# Ground state (in-)stability and long-time behavior in multi-dimensional Schrödinger equations

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# Basic setting

A canonical model in the description of dispersive, weakly nonlinear waves is the nonlinear Schrödinger equation. In its most basic form:

$$i\partial_t u = -\Delta u + \lambda |u|^{p-1} u, \quad u_{|t=0} = u_0. \tag{NLS}$$

Here $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, for $d = 1, 2, 3$ and $p > 1$, while $\lambda = \pm 1$ distinguished between the defocusing and focusing case.

This model appears in, e.g., nonlinear fiber optics, or Bose-Einstein condensation. By scaling invariance, if $u(t, x)$ is a solution then so is

$$u_\kappa(t, x) = \kappa^{2/(p-1)} u\big(\kappa^2 t, \kappa x\big), \quad \kappa > 0.$$

This scaling leaves the $\dot{H}^s$-norm invariant for critical $s_c = \frac{d}{2} - \frac{2}{p-1}$,

# Basic conservation laws

- Mass: $M(u) = \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} = M(u_0)$
- Momentum:

$$P(u) = \operatorname{Im} \int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)dx = P(u_0)$$

- Energy:

$$E(u) = \frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \frac{\lambda}{p+1}\|u(t,\cdot)\|^{p+1}_{L^{p+1}} = E(u_0)$$

The natural energy-space for solutions is the Sobolev space

$$H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d), \text{ for } p \leqslant \tfrac{2d}{(d-2)_+}.$$

# Solitary waves

In the focusing case $\lambda = -1$, the competition between dispersion and nonlinearity allows for the existence of standing waves:

**Definition (Standing wave)**

A standing wave of (NLS) is a solution $u = e^{i\omega t}\phi_\omega(x)$, with $\omega \in \mathbb{R}$ and

$$-\Delta\phi_\omega + \omega\phi_\omega - |\phi_\omega|^{p-1}\phi_\omega = 0, \quad \phi_\omega \in H^1(\mathbb{R}^d) \setminus \{0\}.$$

By Noether's theorem:

conservation laws $\Leftrightarrow$ symmetries of the NLS.

Using the Galilei-invariance, one obtains moving solitary waves:

$$u(t,x) = \phi_\omega(x - p_0 t - x_0)e^{i(\omega t + p_0\cdot x - |p_0|^2 t/2)}, \ x_0, p_0 \in \mathbb{R}^d.$$

# Finite time blow-up

In the focusing case $\lambda = -1$, we also face the possibility of finite-time blow-up[1], i.e. the existence of a $T = T(u_0) < \infty$, such that

$$\lim_{t \to T} \|\nabla u(t, \cdot)\|_{L^2} = +\infty.$$

In particular, for $d = 2$ and $p = 3$ (the mass-critical case in 2D), finite time blow-up is possible, as soon as

$$M(u_0) \geqslant M(Q),$$

where $Q \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ the unique positive standing wave solution

$$-\tfrac{1}{2}\Delta Q + \omega Q - Q^3 = 0, \quad \text{with } \omega = 1,$$

called 2D nonlinear ground state[2].

[1] Zakharov-Shabat '72

[2] Weinstein '83, Merle-Raphael '04...

# Scattering

$Q$ is known to yield the best constant in the sharp Gagliardo-Nirenberg inequality[3], i.e.

$$\|u\|_{L^4}^4 \leqslant C_{\mathrm{GN}} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2, \quad C_{\mathrm{GN}} = \|Q\|_{L^2}^{-2}.$$

On the other hand, for $d = 2$, $p = 3$ and $M(u_0) < M(Q)$, the solution $u$ to (NLS) is known to behave asymptotically linear[4], i.e. $\exists u_\pm \in H^1(\mathbb{R}^2)$ s.t.

$$\lim_{t \to \pm\infty} \|u(t, \cdot) - e^{i\frac{t}{2}\Delta} u_\pm\|_{H^1} = 0.$$

Q: What is the dynamics in situations where blow-up is prohibited, but standing waves are still present?

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[3] Weinstein '83

[4] Dodson '15

# Basic properties of cubic-quintic NLS

To avoid finite time blow-up, we shall regularize the (NLS) by considering a combination of competing focusing and defocusing nonlinearities,

$$i\partial_t u + \Delta u = -|u|^{p-1}u + |u|^{q-1}u, \quad q > p.$$

We thereby lose the scaling invariance of solutions!
For simplicity, we will mainly look at the case of the cubic-quintic NLS in dimensions $d \leqslant 3$, i.e.

$$i\partial_t u + \Delta u = -|u|^2 u + |u|^4 u, \quad u_{|t=0} = u_0. \tag{cqNLS}$$

This equation models, e.g., dense Bose-Einstein condensates with combined two- and three-particle interactions.

# Global well-posedness

## Proposition (Global well-posedness)

*Let* $d \leqslant 3$. *For any* $u_0 \in H^1(\mathbb{R}^d)$, (cqNLS) *has a unique global solution* $u \in C(\mathbb{R}; H^1(\mathbb{R}^d))$. *The solution obeys the conservation laws of mass, momentum, and energy, i.e.*

$$E(u) = \frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R}^d)} + \frac{1}{6}\|u(t,\cdot)\|^6_{L^6(\mathbb{R}^d)}.$$

*and if, in addition,* $u_0 \in \Sigma := H^1(\mathbb{R}^d) \cap \mathcal{F}(H^1(\mathbb{R}^d))$, *then* $u \in C(\mathbb{R}; \Sigma)$.

Energy-conservation, combined with $\|u\|^4_{L^4(\mathbb{R}^d)} \leqslant \|u\|_{L^2(\mathbb{R}^d)}\|u\|^3_{L^6(\mathbb{R}^d)}$, shows that the focusing part cannot obstruct global existence[5].

[5] The problem is energy critical in $d = 3$, cf. Zhang '06

Numerically, one observes a oscillatory behavior within the solution $u$.

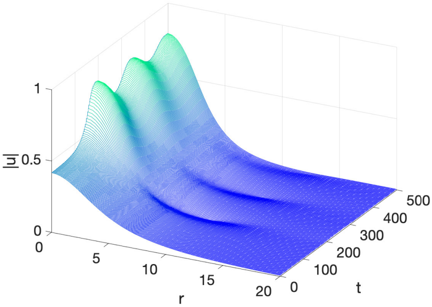



Figure: The time-evolution for a radial solutions $u = u(t, |x|)$ in $d = 3$.

# Basic properties of solitary waves

We look for standing wave solutions to the cubic-quintic model:

$$-\Delta\phi_\omega + \omega\phi_\omega - |\phi_\omega|^2\phi_\omega + |\phi_\omega|^4\phi_\omega = 0, \quad \phi_\omega \in H^1(\mathbb{R}^d) \setminus \{0\}. \quad \text{(sNLS)}$$

**Lemma (A priori estimates for solitary waves)**

*Any $\phi_\omega \in H^1(\mathbb{R}^d)$ solution to (sNLS), satisfies the Pohozaev identities:*

$$\int_{\mathbb{R}^d} |\nabla\phi_\omega|^2\,dx + \int_{\mathbb{R}^d} |\phi_\omega|^6\,dx + \omega\int_{\mathbb{R}^d} |\phi_\omega|^2\,dx = \int_{\mathbb{R}^d} |\phi_\omega|^4\,dx,$$

$$\frac{d-2}{2}\int_{\mathbb{R}^d} |\nabla\phi_\omega|^2\,dx + \frac{d}{6}\int_{\mathbb{R}^d} |\phi_\omega|^6\,dx + \omega d\int_{\mathbb{R}^d} |\phi_\omega|^2\,dx = \frac{d}{4}\int_{\mathbb{R}^d} |\phi_\omega|^4\,dx.$$

1. A first consequence of these identities is that if $\phi_\omega \not\equiv 0$, then

$$0 < \omega < \frac{3}{16},$$

the admissible frequency range.

2. A second consequence is that in $d = 2$:

$$\|\phi_\omega\|_{L^2} > \|Q\|_{L^2},$$

where $Q$ is the cubic ground state solution. Thus, the mass $M(\phi_\omega)$ is strictly bigger than in the cubic case. In 3D, one even knows[6] that $M(\phi_\omega) \to +\infty$, as $\omega \to 0$.

[6] Killip et al. '17

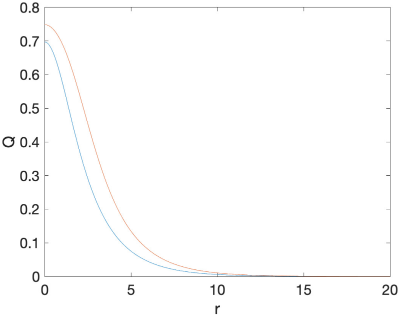



Figure: Ground state solutions $Q_{\omega=0.1}$ to the cubic NLS in blue and the cubic-quintic NLS in red: on the left for $d = 2$ and on the right for $d = 3$.

The strict lower bound $M(\phi_\omega) > M(Q)$ holds more generally (e.g. the cubic-quartic case[7]) and changes the 2D scattering threshold in comparison to the cubic case.

**Theorem (Carles - S. '21)**

*In $d = 2$, if $M_0 \leqslant \|Q\|_{L^2}^2$, then the solution $u \in C(\mathbb{R}; \Sigma)$ to (cqNLS) satisfies*

$$\|e^{-i\frac{t}{2}\Delta}u(t,\cdot) - u_\pm\|_\Sigma \underset{t\to\pm\infty}{\longrightarrow} 0.$$

The proof is based pseudo-conformal conservation law

$$\frac{d}{dt}\left(\frac{1}{2}\|J(t)u\|_{L^2}^2 - \frac{t^2}{2}\|u\|_{L^4}^4 + \frac{t^2}{3}\|u\|_{L^6}^6\right) = -\frac{2t}{3}\|u\|_{L^6}^6. \qquad (1)$$

where $J(t) = x + it\nabla$.

[7] Arora-S. '23

Rigidity results for $L^2$-critical NLS[8] allow us to infer

$$J(t)u \in L^\infty(\mathbb{R}_t; L^2(\mathbb{R}^2)),$$

and general Gagliardo–Nirenberg inequalities, for $2 \leqslant r < \infty$, yield

$$\|u(t)\|_{L^r(\mathbb{R}^2)} \lesssim \|u(t)\|_{L^2(\mathbb{R}^2)}^{1-\theta} \left(\frac{1}{t}\|J(t)u\|_{L^2}\right)^\theta, \quad \theta = 1 - \frac{2}{r}. \tag{2}$$

This implies $u \in L^q(\mathbb{R}; L^r(\mathbb{R}^2))$ for all admissible Strichartz pairs $(q, r)$, i.e. the solution is purely dispersive.

For energy-subcritical $q > p$ power-law nonlinearities the assumption $u_0 \in \Sigma$ can be relaxed[9] to $u_0 \in H^1(\mathbb{R}^2)$. In general, however, the situation seems to be rather subtle, since there are examples of nonlinearities, which yield $\phi_\omega$ with arbitrarily small $H^1$-norm[10].

[8] Banica '04

[9] Cheng 20'

[10] Carles-S. '23

# (In-)stability of standing waves

Q: What about the long-time behavior of perturbations of standing waves?

## Definition (Orbital stability)

For all $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if

$$\|u_0 - \phi_\omega\|_{H^1(\mathbb{R}^d)} \leqslant \delta,$$

then the solution to (cqNLS) satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\substack{\theta\in\mathbb{R}\\ y\in\mathbb{R}^d}} \left\| u(t,\cdot) - e^{i\theta}\phi_\omega(\cdot - y)\right\|_{H^1(\mathbb{R}^d)} \leqslant \varepsilon.$$

For this stability statement it is necessary to take into account the symmetries of the (sNLS), i.e. phase-conjugation and spatial shifts.

In $d = 1$, standing waves to the cubic-quintic NLS solve

$$-\phi_\omega'' + \omega\phi_\omega - |\phi_\omega|^2\phi_\omega + |\phi_\omega|^4\phi_\omega = 0, \quad \phi_\omega \in H^1(\mathbb{R}^d) \setminus \{0\}.$$

The solution to this ODE can be computed explicitly for $\omega \in (0, \frac{3}{16})$:

**Proposition (Ohta '95)**

*The unique (up to translation and change of sign) positive solution which decays as $|x| \to \infty$ is given by:*

$$\phi_\omega(x) = 2\sqrt{\frac{\omega}{1 + \sqrt{1 - \frac{16\omega}{3}}\cosh\left(2x\sqrt{2\omega}\right)}}.$$

*Moreover, this solution is orbitally stable.*

The proof uses ODE arguments which do not carry over to higher dimensions $d \geqslant 2$.

# Existence of action minimizers

A first approach to the existence of standing waves is based on action-minimizers. For given $\omega \in (0, \frac{3}{16})$, denote the action of $\phi \in H^1(\mathbb{R}^d)$ by

$$S_\omega(\phi) = E(\phi) + \omega M(\phi),$$

and note that standing waves are critical points, i.e., $S'_\omega(\phi_\omega) = 0$.

**Definition**

For $\omega \in (0, \frac{3}{16})$, a solution $\phi_\omega$ to (sNLS) is called an action-minimizing ground state, if it minimizes $S_\omega(\phi)$ among all solutions $0 \neq \phi \in H^1(\mathbb{R}^d)$.

Existence of such action minimizers has been proved using various variational techniques[11].

[11] Cazenave-Lions '81, Byeon and Jeanjean '20, ...

By combining results by Berestycki et al. '83, Serrin-Tang '00, and Jang '10, one finds:

## Proposition (Existence and uniqueness of action ground states)

*Let $\omega \in (0, \frac{3}{16})$. Then $\exists$ a unique real-valued solution $\phi_\omega \in C^2(\mathbb{R}^d)$, s.t.:*

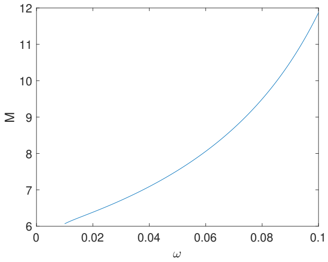
1. $\phi_\omega > 0$ *on* $\mathbb{R}^d$.
2. $\phi_\omega$ *is radially symmetric and non-increasing.*
3. *Derivatives of* $\phi_\omega$ *up to order two decay exponentially as* $|x| \to \infty$.
4. *For every solution* $\phi$ *to* (sNLS)*:* $0 < S_\omega(\phi_\omega) \leqslant S_\omega(\phi)$.
5. *Every action-minimizer is of the form*

$$\phi(x) = e^{i\theta}\phi_\omega(x - x_0).$$

# A first approach to (in-)stability

Let $M(\phi_\omega) = \|\phi_\omega\|_{L^2}^2$ be the ground state mass. By studying the Hessian of $S(\phi)$ at $\phi = \phi_\omega$, one finds[12]:

1. If $\frac{\partial}{\partial\omega} M(\phi_\omega) > 0$, then $e^{i\omega t}\phi_\omega(x)$ is orbitally stable.
2. If $\frac{\partial}{\partial\omega} M(\phi_\omega) < 0$, then $e^{i\omega t}\phi_\omega(x)$ is unstable.



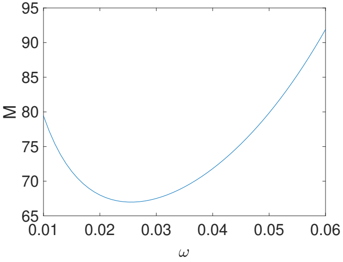




Figure: A numerical plot of $M(\phi_\omega)$ for $d = 2$ and $d = 3$.

[12] Weinstein '85, Grillakis-Shatah-Strauss '87

# A partial stability result

In $d = 2$, we expect all ground states to be orbitally stable. Unfortunately, we can only prove that $\omega \mapsto M(\phi_\omega)$ is increasing asymptotically near $\omega = 0$ and near $\omega = \frac{3}{16}$.

**Theorem (Carles-S. '20)**

*Let $d = 2$. There exist $0 < \omega_0 \leqslant \omega_1 < \frac{3}{16}$ s.t. for $\omega \in (0, \omega_0) \cup (\omega_1, \frac{3}{16})$, $\phi_\omega$ is orbitally stable.*

Similarly, for $d = 3$, Killip et al. '17 have shown that $\omega \mapsto M(\phi_\omega)$ is decreasing near $\omega = 0$ and increasing near $\omega = \frac{3}{16}$. Hence $\phi_\omega$ is unstable for $\omega \in (0, \tilde{\omega}_0)$ and stable for $\omega \in (\tilde{\omega}_1, \frac{3}{16})$.

It is conjectured, that $\tilde{\omega}_0 = \tilde{\omega}_1 = \omega_* \in (0, \frac{3}{16})$.

# Asymptotic analysis

To prove the first part, we turn the singular limit $\omega \to 0$ into a regular one, by rescaling: $\psi_\omega(x) = \frac{1}{\sqrt{\omega}}\phi_\omega\left(\frac{x}{\sqrt{\omega}}\right)$. Then (sNLS) becomes

$$-\Delta\psi_\omega + \psi_\omega - \psi_\omega^3 + \omega\psi_\omega^5 = 0.$$

Invoking uniqueness of $Q = Q_{\omega=1}$ and the implicit function theorem, we have

$$\psi_\omega(x) = Q(x) - \omega\left(L^{-1}Q^5\right)(x) + \mathcal{O}_{H^1}(\omega^2), \text{ as } \omega \to 0.$$

In particular, as $\omega \to 0$, one finds:

$$M(\phi_\omega) \equiv M(\psi_\omega) = M(Q) + \frac{2\omega}{3}\|Q\|^6_{L^6(\mathbb{R}^d)} + \mathcal{O}(\omega^2).$$

# The concentration-compactness approach

A second approach, pioneered by Lions & Berestycki '84, considers the set of constrained energy minimizers: For a fixed mass $\rho > 0$, denote

$$\Gamma(\rho) = \Big\{ u \in H^1(\mathbb{R}^d), \ M(u) = \rho \Big\}.$$

**Definition (Energy minimizers)**

Assuming that the minimization problem

$$E(u) = \inf\{E(v) \ ; \ v \in \Gamma(\rho)\} \tag{EM}$$

has a solution, we denote by $\mathcal{E}(\rho)$ the set of all (constrained) energy-minimizing ground states.

Note: energy-minimizer are not necessarily unique, i.e., $\mathcal{E}(\rho) \neq \{\varphi\}$.

If $\varphi \in \mathcal{E}(\rho)$, then there exists a Lagrange multiplier $\mu = -\omega$ such that

$$E'(\varphi) = \mu M'(\varphi) \;\Leftrightarrow\; E'(\varphi) + \omega M'(\varphi) = 0,$$

and thus, $\varphi$ solves the stationary Schrödinger equation (sNLS) for some (unknown) $\omega = -\mu \in (0, \frac{3}{16})$.

### Definition

We call the set $\mathcal{E}(\rho)$ orbitally stable, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if $u_0 \in H^1(\mathbb{R}^2)$ satisfies

$$\inf_{\varphi \in \mathcal{E}(\rho)} \|u_0 - \varphi\|_{H^1} \leqslant \delta,$$

then the solution to (cqNLS) with $u_{|t=0} = u_0$ satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\varphi \in \mathcal{E}(\rho)} \|u(t, \cdot) - \varphi\|_{H^1} \leqslant \varepsilon.$$

# Orbital stability of energy minimizers

Using the conservation laws of energy and mass, implies that energy minimizers are automatically stable (as a set)[13].

**Theorem (Carles-S. '20)**

1. *Let $d = 2$. Then, for any $\rho > \|Q\|_{L^2}^2$,*

$$E_{\min}(\rho) := \inf_{\Gamma_\rho} E(v) < 0,$$

*implying that the set $\mathcal{E}(\rho) \neq \emptyset$ and orbitally stable.*

2. *If $d = 3$, $\exists\, \rho_1 > \rho_0$ such that for $\rho \geqslant \rho_1$: $E_{\min}(\rho) < 0$, $\mathcal{E}(\rho) \neq \emptyset$ and energy minimizers are $\mathcal{E}(\rho)$-orbitally stable.*

The minimal mass $\rho_0$ in 3D, is related to a Sobolev-minimizer, which, unlike in 2D, cannot be described via a stationary solution $Q$[14].

[13] Cazenave-Lions '82

[14] Killip-Visan '17

# Equivalence of ground states?

We already know that for given $\rho > 0$, all energy minimizers $\varphi \equiv \varphi(\rho)$ satisfy

$$-\Delta\varphi + \omega\varphi - |\varphi|^2\varphi + |\varphi|^4\varphi = 0,$$

for some $\omega \in (0, \frac{3}{16})$, the Lagrange multiplier associated to $\rho$.

Using re-arrangement inequalities, one can show that $\varphi$ is real-valued and radially decreasing.

Hence, one might think that the two notions of action-ground states and energy-ground states are equivalent. However:

**Theorem (Carles-Klein-S. '23)**

*In $d = 3$, not all action ground states are energy minimizers.*

Recall the rescaling $\psi_\omega(x) = \frac{1}{\sqrt{\omega}}\phi_\omega\left(\frac{x}{\sqrt{\omega}}\right)$, and the fact that

$$\psi_\omega = Q + \mathcal{O}_{H^1}(\omega), \quad \text{as } \omega \to 0.$$

In $d = 3$, Pohozaev identities imply that the energy satisfies

$$E(\phi_\omega) = \sqrt{\omega}\|Q\|^2_{L^2(\mathbb{R}^3)} + \mathcal{O}(\omega).$$

Thus $\exists\, \omega^* > 0$ such that $E(\phi_\omega) > 0$ for all $\omega \in (0, \omega^*)$. Since[15]

$$M(\phi_\omega) = \frac{1}{\sqrt{\omega}}M(Q) + \frac{\sqrt{\omega}}{2}\|Q\|^2_{L^6(\mathbb{R}^3)} \underset{\omega\to 0}{\longrightarrow} +\infty,$$

this shows that there exists 3D action ground states with positive energy and arbitrarily large mass. However, we know that for mass $\rho \geqslant \rho_1$, all energy minimizers $\varphi$ satisfy $E_{\min}(\rho) < 0$.

[15] Killip et al. '17

Two problems:

1. A-priori nothing guarantees that an element of $\mathcal{E}(\rho)$ minimizes the action.
2. A more subtle problem: consider a least action ground state $\phi_\omega$, and let $\rho = M(\phi_\omega)$. It is not obvious, and not necessarily true, that $\phi_\omega \in \mathcal{E}(\rho)$. In particular, the map

$$\Lambda : \rho \mapsto \omega \subset (0, \tfrac{3}{16})$$

may not be one-to-one (even ran $\Lambda$ is unclear at this point).

Indeed: Equivalence of the two notions is only known for NLS with a single power law nonlinearity $|u|^{p-1}u$ and $p < 1 + \frac{4}{d}$. The proof is based on a scaling argument which requires homogeneity of the nonlinearity[16].

[16] Cazenave '03

Recently, Jeanjean-Lu '21 have shown for a large class of nonlinearities (including the cubic-quintic case), that every energy minimizer $\varphi$ is a least action ground states $\phi_\omega$ for $\omega = \omega(\varphi) > 0$.

They also prove that if $\omega$ is obtained as the Lagrange multiplier associated to the mass constrained $M(u) = \rho$, then any least action solution of (sNLS) at this value of $\omega$ is a constrained energy minimizer with the same mass $\rho$.

### Conjecture (Cubic-quintic stability in 2D)

*In view of our numerics* (20)*, we conjecture that in $d = 2$ there is full equivalence between energy- and action-ground states, and that all of them are orbitally stable*[a]*.*

[a] Lewin & Rotar-Nodari '20, Carles-Klein-S.'21

# Numerical results

In $d = 3$ however, there are potentially unstable action-ground states which are not energy minimizers.

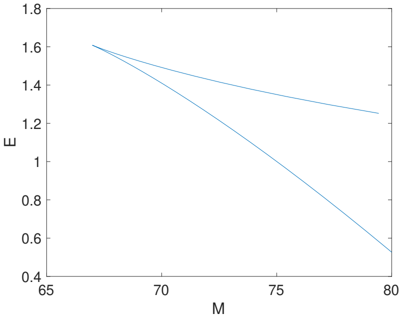



Figure: $E(\phi_\omega)$ as a function of $M(\phi_\omega)$ for cubic-quintic ground states $d = 3$.

The numerical simulations we did for unstable 3D ground states under radial perturbations, yield:

**Conjecture (Cubic-quintic (in-)stability in 3D)**

*For $\omega < \omega_*$, i.e. on the unstable branch, consider*

$$u_0(x) = \phi_\omega(x) + \chi(|x|), \quad \textit{with } \|\chi\|_{H^1} \ll 1.$$

1. *If $M(u_0) < M(\phi_\omega)$, the solution $u(t,\cdot)$ scatters.*
2. *If $M(u_0) > M(\phi_\omega)$, the solution $u(t,\cdot) \underset{t\to\infty}{\sim} e^{i\underline{\omega} t}\phi_{\underline{\omega}}(x)$, where $\phi_{\underline{\omega}}$ is some stable ground state with mass $M(\phi_{\underline{\omega}}) < M(\phi_\omega)$[a].*

[a] Carles-Klein-S.'21

Q: What is the selection principle for $\phi_{\underline{\omega}}$?

Thank you for your attention!