Logarithmic decay for linear damped hypoelliptic wave and Schrödinger equations

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Abstract

We consider linear damped wave (resp. Schrödinger and plate) equations driven by a hypoelliptic "sum of squares" operator $\mathcal{L}$ on a compact manifold $\mathcal{M}$ and a damping function $b(x)$. We assume the Chow-Rashevski-Hörmander condition at rank k (at most k Lie brackets are needed to span the tangent space) together with analyticity of $\mathcal{M}$ and the coefficients of $\mathcal{L}$. We prove that the energy decays at rate $\log(t)^{-\frac{1}{k}}$ (resp. $\log(t)^{-\frac{2}{k}}$) for data in the domain of the generator of the associated group. We show that this decay is optimal on a family of Baouendi–Grushin-type operators. This result follows from a perturbative argument (of independent interest) showing, in a general abstract setting, that quantitative approximate observability/controllability results for wave-type equations imply *a priori* decay rates for associated damped wave, Schrödinger and plate equations. The adapted quantitative approximate observability/controllability theorem for hypoelliptic waves is obtained by the authors in [LL19, LL17].

Keywords

Stability estimates, hypoelliptic operators, wave equation, resolvent estimates, approximate observability.

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1 Introduction and statements

1.1 Damped hypoelliptic evolution equations

We consider a smooth compact connected d-dimensional manifold $\mathcal{M}$, endowed with a smooth positive density measure ds. We denote by $L^2 = L^2(\mathcal{M}) = L^2(\mathcal{M}, ds; \mathbb{C})$ the space of complex-valued square integrable functions with respect to this measure. Given a smooth vector field X, we define by X^* its formal adjoint in $L^2(\mathcal{M})$, that is,

$$\int_{\mathcal{M}} X^*(u)(x)\overline{v(x)}ds(x) = \int_{\mathcal{M}} u(x)\overline{X(v)(x)}ds(x), \quad \text{for any } u, v \in C^\infty(\mathcal{M}).$$

Given $m \in \mathbb{N}$ and m smooth real vector fields $X_1, \cdots, X_m$, we consider the (Hörmander type I) hypoelliptic operator (also called sub-Riemannian Laplacian, see e.g. [LL17, Remark 1.30])

$$\mathcal{L} = \sum_{i=1}^{m} X_i^* X_i. \tag{1.1}$$

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Note that $\mathcal{L}$ is symmetric and nonnegative since $(\mathcal{L}u, v)_{L^2(\mathcal{M})} = \sum_{i=1}^m (X_i u, X_i v)_{L^2(\mathcal{M})}$ for all $u, v \in C^\infty(\mathcal{M})$. Given a nonnegative (so-called damping) function $b \in L^\infty(\mathcal{M}; \mathbb{R}_+)$, we are interested in the first place in asymptotic properties of the linear damped wave equation associated to $(\mathcal{L}, b)$, namely

$$\begin{cases} (\partial_t^2 + \mathcal{L} + b\partial_t)u = 0, & \text{on } (0, +\infty) \times \mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}. \end{cases} \tag{1.2}$$

Solutions of (1.2) enjoy formally the following dissipation identity (obtained by taking the inner product of (1.2) with $\partial_t u$ and integrating on $(0, T)$):

$$E(u(T)) - E(u(0)) = -\int_0^T \int_{\mathcal{M}} b(x)|\partial_t u(t,x)|^2 ds(x)\, dt, \quad E(u) = \frac{1}{2}\left(\sum_{i=1}^m \|X_i u\|_{L^2(\mathcal{M})}^2 + \|\partial_t u\|_{L^2(\mathcal{M})}^2\right).$$

We are also interested in the linear damped Schrödinger equation associated to $(\mathcal{L}, b)$

$$\begin{cases} (i\partial_t + \mathcal{L} + ib)u = 0, & \text{on } (0, +\infty) \times \mathcal{M}, \\ u|_{t=0} = u_0, & \text{on } \mathcal{M}, \end{cases} \tag{1.3}$$

for which the L^2 norm is a dissipated quantity (obtained by taking imaginary part of the inner product of (1.3) with u and integrating on $(0, T)$):

$$\frac{1}{2}\|u(T)\|_{L^2(\mathcal{M})}^2 - \frac{1}{2}\|u_0\|_{L^2(\mathcal{M})}^2 = -\int_0^T \int_{\mathcal{M}} b(x)|u(t,x)|^2 ds(x)\, dt.$$

Hence, in both situations, an "energy" decays, and an interesting question is to understand if it converges to zero, and if so, at which rate.

We shall always assume throughout the paper that the family (X_i) satisfies the Chow-Rashevski-Hörmander condition (or is "bracket generating").

For a family $\mathcal{F}$ of smooth vector fields on $\mathcal{M}$ and $\ell \in \mathbb{N}^*$, we define $\text{Lie}^\ell(\mathcal{F})$, the Lie algebra at rank ℓ of the vector fields as:

- $\text{Lie}^1(\mathcal{F}) = \text{span}(\mathcal{F})$;
- $\text{Lie}^{\ell+1}(\mathcal{F}) = \text{span}\left(\text{Lie}^\ell(\mathcal{F}) \cup \left\{[X, Y]; X \in \mathcal{F}, Y \in \text{Lie}^\ell(\mathcal{F})\right\}\right)$.

Assumption 1.1. There exists $\ell \geq 1$ so that for any $x \in \mathcal{M}$, $\text{Lie}^\ell(X_1, \cdots, X_m)(x) = T_x\mathcal{M}$. Denote then by $k \in \mathbb{N}^*$ the minimal ℓ for which this holds.

The integer k is sometimes referred to as the *hypoellipticity index* of $\mathcal{L}$. In our notation, $\text{Lie}^1(X_1, \cdots, X_m)(x) = \text{span}(X_1, \cdots, X_m)(x)$. Hence, elliptic operators correspond to (1.1) with $k = 1$, Baouendi–Grushin and Heisenberg operators correspond to (1.1) with $k = 2$. We refer e.g. to [LL17, Section 1.1] for other detailed examples.

Under Assumption 1.1, the celebrated Hörmander [Hör67] and Rothschild-Stein [RS76] theorems (see also [BCN82] for a simpler proof) state that $\mathcal{L}$ is subelliptic of order $\frac{1}{k}$, that is: there is $C > 0$ such that for any $u \in C^\infty(\mathcal{M})$, we have

$$\|u\|_{H^{\frac{2}{k}}(\mathcal{M})}^2 \leq C\|\mathcal{L}u\|_{L^2(\mathcal{M})}^2 + C\|u\|_{L^2(\mathcal{M})}^2. \tag{1.4}$$

As a consequence, the operator $\mathcal{L}$ is selfadjoint on $L^2(\mathcal{M})$ with domain $\mathcal{L} : D(\mathcal{L}) \subset L^2(\mathcal{M}) \to L^2(\mathcal{M})$. Since $H^2(\mathcal{M}) \subset D(\mathcal{L}) \subset H^{\frac{2}{k}}(\mathcal{M})$, $\mathcal{L}$ has compact resolvent and thus admits a Hilbert basis of eigenfunctions $(\varphi_j)_{j\in\mathbb{N}}$, associated with the real eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$, sorted increasingly, that is

$$\mathcal{L}\varphi_i = \lambda_i\varphi_i, \qquad (\varphi_i, \varphi_j)_{L^2(\mathcal{M})} = \delta_{ij}, \qquad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty. \tag{1.5}$$

This allows in particular to define adapted Sobolev spaces:

$$\mathcal{H}_{\mathcal{L}}^s = \{u \in \mathcal{D}'(\mathcal{M}),\ (1+\mathcal{L})^{\frac{s}{2}}u \in L^2(\mathcal{M})\}, \quad \|u\|_{\mathcal{H}_{\mathcal{L}}^s} = \left\|(1+\mathcal{L})^{\frac{s}{2}}u\right\|_{L^2(\mathcal{M})}, \quad s \in \mathbb{R},$$

where $f(\mathcal{L})u = \sum_{j\in\mathbb{N}} f(\lambda_j)(u, \varphi_j)_{L^2(\mathcal{M})}\varphi_j$.

In addition to Assumption 1.1, we shall also make the following analyticity assumption.

Assumption 1.2. The manifold $\mathcal{M}$, the density ds, and the vector fields X_i are real-analytic.

A non-exhaustive list of classical examples of operators $\mathcal{L}$ encompassed by this framework is provided in [LL17, Section 1.1]. Note that the damping function b does not need to be analytic but only L^∞; in particular our results work for $b = \mathbb{1}_\omega$ if ω is a non-empty open subset of $\mathcal{M}$.

Motivations for studying propagation and unique continuation properties for hypoelliptic operators arise in different physical situations. For instance, wave-type or Helmholtz-type equations involving a hypoelliptic operator of the form (1.1) appear in the modeling of metamaterials, which are characterized by the fact that some eigenvalues of the material parameter tensor may vanish at places. The modeling of such materials is described for instance in [GKK+18] in connection with sub-Riemannian optics (and with applications to antenna design and energy harvesting). We refer to this article for other related interesting applications to ideal and approximate sub-Riemannian optics designs. Subelliptic operators of the form (1.1) also naturally appear in several other physical contexts; we refer to [Bra14, Chapter 2] for a presentation of some of them.

On the space $\mathcal{H}^1_{\mathcal{L}} \times L^2$, the operator $\mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathcal{L} & -b(x) \end{pmatrix}$ with $D(\mathcal{A}) = \mathcal{H}^2_{\mathcal{L}} \times \mathcal{H}^1_{\mathcal{L}}$ generates a bounded semigroup (from the Hille-Yosida theorem) and (1.2) admits a unique solution $u \in C^0(\mathbb{R}^+; \mathcal{H}^1_{\mathcal{L}}) \cap C^1(\mathbb{R}^+; L^2)$. Our main results for damped hypoelliptic waves are summarized in the following two theorems.

Theorem 1.1 (Decay rates for damped hypoelliptic waves)**.** *Assume that $b \in L^\infty(\mathcal{M})$ is such that $b \geq \delta > 0$ a.e. on a nonempty open set, together with Assumptions 1.1 and 1.2. Then, for all $(u_0, u_1) \in \mathcal{H}^1_{\mathcal{L}} \times L^2$, the associated solution to (1.2) satisfies $E(u(t)) \to 0$. Moreover, for all $j \in \mathbb{N}^*$, there exists $C_j > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A}^j)$, the associated solution to (1.2) satisfies*

$$E(u(t))^{\frac{1}{2}} \leq \frac{C_j}{\log(t+2)^{j/k}} \left\| \mathcal{A}^j(u_0, u_1) \right\|_{\mathcal{H}^1_{\mathcal{L}} \times L^2}, \quad \text{for all } t \geq 0. \tag{1.6}$$

Theorem 1.1 is actually a consequence of the following result, together with [BD08].

Theorem 1.2 (Spectral properties for damped hypoelliptic waves)**.** *Assume that $b \geq \delta > 0$ a.e. on a nonempty open set, together with Assumptions 1.1 and 1.2. Then, the spectrum of $\mathcal{A}$ contains only isolated eigenvalues with finite multiplicity, and satisfies:*

1. $\overline{\mathrm{Sp}(\mathcal{A})} = \mathrm{Sp}(\mathcal{A})$ *and* $\ker(\mathcal{A}) = \mathrm{span}\{(1, 0)\}$ *(where 1 denotes the constant function),*
2. $\mathrm{Sp}(\mathcal{A}) \subset \left(\left(-\frac{1}{2}\|b\|_{L^\infty(\mathcal{M})}, 0 \right) + i\mathbb{R} \right) \cup \left([-\|b\|_{L^\infty(\mathcal{M})}, 0] + 0i \right)$,
3. *there exist $C, \nu > 0$ such that $\left\| (is - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H}^1_{\mathcal{L}} \times L^2)} \leq Ce^{\nu |s|^k}$ for all $|s| \geq 1$,*
4. *there exist $\varepsilon, \nu > 0$ such that $\mathrm{Sp}(\mathcal{A}) \cap \Gamma_k(\varepsilon, \nu) = \{0\}$, where $\Gamma_k(\varepsilon, \nu) = \{z \in \mathbb{C}, \mathrm{Re}(z) \geq -\varepsilon e^{-\nu |\mathrm{Im}(z)|^k}\}$.*

The first two points are rather standard, see [Leb96]. Item 3 is the key information in the Theorem, and is a consequence of the main theorem in [LL17, Theorem 1.15]. The last point of the theorem states an exponentially small spectral gap, and is a consequence of Item 3.

Combined together, Theorems 1.1 and 1.2 are the counterparts to [Leb96, Théorème 1] in the case of the usual wave equation ($k = 1$, in which case no analyticity is required, and boundary conditions can be dealt with).

Note that the fact that $\mathrm{Sp}(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ in Item 2 (which, in turn, implies that $E(u(t)) \to 0$ in Theorem (1.1) for all solutions to (1.2)) is actually a consequence of the *qualitative* uniqueness:

$$\left(\varphi \in \mathcal{H}^2_{\mathcal{L}}, \quad z \in \mathbb{C}, \quad \mathcal{L}\varphi = z\varphi \text{ on } \mathcal{M}, \quad \varphi = 0 \text{ on } \omega \right) \implies \varphi \equiv 0 \text{ on } \mathcal{M}, \tag{1.7}$$

proved by Bony [Bon69], as a consequence of the Holmgren-John theorem. Even this weaker property is not well understood for general hypoelliptic operators if we drop Assumption 1.2, see [Bah86]. Here the key point is the quantification of the Holmgren-John theorem proved in [LL19, LL17] (see also [LL20b] for a survey).

We present analogue results in the case of the damped hypoelliptic Schrödinger equation. We set $\mathcal{A}_S := i\mathcal{L} - b$ with $D(\mathcal{A}_S) = D(\mathcal{L})$, so that (1.3) reformulates as $(\partial_t - \mathcal{A}_S)u = 0$. Note that $\mathcal{A}_S$ generates a contraction semigroup (from the Hille-Yosida theorem) and (1.3) admits a unique solution $u \in C^0(\mathbb{R}^+; L^2(\mathcal{M}))$. Our main results for the damped hypoelliptic Schrödinger equation are summarized in the following two theorems.

Theorem 1.3 (Decay rates for the damped hypoelliptic Schrödinger equation)**.** *Assume that $b \in L^\infty(\mathcal{M})$ is such that $b \geq \delta > 0$ a.e. on a nonempty open set, together with Assumptions 1.1 and 1.2. Then, for all $u_0 \in L^2(\mathcal{M})$, the associated solution to (1.3) satisfies $u(t) \to 0$ in $L^2(\mathcal{M})$. Moreover, for all $j \in \mathbb{N}^*$, there exists $C_j > 0$ such that for all $u_0 \in D(\mathcal{A}_S^j)$, the associated solution to (1.3) satisfies*

$$\|u(t)\|_{L^2(\mathcal{M})} \leq \frac{C_j}{\log(t+2)^{2j/k}} \left\|\mathcal{A}_S^j u_0\right\|_{L^2(\mathcal{M})}, \quad \text{for all } t \geq 0. \tag{1.8}$$

Note that when comparing (1.8) to (1.6), the decay rate looks better ($\log(t+2)^{-2j/k}$ instead of $\log(t+2)^{-j/k}$) but actually consumes more derivatives: for a smooth damping function b, $\left\|\mathcal{A}_S^j u_0\right\|_{L^2(\mathcal{M})} \simeq \|u_0\|_{\mathcal{H}_{\mathcal{L}}^{2j}}$ whereas $\left\|\mathcal{A}^j U_0\right\|_{L^2(\mathcal{M})} \simeq \|U_0\|_{\mathcal{H}_{\mathcal{L}}^{j} \times \mathcal{H}_{\mathcal{L}}^{j-1}}$. Hence both decay rates essentially coincide for data having the same regularity. Theorem 1.3 is a consequence of the following result, together with [BD08].

Theorem 1.4 (Spectral properties for the damped hypoelliptic Schrödinger equation)**.** *Assume that $b \geq \delta > 0$ a.e. on a nonempty open set, together with Assumptions 1.1 and 1.2. Then, the spectrum of $\mathcal{A}_S$ contains only isolated eigenvalues with finite multiplicity, and satisfies:*

1. $\mathrm{Sp}(\mathcal{A}_S) \subset \big[-\|b\|_{L^\infty(\mathcal{M})}, 0\big) + i[0, +\infty)$,
2. *there exist $C, \nu > 0$ such that $\left\|(is - \mathcal{A}_S)^{-1}\right\|_{\mathcal{L}(L^2)} \leq Ce^{\nu|s|^{k/2}}$ for all $s \in \mathbb{R}$,*
3. *there exist $\varepsilon, \nu > 0$ such that $\mathrm{Sp}(\mathcal{A}_S) \cap \Gamma_{k,S}(\varepsilon, \nu) = \emptyset$, where $\Gamma_{k,S}(\varepsilon, \nu) = \{z \in \mathbb{C}, \mathrm{Re}(z) \geq -\varepsilon e^{-\nu|\operatorname{Im}(z)|^{k/2}}\}$.*

Note that in the elliptic case $k = 1$, the results of Theorems 1.3, 1.4 are more or less classical, even though we did not see them written explicitely in the literature. In this situation, analyticity is not necessary and boundary value problems can be dealt with. Our abstract perturbative proof below works as well, as a consequence of [LL19] (with Dirichlet boundary conditions). One can however start from the seminal Lebeau-Robbiano estimates in this situation, see [LR95, Leb96] (see also [LRL12] for a survey) for Dirichlet conditions and [LR97] for Neumann boundary conditions.

A similar result holds for the damped plate equation associated to $(\mathcal{L}, b)$

$$\begin{cases} (\partial_t^2 + \mathcal{L}^2 + b\partial_t)u = 0, & \text{on } (0, +\infty) \times \mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}. \end{cases} \tag{1.9}$$

Solutions of (1.9) also enjoy formally a similar dissipation identity

$$E_P(u(T)) - E_P(u(0)) = -\int_0^T \int_{\mathcal{M}} b(x)|\partial_t u(t,x)|^2 ds(x)\, dt, \quad E_P(u) = \frac{1}{2}\left(\|\mathcal{L}u\|_{L^2(\mathcal{M})}^2 + \|\partial_t u\|_{L^2(\mathcal{M})}^2\right).$$

The framework is quite similar to that of the wave equation. We work on the space $\mathcal{H}_{\mathcal{L}}^2 \times L^2$ with the operator $\mathcal{A}_P = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathcal{L}^2 & -b(x) \end{pmatrix}$ with $D(\mathcal{A}_P) = \mathcal{H}_{\mathcal{L}}^4 \times \mathcal{H}_{\mathcal{L}}^2$. It generates a bounded semigroup and (1.9) admits a unique solution $u \in C^0(\mathbb{R}^+; \mathcal{H}_{\mathcal{L}}^2) \cap C^1(\mathbb{R}^+; L^2)$.

Theorem 1.5 (Decay rates for damped hypoelliptic plates)**.** *Assume that $b \in L^\infty(\mathcal{M})$ is such that $b \geq \delta > 0$ a.e. on a nonempty open set, together with Assumptions 1.1 and 1.2. Then, for all $(u_0, u_1) \in \mathcal{H}_{\mathcal{L}}^2 \times L^2$, the associated solution to (1.9) satisfies $E_P(u(t)) \to 0$. Moreover, for all $j \in \mathbb{N}^*$, there exists $C_j > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A}_P^j)$, the associated solution to (1.9) satisfies*

$$E_P(u(t))^{\frac{1}{2}} \leq \frac{C_j}{\log(t+2)^{2j/k}} \left\|\mathcal{A}_P^j (u_0, u_1)\right\|_{\mathcal{H}_{\mathcal{L}}^2 \times L^2}, \quad \text{for all } t \geq 0. \tag{1.10}$$

Similar spectral statements as Theorems 1.2 and 1.4 hold for the plate equation. We leave the details to the reader. Again, using the result of [LL19], we could also obtain a logarithmic decay in the elliptic case $k = 1$ for a compact manifold with boundary and with Dirichlet boundary conditions. We do not know if this result is new in this context. There is an important literature on the subject, and we refer to [Leb92] and [Kom92] for exact control results (implying exponential decay of the damped equation) and e.g. to [ADZ14] for a spectral analysis of the decay rate.

Finally, we show that the results of Theorems 1.1, 1.2, 1.3, 1.4 are optimal in general (in case $k > 1$; this is already known in the elliptic case $k = 1$, see [Leb96, LR97]). This is also the case for Theorem 1.5 (and the associated spectral statement); we do not state the result for the sake of brevity.

Proposition 1.6. *Consider the manifold with boundary $\mathcal{M} = [-1, 1] \times (\mathbb{R}/\mathbb{Z})$, endowed with the Lebesgue measure dx, and for $k \in (1, +\infty)$, define the operator $\mathcal{L} = -\left(\partial_{x_1}^2 + x_1^{2(k-1)}\partial_{x_2}^2\right)$, with Dirichlet conditions on $\partial\mathcal{M}$. Assume that $\operatorname{supp}(b) \cap \{x_1 = 0\} = \emptyset$. Then, there exist $C, \nu > 0$ and a sequence $(s_j)_{j\in\mathbb{N}}$ with $s_j \to +\infty$ such that*

$$\left\|(is_j - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H}^1_{\mathcal{L}}\times L^2)} \geq Ce^{\nu s_j^k}, \quad \left\|(is_j - \mathcal{A}_S)^{-1}\right\|_{\mathcal{L}(\mathcal{H}^1_{\mathcal{L}}\times L^2)} \geq Ce^{\nu s_j^{k/2}}, \quad \text{for all } j \in \mathbb{N}. \tag{1.11}$$

Moreover, if for all $(u_0, u_1) \in D(\mathcal{A})$, the associated solution to (1.2) satisfies

$$E(u(t))^{\frac{1}{2}} \leq f(t) \left\|\mathcal{A}(u_0, u_1)\right\|_{\mathcal{H}^1_{\mathcal{L}}\times L^2}, \quad \text{for all } t \geq 2,$$

then there is $C > 0$ such that $f(t) \geq \frac{C}{\log(t)^{1/k}}$. Similarly, if for all $u_0 \in \mathcal{H}^1_{\mathcal{L}}$, the associated solution to (1.3) satisfies

$$\|u(t)\|_{L^2(\mathcal{M})} \leq f(t) \left\|\mathcal{A}_S u\right\|_{L^2(\mathcal{M})}, \quad \text{for all } t \geq 2,$$

then there is $C > 0$ such that $f(t) \geq \frac{C}{\log(t)^{2/k}}$.

Recall that for $k \in \mathbb{N}^*$, the operator $\mathcal{L} = -\left(\partial_{x_1}^2 + x_1^{2(k-1)}\partial_{x_2}^2\right)$ satisfies precisely Assumption 1.1. The first statement of the proposition is a consequence of [BCG14, Section 2.3] as reformulated in [LL17, Proposition 1.14]. It proves the optimality in general of Item 3 in Theorem 1.2. The second part of the statement is a corollary of the first one, together with [BD08], and proves optimality of (1.6) and (1.8).

Let us finally mention related known decay results for damped evolution equations driven by a hypoelliptic operator.

First, a reformulation of the result of [Let20] (e.g. combined with [Har89]) in the present context states that if

$$\operatorname{span}(X_1(x), \cdots, X_m(x)) \neq T_x\mathcal{M}$$

for x in a dense subset of $\mathcal{M}$, and $\mathcal{M} \setminus \operatorname{supp}(b) \neq \emptyset$, then uniform decay does not hold: there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(t) \to 0$ such that $E(u(t)) \leq f(t)E(u(0))$. This contrasts with the Riemannian case [RT74, BLR92], and gives in this context a stronger interest to the result of Theorem 1.1 as compared to the Riemannian counterpart. In a genuine sub-Rimannian/hypoelliptic setting, unifrom decay never holds, and the best we can hope for is semi-uniform decay in the sense of [Leb96, BD08], which is precisely what we prove.

Second, one may however notice that logarithmic decay as in Theorem 1.1 is not always optimal. Combining for instance [BS19, Theorem 1] together with [AL14, Theorem 2.3] implies that $\frac{C_j}{\log(t+2)^{j/k}}$ in (1.6) can be replaced by $\frac{C_j}{t^{j/2}}$ (and this is probably not optimal) in the geometric setting of Proposition 1.6 if $b(x_1, x_2) = \mathbb{1}_{(a,b)}(x_2)$, for any $a < b$.

Similarly, logarithmic decay in Theorem 1.3 is not always optimal. For instance [BS19, Theorem 1] (together with classical equivalence between observability for the conservative system and uniform stabilization for the damped system) implies that in the geometric setting of Proposition 1.6 if $b(x_1, x_2) = \mathbb{1}_{(a,b)}(x_2)$ for $a < b$, then uniform decay holds, that is: there are $C, \gamma > 0$ such that $\|u(t)\|_{L^2} \leq Ce^{-\gamma t}\|u_0\|_{L^2}$ for all solutions to (1.3).

Let us finally remark that all proofs below rely on the approximate observability/controllability of the hypoelliptic wave equation with optimal cost. The latter result is proved by the authors in [LL17].

It is interesting to notice that in the elliptic case ($k = 1$ in the discussion above), the approximate observability/controllability of the wave equation (proved in [LL19]) with optimal (exponential) cost allows to recover many known control results obtained with Carleman estimates. In particular, it implies

1. null-controllability of the heat equation with optimal short-time behavior, as proved in [EZ11] and [LL18, Proposition 1.7] (the original result is [LR95, FI96]),
2. approximate observability/controllability of the heat equation with optimal (exponential) cost [LL17, Chapter 4] (the original result is [FCZ00]),
3. optimal logarithmic decay for the damped wave equation, see Theorem 1.1 for $k = 1$ (the original result is [Leb96, LR97]).

Here, we provide a proof of the last point in a general framework presented in Section 1.2 below, and deduce counterparts for hypoelliptic equations using [LL17].

Remark 1.7. All equations considered in this paper are linear. It would be very interesting to extend our results to a nonlinear context. The literature on the nonlinear damped wave equation for the usual Laplacian is huge and we refer e.g. to the recent [JL20] for a survey. In the process of proving a stabilization result for nonlinear hypoelliptic equations, there are however several important obstacles, especially for large data solutions. Most of the results for the usual wave equation rely on very strong geometric assumptions on the damping zone (like multiplier conditions or at least the Geometric Control Condition of [BLR92]). To the authors' knowledge, even in that classical setting, without any further assumption on the damping region, the decay to zero of solutions to nonlinear damped wave equations is an open problem. The article [JL20] deals with related problems for semilinear waves, but in geometric situations in which the decay rate of the *linear* damped wave equation is strong enough and in particular, integrable in time. Unfortunately, the decay rates we obtain in the present paper (without *any* geometric assumption) is of the form $\frac{1}{\log(2+t)^\alpha}$, and hence far from being integrable. Therefore, it does not fit in the abstract framework of [JL20].

1.2 From approximate control to damped waves : abstract setting

As already mentioned, we prove all above results in an abstract operator setting. This allows us to stress links between the cost of approximate controls and a priori decay rates for damped waves. This follows the spirit of e.g. [Har89, BZ04, Phu01, Mil05, Mil06, TW09, EZ11, AL14, CPS+19], exploring the links between different equations and their control properties (i.e. observability, controllability, stabilization...). Here, we follow closely [AL14].

Let H and Y be two Hilbert spaces (resp. the state space and the observation/control space) with norms $\|\cdot\|_H$ and $\|\cdot\|_Y$, and associated inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Y$. We denote by $A : D(A) \subset H \to H$ a *nonnegative* selfadjoint operator with compact resolvent, and $B \in \mathcal{L}(Y;H)$ a bounded control operator. We recall that $B^* \in \mathcal{L}(H;Y)$ is defined by $(B^*h, y)_Y = (h, By)_H$ for all $h \in H$ and $y \in Y$. We define $H_1 = D(A^{\frac{1}{2}})$, equipped with the graph norm $\|u\|_{H_1} := \|(A + \mathrm{Id})^{\frac{1}{2}} u\|_H$, and its dual $H_{-1} = (H_1)'$ (using H as a pivot space) endowed with the norm $\|u\|_{H_{-1}} := \|(A + \mathrm{Id})^{-\frac{1}{2}} u\|_H$.

In applications to Theorems 1.1-1.2-1.3-1.4, we take $H = Y = L^2(\mathcal{M})$, $A = \mathcal{L}$ and $B = B^*$ is multiplication by the function $\sqrt{b}$.

We introduce in this abstract setting the wave equation

$$\begin{cases} \partial_t^2 u + Au = F, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.12}$$

the damped wave equation

$$\begin{cases} \partial_t^2 u + Au + BB^*\partial_t u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{1.13}$$

and the damped Schrödinger equation

$$\begin{cases} i\partial_t u + Au + iBB^* u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.14}$$

Definition 1.8. Given $T > 0$ and a function $G : \mathbb{R}_+ \to \mathbb{R}_+$, we say that the wave equation (1.12) with $F = 0$ is approximately observable from B^* in time T with cost G if there is $\mu_0 > 0$ such that for all $(u_0, u_1) \in H_1 \times H$, the associated solution u to (1.12) with $F = 0$ satisfies

$$\|(u_0, u_1)\|_{H \times H_{-1}} \leq G(\mu) \|B^* u\|_{L^2(0,T;Y)} + \frac{1}{\mu} \|(u_0, u_1)\|_{H_1 \times H}, \quad \text{for all } \mu \geq \mu_0. \tag{1.15}$$

According to [Rob95] or [LL20a, Appendix], this is equivalent to approximate controllability (ε close) with cost $G(1/\varepsilon)$. This is satisfied for the usual wave equation in a general context with $B^* = \mathbb{1}_\omega$, $G(\mu) = Ce^{\nu\mu}$, for all $T > 2\sup_{x \in \mathcal{M}} d_g(x, \omega)$ (where d_g is the Riemannian distance), as proved in [LL19]. For the hypoelliptic wave equation, we proved in [LL17, Theorem 1.15] that this is satisfied for $B^* = \mathbb{1}_\omega$, $G(\mu) = Ce^{\nu\mu^k}$, for all $T > 2\sup_{x \in \mathcal{M}} d_{\mathcal{L}}(x, \omega)$ (where $d_{\mathcal{L}}$ is the appropriate sub-Riemannian, see [LL17, Equation (1.11)] distance and k the hypoellipticity index of $\mathcal{L}$).

Our main results can be divided in several steps. Firstly we have

Proposition 1.9. *Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $G(\mu) \geq \frac{c_0}{\mu} > 0$ for $\mu \geq \mu_0$. Assume that there is $T > 0$ such that the wave equation (1.12) with $F = 0$ is approximately observable from B^* in time T with cost G in the sense of Definition 1.8. Then, we have*

$$\big(\lambda \in \mathbb{C},\ v \in D(A),\ Av = \lambda^2 v,\ B^* v = 0\big) \implies v = 0, \tag{1.16}$$

and there is $\lambda_0 > 0$ such that for all $\alpha > 0$,

$$\|v\|_H \leq \frac{K}{\alpha}(\lambda + \sqrt{2} + \alpha) G(\lambda + \sqrt{2} + \alpha)\big(\|B^* v\|_Y + C \left\|(A - \lambda^2)v\right\|_H \big), \quad \text{for all } v \in D(A), \lambda \geq \lambda_0. \tag{1.17}$$

with $K = \sqrt{T} + c_0^{-1}$ and $C > 0$ a constant depending only on B and T.

Note that in this statement, $\sqrt{2}$ can be replaced by 1 at the cost of a slightly longer proof, and λ_0 is the μ_0 given in the definition of approximate observability. In most applications we have in mind, $G(\mu) \approx e^{\nu\mu^k}$ and the estimate is better for smaller values of α. In a situations in which one would have $G(\mu) \approx \mu^\gamma$, then a better choice of α would be $\alpha \approx \lambda$, so that (1.17) remains a bound of order $G(\lambda)$. Note also that since A is a nonnegative selfadjoint operator with compact resolvent, (1.16) is only interesting for $\lambda^2 \in \mathbb{R}^+$ (but this information is not useful in the proof).

Secondly, we assume that for some function G and some $\lambda_0 > 0$ we have

$$\|v\|_H \leq \mathsf{G}(\lambda)\big(\|B^* v\|_Y + \left\|(A - \lambda^2)v\right\|_H \big), \quad \text{for all } v \in D(A), \lambda \geq \lambda_0. \tag{1.18}$$

This is precisely (1.17) with $\mathsf{G}(\lambda) = \frac{K(1+C)}{\alpha}(\lambda + \sqrt{2} + \alpha)G(\lambda + \sqrt{2} + \alpha)$. From Estimate (1.18), we deduce the sought spectral properties for the damped operators (resolvent estimates and localization of the spectrum linked to the function G). See Section 2.3 for the damped Schrödinger equation and Section 2.4 for the damped wave equation. Note that a direct application of Proposition 1.9 gives in the context of hypoelliptic operators.

Corollary 1.10. *With the notations of Section 1.1, assume that $b \in L^\infty(\mathcal{M})$ is such that $b \geq \delta > 0$ a.e. on a nonempty open set, together with Assumptions 1.1 and 1.2. Then, (1.7) is satisfied and there is $\nu > 0$, $C > 0$ and $\lambda_0 > 0$ such that,*

$$\|v\|_{L^2(\mathcal{M})} \leq Ce^{\nu\lambda^k}\big(\|bv\|_{L^2(\mathcal{M})} + \left\|(\mathcal{L} - \lambda^2)v\right\|_{L^2(\mathcal{M})} \big), \quad \text{for all } v \in \mathcal{H}^2_{\mathcal{L}}, \lambda \geq \lambda_0.$$

This corollary states a stronger version of the Eigenfunction tunneling estimates of [LL17, Theorem 1.12] (which is the same statement for solutions to $(\mathcal{L} - \lambda^2)v = 0$). Note that the constant ν is (essentially) the same as in the cost of approximate controls in [LL17, Theorem 1.15].

Thirdly, we deduce from the spectral properties the sought decay estimates (respectively in Sections 2.3 and 2.4 for the damped Schrödinger and wave equations) using the Batty-Duyckaerts theorem, which we now recall.

Theorem 1.11 (Batty and Duyckaerts [BD08]). *Let $(e^{t\mathcal{B}})_{t\geq 0}$ be a bounded $\mathcal{C}^0$-semigroup on a Banach space $\mathcal{X}$, generated by $\mathcal{B}$.*

Assume that $\left\|e^{t\mathcal{B}}(\mathrm{Id}+\mathcal{B})^{-1}\right\|_{\mathcal{L}(\mathcal{X})} \leq f(t)$ with $f \in C^0([0,+\infty))$ decreasing to 0. Then $i\mathbb{R}\cap \mathrm{Sp}(\mathcal{B}) = \emptyset$ and there are $C, \lambda_0 > 0$ such that

$$\left\|(i\lambda - \mathcal{B})^{-1}\right\|_{\mathcal{L}(\mathcal{X})} \leq 1 + Cf^{-1}\left(\frac{1}{2(|\lambda|+1)}\right), \qquad \text{for all } \lambda \in \mathbb{R}, |\lambda| \geq \lambda_0.$$

Conversely, suppose that $i\mathbb{R}\cap \mathrm{Sp}(\mathcal{B}) = \emptyset$ and

$$\left\|(is - \mathcal{B})^{-1}\right\|_{\mathcal{L}(\mathcal{X})} \leq \mathsf{M}(|s|), \quad s \in \mathbb{R}, \tag{1.19}$$

where $\mathsf{M} : \mathbb{R}_+ \to \mathbb{R}_+^*$ is a non-decreasing function on $\mathbb{R}_+$. Then, setting

$$\mathsf{M}_{\log}(s) = \mathsf{M}(s)\big(\log(1+\mathsf{M}(s)) + \log(1+s)\big), \tag{1.20}$$

for all $j \in \mathbb{N}^*$, there exists $C_j, T_j > 0$ such that,

$$\left\|e^{t\mathcal{B}}\mathcal{B}^{-j}\right\|_{\mathcal{L}(\mathcal{X})} \leq \frac{C_j}{\mathsf{M}_{\log}^{-1}\left(\frac{t}{C_j}\right)^j}, \qquad \text{for } t \geq T_j,$$

where $\mathsf{M}_{\log}^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the inverse of the strictly increasing function $\mathsf{M}_{\log}$.

We refer to [Duy15, CS16] for alternative proofs of the result of [BD08]. Note that on a Hilbert space (which is the case here) $\mathsf{M}_{\log}$ in the result can be replaced by M if it is polynomial at infinity, according to [BT10, Theorem 2.4] (see also [CPS+19] and the references therein for generalizations of [BT10]).

To conclude this introductory section, let us briefly describe the contents of the end of the article, namely Section 2. In Section 2.1, we explain in the abstract functional setting how approximate observability/controllability statements (Definition 1.8) imply "free-resolvent" estimates like (1.18) (proving in particular Proposition 1.9). Then, in Section 2.2, we deduce (still in the abstract functional framework) from these "free-resolvent" estimates a resolvent estimate for damped wave-type or Schrödinger-type operators. The proofs of abstract setting analogues of Theorems 1.4 and 1.3 (resp. Theorems 1.2 or 1.1) for the Schrödinger (resp. wave) equation are completed in Section 2.3 (resp. Section 2.4). Analogue statements and proofs for the damped plate-type equations are deduced in Section 2.5. Finally, the optimality statements of Proposition 1.6 in the case of particular hypoelliptic operators on the square are proved in Section 2.6.

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2 Proof of the results

2.1 From approximate observability of waves to a free resolvent estimate with an observation term: Proof of Proposition 1.9

From approximate observability, we deduce the following (seemingly more general) result, concerning Equation (1.12) with a general right hand-side F.

Proposition 2.1. *Let $T > 0$ and a function $G : \mathbb{R}_+ \to \mathbb{R}_+$. Assume that the wave equation (1.12) with $F = 0$ is approximately observable from B^* in time T with cost G, in the sense of Definition 1.8. Then, there are $\mu_0, C > 0$ such that for all $F \in L^2(0,T;H)$ and $(u_0,u_1) \in H_1 \times H$, the associated solution u to (1.12) satisfies*

$$\|(u_0,u_1)\|_{H\times H_{-1}} \leq G(\mu)\big(\|B^*u\|_{L^2(0,T;Y)} + C\,\|F\|_{L^2(0,T;H)}\big) + \frac{1}{\mu}\,\|(u_0,u_1)\|_{H_1\times H}\,, \quad \text{for all } \mu \geq \mu_0. \tag{2.1}$$

Note that the constant μ_0 is actually the same as in Definition 1.8 and that C depends only on T and $\|B^*\|_{\mathcal{L}(Y;H)}$.

Proof. According to the linearity of (1.12), we decompose u as $u = u^0 + u^F$ where u^0 is the solution to (1.12) for $F = 0$ and u^F is the solution to (1.12) with $(u_0, u_1) = (0, 0)$.

First, according to the assumption, Definition 1.8 applies to the function u^0, so that (1.15) reads:

$$\|(u_0, u_1)\|_{H\times H_{-1}} \le G(\mu) \left\|B^* u^0\right\|_{L^2(0,T;Y)} + \frac{1}{\mu} \|(u_0, u_1)\|_{H_1\times H}\,, \quad \text{for all } \mu \ge \mu_0. \tag{2.2}$$

Second, to estimate u^F, we perform classical energy inequalities for (1.12). We rewrite (1.12) as

$$(\partial_t^2 + A + \mathrm{Id})u^F = u^F + F, \quad (u^F, \partial_t u^F)|_{t=0} = (0, 0). \tag{2.3}$$

Taking the inner product of this equation with $\partial_t u^F$ (assuming at first that $F \in L^2_{\mathrm{loc}}(\mathbb{R}; H_1)$ and thus $u^F \in C^0(\mathbb{R}; D(A)) \cap C^1(\mathbb{R}; H_1) \cap C^2(\mathbb{R}; H)$) implies

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\partial_t u^F\right\|_H^2 + \left\|u^F\right\|_{H_1}^2\right) \le \left(\left\|u^F\right\|_H + \|F\|_H\right)\left\|\partial_t u^F\right\|_H.$$

Writing $\tilde{E}(t) = \frac{1}{2}\left(\left\|\partial_t u^F\right\|_H^2 + \left\|u^F\right\|_{H_1}^2\right)$, this yields $\tilde{E}'(t) \le 2\tilde{E}(t) + \|F\|_H^2$. The Gronwall lemma together with the vanishing initial data in (2.3) imply

$$\sup_{t\in[0,T]} \left\|u^F(t)\right\|_H^2 \le \sup_{t\in[0,T]} \tilde{E}(t) \le C_T \|F\|_{L^2(0,T;H)}^2.$$

As a consequence, boundedness of B^* yields

$$\left\|B^* u^F\right\|_{L^2(0,T;Y)} \le \|B^*\|_{\mathcal{L}(Y;H)} \left\|u^F\right\|_{L^2(0,T;H)} \le \|B^*\|_{\mathcal{L}(Y;H)} C_T \|F\|_{L^2(0,T;H)}.$$

Recalling that $u^0 = u - u^F$ and combining this estimate with (2.2) yields for all $\mu \ge \mu_0$

$$\begin{aligned}\|(u_0, u_1)\|_{H\times H_{-1}} &\le G(\mu)\left\|B^*(u - u^F)\right\|_{L^2(0,T;Y)} + \frac{1}{\mu}\|(u_0, u_1)\|_{H_1\times H} \\ &\le G(\mu)\left(\|B^* u\|_{L^2(0,T;Y)} + C_{B,T}\|F\|_{L^2(0,T;H)}\right) + \frac{1}{\mu}\|(u_0, u_1)\|_{H_1\times H},\end{aligned}$$

which concludes the proof of the proposition. ☐

From this result, we deduce a proof of Proposition 1.9 as a direct corollary.

Proof of Proposition 1.9. For $v \in D(A)$ and $\lambda \in \mathbb{C}$, we may apply the result of Proposition 2.1 to the function $u(t) = \cos(\lambda t)v$ which satisfies (1.12) with

$$u_0 = v, \quad u_1 = 0, \quad F(t) = \cos(\lambda t)(-\lambda^2 + A)v.$$

We first remark that the assumption of (1.16) implies $F = 0$ and $B^* u = 0$, and hence (2.1) reads $\|v\|_H \le \frac{1}{\mu}\|v\|_{H_1}$ for all $\mu \ge \mu_0$. Letting μ converges to $+\infty$ yields the conclusion of (1.16).

Let us now prove (1.17). Still for $u(t) = \cos(\lambda t)v$, we have

$$\|B^* u\|_{L^2(0,T;Y)}^2 \le T\|B^* v\|_Y^2, \quad \|F\|_{L^2(0,T;H)}^2 \le T\left\|(-\lambda^2 + A)v\right\|_H^2.$$

Estimate (2.1) thus implies for all $\lambda \ge 0$, $\mu \ge \mu_0$

$$\|v\|_H \le G(\mu)\sqrt{T}\big(\|B^* v\|_Y + C\left\|(A - \lambda^2)v\right\|_H\big) + \frac{1}{\mu}\|v\|_{H_1}. \tag{2.4}$$

We now remark that

$$(Av, v)_H - \lambda^2\|v\|_H^2 = \left((A - \lambda^2)v, v\right)_H \le \left\|(A - \lambda^2)v\right\|_H \|v\|_H.$$

Hence, we deduce

$$\|v\|_{H_1}^2 = ((A+1)v,v)_H \leq (\lambda^2+1)\|v\|_H^2 + \left\|(A-\lambda^2)v\right\|_H \|v\|_H$$
$$\leq (\lambda^2+2)\|v\|_H^2 + \left\|(A-\lambda^2)v\right\|_H^2.$$

Plugging this into (2.4) yields, for all $\mu \geq \mu_0$ and $\lambda \geq 0$,

$$\|v\|_H \leq G(\mu)\sqrt{T}\big(\|B^*v\|_Y + C\left\|(A-\lambda^2)v\right\|_H\big) + \frac{1}{\mu}\left(\left\|(A-\lambda^2)v\right\|_H + (\lambda+\sqrt{2})\|v\|_H\right).$$

We let $\alpha > 0$ and choose $\mu = \mu(\lambda) = \max\{\lambda + \sqrt{2} + \alpha, \mu_0\}$ so that to absorb the last term in the right handside, implying for all $\lambda \geq 0$,

$$\left(1 - \frac{\lambda+\sqrt{2}}{\lambda+\sqrt{2}+\alpha}\right)\|v\|_H \leq G(\mu(\lambda))\sqrt{T}\big(\|B^*v\|_Y + C\left\|(A-\lambda^2)v\right\|_H\big) + \frac{1}{\mu(\lambda)}\left\|(A-\lambda^2)v\right\|_H.$$

We then take $\lambda \geq \mu_0$ so that $\mu(\lambda) = \lambda+\sqrt{2}+\alpha \geq \mu_0$. This implies $\frac{1}{\mu(\lambda)}\left\|(A-\lambda^2)v\right\|_H \leq c_0^{-1}G(\mu(\lambda))\left\|(A-\lambda^2)v\right\|_H$ and thus, for $\lambda \geq \mu_0$,

$$\frac{\alpha}{\mu(\lambda)}\|v\|_H \leq G(\mu(\lambda))\sqrt{T}\big(\|B^*v\|_Y + C\left\|(A-\lambda^2)v\right\|_H\big) + c_0^{-1}G(\mu(\lambda))\left\|(A-\lambda^2)v\right\|_H.$$

This concludes the proof of the proposition. □

We finally give a proof of Corollary 1.10.

Proof of Corollary 1.10. By assumption, $b \geq \delta > 0$ on a non empty open set ω. Since $\mathcal{M}$ is compact, $\sup_{x\in\mathcal{M}} d_{\mathcal{L}}(x,\omega)$ is finite. For the hypoelliptic wave equation on $H = Y = L^2(\mathcal{M})$, we proved in [LL17, Theorem 1.15] that (1.15) is satisfied for $A = \mathcal{L}$, $B_\omega = B_\omega^* =$ multiplication by $\mathbb{1}_\omega$, $G(\mu) = Ce^{\nu\mu^k}$, for all $T > 2\sup_{x\in\mathcal{M}} d_{\mathcal{L}}(x,\omega)$ (where $d_{\mathcal{L}}$ is the appropriate sub-Riemannian distance and k the hypoellipticity index of $\mathcal{L}$). Since $\|\mathbb{1}_\omega u\|_{L^2(\mathcal{M})} \leq \delta^{-1}\|bu\|_{L^2(\mathcal{M})}$, the same inequality with different constants remains true with $B = B^* =$ multiplication by b. Thus, we deduce from Proposition 1.9 that (1.18) is satisfied (after having fixed $\alpha = 2-\sqrt{2}$) with $\mathsf{G}(\lambda) = K(1+C)(\lambda+2)G(\lambda+2) = C(\lambda+2)e^{\nu(\lambda+2)^k}$. □

2.2 From the free resolvent estimate with an observation term to damped resolvent estimates

In this section, we start from an estimate for A with an observation term like (1.17), and deduce associated estimates for damped operators.

For later use (see Sections 2.3 and 2.4 below), we introduce the operators:

$$Q_\lambda = -i(\mathcal{A}_S - i\lambda) = A - \lambda + iBB^*,$$
$$P_\lambda = P(i\lambda) = A - \lambda^2 + i\lambda BB^*,$$

both with domain $D(Q_\lambda) = D(P_\lambda) = D(A)$.

Proposition 2.2. *Let $G_1, G_2 \geq 0$, $\lambda > 0$, and $v \in D(A)$, and assume*

$$\|v\|_H \leq G_1\|B^*v\|_Y + G_2\left\|(A-\lambda^2)v\right\|_H. \tag{2.5}$$

Then we have

$$\|v\|_H \leq \left((G_1\lambda^{-\frac{1}{2}} + G_2\sqrt{2}\|B\|_{\mathcal{L}(Y;H)})^2 + 2\sqrt{2}G_2\right)\|P_\lambda v\|_H, \tag{2.6}$$

$$\|v\|_H \leq \left((G_1 + G_2\sqrt{2}\|B\|_{\mathcal{L}(Y;H)})^2 + 2\sqrt{2}G_2\right)\|Q_{\lambda^2}v\|_H. \tag{2.7}$$

In particular, given $\mathsf{G} : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mathsf{G}(\mu) \geq c_0 > 0$ on $\mathbb{R}_+$ and $\lambda_0 \geq 1$, if (1.18) is satisfied, then writing $K = (1+\sqrt{2}\|B\|_{\mathcal{L}(Y;H)})^2 + 2\sqrt{2}c_0^{-1}$, we have

$$\|v\|_H \leq K\mathsf{G}(|\lambda|)^2\|P_\lambda v\|_H, \quad \text{for all } v \in D(A), \lambda \in \mathbb{R}, |\lambda| \geq \lambda_0, \tag{2.8}$$

$$\|v\|_H \leq K\mathsf{G}(\sqrt{\lambda})^2\|Q_\lambda v\|_H, \quad \text{for all } v \in D(A), \lambda \geq \lambda_0^2. \tag{2.9}$$

Note that when passing from (1.17) to (2.8) and (2.9), we change G to G^2, which is a loss in general; this is linked to the fact that the proof of Proposition 2.2 consists only in a very rough estimate, treating the damping terms iBB^* and $i\lambda BB^*$ as remainders.

Proof of Proposition 2.2. We only prove the result for P_λ, the analogue proof for Q_λ is identical.

First, we remark that, under the above assumptions, we have

$$\lambda \|B^*v\|_Y^2 = \lambda (BB^*v, v)_H = \operatorname{Im}(P_\lambda v, v)_H \leq \|P_\lambda v\|_H \|v\|_H . \tag{2.10}$$

Second, we notice that $(A-\lambda^2)v = P_\lambda v - i\lambda BB^*v$ and thus, using (2.10),

$$\begin{aligned}\left\|(A-\lambda^2)v\right\|_H^2 &\leq 2\|P_\lambda v\|_H^2 + 2\lambda \|BB^*v\|_H^2 \leq 2\|P_\lambda v\|_H^2 + 2\|B\|_{\mathcal{L}(Y;H)}^2 \lambda \|B^*v\|_Y^2 \\ &\leq 2\|P_\lambda v\|_H^2 + 2\|B\|_{\mathcal{L}(Y;H)}^2 \|P_\lambda v\|_H \|v\|_H .\end{aligned}$$

Plugging the last two estimates in (2.5) yields

$$\|v\|_H \leq (G_1\lambda^{-\frac{1}{2}} + G_2\sqrt{2}\|B\|_{\mathcal{L}(Y;H)}) \|P_\lambda v\|_H^{\frac{1}{2}} \|v\|_H^{\frac{1}{2}} + G_2\sqrt{2}\|P_\lambda v\|_H .$$

Writing

$$(G_1\lambda^{-\frac{1}{2}} + G_2\sqrt{2}\|B\|_{\mathcal{L}(Y;H)}) \|P_\lambda v\|_H^{\frac{1}{2}} \|v\|_H^{\frac{1}{2}} \leq \frac{1}{2}(G_1\lambda^{-\frac{1}{2}} + G_2\sqrt{2}\|B\|_{\mathcal{L}(Y;H)})^2 \|P_\lambda v\|_H + \frac{1}{2}\|v\|_H ,$$

allows to absorb the last term in the left hand-side and implies

$$\frac{1}{2}\|v\|_H \leq \frac{1}{2}(G_1\lambda^{-\frac{1}{2}} + G_2\sqrt{2}\|B\|_{\mathcal{L}(Y;H)})^2 \|P_\lambda v\|_H + G_2\sqrt{2}\|P_\lambda v\|_H .$$

This concludes the proof of (2.6), and (2.8) corresponds to the case $G_1 = G_2 = \mathsf{G}(\lambda)$. Also, we notice that for $\lambda \in \mathbb{R}$, $\overline{P_{-\lambda}u} = P_\lambda \overline{u}$, so the statement for $\lambda \geq \lambda_0$ implies that for $\lambda \leq -\lambda_0$. Finally, the proof of (2.7) is similar to that of (2.6) (beware that it should be written for Q_{λ^2} and not Q_λ), and (2.9) follows from changing λ^2 into λ. ∎

Note that another advantage of Proposition 2.2 is that it is flexible enough to support perturbations of the operator A by lower order terms. This was used in [JL20] where the perturbation comes from the linearization of a nonlinear equation. See also [CPS+19, Bur19] for recent related perturbation results.

2.3 Damped Schrödinger-type equations

There are not many references concerning the damped Schrödinger equation. So let us start from the beginning. We set $\mathcal{A}_S := iA - BB^*$ with $D(\mathcal{A}_S) = D(A)$, so that (1.14) reformulates as $(\partial_t - \mathcal{A}_S)u = 0$.

The compact embedding $D(A) \hookrightarrow H$ implies that $\mathcal{A}_S$ has a compact resolvent. First spectral properties of $\mathcal{A}_S$ are described in the following lemma.

Lemma 2.3. *The spectrum of $\mathcal{A}_S$ contains only isolated eigenvalues and we have*

$$\left\|(z\operatorname{Id} - \mathcal{A}_S)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{1}{\operatorname{Re}(z)}, \quad \text{for } \operatorname{Re}(z) > 0, \tag{2.11}$$

$$\left\|(z\operatorname{Id} - \mathcal{A}_S)^{-1}\right\|_{\mathcal{L}(H)} \leq \frac{1}{|\operatorname{Im}(z)|}, \quad \text{for } \operatorname{Im}(z) < 0. \tag{2.12}$$

Moreover, assuming $(Au = zu, B^*u = 0) \implies u = 0$, we have

$$\operatorname{Sp}(\mathcal{A}_S) \subset [-\|B^*\|_{\mathcal{L}(H;Y)}^2, 0) + i[0, +\infty).$$

Proof. The structure of the spectrum comes from the fact that $\mathcal{A}_S$ has a compact resolvent (since so does A, and BB^* is bounded). Now, for a general $z \in \mathbb{C}$, we have

$$\|(z\operatorname{Id} - \mathcal{A}_S)u\|_H \|u\|_H \geq \operatorname{Re}((z\operatorname{Id} - \mathcal{A}_S)u, u)_H = \operatorname{Re}(z)\|u\|_H^2 + \|B^*u\|_H^2 \geq \operatorname{Re}(z)\|u\|_H^2 ,$$

which yields (2.11). The statement (2.12) comes from

$$\|(\mathcal{A}_S - z\,\mathrm{Id})u\|_H\,\|u\|_H \geq \mathrm{Im}\left((\mathcal{A}_S - z\,\mathrm{Id})u, u\right)_H = (Au,u)_H - \mathrm{Im}(z)\,\|u\|_H^2 \geq -\,\mathrm{Im}(z)\,\|u\|_H^2\,.$$

Finally given $z \in \mathrm{Sp}(\mathcal{A}_S)$, there is $u \in D(A)\setminus\{0\}$ such that $\mathcal{A}_S u = zu$. Taking inner product with u yields

$$z\,\|u\|_H^2 = (\mathcal{A}_S u, u)_H = i(Au,u)_H - \|B^* u\|_H^2\,.$$

In particular,

$$\mathrm{Re}(z) = -\frac{\|B^*u\|_H^2}{\|u\|_H^2} \in [-\,\|B^*\|_{\mathcal{L}(H)}^2\,, 0], \quad \mathrm{Im}(z) = \frac{(Au,u)_H}{\|u\|_H^2} \geq 0.$$

Now if $\mathrm{Re}(z) = 0$, this implies $B^*u = 0$ and hence $zu = \mathcal{A}_S u = iAu$. The assumption then yields $u = 0$, which contradicts the fact that u is an eigenvector. Thus $\mathrm{Sp}(\mathcal{A}_S) \cap i\mathbb{R} = \emptyset$. □

We then deduce straightforwardly from Proposition 2.2 and Lemma 2.3 the following result.

Theorem 2.4. *Let $\mathsf{G} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\mathsf{G}(\mu) \geq c_0 > 0$ on $\mathbb{R}_+$, $\lambda_0 \geq 1$, and assume (1.18). Then there exists $K > 1$ (the same as in Proposition 2.2), such that*

$$\begin{aligned}&\|(i\lambda\,\mathrm{Id} - \mathcal{A}_S)^{-1}\|_{\mathcal{L}(H)} \leq K\mathsf{G}\big(\sqrt{\lambda}\big)^2, \qquad \text{for all } \lambda \geq \lambda_0^2,\\ &\mathrm{Sp}(\mathcal{A}_S) \cap \Gamma_{\mathsf{G},S} = \emptyset,\end{aligned}$$

where $\Gamma_{\mathsf{G},S} = \left\{ z \in \mathbb{C}, \mathrm{Im}(z) \geq \lambda_0^2, \mathrm{Re}(z) \geq -\frac{1}{K\mathsf{G}\left(\sqrt{\mathrm{Im}(z)}\right)^2} \right\}$. *Finally, assuming further (1.16), there exists another constant $\widetilde{K} \geq K$ such that*

$$\begin{aligned}&\|(i\lambda\,\mathrm{Id} - \mathcal{A}_S)^{-1}\|_{\mathcal{L}(H)} \leq \widetilde{K}\mathsf{G}\big(\sqrt{|\lambda|}\big)^2, \qquad \text{for all } \lambda \in \mathbb{R},\\ &\mathrm{Sp}(\mathcal{A}_S) \cap \widetilde{\Gamma}_{\mathsf{G},S} = \emptyset,\end{aligned}$$

where $\widetilde{\Gamma}_{\mathsf{G},S} = \left\{ z \in \mathbb{C}, \mathrm{Re}(z) \geq -\frac{1}{\widetilde{K}\mathsf{G}\left(\sqrt{|\mathrm{Im}(z)|}\right)^2} \right\}$.

Proof. The first point is a rewriting of (2.9) in Proposition 2.2. The second point comes from the general fact that

$$\left\|(z\,\mathrm{Id} - \mathcal{A}_S)^{-1}\right\|_{\mathcal{L}(H)} \geq \frac{1}{\mathrm{dist}(z, \mathrm{Sp}(\mathcal{A}_S))}. \tag{2.13}$$

A simple proof of this inequality in the present context uses that the spectrum is discrete and only consists in eigenvalues. Hence, writing $\mathrm{Sp}(\mathcal{A}_S) = \{z_j, j \in \mathbb{N}\}$ and denoting by ψ_j a normalized eigenfunction of $\mathcal{A}_S$ associated to z_j, we have $\left\|(z\,\mathrm{Id} - \mathcal{A}_S)^{-1}\right\|_{\mathcal{L}(H)} \geq \left\|(z\,\mathrm{Id} - \mathcal{A}_S)^{-1}\psi_j\right\|_{\mathcal{L}(H)} = \left\|(z - z_j)^{-1}\psi_j\right\|_{\mathcal{L}(H)} = |z - z_j|^{-1}$, and the result follows from taking the supremum in $j \in \mathbb{N}$. Hence, we have for $\lambda \geq \lambda_0^2$,

$$\mathrm{dist}(i\lambda, \mathrm{Sp}(\mathcal{A}_S)) \geq \left\|(i\lambda\,\mathrm{Id} - \mathcal{A}_S)^{-1}\right\|_{\mathcal{L}(H)}^{-1} \geq \left(K\mathsf{G}\big(\sqrt{\lambda}\big)^2\right)^{-1},$$

which, together with the localization of the spectrum in Lemma 2.3, proves the second point.

For the last point, Lemma 2.3 ensures that $\lambda \mapsto \|(i\lambda\,\mathrm{Id} - \mathcal{A}_S)^{-1}\|_{\mathcal{L}(\mathcal{H})}$ is a well defined continuous function on $\mathbb{R}$, which is bounded by $\frac{1}{|\lambda|}$ for $\lambda < 0$. On the interval $(-\infty, \lambda_0^2]$, it is therefore bounded by a constant $C_0 \leq C_0 c_0^{-2}\mathsf{G}\big(\sqrt{|\lambda|}\big)^2$. This gives the expected estimates for all $\lambda \in \mathbb{R}$ with another $\widetilde{K} = \max\left(K, C_0 c_0^{-2}\right)$.

Again, (2.13) proves the spectral gap near the imaginary axis. □

As a consequence, we deduce the following decay.

Theorem 2.5. *Let $\lambda_0 \geq 1$, $\mathsf{G} : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\mathsf{G}(0) > 0$, and assume (1.16) and (1.18). Then, for all $j \in \mathbb{N}^*$, there are $C_j, T_j > 0$ such that for all $u_0 \in D(\mathcal{A}_S^j)$ and associated solution u of (1.14),*

$$\|u(t)\|_H \leq \frac{C_j}{\mathsf{M}_{\log}^{-1}\left(\frac{t}{C_j}\right)^j} \left\|\mathcal{A}_S^j u_0\right\|_H, \quad \text{for all } t \geq T_j,$$

where $\mathsf{M}_{\log}$ is defined in (1.20) with $\mathsf{M}(\lambda) = \mathsf{G}\big(\sqrt{\lambda}\big)^2$.

Again, $\mathsf{M}_{\log}$ in the result can be replaced by M if it is polynomial at infinity, according to [BT10, Theorem 2.4].

Proof. This is a direct corollary of Theorem 2.4 and Theorem 1.11 applied to the operator $\mathcal{B} = \mathcal{A}_S$ in the Hilbert space $\mathcal{X} = H$. We have also used that if M is a positive nondecreasing function, $K > 0$, and $\mathsf{N} = K\mathsf{M}$, then $\mathsf{N}_{\log} \leq \mathsf{M}_{\log}$ if $K \leq 1$ and $\mathsf{N}_{\log} \leq K\left(1 + \frac{\log(K)}{\log(1+\mathsf{M}(0))}\right)\mathsf{M}_{\log}$ if $K \geq 1$. Changing M into $K\mathsf{M}$ in Theorem 1.11 thus only changes the values of the constants C_j in the result. □

We may now conclude the proofs of Theorems 1.3 and 1.4.

Proof of Theorems 1.3 and 1.4. Corollary 1.10 implies that (1.18) is true with $\mathsf{G}(\mu) = Ce^{\nu\mu^k}$. Then, Theorem 2.4 implies Theorem 1.4. Indeed, taking into account (1.16), we then obtain that the resolvent is bounded on the positive imaginary axis by a constant times $\mathsf{M}(\lambda) = \mathsf{G}\big(\sqrt{\lambda}\big)^2 = Ce^{2\nu^+\lambda^{k/2}}$ (after having changed the constants slightly).

Finally, we obtain

$$\mathsf{M}_{\log}(\lambda) = Ce^{2\nu^+\lambda^{k/2}}\left(\log\left(1 + Ce^{2\nu^+\lambda^{k/2}}\right) + \log(1+\lambda)\right) \leq Ce^{2\nu^+\lambda^{k/2}}$$

(after having changed the constants slightly), and thus $\mathsf{M}_{\log}^{-1}(t) \geq c\log(t)^{2/k}$ for large t. Theorem 2.5 implies Theorem 1.3. □

2.4 Damped wave-type equations: semigroup setting and end of the proofs

We now turn Estimate 2.8 in Proposition 2.2 into a resolvent estimate for the generator of the damped wave group, and then into an energy decay for (1.13). We equip $\mathcal{H} = H_1 \times H$ with the norm

$$\|(u_0, u_1)\|_{\mathcal{H}}^2 = \|(A + \mathrm{Id})^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2,$$

and define the seminorm

$$|(u_0, u_1)|_{\mathcal{H}}^2 = \|A^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2.$$

Of course, if A is coercive on H, $|\cdot|_{\mathcal{H}}$ is a norm on $\mathcal{H}$ equivalent to $\|\cdot\|_{\mathcal{H}}$. We define the energy of solutions of (1.13) by

$$E(u(t)) = \frac{1}{2}\big(\|A^{\frac{1}{2}} u\|_H^2 + \|\partial_t u\|_H^2\big) = \frac{1}{2}|(u, \partial_t u)|_{\mathcal{H}}^2.$$

The damped wave equation (1.13) can be recast on $\mathcal{H}$ as a first order system

$$\begin{cases} \partial_t U = \mathcal{A}U, \\ U|_{t=0} = {}^t(u_0, u_1), \end{cases} \quad U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -A & -BB^* \end{pmatrix}, \quad D(\mathcal{A}) = D(A) \times H_1. \tag{2.14}$$

The compact embeddings $D(A) \hookrightarrow H_1 \hookrightarrow H$ imply that $D(\mathcal{A}) \hookrightarrow \mathcal{H}$ compactly, and that the operator $\mathcal{A}$ has a compact resolvent. First, spectral properties of $\mathcal{A}$ are described in the following lemma borrowed from [Leb96, AL14]. We define the following quadratic family of operator

$$P(z) = A + z^2 \,\mathrm{Id} + zBB^*, \quad z \in \mathbb{C}, \quad D(P(z)) = D(A). \tag{2.15}$$

Lemma 2.6 (Lemma 4.2 of [AL14]). *The spectrum of $\mathcal{A}$ contains only isolated eigenvalues and, provided (1.16) is satisfied, we have*

$$\mathrm{Sp}(\mathcal{A}) \subset \Big(\big(-\frac{1}{2}\|B^*\|^2_{\mathcal{L}(H;Y)}, 0\big) + i\mathbb{R} \Big) \cup \Big([-\|B^*\|^2_{\mathcal{L}(H;Y)}, 0] + 0i \Big),$$

with $\ker(\mathcal{A}) = \ker(A) \times \{0\}$. *Moreover, the operator $P(z)$ in (2.15) is an isomorphism from $D(A)$ onto H if and only if $z \notin \mathrm{Sp}(\mathcal{A})$.*

This lemma leads us to introduce the spectral projector of $\mathcal{A}$ onto the spectral subspace of $\mathcal{A}$ associated to the eigenvalue 0, namely

$$\Pi_0 = \frac{1}{2i\pi} \int_\gamma (z \,\mathrm{Id} - \mathcal{A})^{-1} dz \in \mathcal{L}(\mathcal{H}),$$

where γ denotes a positively oriented circle centered on 0 with a radius so small that $\mathrm{Sp}(\mathcal{A}) \cap \gamma = \emptyset$ and 0 is the single eigenvalue of $\mathcal{A}$ in the interior of γ. The projector Π_0 and $\ker(\mathcal{A})$ are linked by the following classical lemma.

Lemma 2.7. *Under the assumptions of Lemma 2.6, we have* $\mathrm{range}(\Pi_0) = \ker(\mathcal{A}) = \ker(A) \times \{0\}$.

Proof. We only need to check that there is no generalized eigenfunction (equivalently, no Jordan block) associated to the eigenvalue 0. Given $\{e_0, \cdots, e_k\}$ a basis of $\ker(A)$, and setting $\psi_j = (e_j, 0)$, the set $\{\psi_0, \cdots, \psi_k\}$ forms a basis of $\ker(\mathcal{A})$ according to Lemma 2.6. Assuming $\ker(\mathcal{A}) \subsetneq \mathrm{range}(\Pi_0)$ implies that there is a generalized eigenfunction $\phi = (u_0, u_1) \in D(\mathcal{A})$ and $j \in \{0, \cdots, k\}$ such that $\mathcal{A}\phi = \psi_j$. Recalling the form of $\mathcal{A}$, this is equivalent to $u_1 = e_j$ and $-Au_0 - BB^*u_1 = 0$. Taking the inner product in H of this with $u_1 = e_j$, this implies

$$0 = -(u_0, Ae_j)_H = -(Au_0, e_j)_H = (BB^*e_j, e_j)_H = \|B^*e_j\|^2_Y.$$

We obtain a contradiction with (1.16) since $e_j \neq 0$. This proves the lemma. □

We set $\dot{\mathcal{H}} = (\mathrm{Id} - \Pi_0)\mathcal{H}$ and equip this space with the norm

$$\|(u_0, u_1)\|^2_{\dot{\mathcal{H}}} := |(u_0, u_1)|^2_{\mathcal{H}} = \|A^{\frac{1}{2}} u_0\|^2_H + \|u_1\|^2_H,$$

and associated inner product. This is indeed a norm on $\dot{\mathcal{H}}$ since $\|(u_0, u_1)\|_{\dot{\mathcal{H}}} = 0$ is equivalent to $(u_0, u_1) \in \ker(A) \times \{0\} = \Pi_0\mathcal{H}$. Besides, we set $\dot{\mathcal{A}} = \mathcal{A}|_{\dot{\mathcal{H}}}$ with domain $D(\dot{\mathcal{A}}) = D(\mathcal{A}) \cap \dot{\mathcal{H}}$. Remark that $\mathrm{Sp}(\dot{\mathcal{A}}) = \mathrm{Sp}(\mathcal{A}) \setminus \{0\}$ and thus $\mathrm{Sp}(\dot{\mathcal{A}}) \cap i\mathbb{R} = \emptyset$.

Lemma 2.8 (Lemma 4.3 of [AL14]). *The operator $\dot{\mathcal{A}}$ generates a contraction C^0-semigroup on $\dot{\mathcal{H}}$, denoted $(e^{t\dot{\mathcal{A}}})_{t\geq 0}$. Moreover, the operator $\mathcal{A}$ generates a bounded C^0-semigroup on $\mathcal{H}$, denoted $(e^{t\mathcal{A}})_{t\geq 0}$ and the unique solution to (1.13) is given by $(u, \partial_t u)(t) = e^{t\mathcal{A}}(u_0, u_1)$. Finally, we have*

$$e^{t\mathcal{A}} = e^{t\dot{\mathcal{A}}}(\mathrm{Id} - \Pi_0) + \Pi_0, \quad \textit{for all } t \geq 0. \tag{2.16}$$

Once we have put the abstract damped wave equation (1.13) in the appropriate semigroup setting, it remains to:

1. deduce from (1.17)-(1.18) a resolvent estimate for $\dot{\mathcal{A}}$,
2. relate this resolvent estimate to a decay estimate for $e^{t\dot{\mathcal{A}}}$, and
3. deduce the decay of the energy for (1.13).

Step 1 is achieved thanks to the following result from [AL14].

Lemma 2.9 (Lemma 4.6 of [AL14]). *There exist $C > 1$ such that for $s \in \mathbb{R}$, $|s| \geq 1$,*

$$C^{-1}\|(is\,\mathrm{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} - \frac{C}{|s|} \leq \|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C\|(is\,\mathrm{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} + \frac{C}{|s|}, \tag{2.17}$$

$$C^{-1}|s|\|P(is)^{-1}\|_{\mathcal{L}(H)} \leq \|(is\,\mathrm{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C\left(1 + |s|\|P(is)^{-1}\|_{\mathcal{L}(H)}\right). \tag{2.18}$$

As a corollary of this together with Proposition 2.2, we deduce the following result.

Theorem 2.10. *Let $\mathsf{G} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\mathsf{G}(\mu) \geq c_0 > 0$ on $\mathbb{R}_+$, $\lambda_0 \geq 1$, and assume (1.18). Then there exists $K > 1$ such that*

$$\begin{aligned}
&\|(i\lambda \operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq K|\lambda|\mathsf{G}(|\lambda|)^2, \qquad \text{for all } \lambda \in \mathbb{R}, |\lambda| \geq \lambda_0,\\
&\|(is \operatorname{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} \leq K|\lambda|\mathsf{G}(|\lambda|)^2, \qquad \text{for all } \lambda \in \mathbb{R}, |\lambda| \geq \lambda_0,\\
&\operatorname{Sp}(\dot{\mathcal{A}}) \cap \Gamma_{\mathsf{G}} = \emptyset, \qquad \operatorname{Sp}(\mathcal{A}) \cap \Gamma_{\mathsf{G}} = \emptyset,
\end{aligned}$$

where $\Gamma_{\mathsf{G}} = \left\{ z \in \mathbb{C}, |\operatorname{Im}(z)| \geq \lambda_0, \operatorname{Re}(z) \geq -\frac{1}{K|\operatorname{Im}(z)|\mathsf{G}(|\operatorname{Im}(z)|)^2} \right\}$.

Finally, assuming further (1.16), there exists another constant $\widetilde{K} \geq K$ such that

$$\begin{aligned}
&\|(is \operatorname{Id} - \dot{\mathcal{A}})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} \leq \widetilde{K} \langle \lambda \rangle \, \mathsf{G}(|\lambda|)^2, \qquad \text{for all } \lambda \in \mathbb{R},\\
&\operatorname{Sp}(\dot{\mathcal{A}}) \cap \widetilde{\Gamma}_{\mathsf{G}} = \emptyset, \qquad \operatorname{Sp}(\mathcal{A}) \cap \Gamma_{\mathsf{G}} = \{0\},
\end{aligned}$$

where $\widetilde{\Gamma}_{\mathsf{G}} = \left\{ z \in \mathbb{C}, \operatorname{Re}(z) \geq -\frac{1}{\widetilde{K}\langle \operatorname{Im}(z)\rangle \mathsf{G}(|\operatorname{Im}(z)|)^2} \right\}$.

Proof of Theorem 2.10. The first two points are corollaries of (2.8) in Proposition 2.2 combined with Lemma 2.9.

The last point comes from $\operatorname{Sp}(\dot{\mathcal{A}}) = \operatorname{Sp}(\mathcal{A}) \backslash \{0\}$, together with the general fact that $\left\|(z \operatorname{Id} - \dot{\mathcal{A}})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \geq \frac{1}{\operatorname{dist}(z, \operatorname{Sp}(\dot{\mathcal{A}}))}$ (see (2.13) in the proof of Theorem 2.4). Hence, we have for $\lambda \in \mathbb{R}$, $|\lambda| \geq \lambda_0$,

$$\operatorname{dist}(i\lambda, \operatorname{Sp}(\dot{\mathcal{A}})) \geq \left\|(i\lambda \operatorname{Id} - \dot{\mathcal{A}})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}^{-1} \geq \left(K|\lambda|\mathsf{G}(|\lambda|)^2\right)^{-1},$$

which, together with the localization of the spectrum in Lemma 2.6, proves the statement about the region free of spectrum. The proof concerning the compact zone follows the same way as in the proof of Theorem 2.4, using that, as already noticed, $\operatorname{Sp}(\dot{\mathcal{A}}) \cap i\mathbb{R} = \emptyset$. □

Step 2 is achieved as a consequence of Theorem 1.11 applied to the operator $\mathcal{B} = \dot{\mathcal{A}}$ in the Hilbert space $\mathcal{X} = \dot{\mathcal{H}}$.

Finally, Step 3 is a consequence of the following elementary Lemma 2.11, linking the energy of solutions to the abstract damped wave equation (1.13) to the norm of the semigroup $\left(e^{t\dot{\mathcal{A}}}\right)_{t\geq 0}$.

Lemma 2.11. *For all $j \in \mathbb{N}^*$, $U_0 \in D(\mathcal{A}^j)$ such that $\Pi_0 U_0 \neq U_0$, and associated solution u of (1.13), we have*

$$\frac{E(u(t))}{\frac{1}{2}|\mathcal{A}^j U_0|_{\mathcal{H}}^2} = \frac{|e^{t\mathcal{A}}U_0|_{\mathcal{H}}^2}{|\mathcal{A}^j U_0|_{\mathcal{H}}^2} = \frac{\|e^{t\dot{\mathcal{A}}}\dot{U}_0\|_{\dot{\mathcal{H}}}^2}{\|\dot{\mathcal{A}}^j \dot{U}_0\|_{\dot{\mathcal{H}}}^2}, \quad \text{where} \quad \dot{U}_0 = (\operatorname{Id} - \Pi_0)U_0.$$

In particular, setting $f_j(t) := \left\|e^{t\dot{\mathcal{A}}}\dot{\mathcal{A}}^{-j}\right\|_{\mathcal{L}(\dot{\mathcal{H}})}$ for $j \in \mathbb{N}^*$, we have for all $U_0 \in D(\mathcal{A}^j)$ and associated solution u of (1.13),

$$E(u(t)) \leq \frac{1}{2} f_j(t)^2 \|\mathcal{A}^j U_0\|_{\mathcal{H}}^2, \qquad \text{for all } t \geq 0.$$

Proof. This is essentially [AL14, Lemma 4.4]. Recalling that $\mathcal{A}U_0 = \dot{\mathcal{A}}\dot{U}_0$, we have

$$\begin{aligned}
&E(u(t)) = \frac{1}{2}\left(\|A^{\frac{1}{2}}u(t)\|_H^2 + \|\partial_t u(t)\|_H^2\right) = \frac{1}{2}|e^{t\mathcal{A}}U_0|_{\mathcal{H}}^2 = \frac{1}{2}|e^{t\dot{\mathcal{A}}}\dot{U}_0 + \Pi_0 U_0|_{\mathcal{H}}^2 = \frac{1}{2}\|e^{t\dot{\mathcal{A}}}\dot{U}_0\|_{\dot{\mathcal{H}}}^2,\\
&\|\dot{\mathcal{A}}^j \dot{U}_0\|_{\dot{\mathcal{H}}}^2 = |\mathcal{A}^j U_0|_{\mathcal{H}}^2,
\end{aligned}$$

which yields the first statement. The second one follows from the fact that $|\cdot|_{\mathcal{H}} \leq \|\cdot\|_{\mathcal{H}}$. □

As a consequence, we deduce the following decay.

Theorem 2.12. *Let $\lambda_0 \geq 1$, $\mathsf{G} : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\mathsf{G}(0) > 0$, and assume (1.16) and (1.18). Then, for all $j \in \mathbb{N}^*$, there are $C_j, T_j > 0$ such that for all $U_0 \in D(\mathcal{A}^j)$ and associated solution u of (1.13),*

$$E(u(t))^{\frac{1}{2}} \leq \frac{C_j}{\mathsf{M}_{\log}^{-1}\left(\frac{t}{C_j}\right)^j} \left\|\mathcal{A}^j U_0\right\|_{\mathcal{H}}, \quad \text{for all } t \geq T_j,$$

where $\mathsf{M}_{\log}$ is defined in (1.20) with $\mathsf{M}(\lambda) = \langle\lambda\rangle\, \mathsf{G}(\lambda)^2$.

Again, $\mathsf{M}_{\log}$ in the result can be replaced by M if it is polynomial at infinity, according to [BT10, Theorem 2.4].

Proof. This is a direct corollary of Theorem 2.10, Theorem 1.11 applied to $\mathcal{X} = \dot{\mathcal{H}}$ and $\mathcal{B} = \dot{\mathcal{A}}$, together with Lemma 2.11 (and a remark in the proof of Theorem 2.5). ☐

We conclude this paragraph with the proofs of Theorems 1.1 and 1.2.

Proof of Theorems 1.1 and 1.2. Again, Corollary 1.10 implies the unique continuation property (1.7) (that is (1.16) in the present context) together with (1.18) with $\mathsf{G}(\mu) = Ce^{\nu\mu^k}$. With this estimate at hand, Theorem 1.1 is an application of Theorem 2.12 with $\mathsf{M}(\lambda) = \langle\lambda\rangle\, \mathsf{G}(\lambda)^2 \leq Ce^{2\nu^+\lambda^k}$ (after having changed the constants slightly), while Theorem 1.2 is implied by Lemma 2.6 and Theorem 2.10. ☐

2.5 Damped plate-type equations

The plate equation actually fits into the "wave-type" framework. Indeed, the abstract plate equation

$$\begin{cases} \partial_t^2 u + A^2 u + BB^* \partial_t u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \tag{2.19}$$

is actually a particular case of the abstract equation (1.13) applied with the operator A^2 (instead of A) which is still nonnegative selfadjoint with compact resolvent. In this case, we define $H_2 = D(A)$, equipped with the graph norm $\|u\|_{H_2} := \|(A^2 + \mathrm{Id})^{\frac{1}{2}} u\|_H$, and its dual $H_{-2} = (H_2)'$ (using H as a pivot space) endowed with the norm $\|u\|_{H_{-2}} := \|(A^2 + \mathrm{Id})^{-\frac{1}{2}} u\|_H$.

The natural space is then $\mathcal{H} = H_2 \times H$ with the norm

$$\|(u_0, u_1)\|_{\mathcal{H}}^2 = \|(A^2 + \mathrm{Id})^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2,$$

and the seminorm

$$|(u_0, u_1)|_{\mathcal{H}}^2 = \|Au_0\|_H^2 + \|u_1\|_H^2.$$

The associated energy is

$$E_P(u(t)) = \frac{1}{2}\big(\|Au\|_H^2 + \|\partial_t u\|_H^2\big) = \frac{1}{2}|(u, \partial_t u)|_{\mathcal{H}}^2.$$

In order to transfer the properties of A to A^2, we will only need the following simple lemma.

Lemma 2.13. *Assume (1.18) is satisfied. Then, we have*

$$\|v\|_H \leq \mathsf{G}(\sqrt{\lambda})\big(\|B^* v\|_Y + \lambda^{-1}\left\|(A^2 - \lambda^2)v\right\|_H\big), \quad \text{for all } v \in D(A^2), \lambda \geq \lambda_0^2. \tag{2.20}$$

Proof. Since A is a nonnegative operator, we have $\left\|(A + \lambda^2)w\right\|_H \geq \lambda^2 \|w\|_H$ for all $w \in D(A)$. Applying this to $w = (A - \lambda^2)v$ gives $\left\|(A^2 - \lambda^4)v\right\|_H \geq \lambda^2 \left\|(A - \lambda^2)v\right\|_H$. This, combined with (1.18) implies

$$\|v\|_H \leq \mathsf{G}(\lambda)\big(\|B^* v\|_Y + \left\|(A - \lambda^2)v\right\|_H\big) \leq \mathsf{G}(\lambda)\left(\|B^* v\|_Y + \frac{1}{\lambda^2}\left\|(A^2 - \lambda^4)v\right\|_H\right). \tag{2.21}$$

This is the expected result up to changing λ into $\sqrt{\lambda}$. ☐

Lemma 2.13 implies that if (1.18) is satisfied, the assumptions of Theorem 2.12 are satisfied for the operator A^2 with $G_P(\lambda) = G(\sqrt{\lambda})$. Moreover, since A is a nonnegative selfadjoint operator with compact resolvent, the eigenfunctions of A^2 are those of A. In particular, if (1.16) is true for A, it is also true for A^2. It directly gives the following result.

Theorem 2.14. *Let $\mathsf{G} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\mathsf{G}(\mu) \geq c_0 > 0$ on $\mathbb{R}_+$, $\lambda_0 \geq 1$, and assume* (1.16) *and* (1.18). *Assume further that G is nondecreasing. Then, for all $j \in \mathbb{N}^*$, there are $C_j, T_j > 0$ such that for all $U_0 \in D(\mathcal{A}^j)$ and associated solution u of* (2.19),

$$E_P(u(t))^{\frac{1}{2}} \leq \frac{C_j}{\mathsf{M}_{\log}^{-1}\left(\frac{t}{C_j}\right)^j} \left\| \mathcal{A}_P^j U_0 \right\|_{\mathcal{H}}, \quad \text{for all } t \geq T_j,$$

where $\mathsf{M}_{\log}$ is defined in (1.20) *with $\mathsf{M}(\lambda) = \langle \lambda \rangle \mathsf{G}(\sqrt{\lambda})^2$.*

Proof of Theorem 1.5. Thanks to Corollary 1.10, (1.18) is true with $\mathsf{G}(\mu) = C(\mu+2)e^{\nu(\mu+2)^k}$. Theorem 1.5 is then an application of Theorem 2.14 with $\mathsf{M}(\lambda) = \langle \lambda \rangle \mathsf{G}(\sqrt{\lambda})^2 \leq Ce^{2\nu^+ \lambda^{k/2}}$ (after having changed the constants slightly). ☐

2.6 Lower bounds: proof of Proposition 1.6

Proof of Proposition 1.6. According to [LL17, Proposition 1.14] (which relies on [BCG14, Section 2.3]), since $\operatorname{supp}(b) \cap \{x_1 = 0\} = \emptyset$, there exist $C, c_0 > 0$ and a sequence $(\lambda_j, \varphi_j) \in \mathbb{R}_+ \times C^\infty(\mathcal{M})$ such that

$$\mathcal{L}\varphi_j = \lambda_j \varphi_j, \quad \varphi_j|_{\partial\mathcal{M}} = 0, \quad \|\varphi_j\|_{L^2(\mathcal{M})} = 1, \quad \lambda_j \to +\infty, \quad \|\varphi_j\|_{L^2(\operatorname{supp}(b))} \leq Ce^{-c_0 \lambda_j^{\frac{k}{2}}}.$$

As a consequence, concerning the damped Schrödinger resolvent, we have

$$\|(\mathcal{A}_S - i\lambda_j)\varphi_j\|_{L^2(\mathcal{M})} = \|(i\mathcal{L} - b - i\lambda_j)\varphi_j\|_{L^2(\mathcal{M})} = \|b\varphi_j\|_{L^2(\mathcal{M})} \leq \|b\|_{L^\infty} Ce^{-c_0\lambda_j^{\frac{k}{2}}}.$$

This implies the second estimate in (1.11) with $s_j = \lambda_j$.

Concerning the damped wave resolvent, recalling the definition of $P(z)$ in (2.15), we write

$$\left\| P(i\sqrt{\lambda_j})\varphi_j \right\|_{L^2} = \left\| (\mathcal{L} - \lambda_j + i\sqrt{\lambda_j} b)\varphi_j \right\|_{L^2} = \left\| \sqrt{\lambda_j} b\varphi_j \right\|_{L^2} \leq \sqrt{\lambda_j} \|b\|_{L^\infty} Ce^{-c_0\lambda_j^{\frac{k}{2}}}.$$

With $s_j = \sqrt{\lambda_j}$, this implies $\left\| P(is_j)\varphi_j \right\|_{L^2} \leq s_j Ce^{-c_0 s_j^k}$, and using (2.18) in Lemma 2.9 proves the first estimate in (1.11).

The last part of the Proposition follows from (1.11) together with the first implication in Theorem 1.11 (and, in case of damped waves, equivalence between the resolvents of $\mathcal{A}$ et $\dot{\mathcal{A}}$ in (2.17) in Lemma 2.9). ☐

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