s(\theta)+\theta\varepsilon-s} \prod_{j=1}^2 \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}} \\ &\leq \left(\frac{N_4}{N_1}\right)^{s+\theta\varepsilon} N_2^{s(\theta)-s-\varepsilon} N_3^{s(\theta)+\theta\varepsilon-s} \prod_{j=1}^2 \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}}. \end{aligned}$$

The series is convergent thanks to $N_4 \leq CN_1$ and after choosing θ small enough such that $s(\theta) + \theta\varepsilon - s < 0$ with $b(\theta) - b' < 0$.

Second case: $N_1 = \max(N_1, N_2) \leq N_3$, and so $N_4 \leq CN_3$. This time, N_3 is a large frequency and we cannot have any loss $N_3^{\theta\varepsilon}$ from the interpolation. We proceed with the same interpolation procedure but between (A.1) and (A.3). After summation in K and a good choice of $b' < 1/2 < b$,

$$\begin{aligned} |I(N_1, \dots, N_4)| &\leq CN_1^{(1-\theta)(s_0+\varepsilon)+\theta\mu-s-\varepsilon} N_4^s N_2^{(1-\theta)s_0-s-\varepsilon} N_3^{-s} \prod_{j=1}^2 \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}} \\ &\leq \left(\frac{N_4}{N_3}\right)^s N_1^{(1-\theta)(s_0+\varepsilon)+\theta\mu-s-\varepsilon} N_2^{(1-\theta)s_0-s-\varepsilon} \prod_{j=1}^2 \|u_j\|_{X^{s+\varepsilon,b'}} \|u_3\|_{X^{s,b'}} \|u_4\|_{X^{-s,b'}}. \end{aligned}$$

We choose θ small enough such that $(1 - \theta)(s_0 + \varepsilon) + \theta\mu - s - \varepsilon \leq s_0 + \theta\mu - s < 0$ and $b(\theta) - b' < 0$. And we conclude by the same summation as in the first case.

Last case: $N_4 \geq C(N_1 + N_2 + N_3)$. This case is trivial in the particular case of \mathbb{T}^3 , S^3 , or $S^2 \times S^1$ since this integral is zero for C large enough. In the general case, we apply the following lemma, which is a variant of Lemma 2.6 in [9].

LEMMA A.4. *There exists $C > 0$ such that if, for any $j = 1, 2, 3$, $C\mu_{k_j} \leq \mu_{k_4}$, then for every $p > 0$ there exists $C_p > 0$ such that for every $w_j \in L^2(M)$, $j = 1, 2, 3, 4$,*

$$\int_M \Pi_{k_1} w_1 \Pi_{k_2} w_2 [(-\Delta)^{\frac{\varepsilon}{2}} \Pi_{k_3} w_3 \overline{\Pi_{k_4} w_4} - \Pi_{k_3} w_3 (-\Delta)^{\frac{\varepsilon}{2}} \overline{\Pi_{k_4} w_4}] \leq C_p \mu_{k_4}^{-p} \prod_{j=1}^4 \|w_j\|_{L^2},$$

where Π_k denotes the orthogonal projection on the eigenfunction e_k associated with the eigenvalue μ_k .

This ends the proof of the first statement of Lemma A.3. The second one is obtained by duality. \square

A.3. S^3 and $S^2 \times S^1$ fulfill Assumption 5.

LEMMA A.5. *Assumption 5 holds true with any $s_0 > 1/2$ on S^3 and any $s_0 > 3/4$ on $S^2 \times S^1$.*

Proof. We first treat the case of S^3 and follow the scheme of Proposition 3 of [23]. We write

$$f_j = \sum_{n_j} H_{n_j}^{(j)},$$

where $H_{n_j}^{(j)}$ are spherical harmonics of degree n_j , and where the sum on n_j bears on the domain

$$(A.4) \quad N_j \leq \sqrt{1 + n_j(n_j + 2)} < 2N_j.$$

Then, the solution u_j is given by

$$u_j(t) = e^{it\Delta} f_j = \sum_{n_j} e^{-itn_j(n_j+2)} H_{n_j}^{(j)}$$

and we have to estimate

$$\begin{aligned} Q(f_1, \dots, f_4, \tau) &= \int_{\mathbb{R}} \int_{S^3} \chi(t) e^{it\tau} u_1 u_2 \left[(-\Delta)^{\varepsilon/2} u_3 \overline{u_4} - u_3 (-\Delta)^{\varepsilon/2} \overline{u_4} \right] dx dt \\ &= \sum_{n_1, \dots, n_4} \widehat{\chi} \left(\sum_{j=1}^4 \varepsilon_j n_j(n_j + 2) - \tau \right) I(H_{n_1}^{(1)}, \dots, H_{n_4}^{(4)}), \end{aligned}$$

with $\varepsilon_j = -1$ or 1 depending on the position of conjugates and

$$I(H_{n_1}^{(1)}, \dots, H_{n_4}^{(4)}) = \left(\sqrt{n_3(n_3 + 2)}^\varepsilon - \sqrt{n_4(n_4 + 2)}^\varepsilon \right) \int_{S^3} H_{n_1}^{(1)} H_{n_2}^{(2)} H_{n_3}^{(3)} \overline{H_{n_4}^{(4)}} dx.$$

We notice that $\int H_{n_1} H_{n_2} H_{n_3} \overline{H_{n_4}} \neq 0$ implies $n_4 \leq n_1 + n_2 + n_3$ and $n_3 \leq n_1 + n_2 + n_4$, that is, $|n_4 - n_3| \leq n_1 + n_2$. Then, using Lemma A.1 and, the fundamental theorem of calculus, we have

$$(A.5) \quad \begin{aligned} \left| \sqrt{n_3(n_3 + 2)}^\varepsilon - \sqrt{n_4(n_4 + 2)}^\varepsilon \right| &\leq \left| \sqrt{n_3(n_3 + 2)} - \sqrt{n_4(n_4 + 2)} \right|^\varepsilon \\ &\leq C |n_4 - n_3|^\varepsilon \leq C(N_1^\varepsilon + N_2^\varepsilon). \end{aligned}$$

Moreover, bilinear eigenfunction estimates (see Theorem 2 of [13] or Theorem 2.5 of [12]) yield

$$\begin{aligned} \left| I(H_{n_1}^{(1)}, \dots, H_{n_4}^{(4)}) \right| &\leq C(N_1^\varepsilon + N_2^\varepsilon) \left| \int_{S^3} H_{n_1}^{(1)} H_{n_2}^{(2)} H_{n_3}^{(3)} \overline{H_{n_4}^{(4)}} dx \right| \\ &\leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \dots, N_4)^{1/2+} \prod_{j=1}^4 \left\| H_{n_j}^{(j)} \right\|_{L^2}. \end{aligned}$$

Using the fast decay of $\widehat{\chi}$ at infinity, we infer that

$$\begin{aligned} |Q(f_1, \dots, f_4, \tau)| &\leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \dots, N_4)^{\frac{1}{2}+} \sum_{l \in \mathbb{Z}} (1 + |l|^2)^{-1} \sum_{\Lambda([\tau]+l)} \prod_{j=1}^4 \left\| H_{n_j}^{(j)} \right\|_{L^2} \\ &\leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \dots, N_4)^{1/2+} \sup_{k \in \mathbb{Z}} \sum_{\Lambda(k)} \prod_{j=1}^4 \left\| H_{n_j}^{(j)} \right\|_{L^2}, \end{aligned}$$

where $\Lambda(k)$ denotes the set of (n_1, \dots, n_4) satisfying (A.4) for $j = 1, 2, 3, 4$ and

$$\sum_{j=1}^4 \varepsilon_j n_j (n_j + 2) = k.$$

Now, we write

$$\{1, 2, 3, 4\} = \{\alpha, \beta, \gamma, \delta\}$$

with $m(N_1, \dots, N_4) = N_\alpha N_\beta$ and we split the sum on $\Lambda(k)$ as

$$|Q(f_1, \dots, f_4, \tau)| \leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \dots, N_4)^{1/2+} \sup_{k \in \mathbb{Z}} \sum_{a \in \mathbb{Z}} S(a) S'(k - a),$$

where

$$\begin{aligned} S(a) &= \sum_{\Gamma(a)} \left\| H_{n_\alpha}^{(\alpha)} \right\|_{L^2} \left\| H_{n_\gamma}^{(\gamma)} \right\|_{L^2}, \quad S'(a') = \sum_{\Gamma'(a')} \left\| H_{n_\beta}^{(\beta)} \right\|_{L^2} \left\| H_{n_\delta}^{(\delta)} \right\|_{L^2}, \\ \Gamma(a) &= \left\{ (n_\alpha, n_\gamma) : \text{(A.4) holds for } j = \alpha, \gamma, \sum_{j=\alpha, \gamma} \varepsilon_j n_j (n_j + 2) = a \right\}, \\ \Gamma'(a') &= \left\{ (n_\beta, n_\delta) : \text{(A.4) holds for } j = \beta, \delta, \sum_{j=\beta, \delta} \varepsilon_j n_j (n_j + 2) = a' \right\}. \end{aligned}$$

Then, we use a number theoretic result involving the ring of Gauss integers (see Lemma 3.2 of [9]).

LEMMA A.6. *Let $\sigma \in \{\pm 1\}$. For every $\eta > 0$, there exists C_η such that, given $M \in \mathbb{Z}$ and a positive integer N ,*

$$\#\{(k_1, k_2) \in \mathbb{N}^2 : N \leq k_1 \leq 2N, k_1^2 + \sigma k_2^2 = M\} \leq C_\eta N^\eta.$$

Noticing that $n_j(n_j + 2) = (n_j + 1)^2 - 1$, we get

$$\sup_a \#\Gamma(a) \leq C_\eta N_\alpha^\eta, \quad \sup_{a'} \#\Gamma'(a') \leq C_\eta N_\beta^\eta,$$

and, consequently, by the Cauchy–Schwarz inequality and the orthogonality of the $H_{n_j}^{(j)}$

$$\begin{aligned} \sum_{a \in \mathbb{Z}} S(a) S'(k - a) &\leq C_\eta (N_\alpha N_\beta)^{\eta/2} \\ &\times \left(\sum_a \sum_{\Gamma(a)} \left\| H_{n_\alpha}^{(\alpha)} \right\|_{L^2}^2 \left\| H_{n_\gamma}^{(\gamma)} \right\|_{L^2}^2 \right)^{1/2} \left(\sum_{a'} \sum_{\Gamma'(a')} \left\| H_{n_\beta}^{(\beta)} \right\|_{L^2}^2 \left\| H_{n_\delta}^{(\delta)} \right\|_{L^2}^2 \right)^{1/2} \\ &\leq C_\eta (N_\alpha N_\beta)^{\eta/2} \prod_{j=1}^4 \|f_j\|_{L^2}. \end{aligned}$$

This completes the proof for S^3 .

For $S^2 \times S^1$, we adapt this argument with some slight modifications. First, the formulae should be changed to

$$u_j(t)(x, y) = e^{it\Delta} f_j = \sum_{n_j, p_j} e^{-itn_j(n_j+1) - ip_j^2 t} H_{n_j, p_j}^{(j)}(x) e^{ip_j y},$$

where $H_{n_j, p_j}^{(j)}$ are spherical harmonics on S^2 of degree n_j . Estimate (A.5) becomes

$$\begin{aligned} & \left| \sqrt{n_3(n_3 + 1) + p_3^2}^\varepsilon - \sqrt{n_4(n_4 + 2) + p_4^2}^\varepsilon \right| \\ & \leq \left| \sqrt{n_3(n_3 + 1) + p_3^2} - \sqrt{n_4(n_4 + 1) + p_4^2} \right|^\varepsilon \\ & \leq \left| \left[\sqrt{n_3(n_3 + 1)} - \sqrt{n_4(n_4 + 1)} \right]^2 + (p_3 - p_4)^2 \right|^{\varepsilon/2} \\ & \leq |C(n_3 - n_4)^2 + (p_3 - p_4)^2|^{\varepsilon/2} \\ & \leq C|(n_1 + n_2)^2 + (p_1 + p_2)^2|^{\varepsilon/2} \leq C(N_1^\varepsilon + N_2^\varepsilon), \end{aligned}$$

where we have used $|n_3 - n_4| \leq |n_1 + n_2|$ and $|p_3 - p_4| \leq |p_1| + |p_2|$ for the integral to be nonzero. Bilinear eigenfunction estimates for S^2 yield

$$\left| I(H_{n_1, p_1}^{(1)}, \dots, H_{n_4, p_4}^{(4)}) \right| \leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \dots, N_4)^{1/4} \prod_{j=1}^4 \|H_{n_j, p_j}^{(j)}\|_{L^2}.$$

We finish the proof similarly, replacing the formula for $\Gamma(a)$ by

$$\Gamma(a) = \left\{ (n_\alpha, p_\alpha, n_\gamma, p_\gamma) : N_j \leq \sqrt{1 + n_j(n_j + 2) + p_j^2} \leq 2N_j, j = \alpha, \gamma, \right. \\ \left. \text{and } \sum_{j=\alpha, \gamma} \varepsilon_j [n_j(n_j + 2) + p_j^2] = a \right\}.$$

In that case, the same number theoretic arguments yield $\sup_a \#\Gamma(a) \leq C_\eta N_\alpha^{1+\eta}$ and, finally, after the Cauchy–Schwarz inequality, we obtain

$$|Q(f_1, \dots, f_4, \tau)| \leq C(N_1^\varepsilon + N_2^\varepsilon) m(N_1, \dots, N_4)^{1/4+(1+\eta)/2} \prod_{j=1}^4 \|f_j\|_{L^2}. \quad \square$$

Appendix B. Unique continuation.

B.1. Carleman estimates. This section is only a variant in the Riemannian setting of some results of Mercado, Osses, and Rosier [35]. We follow their proof very closely, sometimes line by line.

For the sake of simplicity, we will assume that u is supported in a fixed compact K of a Riemannian manifold Ω . Yet, the same reasoning as in [35] would allow us to handle the case of Dirichlet boundary conditions for u . We have changed the notation of the manifold from M to Ω because the Carleman estimates will not be used on the whole compact manifold M but only on some open set Ω .

D denotes the Levi–Civita connection associated with the metric g . Then, it is torsion-free and the Hessians of the functions are symmetric.

$\cdot, |\cdot|, \nabla$, and Δ denote the scalar product, the norm, the gradient, and the Laplacian with respect to the metric g . Moreover, the scalar product will be the real one: if $X = a + ib$ and $Y = c + id$, then $X \cdot Y = a \cdot c - b \cdot d + i(b \cdot c + a \cdot d)$ and $|X|^2 = X \cdot \bar{X}$.

v_g denotes the Riemannian volume form, and all the integrals are defined with this (even if it will often be omitted).

First, we list a few formulae that will be used throughout the proof. For any functions $f, h \in C^\infty(\Omega)$ with h compactly supported and any vector fields X, Y , and Z , we have

$$\begin{aligned} D_Z(X \cdot Y) &= (D_Z X) \cdot Y + X \cdot (D_Z Y), \\ \nabla f \cdot Z &= D_Z f, \\ (D_X \nabla f) \cdot Y &= Hess(f)(X, Y), \\ \int_\Omega (\Delta f) h \, dv_g &= - \int_\Omega \nabla f \cdot \nabla h \, dv_g, \\ \nabla(fh) &= (\nabla f)h + f(\nabla h), \\ \operatorname{div}(fX) &= f \operatorname{div}(X) + X \cdot \nabla f. \end{aligned}$$

For brevity, \iint will denote the integral over $] - T, T[\times \Omega$ and \iint_ω the integral over $] - T, T[\times \omega$, where ω is an open subset of Ω .

Let $\Psi \in C^4(\Omega)$ real valued. We assume that Ψ satisfies the following properties:

$$\begin{aligned} \text{(B.1)} \quad & \nabla \Psi \neq 0 \text{ in } \Omega \setminus \omega, \\ \text{(B.2)} \quad & \Psi(x) \geq 2/3 \|\Psi\|_{L^\infty}. \end{aligned}$$

Inequality (B.2) is technical and is easily fulfilled by replacing Ψ by $\Psi + C$ with C large enough. We distinguish two cases: strong pseudoconvexity and weak pseudoconvexity.

The case of strong pseudoconvexity can be found in Isakov [25] but with local in time estimates; it reads as

$$\text{(B.3)} \quad Hess(\Psi(x))(\xi, \xi) + |\nabla \Psi(x) \cdot \xi|^2 > 0 \quad \forall (x, \xi) \in T\Omega \setminus T\omega,$$

which implies since the support is compact that

$$\text{(B.4)} \quad Hess(\Psi(x))(\xi, \xi) + |\nabla \Psi(x) \cdot \xi|^2 > C |\xi|^2 \quad \forall (x, \xi) \in T\Omega \setminus T\omega, \quad x \in K.$$

Weak pseudoconvexity is defined by

$$\text{(B.5)} \quad Hess(\Psi(x))(\xi, \xi) + |\nabla \Psi(x) \cdot \xi|^2 \geq 0 \quad \forall (x, \xi) \in T\Omega \setminus T\omega.$$

Set $C_\Psi = 2 \|\Psi\|_{L^\infty(\Omega)}$ and

$$\theta(t, x) := \frac{e^{\lambda \Psi(x)}}{(T-t)(T+t)}, \quad \varphi(t, x) := \frac{e^{\lambda C_\Psi} - e^{\lambda \Psi(x)}}{(T-t)(T+t)} \quad \forall (t, x) \in] - T, T[\times \Omega.$$

Denote by $L(q) = i\partial_t q + \Delta q$ the linear Schrödinger operator.

PROPOSITION B.1. *Let $T > 0$. Let Ω be a Riemannian manifold and K a compact subset of Ω . Assume that there exists a function $\Psi \in C^4(\Omega)$ such that (B.1), (B.2), and (B.4) hold for some open set $\omega \subset \Omega$. Then, there exist constants λ_0, s_0 , and C*

such that for all $\lambda \geq \lambda_0$, all $s \geq s_0$, and $q \in L^2(\cdot - T, T[, H^1(\Omega))$, supported in K , with $L(q) \in L^2(\cdot - T, T[\times \Omega)$ we have

$$(B.6) \quad \begin{aligned} & \iint \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda \theta |\nabla q|^2 \right] e^{-2s\varphi} \\ & \leq C \iint |L(q)|^2 e^{-2s\varphi} + C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda \theta |\nabla q|^2 \right] e^{-2s\varphi}. \end{aligned}$$

PROPOSITION B.2. *If in Proposition B.1 we replace assumption (B.4) by (B.5), we obtain the same result with*

$$(B.7) \quad \begin{aligned} & \iint \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda^2 \theta |\nabla \Psi \cdot \nabla q|^2 \right] e^{-2s\varphi} \\ & \leq C \iint |L(q)|^2 e^{-2s\varphi} + C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda \theta |\nabla q|^2 \right] e^{-2s\varphi}. \end{aligned}$$

Proof. Using regularization in a standard way, we are reduced to considering $q \in C^\infty(\cdot - T, T[\times \Omega)$. Denote $u = e^{-s\varphi} q$ and $w = e^{-s\varphi} L(q) = e^{-s\varphi} L(e^{s\varphi} u)$. We notice that u and all its time derivatives vanish at $t = -T$ and $t = T$. Thus, all the integrations by parts in time do not create any boundary term. We compute

$$w = Pu = iu_t + is\varphi_t u + \Delta u + 2s\nabla\varphi \cdot \nabla u + s(\Delta\varphi)u + s^2|\nabla\varphi|^2 u.$$

We decompose $P = P_1 + P_2$ with

$$\begin{aligned} P_1 u &:= is\varphi_t u + 2s\nabla\varphi \cdot \nabla u + s(\Delta\varphi)u, \\ P_2 u &:= iu_t + \Delta u + s^2|\nabla\varphi|^2 u, \end{aligned}$$

$$\|w\|_{L^2(-T, T[\times \Omega)}^2 = \|P_1 u + P_2 u\|^2 = \|P_1 u\|^2 + \|P_2 u\|^2 + 2\Re(P_1 u, P_2 u).$$

As usual in Carleman estimates, we use only

$$2\Re(P_1 u, P_2 u) \leq \|w\|_{L^2(-T, T[\times \Omega)}^2.$$

We also decompose $2\Re(P_1 u, P_2 u) = I_1 + I_2 + I_3$ with

$$\begin{aligned} I_1 &:= 2\Re \iint (2s\nabla\varphi \cdot \nabla u + s(\Delta\varphi)u)(-i\bar{u}_t + \Delta\bar{u} + s^2|\nabla\varphi|^2\bar{u}), \\ I_2 &:= 2\Re \iint is\varphi_t u(-i\bar{u}_t + \Delta\bar{u}), \\ I_3 &:= 2\Re \iint is\varphi_t u(s^2|\nabla\varphi|^2\bar{u}) = 0. \end{aligned}$$

We first deal with I_1 :

$$\begin{aligned} I_1 &= 2\Re \iint (2s\nabla\varphi \cdot \nabla u + s(\Delta\varphi)u)(\Delta\bar{u} + s^2|\nabla\varphi|^2\bar{u}) \\ &\quad - 2\Re \iint i(2s\nabla\varphi \cdot \nabla u + s(\Delta\varphi)u)\bar{u}_t \\ &= I_1^1 + I_1^2. \end{aligned}$$

Set $J = \iint (\nabla\varphi \cdot \nabla u) \Delta \bar{u} = - \iint \nabla \bar{u} \cdot \nabla (\nabla\varphi \cdot \nabla u)$. We have

$$\begin{aligned} \nabla \bar{u} \cdot \nabla (\nabla\varphi \cdot \nabla u) &= D_{\nabla \bar{u}}(\nabla\varphi \cdot \nabla u) = (D_{\nabla \bar{u}}\nabla\varphi) \cdot \nabla u + \nabla\varphi \cdot (D_{\nabla \bar{u}}\nabla u) \\ &= Hess(\varphi)(\nabla u, \nabla \bar{u}) + Hess(u)(\nabla \bar{u}, \nabla\varphi). \end{aligned}$$

Actually

$$\begin{aligned} \nabla\varphi \cdot \nabla |\nabla u|^2 &= D_{\nabla\varphi}(\nabla u \cdot \nabla \bar{u}) = (D_{\nabla\varphi}\nabla u) \cdot \nabla \bar{u} + \nabla u \cdot (D_{\nabla\varphi}\nabla \bar{u}) \\ &= 2\Re(D_{\nabla\varphi}\nabla u) \cdot \nabla \bar{u} = 2\Re Hess(u)(\nabla\varphi, \nabla \bar{u}). \end{aligned}$$

Therefore,

$$2\Re J = -2 \iint Hess(\varphi)(\nabla u, \nabla \bar{u}) + \iint \Delta\varphi |\nabla u|^2.$$

Expanding I_1^1 , we obtain

$$\begin{aligned} I_1^1 &= 2\Re \left\{ 2sJ + \iint s(\Delta\varphi)u\Delta \bar{u} + \iint 2s^3(\nabla\varphi \cdot \nabla u)|\nabla\varphi|^2\bar{u} + \iint s^3(\Delta\varphi)|u|^2|\nabla\varphi|^2 \right\} \\ &= 4s\Re J - 2s\Re \iint ((\nabla\Delta\varphi)u + \Delta\varphi\nabla u) \cdot \nabla \bar{u} \\ &\quad + \iint 2s^3|\nabla\varphi|^2\nabla\varphi \cdot \nabla |u|^2 + 2 \iint s^3(\Delta\varphi)|u|^2|\nabla\varphi|^2, \end{aligned}$$

where we have used $\nabla |u|^2 = 2\Re(\bar{u}\nabla u)$. Then, we remark that

$$\begin{aligned} -2s\Re \iint (\nabla\Delta\varphi)u \cdot \nabla \bar{u} &= -s \iint (\nabla\Delta\varphi) \cdot \nabla |u|^2 \\ &= s \iint (\Delta^2\varphi)|u|^2, \\ 2 \iint s^3(\Delta\varphi)|u|^2|\nabla\varphi|^2 &= -2s^3 \iint \nabla\varphi \cdot (|\nabla\varphi|^2\nabla |u|^2 + |u|^2\nabla|\nabla\varphi|^2). \end{aligned}$$

We simplify to

$$\begin{aligned} I_1^1 &= -4s\Re \iint Hess(\varphi)(\nabla u, \nabla \bar{u}) + 2s \iint \Delta\varphi |\nabla u|^2 \\ &\quad + s \iint (\Delta^2\varphi)|u|^2 - 2s \iint \Delta\varphi |\nabla u|^2 - 2s^3 \iint |u|^2\nabla\varphi \cdot \nabla|\nabla\varphi|^2 \\ &= -4s \iint Hess(\varphi)(\nabla u, \nabla \bar{u}) + s \iint (\Delta^2\varphi)|u|^2 - 2s^3 \iint (\nabla\varphi \cdot \nabla|\nabla\varphi|^2)|u|^2. \end{aligned}$$

Expanding $2\Re a = a + \bar{a}$ for I_1^2 and performing integration by parts in t for the first term, we get

$$\begin{aligned} -I_1^2 &= \iint i(2s\nabla\varphi \cdot \nabla u + s(\Delta\varphi)u)\bar{u}_t - i \iint (2s\nabla\varphi \cdot \nabla \bar{u} + s(\Delta\varphi)\bar{u})u_t \\ &= \iint -i[2s\nabla\varphi_t \cdot \nabla u + 2s\nabla\varphi \cdot \nabla u_t + s(\Delta\varphi_t)u + s(\Delta\varphi)u_t]\bar{u} \\ &\quad - i \iint 2s(\nabla\varphi \cdot \nabla \bar{u})u_t - i \iint s(\Delta\varphi)\bar{u}u_t. \end{aligned}$$

Integration by parts in x yields

$$-i \iint 2s(\nabla\varphi \cdot \nabla\bar{u})u_t = 2is \iint (\Delta\varphi)\bar{u}u_t + 2is \iint (\nabla\varphi \cdot \nabla u_t)\bar{u}.$$

As a consequence,

$$\begin{aligned} -I_1^2 &= \iint -i2s(\nabla\varphi_t \cdot \nabla u)\bar{u} - is \iint (\Delta\varphi_t)|u|^2 \\ &= \iint -i2s(\nabla\varphi_t \cdot \nabla u)\bar{u} + is \iint \nabla\varphi_t \cdot \nabla|u|^2 \\ &= i \iint s\nabla\varphi_t \cdot (u\nabla\bar{u} - \bar{u}\nabla u) = 2s\Re i \iint \nabla\varphi_t \cdot (u\nabla\bar{u}). \end{aligned}$$

Finally,

$$\begin{aligned} I_1 &= -4s\Re \iint Hess(\varphi)(\nabla u, \nabla\bar{u}) + s \iint (\Delta^2\varphi)|u|^2 \\ &\quad - 2s^3 \iint \nabla\varphi \cdot \nabla|\nabla\varphi|^2|u|^2 - 2s\Re i \iint \nabla\varphi_t \cdot (u\nabla\bar{u}). \end{aligned}$$

On the other hand, we have

$$\nabla\varphi \cdot \nabla|\nabla\varphi|^2 = D_{\nabla\varphi}(\nabla\varphi \cdot \nabla\varphi) = 2D_{\nabla\varphi}\nabla\varphi \cdot \nabla\varphi = 2Hess(\varphi)(\nabla\varphi, \nabla\varphi).$$

We now turn to the other term I_2 :

$$\begin{aligned} I_2 &= 2\Re \iint is\varphi_t u(-i\bar{u}_t + \Delta\bar{u}) = s \iint \varphi_t \partial_t |u|^2 + 2s\Re i \iint \varphi_t u \Delta\bar{u} \\ &= -s \iint \varphi_{tt}|u|^2 - 2s\Re i \iint (\nabla\varphi_t u + \varphi_t \nabla u) \cdot \nabla\bar{u} \\ &= -s \iint \varphi_{tt}|u|^2 - 2s\Re \iint i(\nabla\varphi_t \cdot \nabla\bar{u})u. \end{aligned}$$

Consequently, our final result is

$$(B.8) \quad 2\Re(M_1 u, M_2 u) = \iint [-4s^3 Hess(\varphi)(\nabla\varphi, \nabla\varphi) - s\varphi_{tt} + s(\Delta^2\varphi)] |u|^2$$

$$(B.9) \quad -4s\Re \iint Hess(\varphi)(\nabla u, \nabla\bar{u})$$

$$(B.10) \quad -4s\Re \iint iu\nabla\varphi_t \cdot \nabla\bar{u}.$$

Equations (B.8) and (B.9) are the main parts in $|u|^2$ and $|\nabla u|^2$, respectively. Equation (B.10) is a remainder term that will be estimated from above.

In what follows, $\varepsilon > 0$ denotes small constants (used in estimates from below) and C large ones (used for estimates from above). We observe the following identities, which will be used throughout the proof:

$$\nabla\varphi = -\lambda\theta\nabla\Psi,$$

$$\begin{aligned} Hess(\varphi)(X, Y) &= (D_X \nabla\varphi) \cdot Y \\ &= -\lambda D_X(\theta\nabla\Psi) \cdot Y = -\lambda\theta(D_X \nabla\Psi) \cdot Y - \lambda d\theta(X)\nabla\Psi \cdot Y \\ &= -\lambda\theta Hess(\Psi)(X, Y) - \lambda^2\theta(\nabla\Psi \cdot X)(\nabla\Psi \cdot Y) \\ &= -\theta\lambda [Hess(\Psi)(X, Y) + \lambda(\nabla\Psi \cdot X)(\nabla\Psi \cdot Y)]. \end{aligned}$$

First, we estimate term (B.10):

$$\begin{aligned}
|(B.10)| &\leq Cs \iint |\nabla\varphi_t \cdot \nabla u| |u| \leq Cs \iint \frac{t\lambda e^{\lambda\Psi}}{(T^2-t^2)^2} |\nabla\Psi \cdot \nabla u| |u| \\
&\leq Cs \iint \frac{e^{\lambda\Psi}}{(T^2-t^2)} |\nabla\Psi \cdot \nabla u|^2 + Cs \iint \frac{(T\lambda)^2 e^{\lambda\Psi}}{(T^2-t^2)^3} |u|^2 \\
(B.11) \quad &\leq Cs \iint \theta |\nabla\Psi \cdot \nabla u|^2 + Cs\lambda^{-1} \iint |\nabla\varphi|^3 |u|^2 + Cs \iint_{\omega} \lambda^2 \theta^3 |u|^2.
\end{aligned}$$

Then, we estimate term (B.8) using assumptions (B.1) and (B.5) (or (B.4)). On $(\Omega \setminus \omega) \cap K$, we have

$$\begin{aligned}
-4s^3 \text{Hess}(\varphi)(\nabla\varphi, \nabla\varphi) &= 4s^3 \lambda \theta \left[\text{Hess}(\Psi)(\nabla\varphi, \nabla\varphi) + \lambda |\nabla\Psi \cdot \nabla\varphi|^2 \right] \\
&\geq 4s^3 \lambda \theta (\lambda - 1) |\nabla\Psi \cdot \nabla\varphi|^2 \geq s^3 \lambda^4 \theta^3 |\nabla\Psi|^4 \geq \varepsilon s^3 \lambda |\nabla\varphi|^3.
\end{aligned}$$

Assumption (B.2) gives $\Psi(x) \leq C_{\Psi} \leq 3\Psi(x)$, and then we have on $(\Omega \setminus \omega) \cap K$

$$|s\varphi_{tt}| \leq Cs \frac{e^{\lambda C_{\Psi}}}{(T^2-t^2)^3} \leq Cs \frac{e^{3\lambda\Psi(x)}}{(T^2-t^2)^3} \leq Cs |\nabla\varphi|^3.$$

Moreover, on $(\Omega \setminus \omega) \cap K$ we have

$$|s\Delta^2\varphi| \leq Cs\theta\lambda^4 \leq Cs\lambda |\nabla\varphi|^3.$$

Finally, for λ and s large enough

$$\iint_{\Omega \setminus \omega} [-4s^3 \text{Hess}(\varphi)(\nabla\varphi, \nabla\varphi) - s\varphi_{tt} + s(\Delta^2\varphi)] |u|^2 \geq \iint_{\Omega \setminus \omega} \varepsilon s^3 \lambda |\nabla\varphi|^3 |u|^2.$$

For the domain ω , we have the estimate

$$\left| \iint_{\omega} [-4s^3 \text{Hess}(\varphi)(\nabla\varphi, \nabla\varphi) - s\varphi_{tt} + s(\Delta^2\varphi)] |u|^2 \right| \leq C \iint_{\omega} s^3 \lambda^4 \theta^3 |u|^2.$$

The final estimate for (B.8) is

$$(B.12) \quad (B.8) \geq \iint_{\Omega \setminus \omega} \varepsilon s^3 \lambda |\nabla\varphi|^3 |u|^2 - C \iint_{\omega} s^3 \lambda^4 \theta^3 |u|^2.$$

Now, let us estimate (B.9). We begin with the integral on ω :

$$\begin{aligned}
-4s\Re \iint_{\omega} \text{Hess}(\varphi)(\nabla u, \nabla \bar{u}) &= 4s\Re \iint_{\omega} \theta \lambda \left[\text{Hess}(\Psi)(\nabla u, \nabla \bar{u}) + \lambda |\nabla\Psi \cdot \nabla u|^2 \right] \\
&\geq -Cs\lambda \iint_{\omega} \theta |\nabla u|^2 + 4s \iint_{\omega} \theta \lambda^2 |\nabla\Psi \cdot \nabla u|^2 \\
&\geq -Cs\lambda \iint_{\omega} \theta |\nabla u|^2.
\end{aligned}$$

Now, for the integral on $\Omega \setminus \omega$, we distinguish the two cases described above.

Strong pseudoconvexity: End of the proof of Proposition B.1. Using assumption (B.4), we can estimate the part of (B.9) on $\Omega \setminus \omega$ by

$$\begin{aligned} -4s\Re \iint_{\Omega \setminus \omega} Hess(\varphi)(\nabla u, \nabla \bar{u}) &= 4s\Re \iint_{\Omega \setminus \omega} \theta \lambda \left[Hess(\Psi)(\nabla u, \nabla \bar{u}) + \lambda |\nabla \Psi \cdot \nabla u|^2 \right] \\ &\geq \varepsilon s \lambda \iint_{\Omega \setminus \omega} \theta |\nabla u|^2. \end{aligned}$$

The final estimate for (B.9) is

$$(B.13) \quad (B.9) \geq \varepsilon s \lambda \iint_{\Omega \setminus \omega} \theta |\nabla u|^2 - Cs \lambda \iint_{\omega} \theta |\nabla u|^2.$$

Putting together (B.11), (B.12), and (B.13), we get for s, λ large enough

$$\begin{aligned} (B.8) + (B.9) + (B.10) &\geq \varepsilon \iint_{\Omega \setminus \omega} s^3 \lambda |\nabla \varphi|^3 |u|^2 - C \iint_{\omega} s^3 \lambda^4 \theta^3 |u|^2 - Cs \lambda \iint_{\omega} \theta |\nabla u|^2 \\ &\quad + \varepsilon s \lambda \iint_{\Omega \setminus \omega} \theta |\nabla u|^2 - Cs \iint_{\omega} \theta |\nabla \Psi \cdot \nabla u|^2 \\ &\quad - Cs \lambda^{-1} \iint_{\Omega \setminus \omega} |\nabla \varphi|^3 |u|^2 - Cs \iint_{\omega} \lambda^2 \theta^3 |u|^2 \\ &\geq \varepsilon \iint_{\Omega \setminus \omega} s^3 \lambda^4 \theta^3 |u|^2 + \varepsilon s \lambda \iint_{\Omega \setminus \omega} \theta |\nabla u|^2 \\ (B.14) \quad &\quad - C \iint_{\omega} s^3 \lambda^4 \theta^3 |u|^2 - Cs \lambda \iint_{\omega} \theta |\nabla u|^2, \end{aligned}$$

where we have used the decomposition $\iint_{\Omega \setminus \omega} = \iint - \iint_{\omega}$ for the second inequality. Replacing u by $e^{-s\varphi}q$ and computing $\nabla q = e^{s\varphi}[\nabla u - s\lambda\theta u \nabla \Psi]$, we have, after absorption,

$$\begin{aligned} &\iint \left[s^3 \lambda^4 \theta^3 |q|^2 + s\lambda\theta |\nabla q|^2 \right] e^{-2s\varphi} \\ &\leq C \iint \left[s^3 \lambda^4 \theta^3 |u|^2 + s\lambda\theta |\nabla u|^2 + s^3 \lambda^3 \theta^3 |\nabla \psi|^2 |u|^2 \right] \\ (B.15) \quad &\leq C \iint \left[s^3 \lambda^4 \theta^3 |u|^2 + s\lambda\theta |\nabla u|^2 \right], \end{aligned}$$

$$\begin{aligned} &\iint_{\omega} s^3 \lambda^4 \theta^3 |u|^2 + s\lambda \iint_{\omega} \theta |\nabla u|^2 \\ &\leq C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s\lambda\theta |\nabla q|^2 + s^3 \lambda^3 \theta^3 |\nabla \psi|^2 |q|^2 \right] e^{-2s\varphi} \\ (B.16) \quad &\leq C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s\lambda\theta |\nabla q|^2 \right] e^{-2s\varphi}. \end{aligned}$$

Combining (B.14), (B.15), and (B.16), we get the expected result:

$$\begin{aligned} &\iint \left[s^3 \lambda^4 \theta^3 |q|^2 + s\lambda\theta |\nabla q|^2 \right] e^{-2s\varphi} \\ &\leq C \iint |i\partial_t q + \Delta q|^2 e^{-2s\varphi} + C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s\lambda\theta |\nabla q|^2 \right] e^{-2s\varphi}. \end{aligned}$$

Weak pseudoconvexity: End of the proof of Proposition B.2. Assumption (B.5) yields that for λ large enough

$$-4s\Re \iint_{\Omega \setminus \omega} Hess(\varphi)(\nabla u, \nabla \bar{u}) \geq \varepsilon s \iint_{\Omega \setminus \omega} \theta \lambda^2 |\nabla \Psi \cdot \nabla u|^2.$$

We finish the proof similarly to get

$$\begin{aligned} \text{(B.8) + (B.9) + (B.10)} &\geq \iint_{\Omega \setminus \omega} \varepsilon s^3 \lambda |\nabla \varphi|^3 |u|^2 - C \iint_{\omega} s^3 \lambda^4 \theta^3 |u|^2 - Cs \lambda \iint_{\omega} \theta |\nabla u|^2 \\ &\quad + \varepsilon s \iint_{\Omega \setminus \omega} \theta \lambda^2 |\nabla \Psi \cdot \nabla u|^2 - Cs \iint \theta |\nabla \Psi \cdot \nabla u|^2 \\ &\quad - Cs \lambda^{-1} \iint |\nabla \varphi|^3 |u|^2 - Cs \iint_{\omega} \lambda^2 \theta^3 |u|^2 \\ &\geq \varepsilon \iint s^3 \lambda^4 \theta^3 |u|^2 + \varepsilon s \lambda^2 \iint \theta |\nabla \Psi \cdot \nabla u|^2 \\ &\quad - C \iint_{\omega} s^3 \lambda^4 \theta^3 |u|^2 - Cs \lambda \iint_{\omega} \theta |\nabla u|^2 \end{aligned}$$

and then

$$\begin{aligned} &\iint \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda^2 \theta |\nabla \Psi \cdot \nabla q|^2 \right] e^{-2s\varphi} \\ &\leq C \iint |i\partial_t q + \Delta q|^2 e^{-2s\varphi} + C \iint_{\omega} \left[s^3 \lambda^4 \theta^3 |q|^2 + s \lambda \theta |\nabla q|^2 \right] e^{-2s\varphi}. \quad \square \end{aligned}$$

B.2. Carleman estimates with potential $L^\infty([-T, T], L^3)$. The following result proves that the strong pseudoconvexity allows one to absorb some potential terms in $L^\infty([-T, T], L^3)$. This is in contrast with the weak pseudoconvexity, which absorbs only terms in $L^\infty([-T, T] \times \Omega)$.

PROPOSITION B.3. *Assume $\dim(\Omega) \leq 3$. Let $V_1, V_2 \in L^\infty([-T, T], L^3)$. Then, Proposition B.1 holds with L replaced by $L(q) = i\partial_t q + \Delta q + V_1 q + V_2 \bar{q}$.*

Proof. We use the notation of Proposition B.1. We write

$$\begin{aligned} \iint |i\partial_t q + \Delta q|^2 e^{-2s\varphi} &\leq 4 \|e^{-s\varphi} L(q)\|_{L^2([0, T], L^2)}^2 + 4 \|e^{-s\varphi} (V_1 q)\|_{L^2([0, T], L^2)}^2 \\ &\quad + 4 \|e^{-s\varphi} (V_2 \bar{q})\|_{L^2([0, T], L^2)}^2. \end{aligned}$$

But, by the Hölder inequality and Sobolev embedding, we have for $s > 1$

$$\begin{aligned} \|e^{-s\varphi} V_1 q\|_{L^2([0, T], L^2)}^2 &\leq C \|V_1\|_{L^\infty(L^3)}^2 \|e^{-s\varphi} q\|_{L^2(L^6)}^2 \\ &\leq C \left(\|e^{-s\varphi} q\|_{L^2(L^2)}^2 + \|\nabla(e^{-s\varphi} q)\|_{L^2(L^2)}^2 \right) \\ &\leq C \left(\|e^{-s\varphi} q\|_{L^2(L^2)}^2 + \|e^{-s\varphi} \nabla q\|_{L^2(L^2)}^2 \right. \\ &\quad \left. + s^2 \lambda^2 \|\theta(\nabla \Psi) e^{-s\varphi} q\|_{L^2(L^2)}^2 \right) \\ &\leq C \left(\iint \left[s^2 \lambda^2 \theta^3 |q|^2 + \theta |\nabla q|^2 \right] e^{-2s\varphi} \right), \end{aligned}$$

where we have used $\theta \geq C$. We get the desired result using estimate (B.6) of Proposition B.1 for s large enough. \square

REMARK B.1. *The uniqueness results we will obtain from the former proposition are not optimal with respect to the regularity of the potential. Indeed, some recent papers (see the work of Koch and Tataru [27] or Dos Santos Ferreira [19]) establish Carleman-type estimates in L^p which are much better than what we get. They are more complicated and not required for our purpose. Yet, they would become necessary if we considered nonlinearities $|u|^\alpha u$ with $\alpha > 2$.*

B.3. Application to uniqueness.

PROPOSITION B.4. *Let Ω, T, ω, Ψ fulfill the same assumptions as in Proposition B.1. Let $q \in L^\infty([-T, T], H^1(\Omega))$ be a compactly supported solution of $i\partial_t q + \Delta q + V_1 q + V_2 \bar{q} = 0$ with $V_i \in L^\infty([-T, T], L^3)$. Let D be an open subset of Ω such that $\tilde{m} = \inf_{x \in D} \{\Psi(x)\} > \sup_{x \in \omega} \{\Psi(x)\} = m$. Then, $q = 0$ on $] - T, T[\times D$.*

REMARK B.2. *By considering the maximum of Ψ , we see that the assumptions of Proposition B.4 cannot be fulfilled on a compact manifold. Therefore, we will apply this result only on an open set Ω of M , and the compact support of u becomes important.*

Since the previous Carleman estimates hold for every time interval (with constants depending on its length), we are reduced to the following lemma.

LEMMA B.5. *Under the assumptions of Proposition B.4, there exists one $\eta > 0$ such that $q = 0$ on $] - \eta, \eta[\times D$.*

Proof. Fix $\lambda \geq \lambda_0 > 1$ (the next constants could depend on λ but not on s). Let $T \geq \eta > 0$ be chosen later. Denote $\lambda_1 = e^{\lambda C_\Psi} - e^{\lambda \tilde{m}}$ and $\lambda_1 + \varepsilon = e^{\lambda C_\Psi} - e^{\lambda m}$ with $\lambda_1 > 0$ and $\varepsilon > 0$. By definition of \tilde{m} and m , we have for $s \geq 0$

$$e^{-2s\varphi} \leq e^{-2s \frac{\lambda_1 + \varepsilon}{T^2 - t^2}} \quad \forall (t, x) \in] - T, T[\times \omega,$$

$$e^{-2s \frac{\lambda_1}{T^2 - \eta^2}} \leq e^{-2s\varphi} \quad \forall (t, x) \in] - \eta, \eta[\times D.$$

Moreover, once λ_1 and ε are fixed, there exists some constant C such that $y^3 e^{-2(\lambda_1 + \varepsilon)y} \leq C e^{-2(\lambda_1 + \varepsilon/2)y}$ for $y \geq 0$. Therefore, for every $(t, x) \in] - T, T[\times \Omega$ with $x \in \text{Supp } q = K$, we have

$$(s\theta)^3 e^{-2s \frac{\lambda_1 + \varepsilon}{T^2 - t^2}} \leq C \left(\frac{s}{T^2 - t^2} \right)^3 e^{-2s \frac{\lambda_1 + \varepsilon}{T^2 - t^2}} \leq C e^{-2s \frac{\lambda_1 + \varepsilon/2}{T^2 - t^2}} \leq C e^{-2s \frac{\lambda_1 + \varepsilon/2}{T^2}}.$$

Here, the constant C does not depend on s . Then, using the Carleman estimate and $\theta \geq C > 0$, we get

$$\iint_{] - \eta, \eta[\times D} s^3 |q|^2 e^{-2s \frac{\lambda_1}{T^2 - \eta^2}} \leq C \iint_{] - T, T[\times \omega} \left[|q|^2 + |\nabla q|^2 \right] e^{-2s \frac{\lambda_1 + \varepsilon/2}{T^2}}.$$

Therefore,

$$s^3 e^{-2s \frac{\lambda_1}{T^2 - \eta^2}} \iint_{] - \eta, \eta[\times D} |q|^2 \leq C e^{-2s \frac{\lambda_1 + \varepsilon/2}{T^2}} \|q\|_{L^2(H^1)}^2.$$

Then, to finish the proof, we just have to choose η such that $-2 \frac{\lambda_1}{T^2 - \eta^2} > -2 \frac{\lambda_1 + \varepsilon/2}{T^2}$, that is, $\eta^2 < \frac{T^2 \varepsilon/2}{\lambda_1 + \varepsilon/2}$, and let s tend to $+\infty$. \square

B.4. Geometrical examples. We give some geometrical examples where Proposition B.4 applies. Denote $q \in L^\infty([-T, T], H^1(\Omega))$ a solution of $i\partial_t q + \Delta q + V_1 q + V_2 \bar{q} = 0$ with $V_i \in L^\infty([-T, T], L^3)$. In these following cases, Assumptions 2 and 4 are fulfilled. For the convenience of the reader, we recall the problem.

PROPOSITION B.6. *Let $(M, \tilde{\omega})$ be either*

- $(\mathbb{T}^3, \{x \in \mathbb{R}^3 / (\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z}) \mid \exists i \in \{1, 2, 3\}, x_i \in]-\varepsilon, \varepsilon[+ \theta_i\mathbb{Z}\})$,
- $(S^3, \tilde{\omega})$, where $\tilde{\omega}$ is a neighborhood of $S^3 \cap \{x_4 = 0\}$ in $S^3 \subset \mathbb{R}^4$,
- $(S^2 \times S^1, (\omega_1 \times S^1) \cup (S^2 \times]0, \varepsilon[))$, where ω_1 is a neighborhood of the equator of S^2 .

For every $T > 0$, the only solution in $C([0, T], H^1)$ to the system

$$(B.17) \quad \begin{cases} i\partial_t q + \Delta q + b_1(t, x)q + b_2(t, x)\bar{q} = 0 & \text{on } [0, T] \times M, \\ q = 0 & \text{on } [0, T] \times \tilde{\omega}, \end{cases}$$

where $b_1(t, x)$ and $b_2(t, x) \in L^\infty([0, T], L^3)$, is the trivial one $q \equiv 0$.

B.4.1. $M = \mathbb{T}^3$. We assume $q = 0$ on $\tilde{\omega} = \{x \in \mathbb{R}^3 / (\theta_1\mathbb{Z} \times \theta_2\mathbb{Z} \times \theta_3\mathbb{Z}) \mid \exists i \in \{1, 2, 3\}, x_i \in]-\varepsilon, \varepsilon[+ \theta_i\mathbb{Z}\}$. We define \tilde{q} on \mathbb{R}^3 by $\tilde{q}(x) = q(x)$ if $x \in [0, \theta_1] \times [0, \theta_2] \times [0, \theta_3]$ and $\tilde{q}(x) = 0$ otherwise. \tilde{q} satisfies the same Schrödinger equation on \mathbb{R}^3 with compact support K . By translation, we can assume that 0 is the center of the rectangle.

We use the function $\Psi = \|(x, y, z)\|^2 + C$. C is chosen large enough so that (B.2) is fulfilled on K . Let $\delta > 0$ small. Outside of $\omega = B(0, \delta)$, Ψ is strictly convex (that is, strongly pseudoconvex for the flat metric inherited from \mathbb{R}^3) and $\nabla\Psi \neq 0$. Then, assumptions (B.1) and (B.4) are fulfilled.

We can apply Proposition B.4 with $\Omega = \mathbb{R}^3$, $\omega = B(0, \delta)$, and $D = B(0, 2\delta)^c$. As δ is arbitrary, we get $\tilde{q} = 0$ everywhere and so $q = 0$.

B.4.2. $M = S^3$.

LEMMA B.7. *Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. Then, the function $h : (x_1, \dots, x_{n+1}) \mapsto x_{n+1}$ restricted to $S^n \cap \{x_{n+1} < 0\}$ has a strictly positive Hessian for the metric induced by \mathbb{R}^{n+1} .*

Proof. h defined on \mathbb{R}^{n+1} is linear. Then, using Exercise 2.65(b) of [20], we get $Hess(h) = -hg$, where g is the bilinear form of the Riemannian structure. Then, $Hess(h)$ is positive definite if and only if $h < 0$. \square

We assume $q = 0$ on a neighborhood of $x_4 = 0$. Let $\delta > 0$ be small. We choose $\Omega = \{x \in S^3 \mid x_4 < 0\}$, $D = \{x \in S^3 \mid x_4 \in]-1 + 2\delta, 0[\}$, and $\omega = S^3 \cap \{x_4 \in [-1, -1 + \delta[\}$. We use the function $\Psi = x_4 + C$. C is chosen large enough so that (B.2) is fulfilled on the support of q . On $\Omega \setminus \omega$, Ψ is strictly convex thanks to Lemma B.7 and $\nabla\Psi \neq 0$. Therefore, assumptions (B.1) and (B.4) are fulfilled. As the support of q is compact in Ω , Proposition B.4 applies and we get $q = 0$ on D . Since δ is arbitrary, we get $q = 0$ on $S^3 \cap \{x_4 < 0\}$. The symmetry of the problem gives $q = 0$ on S^3 .

B.4.3. $M = S^2 \times S^1$.

Let $\omega_1 \subset S^2$ be a neighborhood of the equator $\{x_3 = 0\}$ and $\varepsilon > 0$. We assume $q = 0$ on $(\omega_1 \times S^1) \cup (S^2 \times]-\varepsilon, \varepsilon[)$.

The geometric situation is quite similar to the case of \mathbb{T}^3 : this is a product of manifolds, and the weight function Ψ will be the sum of two pseudoconvex weights in each coordinate.

The current point x of S^2 will be denoted by its coordinates in \mathbb{R}^3 and the current point y of $S^1 = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ by its coordinates in \mathbb{R} . Then, we can define \tilde{q} on the open set $\Omega = \{x \in S^2 \mid x_3 < 0\} \times \mathbb{R}$ by $\tilde{q}(x, y) = q(x, y)$ if $y \in [0, 1]$ and 0 otherwise. \tilde{q} is then compactly supported and is the solution of the same Schrödinger equation.

We choose $\Psi(x, y) = x_3 + y^2 + C$ with C large enough. Ψ is positive definite everywhere and nonsingular everywhere outside of any $\omega = \{(x, y) \in S^2 \times \mathbb{R} \mid x_3 \in$

$[-1, -1 + \delta[$ and $y^2 < \delta\}$ for $\delta > 0$. Then, choosing

$$D = \{(x, y) \in S^2 \times \mathbb{R} \mid x_3 \in]-1 + 3\delta, 0[\text{ or } y^2 > 3\delta\}$$

and applying Proposition B.4 we get $\tilde{q} = 0$ on D . Therefore, $q = 0$ on $S^2 \times S^1$.

Acknowledgments. The author deeply thanks his adviser Patrick Gérard for attracting his attention to this problem and for helpful discussions and encouragement. He also thanks Belhassen Dehman for some explanations about his article [17] and David Dos Santos Ferreira for enlightenment about unique continuation in low regularity.

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