THE GLOBAL NONLINEAR STABILITY OF SLOWLY ROTATING KERR BLACK HOLES

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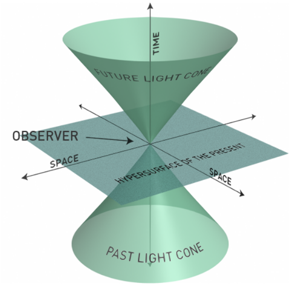
GEOMETRY

- MANIFOLDS $M = M^n, \quad T_p(M), \quad (U, x^\alpha)$
- METRIC $g = g_{\alpha\beta} dx^\alpha dx^\beta,$
- CONNECTION $\Gamma = \Gamma^\gamma_{\alpha\beta},$ Levi-Civitta
- CURVATURE $R = R_{\alpha\beta\gamma\delta},$ Gauss, Riemann
- RICCI TENSOR $\mathrm{Ric} = R_{\alpha\gamma} = g^{\beta\delta} R_{\alpha\beta\gamma\delta}$

- Euclid $\mathbb{R}^n,$ Riemannian Geometry
- Minkowski $\mathbb{R}^{1+n},$ Lorentzian Geometry

GEOMETRIC FRAMEWORK OF GR

1. CAUSALITY LORENTZIAN MANIFOLDS (M, g)



- Inertia–$T_p(M)$ = Minkowski
- Events≡ points in M
- Observers≡ timelike curves
- Light rays≡ null geodesics
- Equiv. Pr. ≡ Gen. covariance
- Tidal forces ≡ curvature
- Isolated system≡ Asympt. flat

2. FIELD EQUATIONS

$$\mathbf{G}_{\alpha\beta} := \boxed{R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = T_{\alpha\beta}}.$$

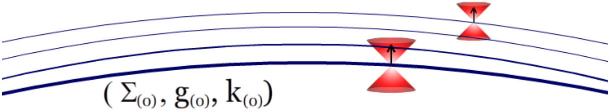
3. VACUUM EQUATIONS(EVE) $R_{\alpha\beta} = 0.$

4. GENERAL COVARIANCE $g \equiv \Phi^* g, \quad \Phi : M \longrightarrow M.$

INITIAL VALUE FORMULATION(EVE)

HYPERBOLICITY. Wave coordinates

INITIAL DATA SETS. $(\Sigma_{(0)}, g_{(0)}, k_{(0)})$ $+$ Constraints



THEOREM(Bruhat-Geroch) *Smooth IDS admit unique, smooth, maximal future globally hyperbolic developments (MFGHD).*

KERR FAMILY $\mathcal{K}(a, m)$

2-parameter family of stationary, asympt. flat (AF), solutions of

$$\mathrm{Ric}(\mathbf{g}) = 0. \qquad (EVE)$$

- MINKOWSKI (1907). $a = m = 0.$
- SCHWARZSCHILD (1915). $a = 0, m \neq 0$
- KERR(1963). $0 < |a| \leq m.$

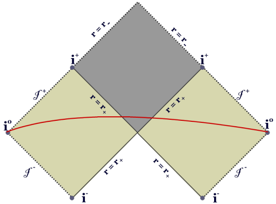
KERR FAMILY $\mathcal{K}(a, m)$

$$-\frac{|q|^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{|q|^2}\left(d\varphi - \frac{2amr}{\Sigma^2}dt\right)^2 + \frac{|q|^2}{\Delta}(dr)^2 + |q|^2(d\theta)^2$$

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ |q|^2 = r^2 + a^2(\cos\theta)^2; \\ \Sigma^2 = (r^2 + a^2)^2 - a^2(\sin\theta)^2\Delta. \end{cases}$$

STATIONARY, AXISYMMETRIC. $\partial_t,\ \partial_\varphi$ KILLING

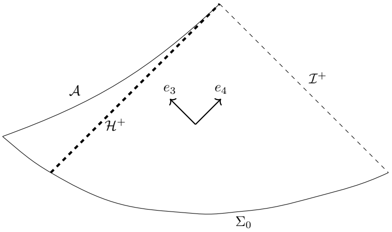
ASYMPTOTICALLY FLAT. Approaches Minkowski as $r \to \infty$.



KERR FAMILY $\mathcal{K}(a, m)$

PRINCIPAL NULL PAIR. $\{e_3, e_4\}$.

- Diagonalizes the curvature tensor.
- Horizontal structure $\mathcal{H} = \{e_3, e_4\}^\perp$ is non-integrable if $a \neq 0$.

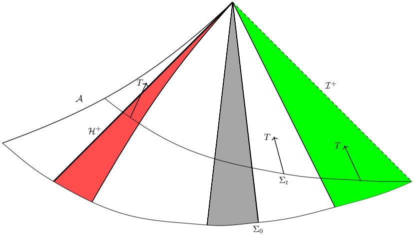


KERR FAMILY $\mathcal{K}(a, m)$

- HORIZON. $\Delta(r) = 0, \quad \mathrm{r}{=}r_+ = m + \sqrt{m^2 - a^2}.$
- ERGOREGION. $\mathbf{g}(T, T) = \frac{\Delta - a^2 \sin^2\theta}{|q|^2} > 0,$
- TRAPPING. $\mathcal{T} = \mathcal{T} = r^3 - 3mr^2 + a^2 r + ma^2$

$$\mathcal{M}_{trap} := \mathcal{M} \cap \left\{ \frac{|\mathcal{T}|}{r^3} \leq \delta_{trap} \right\}.$$

- NULL INFINITY. $r \to \infty$
- NON-INTEGRABILITY.



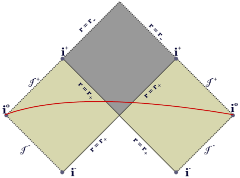
MATHEMATICAL TESTS OF REALITY

1. COLLAPSE. Can black holes (trapped surfaces) form from reasonable initial data configurations?

2. RIGIDITY. Does the Kerr family $\mathcal{K}(a, m)$, $0 \leq a \leq m$, exhaust all possible stationary solutions?

3. STABILITY. Is the Kerr family stable for arbitrary small perturbations?

4. COSMIC CENSORSHIP



FINAL STATE CONJECTURE

FINAL STATE CONJECTURE. The MFHD's of generic IDS behave, asymptotically, like a finite number of Kerr black holes, moving away from each other, plus a radiative decaying term.

1. Small data don't concentrate, i.e. lead to pure, *slowly decaying*, gravitational waves. Stability of Minkowski space.
2. Large data may concentrate to produce stationary states, i.e. BHs. Collapse.
3. All stationary states are Kerr. Rigidity.
4. Kerr solutions are stable. Stability.

FINAL STATE CONJECTURE

5. There can be no singularities outside BHs. Cosmic Censorship Conjecture.

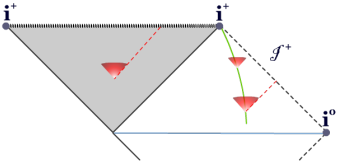


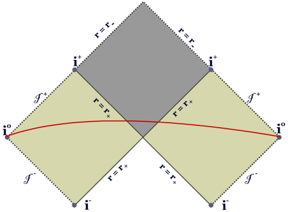
Figure: Null geodesics outside and inside the black hole

6. Two (and more) body problem.

STABILITY OF KERR

CONJECTURE[Stability of (external) Kerr].

Small perturbations of a given exterior Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.



THEOREM "True" if $|a|/m \ll 1$.

- MAIN[K-Szeftel(2021)]
- GCM PAPERS[K-Szeftel(2019), Shen(2022)]
- WAVE PAPER[Giorgi-K-Szeftel(2022)]

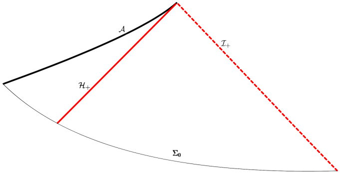
MAIN DIFFICULTIES

- E.V. strongly coupled, tensorial, hyperbolic, nonlinear.
- GAUGE GROUP= All diffeomorphisms $\mathbf{g} \equiv \Phi^* \mathbf{g}$.
- MODULATION IN INFINITE DIMENSIONS.
- NON-TRIVIAL CHARACTER OF $\mathcal{K}(a, m)$
  - horizon,
  - ergoregion,
  - trapping,
  - null infinity,
  - non-integrability.
- DECAY. Decay of waves in Kerr
- FINAL PARAMETERS? GAUGE? Emerge in the limit!
- LOW RATES OF DECAY TO THE FINAL STATE.

STABILITY OF SLOWLY ROTATING KERR

THEOREM[Kl-Szeftel(2021)] *The MFD of a general IDS, close to the IDS of a* $\mathcal{K}(a_0, m_0)$, $|a_0|/m_0 \ll 1$

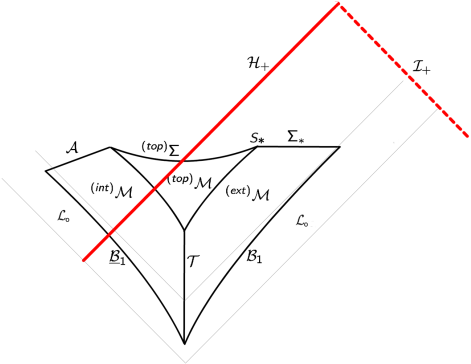
- *Has a complete future null infinity* $\mathcal{I}^+$
- *Converges in* $\mathcal{J}^{-1}(\mathcal{I}^+)$ *to a nearby* $\mathcal{K}(a_\infty, m_\infty)$ *with* (a_∞, m_∞) *close to* (a_0, m_0).
- *Has an event horizon* $\mathcal{H}^+$.
- Recoil.



- $\mathcal{K}(a_\infty, m_\infty)$ - limit of finite GCM admissible spacetimes.

GCM ADMISSIBLE $\mathcal{M} = {}^{int}\mathcal{M} \cup {}^{ext}\mathcal{M} \cup {}^{top}\mathcal{M}$

- S_* - **GCM** surface.
  - (a, m), "**axis**".
- Σ_*- **GCM** hypers.
  - **Initializes** Φ_f.
- **PG**-structures
  - $({}^{ext}\mathcal{M}, u, r)$
  - $({}^{int}\mathcal{M}, \underline{u}, r)$
- $\mathcal{L}_0$-**Initial Layer**
- **Bootstrap.**



$(\mathcal{M}, a, m, \textbf{axis}, \Phi_f)$ are continuously **upgraded**.

REMARK. Gauge is initialized from the **future** with no reference to the initial data! Modifies the initial layer foliation!-Recoil!

MAIN NEW IDEAS

- General Covariant Modulated (GCM) spheres.
- Choice of the last slice Σ_*.
- Non integrability if $a \neq 0$.
- Control of Teukolsky variables.
- Recoil

KERR STABILITY-SHORT HISTORY

1. Discovery of Kerr[1963].

2. Linear mode stability[1963-1975].
   - Regge-Wheeler[1957]. metric perturbations
   - Newmann-Penrose[1962]. **curvature perturbations**
   - Teukolsky equations[1973]. curvature perturbations
   - **Chandrasekhar transform**[1975].
   - Whiting[1989].

3. Global Stability of Minkowski space [1993]
   - **Vectorfield method**
     - Local Energy Decay[1961].
     - Pointwise Decay[1985].
   - **Null condition**[1983, 1986].

KERR STABILITY. SHORT HISTORY

4. Robust decay for scalar waves [2003-2014]
   - $a = 0$, $m > 0$. **Soffer, Blue-S.** Morawetz monotonicity!
   - $a = 0$, $m > 0$. B-Sterbenz, Daf-Rodn, Marzuola-Metcalf-Tataru-Tohaneanu
   - $a \ll m$. D-R, T-T, **Andersson-Blue**

5. Robust decay for spin-2 waves [2016-2019]
   - $a = 0$. **D-Holzegel-R**.
   - $|a| \ll m$. **Ma**, D-H-R

6. Linear stability
   - $a = 0$. D-H-R[2016], Hung-Keller- M.T.Wang.
   - $a \ll m$. A-Bäckdahl-Blue-Ma[2019], Hintz-Vasy[2021].

SHORT HISTORY

6. Nonlinear stability of Schwarzschild
   - Polarized case **K-Szeftel**[2018].
   - Codim 3 Data DHR+Taylor[2021].

7. GCM Spheres and hypersurfaces in perturbations of Kerr
   - Construction of GCM spheres [K-S(2018)].
   - Intrinsic GCM spheres [K-S(2019)].
   - Construction of GCM hypersurfaces [Shen(2022)].

8. Nonlinear stability of slowly rotating Kerr
   - Kerr stability for small angular momentum[K-S(2021)].
   - Wave equations estimates and the nonlinear stability of slowly rotating Kerr black holes[Giorgi-K-S(2022)].