Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients

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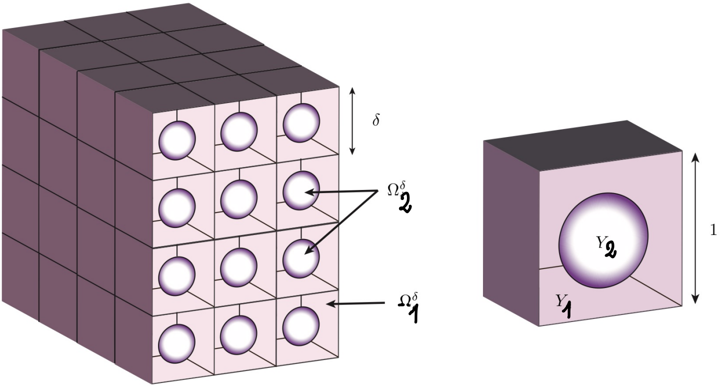
Geometrical setting

Example of a 3-D domain Ω with two periodic components Ω_1^δ and Ω_2^δ, and interface Γ^δ; reference cell $Y = (0,1)^3$, $Y = Y_1 \cup Y_2 \cup \Gamma$.



Plan of the presentation

1. Sign-Changing coefficients Maxwell problem
2. Positive coefficients Maxwell problem - homogenization
3. Sign-changing coefficients Maxwell problem at fixed δ
4. Sign-changing coefficients Maxwell problem – homogenization
5. Conclusions

Sign-changing coefficients Maxwell problem

For a given frequency $\omega \neq 0$, we consider time harmonic Maxwell's equations

$$\mathbf{curl}\, \boldsymbol{E}^\delta - i\omega\, \mu^\delta \boldsymbol{H}^\delta = 0 \qquad \text{and} \qquad \mathbf{curl}\, \boldsymbol{H}^\delta + i\omega\, \varepsilon^\delta \boldsymbol{E}^\delta = \boldsymbol{J} \ \text{ in } \Omega$$

with the boundary conditions

$$\boldsymbol{E}^\delta \times n = 0 \qquad \text{and} \qquad \mu^\delta \boldsymbol{H}^\delta \cdot n = 0 \ \text{ on } \partial\Omega,$$

where n denotes the unit outward normal vector field to $\partial\Omega$.

- $\boldsymbol{E}^\delta$ is the electric component of the electromagnetic field
- $\boldsymbol{H}^\delta$ is the magnetic components of the electromagnetic field
- the source term $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ is the current density
- the medium Ω is surrounded by a perfect conductor

J.-C. Nédélec, Acoustic and electromagnetic equations, Applied Mathematical Sciences, vol. 144, Springer, 2001.

Sign-changing coefficients Maxwell problem

- ϵ^δ is the electric permittivity and μ^δ is the magnetic permeability:

$$\epsilon^\delta(x) = \epsilon\left(\frac{x}{\delta}\right),\ \mu^\delta(x) = \mu\left(\frac{x}{\delta}\right),$$

$$\epsilon(y) = (\epsilon_1 \mathbf{1}_{Y_1}(y) + \epsilon_2 \mathbf{1}_{Y_2}(y))Id_3,\ \mu(y) = (\mu_1 \mathbf{1}_{Y_1}(y) + \mu_2 \mathbf{1}_{Y_2}(y))Id_3,$$

$\mathbf{1}_D$ is the characteristic function of the domain D,

$$\epsilon_1 > 0,\ \epsilon_2 < 0; \qquad \mu_1 > 0,\ \mu_2 < 0. \tag{1}$$

$$(\mathcal{P}^\delta)\ \left|\ \begin{array}{l} \text{Find } \boldsymbol{E}^\delta \in \mathbf{H}_n(\mathbf{curl}) \text{ s.t. for every } \boldsymbol{E}' \in \mathbf{H}_n(\mathbf{curl}): \\ ((\mu^\delta)^{-1}\mathbf{curl}\,\boldsymbol{E}^\delta, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\epsilon^\delta \boldsymbol{E}^\delta, \boldsymbol{E}') = i\omega\,(\boldsymbol{J}, \boldsymbol{E}'), \end{array}\right. \tag{2}$$

where
$\mathbf{H}_n(\mathbf{curl}) = \{\boldsymbol{E} \in \mathbf{H}(\mathbf{curl}) \mid \boldsymbol{E} \times n = 0 \text{ on } \partial\Omega\}$,
$\mathbf{H}(\mathbf{curl}) = \{\boldsymbol{H} \in \boldsymbol{L}(\Omega) \mid \mathbf{curl}\,\boldsymbol{H} \in \boldsymbol{L}(\Omega)\}$,
$\boldsymbol{L}(\Omega) = (L^2(\Omega))^3$.

Sign-changing coefficients Maxwell problem

Motivation (negative materials)

• for metals at optical frequencies one has $\epsilon < 0$ and $\mu > 0$;
• artificial metamaterials have been realized which can be modelled for certain frequencies by $\epsilon < 0$ and $\mu < 0$.

The negative refraction at the interface dielectric/metamaterial could allow the realization of perfect lenses, photonic trap.

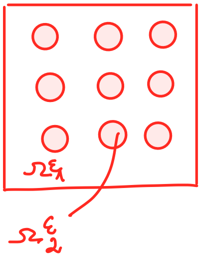
The homogenization method at a glance ($\delta \to 0$)

$(\mathcal{P}^\delta)$

E^δ

$\epsilon^\delta(x)$

$\mu^\delta(x)$



$\Omega = \Omega^\varepsilon_1 \cup \Omega^\varepsilon_2 \cup \Gamma^\varepsilon$

$\delta \to 0$

$(\mathcal{P}^\delta)$ well-posed

E^δ verifies uniform estimates

E^δ "converges" to E^{hom}

$(\mathcal{P}^{hom})$ well-posed

J.L. Lions; E. Sanchez-Palencia; F. Murat; L. Tartar; G. Nguetseng; G. Allaire; D. Cioranescu, G. Griso, A. Damlamian

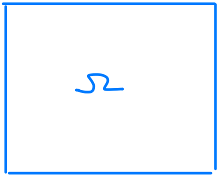
$(\mathcal{P}^{hom})$ Ω

E^{hom}

$\mathcal{H}(\epsilon)$ constant

$\mathcal{H}(\mu)$ constant

$(\mathcal{P}^{loc})$ Y



$Y = Y_1 \cup Y_2 \cup \Gamma$

Sign-changing coefficients Maxwell problem

$$(\mathcal{P}^\delta) \quad \left| \begin{array}{l} \text{Find } \boldsymbol{E}^\delta \in \mathbf{H}_n(\mathbf{curl}) \text{ s.t. for every } \boldsymbol{E}' \in \mathbf{H}_n(\mathbf{curl}): \\ ((\mu^\delta)^{-1}\mathbf{curl}\,\boldsymbol{E}^\delta, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\epsilon^\delta \boldsymbol{E}^\delta, \boldsymbol{E}') = i\omega\,(\boldsymbol{J}, \boldsymbol{E}'). \end{array} \right.$$

Questions we address

Q1 Well-posedness of the initial problem $(\mathcal{P}^\delta)$.
difficulties: loss of coercivity in the first term and not compactness of the second term

Q2 Passage to the limit with respect to δ in the initial problem $(\mathcal{P}^\delta)$ and derivation of the homogenized problem, denoted $(\mathcal{P}^{\mathrm{hom}})$.
difficulties: we can not derive uniform energy estimates for the solution $\boldsymbol{E}^\delta$

Q3 Well-posedness of the homogenized problem, denoted $(\mathcal{P}^{\mathrm{hom}})$.
difficulties: ?

Positive coefficients Maxwell problem - homogenization

$$\epsilon_1 > 0,\ \epsilon_2 > 0; \qquad \mu_1 > 0,\ \mu_2 > 0.$$

A. Bensoussan, J.-L. Lions, G. Papanicolaou, Asymptotic Analysis for Periodic Structures, AMS Chelsea Publishing, Providence, RI, 2011.

Theorem (Ciarlet, Fliss, Stohrer, *Comput. Math. Appl.*, 2017)

Assume that $\omega^2 \notin \Lambda^{\mathrm{hom}}$ where Λ^{hom} is a discrete subset of $[0, +\infty)$. Then there exists δ_0 such that for $\delta \in (0, \delta_0]$, the initial problem $(\mathcal{P}^\delta)$ is well defined and the sequence of solutions E^δ satisfies

$$\boldsymbol{E}^\delta \rightharpoonup \boldsymbol{E}^{\mathrm{hom}} \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{E}^\delta \rightharpoonup \mathbf{curl}\, \boldsymbol{E}^{\mathrm{hom}} \quad \text{in } \boldsymbol{L}^2(\Omega) \text{ weak},$$

where $\boldsymbol{E}^{\mathrm{hom}}$ is the unique solution of the homogenized problem $(\mathcal{P}^{\mathrm{hom}})$ defined below.

Λ^{hom} is the set of eigenvalues associated to the homogenized problem.

Positive coefficients Maxwell problem - homogenization

We associate to the electric field problem the bilinear form $a_\omega^\delta(\cdot,\cdot)$

$$a_\omega^\delta(\boldsymbol{E},\boldsymbol{E}') = ((\mu^\delta)^{-1}\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{E}') - \omega^2(\epsilon^\delta \boldsymbol{E},\boldsymbol{E}'), \qquad \forall \boldsymbol{E},\boldsymbol{E}' \in \mathbf{H}_n(\mathbf{curl}).$$

We define the homogenized problem

$$\left|\begin{array}{l} \text{Find } \boldsymbol{E}^{\mathrm{hom}} \in \mathbf{H}_n(\mathbf{curl}) \text{ s.t.} \\ \mathbf{curl}\,((\mathcal{H}(\mu))^{-1}\mathbf{curl}\,\boldsymbol{E}^{\mathrm{hom}}) - \omega^2\mathcal{H}(\epsilon)\boldsymbol{E}^{\mathrm{hom}} = i\omega \boldsymbol{J} \end{array}\right.$$

and the corresponding variational formulation

$$(\mathcal{P}^{\mathrm{hom}}) \quad \left|\begin{array}{l} \text{Find } \boldsymbol{E}^{\mathrm{hom}} \in \mathbf{H}_n(\mathbf{curl}) \text{ s.t. for every } \boldsymbol{E}' \in \mathbf{H}_n(\mathbf{curl}) \\ a_\omega^{\mathrm{hom}}(\boldsymbol{E}^{\mathrm{hom}},\boldsymbol{E}') = i\omega\,(\boldsymbol{J},\boldsymbol{E}'), \end{array}\right.$$

where $a_\omega^{\mathrm{hom}}(\cdot,\cdot)$ is the bilinear form defined on $\mathbf{H}_n(\mathbf{curl})$ by

$$a_\omega^{\mathrm{hom}}(\boldsymbol{E},\boldsymbol{E}') = ((\mathcal{H}(\mu))^{-1}\mathbf{curl}\,\boldsymbol{E},\mathbf{curl}\,\boldsymbol{E}') - \omega^2(\mathcal{H}(\epsilon)\boldsymbol{E},\boldsymbol{E}').$$

Positive coefficients Maxwell problem - homogenization

The **homogenized matrix** $\mathcal{H}(\epsilon) = (\mathcal{H}_{jk}(\epsilon))_{1\leq j,k\leq 3}$ is given by

$$\mathcal{H}_{jk}(\epsilon) = \frac{1}{|Y|}\int_Y \epsilon\nabla(y_j - \chi_j^\epsilon)\nabla(y_k - \chi_k^\epsilon)\,\mathrm{d}y, \qquad (3)$$

where, for $j = 1, 2, 3$, the functions $\chi_j^\epsilon \in \mathrm{H}^1_{\mathrm{per},0}(Y)$ are solutions of the local **scalar** problems $(\mathcal{P}^{\mathrm{loc}})_j$, defined by

$$(\mathcal{P}^{\mathrm{loc}})_j \qquad (\epsilon\nabla\chi_j^\epsilon, \nabla\xi)_Y = (\epsilon\nabla y_j, \nabla\xi)_Y, \qquad \forall\xi \in \mathrm{H}^1_{\mathrm{per},0}(Y). \qquad (4)$$

The homogenized matrix $\mathcal{H}(\mu)$ is defined analogously.

Remark

Matrices $\mathcal{H}(\epsilon)$ and $\mathcal{H}(\mu)$ are positive definite in the positive coefficients case.

Positive coefficients Maxwell problem - homogenization

Ideas of the proof

• Suppose that $(\boldsymbol{E}^\delta)$ is a sequence of solutions for the initial problem $(\mathcal{P}^\delta)$, satisfying the uniform estimate

$$\exists C > 0,\ \forall \delta \in (0;1], \qquad \|\boldsymbol{E}^\delta\| + \|\mathbf{curl}\, \boldsymbol{E}^\delta\| \leq C.$$

Then for $\delta \to 0$, one has

$$\boldsymbol{E}^\delta \rightharpoonup \boldsymbol{E}^{\mathrm{hom}} \qquad \text{and} \qquad \mathbf{curl}\, \boldsymbol{E}^\delta \rightharpoonup \mathbf{curl}\, \boldsymbol{E}^{\mathrm{hom}} \qquad \text{in } \boldsymbol{L}^2(\Omega) \text{ weak},$$

where $\boldsymbol{E}^{\mathrm{hom}}$ is solution of the homogenized problem $(\mathcal{P}^{\mathrm{hom}})$.

This step is independent on the sign of the coefficients.

Positive coefficients Maxwell problem - homogenization

Ideas of the proof

• If $\omega^2 \notin \Lambda^{\mathrm{hom}}$, then the homogenized problem $(\mathcal{P}^{\mathrm{hom}})$ is well-posed.

This step depends on the matrix $\mathcal{H}(\mu)$; it is positive definite in the positive coefficients case.

• If $\omega^2 \notin \Lambda^{\mathrm{hom}}$, then there is $0 < \delta_0$ such that for $\delta \in (0, \delta_0]$, problem $(\mathcal{P}^{\delta})$ is well-posed and the solutions E^{δ} are uniformly bounded in $\mathbf{H}_n(\mathbf{curl})$.

A spectral decomposition argument is applied. This argument strongly uses the positivity of the coefficients, for both $(\mathcal{P}^{\delta})$ and $(\mathcal{P}^{\mathrm{hom}})$ problems.

Sign-changing coefficients Maxwell at fixed δ

We define the linear and continuous operator $\mathcal{A}_n^\delta(\omega) : \mathbf{H}_n(\mathbf{curl}) \to \mathbf{H}_n(\mathbf{curl})$ s.t. for all $\omega \in \mathbb{C}$ and $\boldsymbol{E}, \boldsymbol{E}' \in \mathbf{H}_n(\mathbf{curl})$, one has

$$(\mathcal{A}_n^\delta(\omega)\boldsymbol{E}, \boldsymbol{E}')_{\mathbf{curl}} = ((\mu^\delta)^{-1}\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{E}') - \omega^2(\epsilon^\delta \boldsymbol{E}, \boldsymbol{E}'). \tag{5}$$

Difficulties : (NOT coercive) + (NOT compact)

For fixed δ, the properties of $\mathcal{A}_n^\delta(\omega)$ are strongly related to the properties of the **scalar** linear and continuous sign-changing operators $A_\epsilon^\delta : H_0^1(\Omega) \to H_0^1(\Omega)$ and $B_\mu^\delta : H_\#^1(\Omega) \to H_\#^1(\Omega)$ s.t.

$$\begin{aligned} (\nabla(A_\epsilon^\delta\varphi), \nabla\varphi') &= (\epsilon^\delta\nabla\varphi, \nabla\varphi'), && \forall\varphi, \varphi' \in H_0^1(\Omega) \\ (\nabla(B_\mu^\delta\varphi), \nabla\varphi') &= (\mu^\delta\nabla\varphi, \nabla\varphi'), && \forall\varphi, \varphi' \in H_\#^1(\Omega). \end{aligned}$$

where

$$H_\#^1(\Omega) = \left\{\varphi \in H^1(\Omega) \,|\, \int_\Omega \varphi \,\mathrm{dx} = 0\right\}.$$

Sign-changing coefficients Maxwell at fixed δ

Theorem 6.1 of Bonnet, Chesnel, Ciarlet (*Commun. Part. Diff. Eq.* 2014) writes as follows.

Theorem

Assume that the scalar operators $A^\delta_\epsilon : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ *and* $B^\delta_\mu : \mathrm{H}^1_\#(\Omega) \to \mathrm{H}^1_\#(\Omega)$ *are isomorphisms.*
Then $\mathcal{A}^\delta_N(\omega) : \mathbf{H}_n(\mathbf{curl}) \to \mathbf{H}_n(\mathbf{curl})$ *is an isomorphism if it is injective.*

Key point: find conditions ensuring that A^δ_ϵ and B^δ_μ are isomorphisms.

Difficulties: prove the uniform invertibility of these operators as δ tends to zero.

Sign-changing coefficients problem – homogenization

$$\epsilon_1 > 0, \epsilon_2 < 0; \quad \mu_1 > 0, \mu_2 < 0$$

Theorem (B., Chesnel, Ramdani, Rihani, 2021)

Let κ_ϵ and κ_μ belong to $(-\infty; -1/m) \cup (-1/M; 0)$, where m and M are two positive constants related to the geometry of the periodicity cell Y. Assume that $\omega^2 \notin \Lambda^{\mathrm{hom}}$ where Λ^{hom} is a discrete subset of $[0, +\infty)$. Then there exists δ_0 such that for $\delta \in (0, \delta_0]$, the solution $(\boldsymbol{E}^\delta)$ of the initial problem $(\mathcal{P}^\delta)$ (which is well defined) satisfies

$$\boldsymbol{E}^\delta \rightharpoonup \boldsymbol{E}^{\mathrm{hom}} \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{E}^\delta \rightharpoonup \mathbf{curl}\, \boldsymbol{E}^{\mathrm{hom}} \quad \text{in } \boldsymbol{L}^2(\Omega) \text{ weak,}$$

where $\boldsymbol{E}^{\mathrm{hom}}$ is the unique solution of the homogenized problem $(\mathcal{P}^{\mathrm{hom}})$.

Proof

Step 1

We suppose that $(\boldsymbol{E}^{\delta})$ is a sequence of solutions for the initial problem $(\mathcal{P}^{\delta})$, satisfying the uniform estimate

$$\exists C > 0,\ \forall \delta \in (0;1], \qquad \|\boldsymbol{E}^{\delta}\| + \|\mathbf{curl}\,\boldsymbol{E}^{\delta}\| \leq C. \tag{6}$$

Then for $\delta \to 0$, one has

$$\boldsymbol{E}^{\delta} \rightharpoonup \boldsymbol{E}^{\mathrm{hom}} \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{E}^{\delta} \rightharpoonup \mathbf{curl}\,\boldsymbol{E}^{\mathrm{hom}} \quad \text{in } \boldsymbol{L}^2(\Omega) \text{ weak},$$

where $\boldsymbol{E}^{\mathrm{hom}}$ is solution of the homogenized problem $(\mathcal{P}^{\mathrm{hom}})$.

Proof

Step 2

We study the well-posedness of the homogenized problem $(\mathcal{P}^{\text{hom}})$.

$$a_\omega^{\text{hom}}(\boldsymbol{E},\boldsymbol{E}') = ((\mathcal{H}(\mu))^{-1}\textbf{curl}\,\boldsymbol{E},\textbf{curl}\,\boldsymbol{E}') - \omega^2(\mathcal{H}(\epsilon)\boldsymbol{E},\boldsymbol{E}').$$

Difficulty: find conditions on the contrasts κ_ϵ and κ_μ such that the homogenized matrices $\mathcal{H}(\mu)$ and $\mathcal{H}(\epsilon)$ are (positive/negative) definite.

We recall that the homogenized matrices depend on the solution of sign-changing local scalar problems $(\mathcal{P}^{\text{loc}})_j$.

Once we find the conditions on the contrasts κ_ϵ and κ_μ, one can prove that the homogenized problem $(\mathcal{P}^{\text{hom}})$ admits a unique solution for $\omega^2 \notin \Lambda^{\text{hom}}$, where Λ^{hom} is a discrete subset of $[0,+\infty)$.

Proof

Step 3

We prove that there exists $\delta_0 > 0$ s.t. for all $\delta \in (0; \delta_0]$, the initial problem $(\mathcal{P}^\delta)$ has a unique solution $\boldsymbol{E}^\delta$. Moreover, one has the estimate

$$\|\boldsymbol{E}^\delta\| + \|\mathbf{curl}\, \boldsymbol{E}^\delta\| \leqslant C,$$

where $C > 0$ is independent of $\delta \in (0; \delta_0]$.

In order to prove this result we proceed by contradiction.
We make the link with the results known for the case of a fixed δ.
Difficulties:
- prove the uniform (in δ) invertibility of the scalar operators A^δ_ϵ and B^δ_μ ;
- prove a uniform compactness property.

Operators $A^\delta_\epsilon : H^1_0(\Omega) \to H^1_0(\Omega)$ and $B^\delta_\mu : H^1_\#(\Omega) \to H^1_\#(\Omega)$ are s.t.

$$\begin{array}{rcll} (\nabla(A^\delta_\epsilon \varphi), \nabla\varphi') & = & (\epsilon^\delta \nabla\varphi, \nabla\varphi'), & \forall \varphi, \varphi' \in H^1_0(\Omega), \\ (\nabla(B^\delta_\mu \varphi), \nabla\varphi') & = & (\mu^\delta \nabla\varphi, \nabla\varphi'), & \forall \varphi, \varphi' \in H^1_\#(\Omega). \end{array}$$

Step 3: the scalar operators A_ϵ^δ and B_μ^δ

Theorem (B., Chesnel, Ramdani, Rihani, AFST, 2021)

There exist two positive constants m and M such that.
When $\kappa_\epsilon \in (-\infty; -1/m) \cup (-1/M; 0)$, $A_\epsilon^\delta : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ is uniformly invertible as $\delta \to 0$.
When $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$, $B_\mu^\delta : \mathrm{H}_\#^1(\Omega) \to \mathrm{H}_\#^1(\Omega)$ is uniformly invertible as $\delta \to 0$.

1. Case of operator A_ϵ^δ and $\kappa_\epsilon \in (-1/M; 0)$: **B.**, Ramdani 2016
2. Case of operator A_ϵ^δ: Bonnetier, Dapogny, Triki, 2019
3. Case of operators A_ϵ^δ and B_μ^δ: **B.**, Chesnel, Ramdani, Rihani, 2021
4. Case of operator A_ϵ^δ and $\kappa_\epsilon \in (-\infty; -1/m)$: **B.**, Ramdani, Timofte, 2021

Remark: The constants m and M depend on the geometry of the elementary cell Y.

Step 3: the scalar operator A_ϵ^δ ($f \in L^2(\Omega)$ given)

Consider the scalar problem : find $\varphi^\delta \in H_0^1(\Omega)$ such that

$$b(\varphi^\delta, \varphi') = (f, \varphi'), \quad \forall \varphi' \in H_0^1(\Omega),$$

$$\text{where } b(\varphi^\delta, \varphi') = (\epsilon^\delta \nabla \varphi^\delta, \nabla \varphi')$$

There are different methods to study this problem:

- the integral approach
- the T-coercivity approach
- the Neumann-Poincaré operator approach

Step 3: the scalar operator A_ϵ^δ

The T-coercivity approach allows to present the sign-changing coefficient problem as a coercive variational formulation. For the local problem $(\mathcal{P}^{\rm loc})$, this result is due to Bonnet-Ben Dhia, Ciarlet, Zwölf (*J. Comput. Appl. Math.*, 2008).

Definition (T-coercivity)

Let V be a Hilbert space endowed with the norm $\|\cdot\|$ and let $\mathbf{T} \in \mathcal{L}(V)$ a bounded operator on V. A bilinear form $b(\cdot,\cdot)$ defined on $V \times V$ is T-coercive if there exists $\alpha > 0$ such that

$$|b(v, \mathbf{T}v)| \geq \alpha \|v\|^2, \quad \forall v \in V.$$

Difficulty: the construction of the T-coercivity operators $\mathcal{T}^\delta$.

Step 3: The T-coercivity operators $\mathcal{T}^\delta$

- $\kappa_\epsilon \in (-1/M;0)$; $\mathcal{T}^\delta : H_0^1(\Omega) \to H_0^1(\Omega)$ such that

$$\mathcal{T}^\delta \varphi = \left| \begin{array}{ll} \varphi & \text{in } \Omega_1^\delta \\ -\varphi + 2P^\delta\varphi & \text{in } \Omega_2^\delta \end{array} \right.$$

where P^δ denotes the harmonic extension operator from Ω_1^δ to Ω_2^δ i.e. the operator such that $P^\delta\varphi$ solves the problem

$$\left| \begin{array}{rcll} \Delta(P^\delta\varphi) & = & 0 & \text{in } \Omega_2^\delta \\ P^\delta\varphi & = & \varphi & \text{on } \partial\Omega_2^\delta. \end{array} \right.$$

We have $\mathcal{T}^\delta \circ \mathcal{T}^\delta = \mathrm{Id}$, which shows that $\mathcal{T}^\delta$ is an isomorphism of $H_0^1(\Omega)$.
One has $M = [1 + (C_{PW}(Y_1))^2] \|P\|_{\mathcal{L}(H^1(Y_1);H^1(Y))}$.

- $\kappa_\epsilon \in (-\infty; -1/m)$: we need to construct an extension operator from the interior set Ω_2^δ to the exterior set Ω_1^δ ; the proof is more involved.

Conclusions

For the sign-changing Maxwell problem with periodic oscillating coefficients, stated in a two-component domain, we proved that
if κ_ϵ and κ_μ belong to $\mathrm{I} = (-\infty; -1/m) \cup (-1/M; 0)$
then we can homogenize, the initial and the homogenized problems are well-posed.

The positive constants m and M are related to the geometry of the periodicity cell Y.

Remark

One has $-1 \notin \mathrm{I}$.

References

A.Bonnet-Ben Dhia, L.Chesnel, P.Ciarlet, *T-coercivity for the Maxwell problem with sign-changing coefficients*, Commun. Part. Diff. Eq., 39, 1007-1031, 2014.
P.Ciarlet Jr., S.Fliss, C.Stohrer, *On the approximation of electromagnetic fields by edge finite elements. Part 2 : a heterogeneous multiscale method for Maxwell's equations*, Comput. Math. Appl., 73, 1900-1919, 2017.

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B., K.Ramdani, *Homogenization of materials with sign-changing coefficients*, Comm. Math. Sci., 14, 132, 1137-1154, 2016.
E. Bonnetier, C.Dapogny, F.Triki, *Homogenization of the eigenvalues of the Neumann-Poincaré operator*, Arch. Rational Mech. Anal., 234, 777-855, 2019.
B., L.Chesnel, K.Ramdani, M.Rihani, *Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients*, Ann. de la Faculté des Sciences de Toulouse : Mathématiques, 6, 30 (5), 1075-1119, 2021.
B., K.Ramdani, C.Timofte, *T-coercivity for the asymptotic analysis of scalar problems with sign-changing coefficients in thin periodic domains*, EJDE, 59, 1-22, 2021.

Thank you for your attention.