A stability analysis technique called trajectory-based approach

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Outline

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Background and Motivation

Background and Motivation

→ Stability analysis of nonlinear systems is not always an easy task.

Especially when the systems are time-varying, have delays, discontinuities, are complicated interconnected systems.

→ Lyapunov functions are fundamental tools.

Lyapunov-Krasovskii functionals: natural analog of Lyapunov functions.

Razumikhin's theorem: crucial result, (especially when the delays are time-varying).

Background and Motivation

$\rightarrow$ Attention ! Frequently, Lyapunov-Krasovskii techniques in general *do not* apply to systems with **time-varying** delays.

The same is true for the **frequency domain** techniques.

$\rightarrow$ Then what can be done ?

Background and Motivation

Difficult case: delays with **discontinuities**.

$\rightarrow$ Motivations: sampling, networked systems, biomedical models.

Alternative techniques:

- **Trajectory based approach.**
- **Result based on Halanay's inequality.**

Background and Motivation

→ They can be applied to many families of systems: ODE coupled with difference equations, interconnected and networked systems with time-varying delay, time-varying linear systems.

→ They help to solve both stability analyzes and stabilization problems.

→ They help overcome the difficulties of finding Lyapunov functionals.

Part I: Trajectory based approach: key result

Result

Theorem 1

Let $T^* > 0$*. Consider* $w : [-T^*, +\infty) \to [0, +\infty)$*,* $d : [0, +\infty) \to [0, +\infty)$ *and* $\rho \in (0, 1)$ *s. t.*

$$w(t) \leq \rho \sup_{\ell \in [t-T^*, t]} w(\ell) + d(t) \ , \ \forall t \geq 0. \tag{1}$$

Then

$$w(t) \leq \sup_{\ell \in [-T^*, 0]} w(\ell) e^{\frac{\ln(\rho)}{T^*} t} + \frac{1}{(1-\rho)^2} \sup_{\ell \in [0, t]} d(\ell) \ , \ \forall t \geq 0. \tag{2}$$

Result

$\rightarrow$ Since $\rho \in (0,1)$, the function $e^{\frac{\ln(\rho)}{T^*}t}$ goes **exponentially** to zero.

$\rightarrow$ The inequality (2) is an **ISS inequality**.

Academic examples

Consider the system with delay

$$\dot{x}(t) = -x(t) + bx(t-\tau) \tag{3}$$

with $b \in (0,1)$ and $\tau > 0$.

1) Let $T > 0$. We **integrate** to get

$$x(t) = e^{-T}x(t-T) + b\int_{t-T}^{t} e^{m-t}x(m-\tau)\mathrm{d}m.$$

2) Then

$$|x(t)| \leq e^{-T}|x(t-T)| + b\left(1-e^{-T}\right)\sup_{\ell\in[t-T-\tau,t-\tau]}|x(\ell)|. \tag{4}$$

Academic examples

3) As an immediate consequence,

$$|x(t)| \leq \left[e^{-T} + b\left(1 - e^{-T}\right)\right] \sup_{\ell \in [t-T-\tau, t-\tau]} |x(\ell)|. \qquad (5)$$

4) Since $b < 1$, $e^{-T} + b\left(1 - e^{-T}\right) < 1$ for all $T > 0$.

Conclusion: from Theorem 1, we conclude that the system (3) is UGES to zero.

Academic examples

Consider the system

$$\dot{x}(t) = -x(t) + 9\cos^{2p}(t)x(t-\tau), \tag{6}$$

with $x \in \mathbb{R}$, $p \in \mathbb{N}$ and $\tau \geq 0$.

Remark. When $\tau = 0$ and $p = 1$, $x(t) = e^{\frac{7}{2}t + \frac{9}{2}\sin(t)\cos(t)}x(0)$. Then the system is unstable. But when p is sufficiently large, the system is GUES.

Remark. One cannot establish that the origin of (6) is GUES by applying Razumikhin's theorem: with $V(x) = \frac{1}{2}x^2$,

$$\dot{V}(t) = -x(t)^2 + 9\cos^{2p}(t)x(t)x(t-\tau).$$

Academic examples

1) We integrate to get

$$x(t) = e^{-2\pi}x(t-2\pi) + 9\int_{t-2\pi}^{t} e^{m-t}\cos^{2p}(m)x(m-\tau)\mathrm{d}m.$$

2) Then

$$\begin{array}{rcl} |x(t)| & \leq & e^{-2\pi}|x(t-2\pi)| \\ & & +9\int_{t-2\pi}^{t} e^{m-t}\cos^{2p}(m)\mathrm{d}m \sup_{\ell\in[t-2\pi-\tau,t-\tau]} |x(\ell)|. \end{array} \quad (7)$$

3) Then, when $p > 2$,

$$|x(t)| \leq \frac{e^{-2\pi} + 1}{2} \sup_{\ell \in [t-2\pi-\tau, t-\tau]} |x(\ell)|. \tag{8}$$

Conclusion: from Theorem 1, we conclude that the system (6) is UGES to zero.

Equations with delay

We consider the system

$$\dot{x}(t) = f(t, x(t), \zeta(t, \tau(t)), \delta(t)) \tag{9}$$

where $x \in \mathbb{R}^n$,
$\zeta(t, \tau(t)) = (X_1(t - \tau_1(t)), X_2(t - \tau_2(t)), \ldots, X_L(t - \tau_L(t)))$, each subvector X_i of x has some dimension n_i.

Property: $\tau_i(t) \in [\tau_S, \tau_M]$ for all t and i with $\tau_M \geq \tau_S > 0$.

Equations with delay

We introduce assumptions:

Assumption A1. *There are* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$ *s. t. all trajectories of* $\dot{\xi}(t) = f(t, \xi(t), u(t))$ *for all* u *valued in* $\mathbb{R}^n \times \mathbb{R}^m$ *satisfy*

$$|\xi(t)| \leq \beta(|\xi(t_0)|, t - t_0) + \gamma\left(\sup_{\ell \in [t_0, t]} |u(\ell)|\right) \ , \ \forall t \geq t_0. \quad (10)$$

Assumption A2. *There are* $T > 0$ *and* $\rho_0 \in (0, 1)$ *s. t.*

$$\beta(s, T) + \gamma(s) \leq \rho_0 s \quad , \quad \forall s \geq 0. \quad (11)$$

Equations with delay

Theorem 2

We can build a $\mathcal{L} \in \mathcal{K}_\infty$ *s. t. with*

$$\begin{array}{l} \overline{\beta}(s,t) = \mathcal{L}(s)\left(e^{\frac{\ln(\rho_0)}{2T}(t-2T)} + e^{2T-t}\right) \\ \text{and } \overline{\gamma}(s) = \frac{s\rho_0}{(1-\rho_0)^2} + \mathcal{L}(s)\ , \end{array} \tag{12}$$

the ISS estimate

$$|x(t)| \leq \overline{\beta}\left(\sup_{\ell\in[t_0-\overline{\tau},t_0]} |x(\ell)|, t - t_0\right) + \overline{\gamma}\left(\sup_{\ell\in[t_0,t]} |\delta(\ell)|\right) \tag{13}$$

holds along all trajectories of (9).

Proof.

1) Assumptions A1 and A2 give

$$
\begin{aligned}
|x(t)| \quad &\leq \quad \beta\left(\sup_{\ell\in[t-2T,t]}|x(\ell)| + \sup_{\ell\in[t_0,t]}|\delta(\ell)|, T\right) \\
&\qquad +\gamma\left(\sup_{\ell\in[t-2T,t]}|x(\ell)| + \sup_{\ell\in[t_0,t]}|\delta(\ell)|\right) \qquad (14)\\
&\leq \quad \rho_0\left(\sup_{\ell\in[t-2T,t]}|x(\ell)| + \sup_{\ell\in[t_0,t]}|\delta(\ell)|\right)
\end{aligned}
$$

for all $t \geq t_0 + 2T$ and all δ.

Equations with delay

2) By applying **Theorem 1** with $w(t) = |x(t + 2T + t_0)|$, $\rho = \rho_0$, $T_* = 2T$, and $d(t) = \rho_0 \sup_{\ell \in [t_0, t_0 + 2T + t]} |\delta(\ell)|$ we obtain:

$$|x(t)| \leq \sup_{\ell \in [t_0, t_0 + 2T]} |x(\ell)| e^{\frac{\ln(\rho_0)}{2T}(t - 2T - t_0)} + \frac{\rho_0}{(1 - \rho_0)^2} \sup_{\ell \in [t_0, t]} |\delta(\ell)| \tag{15}$$

for all $t \geq 2T + t_0$.

3) Next, by studying the trajectories over $[t_0, t_0 + 2T]$, we can **conclude**.

Systems with impulses

We consider now a system with **impulses** and **delay**:

$$\begin{cases} \begin{cases} \dot{x}(t) &= f\big(t, x(t), x(t-\tau), z(t)\big) \\ \dot{z}(t) &= g\big(t, x(t), z(t)\big) \quad \forall t \in [t_k, t_{k+1}) \text{ and } k \geq 0 \end{cases} \\ \quad z(t_k) = h\big(z(t_k^-)\big), \quad k \in \mathbb{N} \end{cases} \tag{16}$$

with $\tau \geq 0$.

Stability result: obtained under 2 assumptions:

Systems with impulses

Assumption B1. *There are* $c_1 \geq 0$, $c_2 \geq 0$, *and* $c_3 > 0$ *s. t.*
(a) $|h(z)| \leq c_1 |z|$ *and*
(b) *all solutions of*

$$\dot{\xi}(t) = g(t, v(t), \xi(t)) \tag{17}$$

satisfy

$$|\xi(t)| \leq e^{c_3 (t - t_0)} |\xi(t_0)| + c_2 \sup_{\ell \in [t_0, t]} v(\ell). \tag{18}$$

Also,

$$e^{c_3 \tau} c_1 < 1. \tag{19}$$

Systems with impulses

Assumption B2. *The system*

$$\dot{\xi}(t) = f\big(t, \xi(t), u(t)\big) \tag{20}$$

satisfies **ISS** *for some* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}_\infty$. *There are an integer* $j \geq 1$ *and* $\rho \in (0, \frac{1}{1+c_2})$ *s. t.*

$$\beta(s, j\tau) + \gamma(s) \leq \rho s\ ,\ \forall s \geq 0 \tag{21}$$

and

$$c_1^j e^{jc_3\tau} + \rho + \sqrt{(c_1^j e^{jc_3\tau} - \rho)^2 + \frac{4\rho c_2(1+c_1 e^{c_3\tau})}{1-c_1 e^{c_3\tau}}} < 1 \tag{22}$$

hold.

Part II: Vector version of the trajectory based approach

Result

$\rightarrow$ Let $S \in \mathbb{R}^{l \times l}$ be a **positive** and **Schur stable** matrix.

Bearing in mind **Perron-Frobenius** theorem, let $V \in [1 + \infty)^n$ be s. t.

$$SV = qV \tag{23}$$

with $q \in (0, 1)$.

$\rightarrow$ Let $\Delta : [0, +\infty) \rightarrow [0, +\infty)^n$ be s. t. each component Δ_i is piecewise continuous and **nondecreasing**.

$\rightarrow$ Let

$$\rho(t) = -(-I + S)^{-1}\Delta(t). \tag{24}$$

Result

Theorem 3

Let $T_\star > 0$ *and* n *functions* $z_i : [-T_\star + \infty) \to [0 + \infty)$ *be s. t.*

$$\begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix} \leq S \begin{pmatrix} \sup_{s\in[t-T_\star,t]} z_1(s) \\ \vdots \\ \sup_{s\in[t-T_\star,t]} z_n(s) \end{pmatrix} + \Delta(t). \qquad (25)$$

Then

$$\begin{pmatrix} z_1(t) \\ \vdots \\ z_n(t) \end{pmatrix} \leq \sum_{j=1}^{n} \sup_{\ell\in[-T_\star,0]} z_j(\ell) e^{\frac{\ln(q)}{T_\star} t} V + \rho(t). \qquad (26)$$

Result

$\rightarrow$ Since $q \in (0,1)$, the function $e^{\frac{\ln(q)}{T^*}t}$ goes **exponentially** to zero.

$\rightarrow$ From the vector inequality (26), an **ISS inequality** can be deduced.

$\rightarrow$ Since $-I + S$ is Metzler and Hurwitz, ρ is **nonnegative** and **nondecreasing**.

Academic example

We consider the system:

$$\left\{ \begin{array}{lcl} \dot{x}_1(t) & = & -2x_1(t) - x_2(t-\tau_1) \\ x_2(t) & = & \sin(x_1(t)) + \frac{1}{4}x_2(t-\tau_2) \end{array} \right. \tag{27}$$

with $\tau_1 \geq 0$ and $\tau_2 > 0$.

Then, for all $R > 0$,

$$\begin{array}{rcl} x_1(t) & = & e^{-2R}x_1(t-R) - \int_{t-R}^{t} e^{2(\ell-t)}x_2(\ell-\tau_1)d\ell \\ |x_2(t)| & \leq & |x_1(t)| + \frac{1}{4}|x_2(t-\tau_2)|. \end{array} \tag{28}$$

Academic example

Consequently, for all $R \geq \tau_2$:

$$\begin{array}{rcl} |x_1(t)| & \leq & e^{-2R}|x_1(t-R)| + \frac{1-e^{-2R}}{2} \sup_{\ell \in [t-\tau_1-R, t]} |x_2(\ell)| \\ |x_2(t)| & \leq & |x_1(t)| + \frac{1}{4} \sup_{\ell \in [t-\tau_1-R, t]} |x_2(\ell)| \end{array} \quad (29)$$

for all $t \geq \tau_1 + R$.

Let

$$\Upsilon_R = \left[\begin{array}{cc} e^{-2R} & \frac{1-e^{-2R}}{2} \\ 1 & \frac{1}{4} \end{array} \right]. \quad (30)$$

Academic example

Then

$$\begin{pmatrix} |x_1(t)| \\ |x_2(t)| \end{pmatrix} \leq \Upsilon_R \begin{pmatrix} \sup\limits_{\ell \in [t-\tau_1-R,t]} |x_1(\ell)| \\ \sup\limits_{\ell \in [t-\tau_1-R,t]} |x_2(\ell)| \end{pmatrix} \qquad (31)$$

for all $t \geq \tau_1 + R$.

Question: is Υ_R a Schur stable matrix for some constant R ?

Academic example

The positive eigenvalue of:

$$\Upsilon_{\ln(10)} = \begin{bmatrix} \frac{1}{100} & \frac{99}{200} \\ 1 & \frac{1}{4} \end{bmatrix} \tag{32}$$

is $\frac{13}{100} + \frac{1}{100}\sqrt{5094} < 1$.

It follows that $\Upsilon_{\ln(10)}$ is **Schur stable**.

We conclude from Theorem 3 that the origin of (27) is UGES.

Observer

Consider

$$\begin{cases} \dot{x}(t) = [A + \delta(t)]x(t) \\ y(t) = Cx(t) \end{cases} \tag{33}$$

where x is valued in $\mathbb{R}^n$ and y is valued in $\mathbb{R}^p$.

Assumption C1. δ *is known and for all* (i, j),

$$|\delta_{i,j}(t)| \leq \overline{\delta} \tag{34}$$

for all t.

Assumption C2. (A, C) *is observable.*

Objective: construction of a **finite-time** or **an almost finite-time** observer under Assumptions C1 and C2.

Observer

Difficulty to construct a finite-time observer: δ is time-varying.

Let $\tau > 0$ and

$$E = \int_{-\tau}^{0} e^{A^\top s} C^\top C e^{As} ds \tag{35}$$

Assumption C2 ensures that E is **invertible**.

One can prove that

$$\begin{array}{rcl} x(t) & = & E^{-1} \int_{t-\tau}^{t} e^{A^\top (s-t)} C^\top y(s) ds \\ & & + E^{-1} \int_{t-\tau}^{t} e^{A^\top (s-t)} C^\top \int_{s}^{t} e^{A(s-m)} \delta(m) x(m) dm \end{array} \tag{36}$$

for all $t \geq \tau$.

Observer

We introduce the observer:

$$\begin{array}{rcl} \hat{x}(t) & = & E^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\top y(s)ds \\ & & +E^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\top \int_{s}^{t} e^{A(s-m)}\delta(m)\hat{x}(m)dm. \end{array} \quad (37)$$

Then

$$\tilde{x}(t) = E^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\top \int_{s}^{t} e^{A(s-m)}\delta(m)\tilde{x}(m)dm \quad (38)$$

with $\tilde{x}(t) = \hat{x}(t) - x(t)$.

Observer

Assumption C1 ensures that there is a matrix $P > 0$ s. t.

$$\begin{pmatrix} |\tilde{x}_1(t)| \\ \vdots \\ |\tilde{x}_n(t)| \end{pmatrix} \leq \tau^2 \overline{\delta} P \begin{pmatrix} \sup_{m \in [t-\tau, t]} |\tilde{x}_1(m)| \\ \vdots \\ \sup_{m \in [t-\tau, t]} |\tilde{x}_n(m)| \end{pmatrix}. \quad (39)$$

The matrix $\tau^2 \overline{\delta} P$ is Schur stable when $\overline{\delta}$ is sufficiently small.

We deduce from Theorem 3 that the convergence rate of observer is proportional to $-\ln(\overline{\delta})$.

In other words, we have an almost finite-time observer.

Remarks

We have established a vector discrete-time version of the trajectory based approach.

Let $S > 0$, $p \in (0,1)$ and $U \in [1,+\infty)^n$ be such that

$$SU = pU \tag{40}$$

and

$$V_i = \begin{pmatrix} v_{1,i} \\ \vdots \\ v_{n,i} \end{pmatrix} \in [0,+\infty)^n \tag{41}$$

be such that

Remarks

$$V_{i+1} \leq S \begin{pmatrix} \max\{v_{1,i}, ..., v_{1,i-r+1}\} \\ \vdots \\ \max\{v_{n,i}, ..., v_{n,i-r+1}\} \end{pmatrix} + \Delta_i \tag{42}$$

Then

$$V_i \leq \Gamma_i \tag{43}$$

with

$$\Gamma_i = cp^{\frac{i}{r}} U + (I - S)^{-1} \Delta_i \tag{44}$$

and

$$c > \frac{1}{p} \max_{\{-r,...,0\}} \{\max\{v_{1,i}, ..., v_{n,i}\}\} \tag{45}$$

Remarks

Potential advantages of our **trajectory based methods**:

1. We do not require special regularity on the dynamics such as Lipschitzness conditions.
2. We can study ‘complicated’ interconnected systems.
3. We do not need Lyapunov function formulas for any subsystems.
4. We get explicit comparison functions in the final exponential ISS estimates.

Part III: Trajectory based approach and Halanay's inequality

Classical result

Classical result:

A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, 1966.

Let $v : [-\tau, +\infty) \to [0, +\infty)$ be a scalar function of class C^1 and constants $a > 0$, $b \geq 0$ and $\tau \geq 0$ be s. t.

$$\dot{v}(t) \leq -av(t) + b \sup_{\ell \in [t-\tau, t]} v(\ell). \tag{46}$$

Then if $a > b$, then $v(t)$ **exponentially converges** to zero as $t \to +\infty$.

Time-varying inequality

Can the approach be extended to time-varying inequalities:

$$\dot{v}(t) \leq -a(t)v(t) + b(t) \sup_{\ell \in [t-\tau, t]} v(\ell) \tag{47}$$

and to vector inequalities:

$$\begin{pmatrix} \dot{v}_1(t) \\ \vdots \\ \dot{v}_n(t) \end{pmatrix} \leq M(t) \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} + P(t) \begin{pmatrix} \sup\limits_{s \in [t-\tau, t]} v_1(s) \\ \vdots \\ \sup\limits_{s \in [t-\tau, t]} v_n(s) \end{pmatrix} \tag{48}$$

where M is Metzler and $P \geq 0$?

Time-varying inequality

Possible approach for (47): **the trajectory based approach.**

$\rightarrow$ We integrate (47):

$$v(t) \leq e^{\int_s^t a(\ell)d\ell} v(t-s) + \int_s^t e^{\int_m^t a(\ell)d\ell} b(m) \sup_{\ell \in [m-\tau, m]} v(\ell) dm. \tag{49}$$

$\rightarrow$ Consequently, for any $h > 0$,

$$v(t) \leq \left[e^{\int_{t-h}^t a(\ell)d\ell} + \int_{t-h}^t e^{\int_m^t a(\ell)d\ell} b(m) dm \right] \sup_{\ell \in [t-h-\tau, t]} v(\ell). \tag{50}$$

Time-varying inequality

→ Consequently, if there is $\rho \in (0, 1)$ s. t.

$$e^{\int_{t-h}^{t} a(\ell)d\ell} + \int_{t-h}^{t} e^{\int_m^t a(\ell)d\ell} b(m)dm \leq \rho \tag{51}$$

for all $t \in \mathbb{R}$, Theorem 1 applies.

Then we conclude that v **exponentially** converges to zero as $t \to +\infty$.

Remark. When an additive term is present, an **ISS inequality** can be obtained.

Time-varying inequality

The previous result applies in **many cases**.

However, it has a limitation.

To understand it, we consider **a particular example**.

Time-varying inequality

→ Let $t_i = i\overline{T}$. Let v be s. t.

$$\dot{v}(t) \leq -v(t) + b(t) \sup_{\ell \in [t-\tau, t]} v(\ell) \tag{52}$$

where b is defined by

$$b(t) = \begin{cases} 2, & \text{if } t \in \cup_{i \in \mathbb{Z}_{\geq 0}} [t_i, t_i + T), \\ 0, & \text{if } t \in \cup_{i \in \mathbb{Z}_{\geq 0}} [t_i + T, t_{i+1}), \end{cases} \tag{53}$$

where $T > 0$ is "**small**".

Time-varying inequality

Notice that if $T = 0$, then

$$\dot{v}(t) \leq -v(t) \tag{54}$$

Thus the intuition suggests that if $T > 0$ is **sufficiently small**, then v converges to zero.

Question: Can we establish this result via the trajectory based approach ?

Answer: A direct application of the trajectory based approach does not allow us to conclude.

Time-varying inequality

$\rightarrow$ Integrating (52) over $[t_i, t]$ with $t \in [t_i, t_{i+1})$ yields

$$v(t) \leq e^{t_i - t} v(t_i) + \int_{t_i}^{t} e^{m-t} b(m) \mathrm{d}m \sup_{\ell \in [t_i - T, t]} v(\ell). \quad (55)$$

$\rightarrow$ Thus for any $t \in [t_i, t_i + T)$,

$$v(t) \leq (2 - e^{t_i - t}) \sup_{\ell \in [t_i - T, t]} v(\ell). \quad (56)$$

$\rightarrow$ Since $2 - e^{t_i - t} > 1$ for any $t > t_i$, the trajectory based approach **does not** apply.

Time-varying inequality

We provided an **extension** of Halanay's inequality in the contribution:

F. Mazenc, M. Malisoff, M. Krstic, *Stability and observer designs using new variants of Halanay's inequality.*, Automatica, Vol. 123, Jan. 2021.

Vector inequality

Interconnected systems often lead to the problem of studying inequalities

$$\begin{pmatrix} \dot{v}_1(t) \\ \vdots \\ \dot{v}_n(t) \end{pmatrix} \leq M(t) \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} + P(t) \begin{pmatrix} \sup_{s \in [t-\tau, t]} v_1(s) \\ \vdots \\ \sup_{s \in [t-\tau, t]} v_n(s) \end{pmatrix} \quad (57)$$

where $M(t)$ is Metzler, $P(t)$ is nonnegative, $\tau > 0$ and where each v_i is a **nonnegative real-valued** function.

Vector inequality

We have:

$$\begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} \leq \Phi(t, t_0) \begin{pmatrix} v_1(t_0) \\ \vdots \\ v_n(t_0) \end{pmatrix} + \int_{t_0}^{t} \Phi(t, m) P(m) \begin{pmatrix} \sup_{s \in [m-\tau, m]} v_1(s) \\ \vdots \\ \sup_{s \in [m-\tau, m]} v_n(s) \end{pmatrix} dm \tag{58}$$

for all $t \geq t_0 \geq \tau$, where Φ is the state transition matrix of M.

Vector inequality

Consequently

$$\begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} \leq \Phi(t, t-T) \begin{pmatrix} v_1(t-T) \\ \vdots \\ v_n(t-T) \end{pmatrix} + \int_{t-T}^{t} \Phi(t, m)P(m)dm \begin{pmatrix} \sup_{s\in[t-T-\tau,t]} v_1(s) \\ \vdots \\ \sup_{s\in[t-T-\tau,t]} v_n(s) \end{pmatrix} \quad (59)$$

for any $T > 0$ and $t \geq T + \tau$.

Vector inequality

We obtain the vector inequality

$$\begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} \leq G(t,T) \begin{pmatrix} \sup\limits_{s\in[t-T-\tau,t]} v_1(s) \\ \vdots \\ \sup\limits_{s\in[t-T-\tau,t]} v_n(s) \end{pmatrix} \tag{60}$$

with $G(t,T) = \Phi(t,t-T) + \int_{t-T}^{t} \Phi(t,m)P(m)dm.$

Notice that $G(t,T) \geq 0$

Vector inequality

Assumption D1. *There are* $T > 0$ *and a Schur stable matrix* $S > 0$ *s. t.*

$$G(t, T) \leq S \tag{61}$$

for all $t \geq T$.

Then

$$\begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} \leq S \begin{pmatrix} \sup_{s\in[t-T-\tau,t]} v_1(s) \\ \vdots \\ \sup_{s\in[t-T-\tau,t]} v_n(s) \end{pmatrix}. \tag{62}$$

The **vector version** of the trajectory based approach (Theorem 3) allows us to conclude.

Conclusion

Conclusion

▷ We developed a trajectory based approach to prove **UGES** for nonlinear systems, notably with time-varying delays.

▷ They apply in many cases (ODEs coupled with difference equations, systems with delay of neutral type, and networked control systems).

▷ Using our approach, we proved stabilizability of nonlinear systems in the case where the **delay** could be unknown and not necessarily continuous, and where the system is subjected to additive **uncertainties** on the right side.

Conclusion

→ **Trajectory-based approach:**

F. Mazenc, M. Malisoff. *Trajectory based approach for the stability analysis of nonlinear systems with time delays.* IEEE Trans. Aut. Contr. 60, (6), 2015.

F. Mazenc, M. Malisoff. *Reduction model approach for systems with a time-varying delay.* In Proceedings of the IEEE 54th Conference on Decision and Control, pp. 7723-7727, Osaka, Japan, 2015.

Conclusion

F. Mazenc, M. Malisoff, S-I. Niculescu. *Stability and control design for time-varying systems with time-varying delays using a trajectory based approach.* SIAM Journal Contr. Opt., vol. 55, Issue 1, 2017.

S. Ahmed, F. Mazenc, H. Ozbay, *Dynamic output feedback stabilization of switched linear systems with delay via a trajectory based approach.* Automatica, vol. 93, 2018.

Conclusion

→ Extensions of the **Halanay's inequality**:

F. Mazenc, M. Malisoff, M. Krstic, *Stability analysis for time-varying systems with asynchronous sampling using contractivity approach*, IEEE Control Systems Letters, Vol. 5, Issue 1, pp. 49-54, Jan. 2021.

F. Mazenc, M. Malisoff, M. Krstic, *Stability and observer designs using new variants of Halanay's inequality*, Automatica, Vol. 123, Jan. 2021.

Conclusion

F. Mazenc, M. Malisoff, M. Krstic. *Stability Analysis using Generalized Sup-Delay Inequalities.* IEEE Control Systems Letters, 5(4):1411-1416, 2021.

F. Mazenc, M. Malisoff, M. Krstic. *Vector Extensions of Halanay's Inequality.* IEEE Transactions on Automatic Control. vol. 67, No. 3, 2022.

Thank you for your attention.