Multi-phase Minimizers for the Allen-Cahn System on the plane

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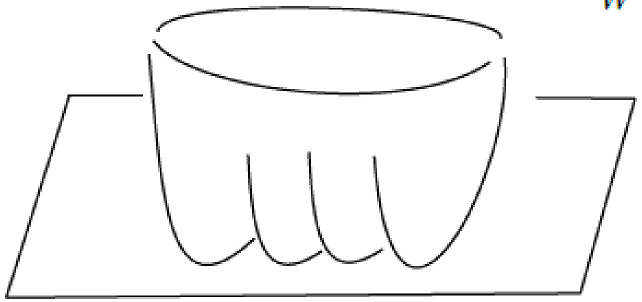
(1) $$J(u;\Omega) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx\ , \quad u\ :\ \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$$

(2) $$\Delta u - W_u(u) = 0 \quad , \quad u : \mathbb{R}^2 \to \mathbb{R}^2$$



<u>Hypotheses on W</u>

(H1) $\{W = 0\} = \{a_1, ..., a_N\} =: A\ ,\ W \geq 0\ ,\ W \in C^2(\mathbb{R}^2)$
$\partial^2 W(a_i) \geq c_1 I\ ,\ W_u \cdot u > 0\ ,\ |u| > M\ ,\ u \in \mathbb{R}^2$

(H2) Existence of all <u>Connections</u>

$$\sigma_{ij} = \min\left\{\int_{\mathbb{R}} (\frac{1}{2}|U'|^2 + W(U))d\eta\ ,\ \lim_{\eta\to+\infty} U(\eta) = a_i\ ,\ \lim_{\eta\to-\infty} U(\eta) = a_j\right\}$$

$(\exists\ U_{ij}\ ,\ i \neq j \in \{1, 2, ..., N\})$

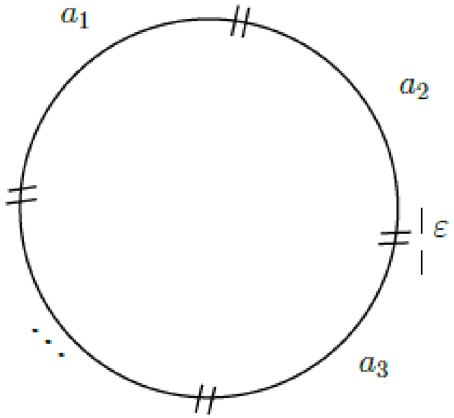
$\sigma_{ij} < \sigma_{ik} + \sigma_{kj}\ \ (i \neq j\ ,\ i, j \in \{1, 2, ..., N\} \setminus \{k\})$

$$\Sigma_N = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} & \cdots \\ \sigma_{12} & 0 & \sigma_{23} & \cdots \\ \vdots & & & \end{pmatrix} \qquad \frac{N(N-1)}{2}$$

(I) Dirichlet Problem

$\Omega = B_1 = \{|x| < 1\}$, $\tilde{A} \subset A$, $|\tilde{A}| = \tilde{N} \leq N$

$$\min_{u=g_\varepsilon} \int_{B_1} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) dx$$



(H3) $g_\varepsilon \in \{\tilde{A}$, except $\varepsilon -$ transitions $\}$, phases on ∂B_1 connected

$$I_{\varepsilon,\gamma} := \{z \in B_1 \mid |u_\varepsilon(z) - a_i| > \gamma \, , \; \forall \; a_i \in A\}$$

$$(\, \gamma = \gamma_\varepsilon \,)$$

$$\Omega_a = \{x \in B_1 \mid |u_\varepsilon - a| \leq \gamma\} \;\; , \; a \in A$$

Problem: Geometric Description of $I_{\varepsilon,\gamma}$ and Ω_a for small $\varepsilon > 0$.

(II) Entire Solution

$\exists \; u : \mathbb{R}^2 \to \mathbb{R}^2$, minimizer, $\Delta u - W_u(u) = 0$, $|u(x)| < M$, connecting the N phases at ∞.

Problem: Geometric Description of $I_{\varepsilon,\gamma}$ and Ω_a In particular, the asymptotic behavior at ∞:

$$\text{(Blow-down)} \;\; u(rx) \xrightarrow[r\to\infty]{L^1_{loc}} u_0(x) = \sum_{i=1}^{N} a_i \chi_{D_i} = \text{minimal cone}$$

$$\mathcal{P} = \{D_1, ..., D_N\} \; , \;\; \text{partitioning} \;\; \mathbb{R}^2$$

Definition : u minimizing if

$$J(u+v;B_R(x_0)) \geq J(u;B_R(x_0)) \quad \forall\ v \in C_c^1(B_R(x_0);\mathbb{R}^2) \quad \forall\ B_R(x_0) \subset \mathbb{R}^2 \tag{3}$$

Definition : $\mathcal{P}$ minimizing partition in $\mathbb{R}^2$

$$\forall\, \Omega \subset \mathbb{R}^2\ ,\ \mathcal{P} \llcorner \Omega = \{D_i \cap \Omega\}_{i=1}^N \quad \text{solves}$$

$\mathcal{P}$ solves

$$\min_{\mathcal{A}} \sum_{0<i<j\leq N} \sigma_{ij} \mathcal{H}^1(\partial(A_i \cap \Omega) \cap \partial(A_j \cap \Omega))$$

$$\text{(no energy associated to } \partial\Omega)$$

$$\mathcal{A} = \{A_i\}_{i=1}^N\ ,\ N-\text{partition}\ ,\ \mathcal{P}\llcorner(\mathbb{R}^2 \setminus \Omega) = \mathcal{A}\llcorner(\mathbb{R}^2 \setminus \Omega)$$

A_i not necessarily connected.

Shortcomings of the general theory

1. No Monotonicity Formula for the Allen-Cahn System

(contrary to the scalar Allen-Cahn)

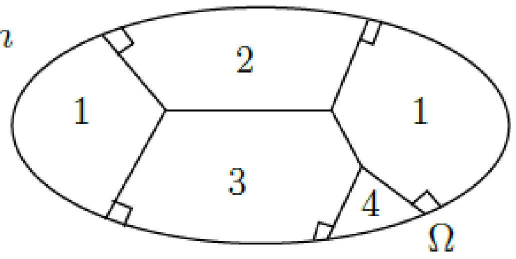
2. No rate of convergence for the

$$\Gamma - \lim_{\varepsilon \to 0} \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx = \sum_{0<i<j<N} \sigma_{ij} \mathcal{H}^1(\partial(A_i \cap \Omega) \cap \partial(A_j \cap \Omega)) \tag{4}$$

On the other hand

The monotonicity formula holds for minimal partitions, that is minimizers of the right hand side of (4).

$$\mathcal{E}(\mathcal{P}) = \sum_{1\le i<j\le N} \sigma_{ij}\mathcal{H}^1(\partial P_i \cap \partial P_j)\ ,\ \mathcal{P}\ partition$$



(phase transition functional)

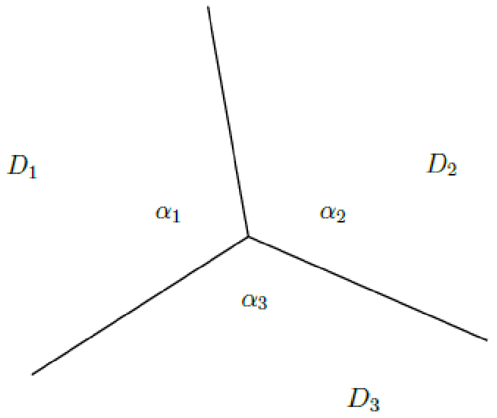
$$\mathcal{F}(\mathcal{P}) = \sum_{j=1}^{N} c_j \mathcal{H}^1(\partial P_j)\ \ ,\ c_j > 0$$

(weighted perimeter functional)

$$\frac{N(N-1)}{2} > N\ \ ,\ for\ N \ge 4$$

$$ONLY\ for\ \sigma_{ij} = \sigma\ \ or\ \ N = 2, 3$$

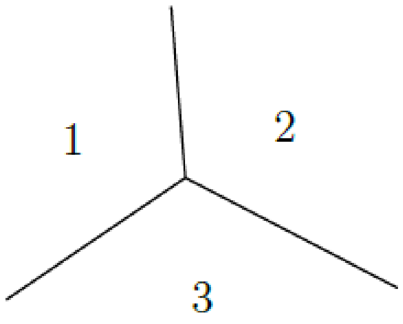
$$\mathcal{E} \equiv \mathcal{F}$$

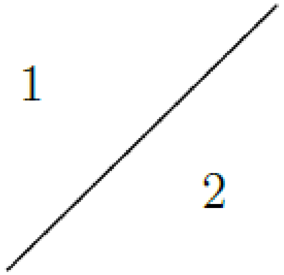


$$\frac{sin\ \alpha_1}{\sigma_{23}} = \frac{sin\ \alpha_2}{\sigma_{13}} = \frac{sin\ \alpha_3}{\sigma_{12}}$$

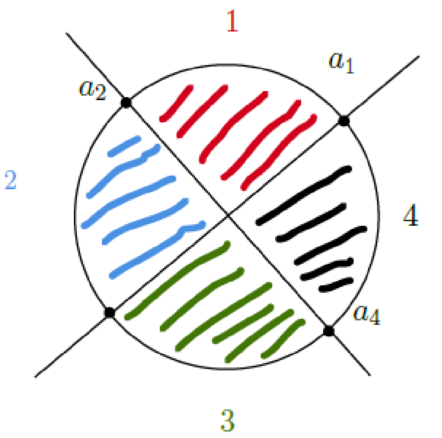
(Young's law)

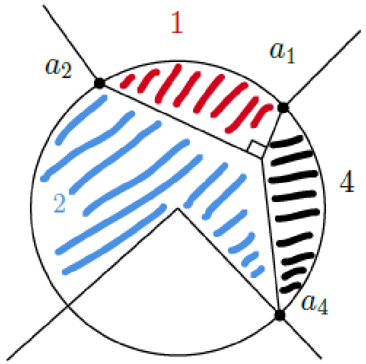
$$\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3}\ ,\ \sigma_{ij} = \sigma$$





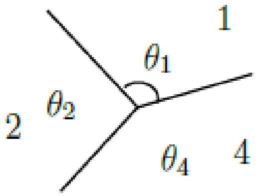
Example of 4-phase Minimal Cone: The Cross





$$\sigma_{12} = \sigma_{23} = \sigma_{34} = \sigma_{41} = 1$$

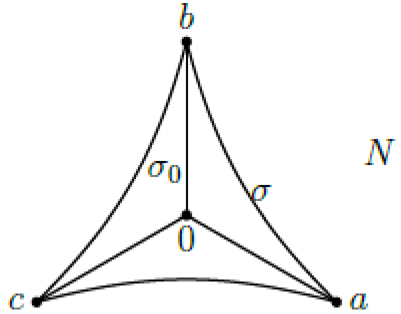
$$\sigma_{24} = \sigma_{13} = \sqrt{2}$$



$$\frac{sin\ \theta_1}{\sigma_{24}} = \frac{sin\ \theta_4}{\sigma_{12}} = \frac{sin\ \theta_2}{\sigma_{14}}$$

$\nexists$ triple junction facing the chord $\overline{a_1 a_2}$ (trigonometry). The cross can not be destabilized.

Proposed List of Minimizers (G.Fusco)
Dirichlet (I) (Three phases on ∂B_1)

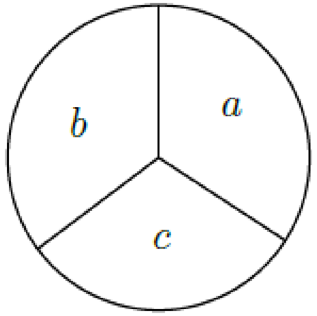


$N = 4\ ,\ \tilde{N} = 3$ Phase Space

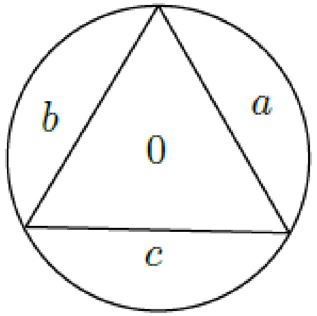
$\{W = 0\} = \{0, a, b, c\}$

$\{W = 0\}$

B_1



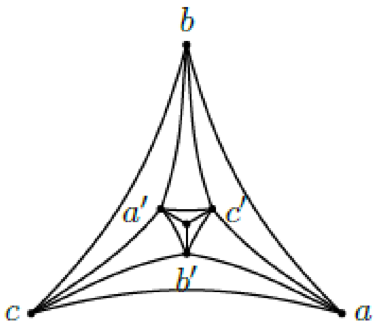
$\frac{2\sigma_0}{\sigma} > \frac{2}{\sqrt{3}}$



B_1

$\frac{2\sigma_0}{\sigma} \in \left(1, \frac{2}{\sqrt{3}}\right)$

0 does not appear on ∂B_1



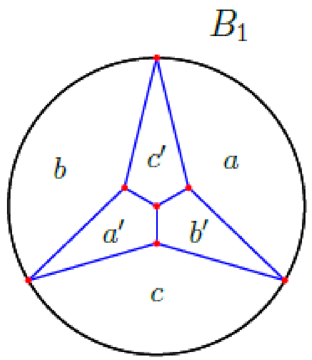
$$N = 7 \qquad \tilde{N} = 3$$

$$\sigma_{ab} = \sigma, \quad \sigma_{a'b'} = \tau$$

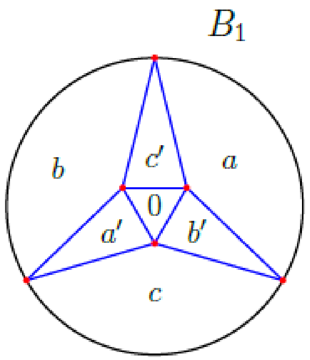
$$\sigma_{a'a} = \sigma_0, \quad \sigma_{0a'} = \tau_0$$

$$\{W = 0\} = \{0, a, b, c, a', b', c'\}$$

$(all \;\; n_3)$



$\frac{2\sigma_0}{\sigma} \in (1, \frac{1}{cos\ \alpha}), \;\; \frac{2\tau_0}{\tau} \in (\frac{2}{\sqrt{3}}, +\infty)$



$\frac{2\sigma_0}{\sigma} \in (1, \frac{1}{cos\ \alpha}), \;\; \frac{2\tau_0}{\tau} \in (1, \frac{2}{\sqrt{3}})$

$(all \;\; n_4)$

n = number of vertices

e = number of edges

f = number of faces

$n - e + f = 2$ (Euler)

$2e = n_1 + 2n_2 + \ldots$

n_j = vertices with j – edges

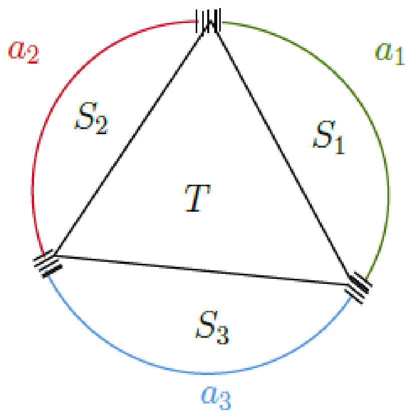
$$n_4^{int} = N_R - \tilde{N} - 1 \qquad n_3^{int} = 2(N_R - 1) - \tilde{N}$$

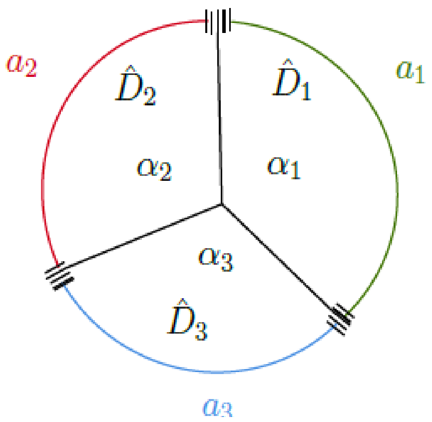
$$e^{int} = 2N_R - \tilde{N} - 2 \qquad e^i = 3(N_R - 1) - \tilde{N}$$

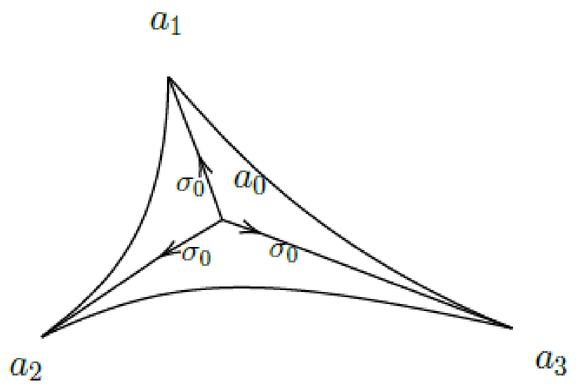
N_R = number of realized phases

Triangle vs Triple Junction - $\{W=0\}=\{a_0,a_1,a_2,a_3\}$

$N=4\ ,\ \tilde{N}=3$







- No Symmetry hypothesis either on W or on Solution
- No Hypothesis of connectedness of the phases Ω_α

$$\frac{\sin\ \alpha_1}{\sigma_{23}}=\frac{\sin\ \alpha_2}{\sigma_{13}}=\frac{\sin\ \alpha_3}{\sigma_{12}}$$

(Young's law)

σ_0 *connections* $a_0\ \to a_i$

σ_{ij} *connections* $a_i\ \to a_j$

$$e:=\frac{1}{2}\max\left\{\frac{sin\alpha_2}{sin\alpha_1},\frac{sin\alpha_3}{sin\alpha_1}\right\}$$

$$c:=2\frac{(sin\frac{\alpha_1}{2})\{1+\frac{sin\frac{\alpha_2}{2}}{sin\frac{\alpha_1}{2}}+\frac{sin\frac{\alpha_3}{2}}{sin\frac{\alpha_1}{2}}\}}{1+\frac{sin\alpha_2}{sin\alpha_1}+\frac{sin\alpha_3}{sin\alpha_1}}$$

THE THEOREMS

(I) Dirichlet

Th 1

(5) $$\begin{cases} \varepsilon\Delta u - \frac{1}{\varepsilon}W_u(u) = 0 \ , \ \{|z|<1\} =: B \\ u = g_\varepsilon \quad , \ z \in \partial B \end{cases}$$

Under (H1),(H2),(H3), (5) admits a classical solution u_ε that minimizes J_ε , $|u_\varepsilon(z)| < M$.

(i) $\left[\frac{\sigma_0}{\sigma_{13}} > \frac{1}{c}\right]$ Triple Junction (A + Geng, ARMA '24)

$$|u_\varepsilon(z) - a_i| \leq Ke^{-\frac{k}{\varepsilon}(d(z,\partial\hat{D}_i\setminus\partial B)-C\varepsilon^{\frac{1}{4}})^+} \ , \ z \in \hat{D}_i$$

(ii) $\left[0 < \frac{\sigma_0}{\sigma_{13}} < \frac{1}{c}\right]$ Triangle (A + Gazoulis)

$$|u_\varepsilon(z) - a_0| \leq Ke^{-\frac{k}{\varepsilon}(d(z,\partial T)-C\varepsilon^{\frac{1}{4}})^+} \ , \ z \in T$$
$$|u_\varepsilon(z) - a_i| \leq Ke^{-\frac{k}{\varepsilon}(d(z,T)-C\varepsilon^{\frac{1}{4}})^+} \ , \ z \in S_i$$

(II) Entire (A + Geng, ARMA '24)

Th 2: Under (H1),(H2), $\frac{\sigma_0}{\sigma_{13}} > \frac{1}{c}$, $\exists \ u : \mathbb{R}^2 \to \mathbb{R}^2$, classical, $|u(x)| < M$

$$\Delta u - W_u(u) = 0$$

u minimizing J ; $\exists \{r_k\} \to +\infty$ s.t.

$$u(r_k x) \to u_0(x) \ , \ \text{in} \ L^1_{loc}(\mathbb{R}^2)$$
$$u_0(x) = \sum a_i \chi_{D_i} \ , \ \mathcal{P} = \{D_1, D_2, D_3\}$$

$\partial\mathcal{P}$ the triod

RELATED WORK

(1) Bronsard-Gui-Schatzman (CPAM, 1996)
Triple Junction in equivariant class (reflection group of e.t.)
(2) G.Fusco (Pisa journal, 2022)
Triple Junction in equivariant class of rotation group of e.t.)
(3) G.Fusco (Calculus of Var. and PDE 2022)
Triangle in the equivariant class (reflection group of e.t.)

Dismissing Symmetry hypothesis

(4) A-Fusco (BHMS, 2023)
Introduction of Lower/Upper bound method at $\varepsilon-$level with simple exs.
(5) E. Sandier- P. Sternberg (2024)
Comparable results to Theorem 2 ($\{W=0\}=\{a_1,a_2,a_3\}$) obtained independently and roughly the same time, with very different methods.

(6) Th (Zhiyan Geng)
Suppose **(H1)**,**(H2)** and u a minimizer, and suppose $\exists\ \{r_k\}\to+\infty$

$$u(r_k x)\xrightarrow{L^1_{loc}} u_0(x)=\sum_{i=1}^{3} a_i\chi_{D_i}\ ,$$

Then

$$u(Rx)\xrightarrow[R\to\infty]{L^1_{loc}} u_0(x)$$

• Minimal Cones

Expect that given an arbitrary cone in $\mathbb{R}^2$ made up of N sectors there is a class of surface tension matrices Σ_N for which it is minimizing for $\mathcal{E}$.

• Entire solutions

Expect existence of corresponding entire solutions, s.t.

$$u(rx) \xrightarrow[r\to+\infty]{L^1_{loc}} \sum a_i \chi_{D_i}$$

• Replace $\mathbb{R}^2$ with $\mathbb{S}^2$.

Expect a variety of minimizing cones in $\mathbb{R}^3$.

(Work in progress with Giorgio Fusco)

Sketch of proof of Th 1 (ii) for equilateral triangle

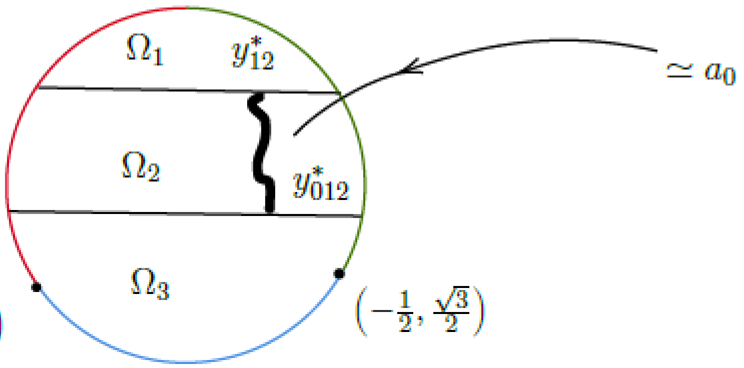
$(\sigma_{12} = \sigma_{13} = \sigma_{23} = \sigma \ , \ \sqrt{3} < (\frac{\sigma}{\sigma_0}) < 2)$

(No hypothesis of symmetry on solution)

Lemma (LOWER/upper BOUND)

$$3\sqrt{3}\sigma_0 - C\varepsilon^{1/3} \le \int_{B_1} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) dx \le 3\sqrt{3}\sigma_0 + C\varepsilon$$

Proof of L.B.



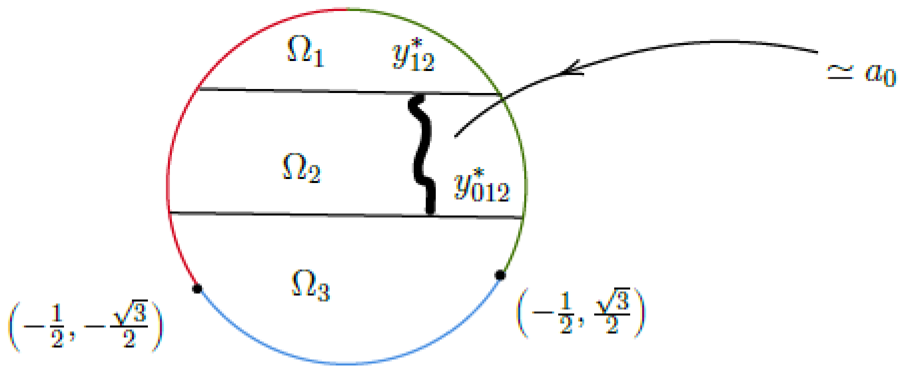
$$\gamma_y = \{(x,y) \mid y \in \mathbb{R}\} \cap B_1$$
$$\lambda_i(y) = \mathcal{L}^1(\gamma_y \cap \{|u(x,y) - a_i| < \varepsilon^{1/6}\}) \ , \ i = 0,1,2,3.$$

$$y^*_{12} = \min\{y \in [-\frac{1}{2} + c_0\varepsilon, 1] \ : \ \lambda_1(y) + \lambda_2(y) \ge \mathcal{L}^1(\gamma_y) - \varepsilon^{1/3}\}$$
$$y^*_{012} = \min\{y \in [-\frac{1}{2} + c_0\varepsilon, 1] \ : \ \lambda_0(y) + \lambda_1(y) + \lambda_2(y) \ge \mathcal{L}^1(\gamma_y) - \varepsilon^{1/3}\}$$

$$\Omega_1 = B_1 \cap \{y \ge y^*_{12}\} \ , \ \Omega_2 = B_1 \cap \{y^*_{012} \le y \le y^*_{12}\} \ , \ \Omega_3 = B_1 \cap \{y < y^*_{012}\}$$

If $y^*_{12} = y^*_{012} \Rightarrow J_\varepsilon(u) \ge 3\sigma - C\varepsilon^{1/3} \ge 3\sqrt{3}\sigma_0 - C\varepsilon^{1/3}$.

w.l.o.g. $y^*_{012} < y^*_{12}$, $\lambda_0(y) > 0$ for $y \in [y^*_{012}, y^*_{12})$.



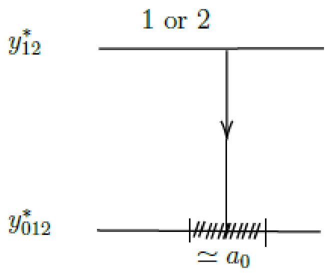
In Ω_1 ($y^*_{12} \le 1 - c_0\varepsilon$ w.l.o.g.)

$$\iint_\Omega \left(\frac{\varepsilon}{2}\left|\frac{\partial u_\varepsilon}{\partial x}\right|^2 + \frac{1}{\frac{1}{\varepsilon}}W(u_\varepsilon)\right) dz \ge \sigma(1 - c_0\varepsilon - y^*_{12})$$

In Ω_2

Horizontal: $$\int_{y^*_{012}}^{y^*_{12}}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{\varepsilon}{2}|\partial_x u|^2 + \frac{cos^2\theta}{\varepsilon}W(u)\right) dxdy \ge cos\theta(2\sigma_0 - 2C_W\varepsilon^{1/3})[y^*_{12} - y^*_{012}]$$

Vertical: $$\int_{\gamma_{y^*_{012}}\cap\{|u-a_0|<\varepsilon^{1/6}\}}\left(\int_{y^*_{012}}^{y^*_{12}}\left(\frac{\varepsilon}{2}|\partial_y u|^2 + \frac{sin^2\theta}{\varepsilon}W(u)\right)dy\right)dx \ge sin\theta(\sigma_0 - C_W\varepsilon^{1/3})\lambda_0(y^*_{012})$$



Optimizing in θ

$$\iint_{\Omega_2}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)dz \ge \sqrt{(2\sigma_0 - 2C_W\varepsilon^{1/3})^2[y^*_{12}-y^*_{012}]^2 + (\sigma_0 - C_W\varepsilon^{1/3})^2\lambda_0^2(y^*_{012})}$$

In Ω_3

- $\mathcal{L}^1\left(\left\{y : -\frac{1}{2}+c_0\varepsilon < y < y^*_{012}\,,\ \lambda_3(y)=0\right\}\right) < C\varepsilon^{1/3}$

Proof

$$\lambda_0(y)+\lambda_1(y)+\lambda_2(y)+\lambda_3(y) < \mathcal{L}^1(\gamma_y) - \varepsilon^{1/3}$$

Hence

$$\min_i |u(x,y)-a_i| \geq \varepsilon^{1/6} \quad \text{on an } \varepsilon^{1/3} \text{ set of } \gamma_y$$

Hence

$$\int_{\gamma_y} W(u)dx \geq (\frac{1}{2}c_W\varepsilon^{1/3})(\varepsilon^{1/3})$$

$$\int_S\int_{\gamma_y} W(u)dxdy \geq (\frac{1}{2}c_W\varepsilon^{1/3})(\varepsilon^{1/3})\mathcal{L}^1(S)\,,$$

S the set of $y's$.

From

$$C > J_\varepsilon(u) \geq \frac{1}{\varepsilon}\iint_{B_1} W(u)dz$$

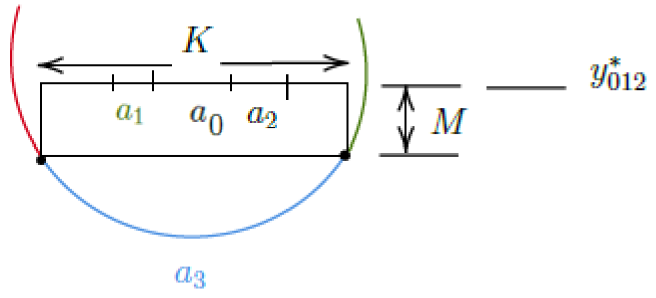
we conclude.

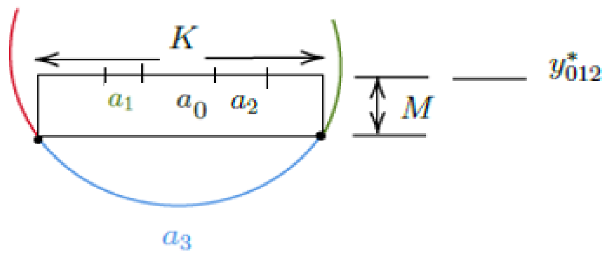
$$K = \left\{x \in [-\frac{\sqrt{3}}{2}+c_0\varepsilon, \frac{\sqrt{3}}{2}-c_0\varepsilon] : |u(x,y^*_{012})-a_i| < \varepsilon^{1/6}\,,\ \text{some } i=0,1,2\right\}$$

$$M = \left\{y \in [-\frac{1}{2}+c_0\varepsilon, y^*_{012}] : \lambda_3(y) > 0\right\}$$

$$\lambda_0(y^*_{012})+\lambda_1(y^*_{012})+\lambda_2(y^*_{012}) = \mathcal{L}^1(\gamma_{y^*_{012}}) - \varepsilon^{1/3}$$

$$\mathcal{L}^1(K) \geq \sqrt{3} - 2c_0\varepsilon - \varepsilon^{1/3}\ ,\ \mathcal{L}^1(M) \geq (y^*_{012}+\frac{1}{2}) - c_0\varepsilon - C\varepsilon^{1/3}$$





Horizontal: $\int_M \int_{\gamma_y} \left(\frac{\varepsilon}{2} |\partial_x u|^2 + \frac{cos^2\phi}{\varepsilon} W(u) \right) dxdy \geq 2cos\phi(\sigma - C_W \varepsilon^{1/3})[(y^*_{012} + \frac{1}{2}) - C\varepsilon^{1/3}]$

Vertical:

$$\int_K \int_{-\sqrt{1-x^2}}^{y^*_{012}} \left(\frac{\varepsilon}{2} |\partial_y u|^2 + \frac{sin^2\phi}{\varepsilon} W(u) \right) dydx$$

$$\geq sin\phi \left\{ \lambda_0(y^*_{012})(\sigma_0 - C_W \varepsilon^{1/3}) + (\sqrt{3} - \lambda_0(y^*_{012}))(\sigma - C_W\varepsilon^{1/3}) - C\varepsilon^{1/3} \right\}$$

Optimizing in ϕ

$$\iint_{\Omega_3} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) dz \geq \sqrt{(2\sigma - 2C_W\varepsilon^{1/3})^2[(y^*_{012} + \frac{1}{2}) - C\varepsilon^{1/3}]^2 + [(\sigma_0 - C_W\varepsilon^{1/3})\lambda_0(y^*_{012}) + (\sigma - C_W\varepsilon^{1/3})(\sqrt{3} - \lambda_0(y^*_{012}))]^2}$$

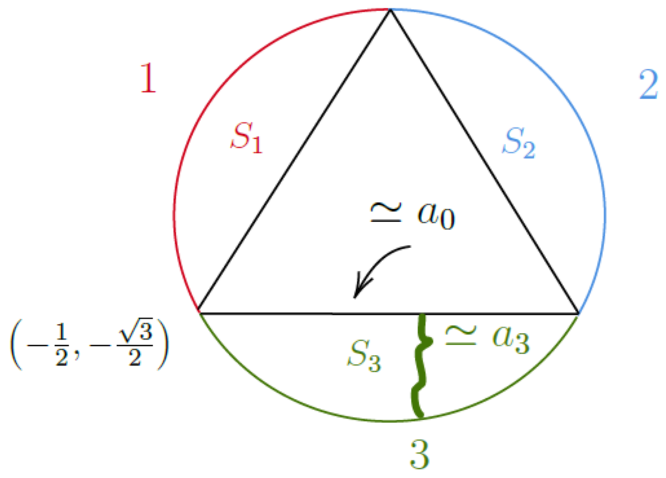
Conclusion $\iint_{B_1} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) dz \geq$

$$\sigma(1 - y^*_{12}) + \sqrt{(2\sigma_0)^2(y^*_{12} - y^*_{012})^2 + \lambda_0^2(y^*_{012})\sigma_0^2} + \sqrt{[(2\sigma)(y^*_{012} + \frac{1}{2})]^2 + [\sqrt{3}\sigma + \lambda_0(y^*_{012})(\sigma_0 - \sigma)]^2} - C\varepsilon^{1/3}$$

Corollary

$$y^*_{12} = 1 - O(\varepsilon^{1/3}) \ , \ y^*_{012} = -\frac{1}{2} + O(\varepsilon^{1/3}) \ , \ \lambda_0(y^*_{012}) = \sqrt{3} - O(\varepsilon^{1/3})$$

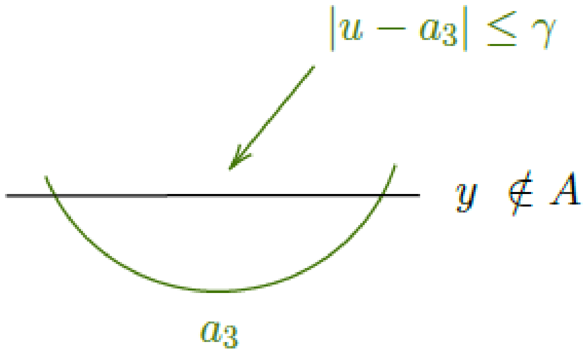
Localization



$$A = \text{set of } y's \text{ in } [-1, -\frac{1}{2} + C\varepsilon^{1/3}] \ , \ \text{s.t.} \ \max_{z \in \gamma_y} |u(z) - a_3| > \gamma$$

$\gamma > 0$, small , arbitrary

$$\mathcal{L}^1(A) < C\varepsilon^{1/3}$$



Then by the variational maximum principle

$$|u - a_3| \leq \gamma \quad \text{in} \ S_3$$

THANK YOU