Non-uniform finite-element meshes defined by ray dynamics for Helmholtz trapping problems

Martin Averseng, Jeff Galkowski, Euan Spence

$$\frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = 0 \quad \xrightarrow{\;U(x,t) = e^{-i\omega t} u(x)\;} \quad -k^2 u - \Delta u = 0$$

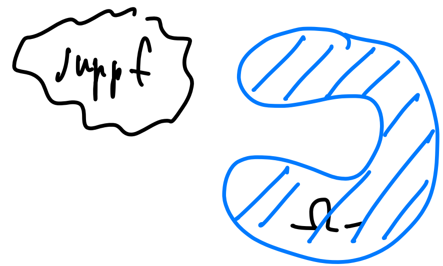
$U(x,t)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}$

$u(x)$, $x \in \mathbb{R}^d$

$k := \frac{\omega}{c}$

model Helmholtz problem

$$\left.\begin{aligned} (-k^{-2}\Delta - 1)u &= f \quad \text{in } \Omega_+ \\ u &= 0 \quad \text{on } \partial\Omega_+ \\ \text{radiation condition} \quad \left(k^{-1}\frac{\partial}{\partial r} - i\right)u &= o\left(\frac{1}{r^{\frac{d-1}{2}}}\right) \\ \text{as } r := |x| &\to \infty \end{aligned}\right\} (*)$$



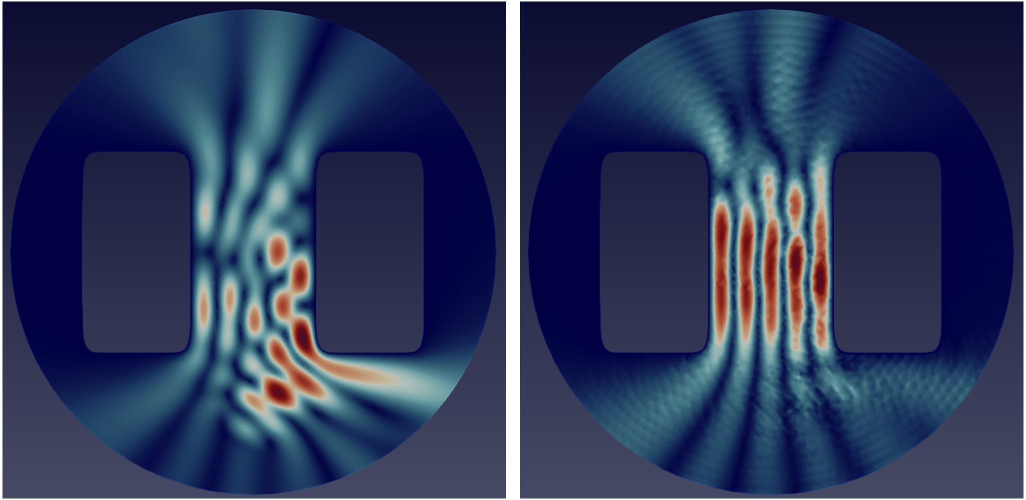
$\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$

goal: compute sol$^{\underline{n}}$ of (*) to arbitrary accuracy for $k \gg 1$

Motivating numerical experiments

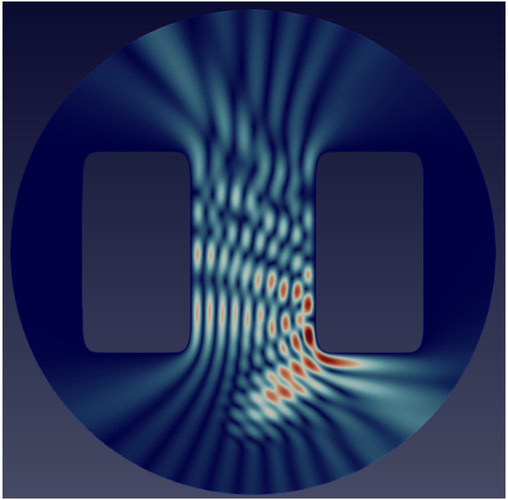
- choose k so that trapped waves can exist between mirrors
- "beam" source
- h-version of FEM with $p=2$, uniform (k-dependent) h
- reference solution $\tilde{u}$ using $p=4$ on same mesh
- computations done with FreeFEM ✓ ☺

$kR \approx 40$

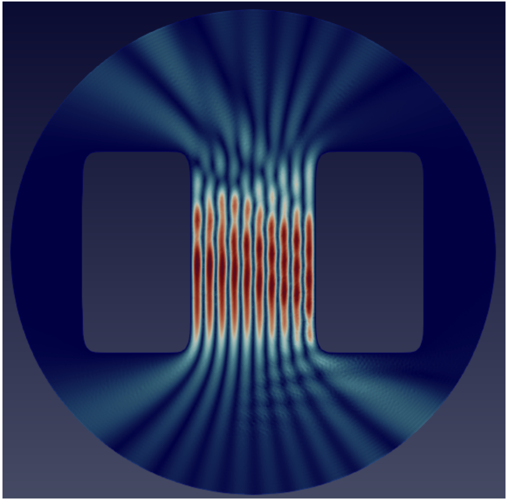


$|\widetilde{u}|$ $|u - u_h|$

$kR \approx 80$

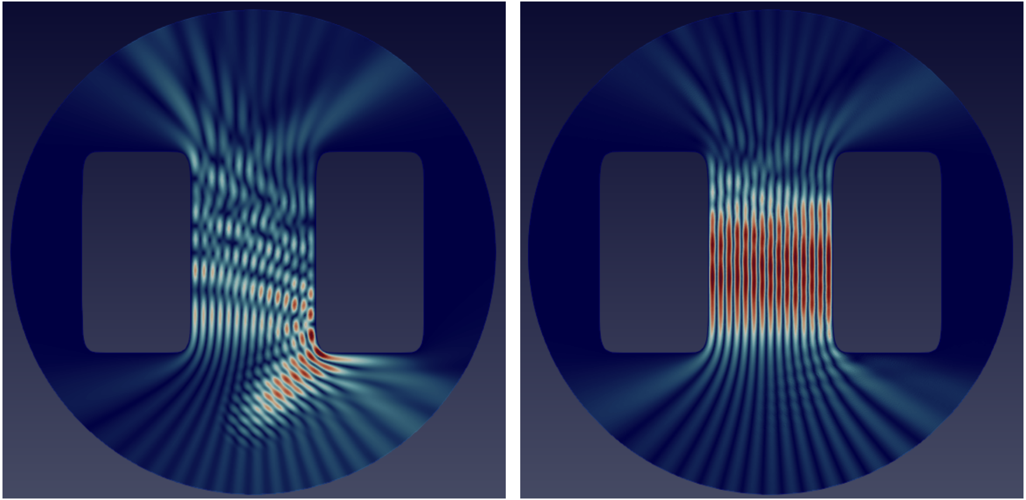


$|\widetilde{u}|$



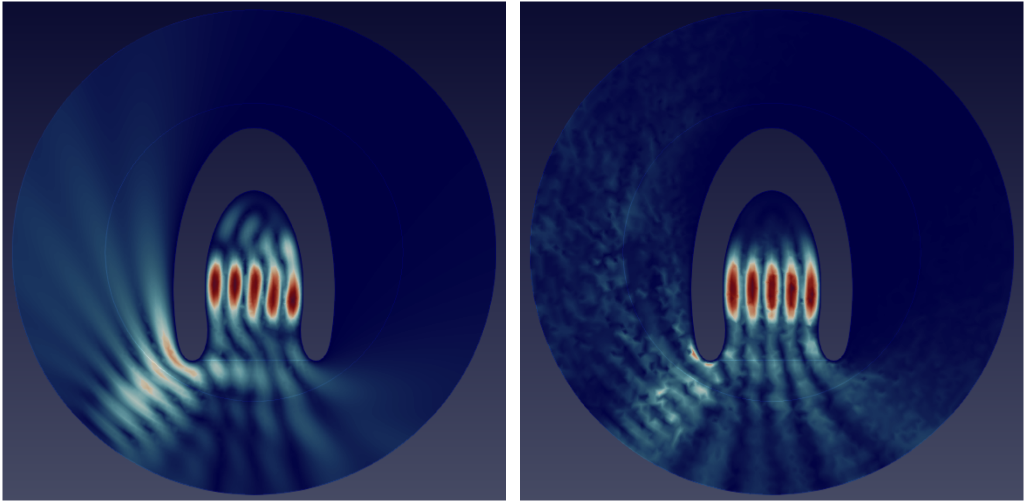
$|\widetilde{u} - u_h|$

$kR \approx 120$



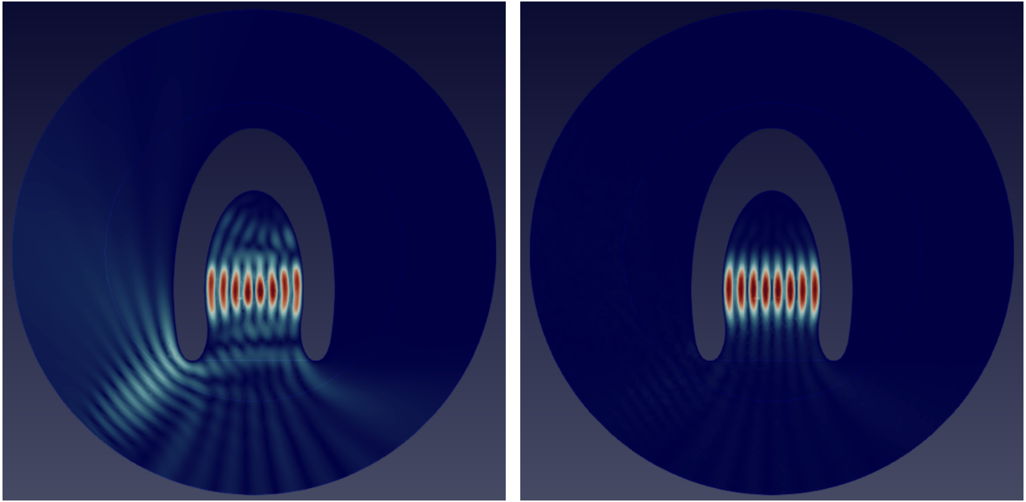
$|\widetilde{u}|$ $|\widetilde{u} - u_h|$

$kR \approx 50$



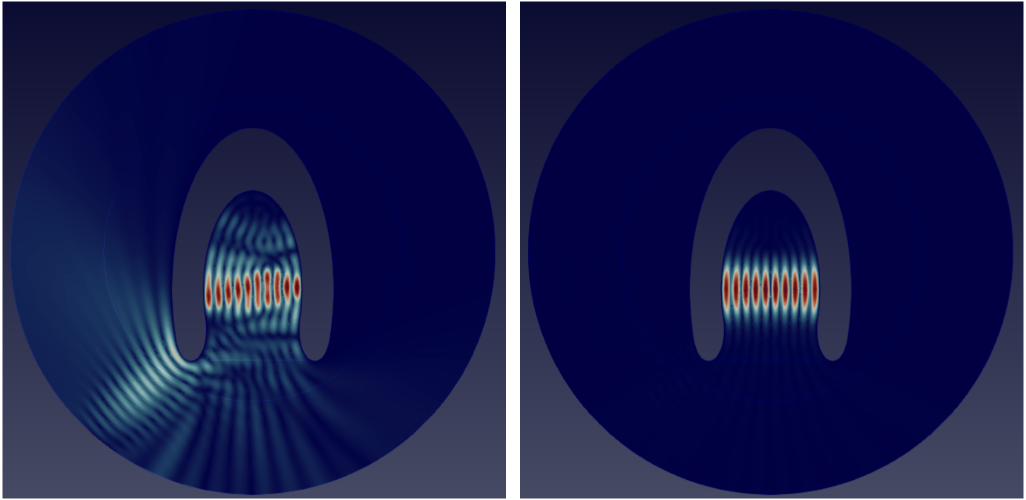
$|\widetilde{u}|$ $\quad$ $|\widetilde{u} - u_h|$

$kR \approx 75$



$|\widetilde{u}|$ $|\widetilde{u} - u_h|$

$kR \approx 100$



$|\widetilde{u}|$ $|\widetilde{u} - u_h|$

Two questions

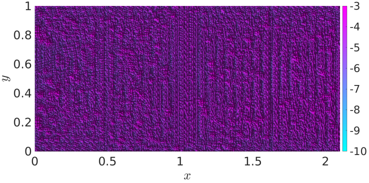
1. For a uniform h chosen as a function of k to maintain accuracy as $k \to \infty$, can one prove that the FEM error is smaller away from trapping?

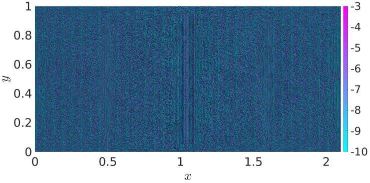
2. Can one choose a non-uniform (k-dependent) h to achieve different goals, e.g., control error in trapping, control error away from trapping?

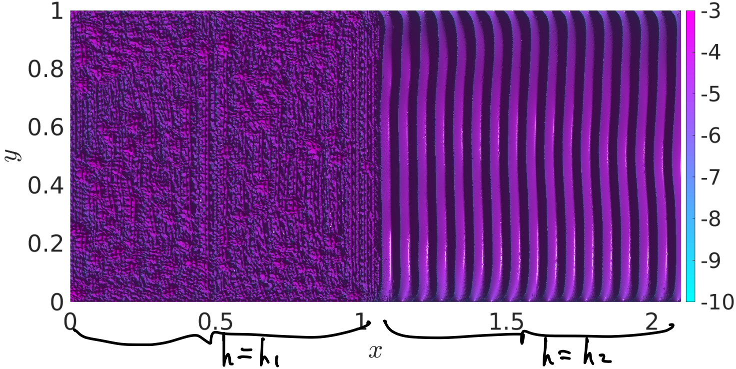
Non-uniform meshes: a cautionary tale

[Averseng, Galkowski, Spence, 2023]

$h = h_1$ $\qquad\qquad$ $h = h_2 \ll h_1$







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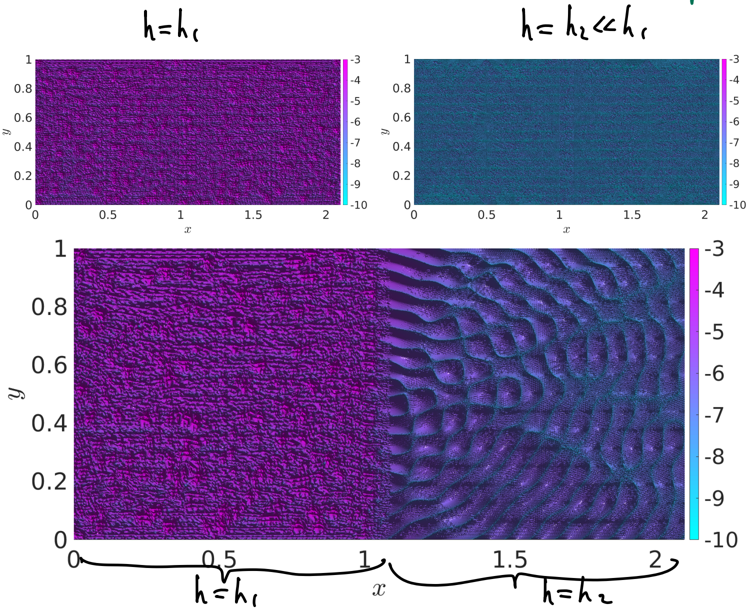

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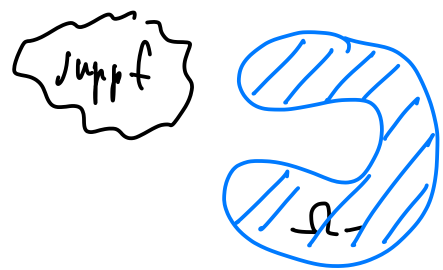

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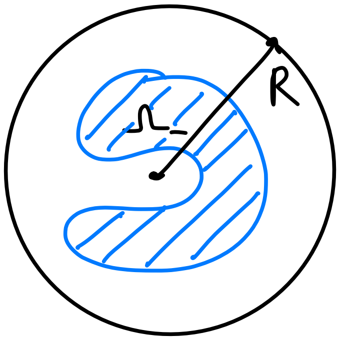
goal: compute solⁿ of (*) to arbitrary accuracy for $k \gg 1$

k-dependence of Helmholtz sol^{n} operator

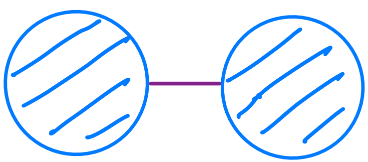


$$\rho(k) := \sup_{f:\, \mathrm{supp}\, f \subset B_R} \frac{\|u\|_{H^1_k(B_R\cap\Omega_+)}}{\|f\|_{L^2(B_R\cap\Omega_+)}}$$

where

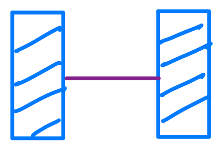
$$\|u\|^2_{H^1_k} := k^{-2}\|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}$$

Ω_- nontrapping $\Rightarrow$ $\rho(k) \sim k$ [Vainberg 1975, Morawetz, Ralston, Strauss 1977]

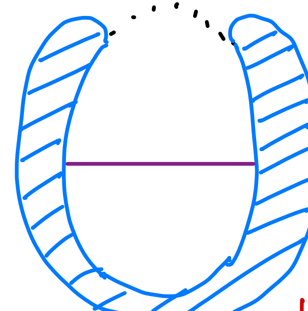


$\rho(k_j) \sim k_j \log k_j$

$0 < k_1 < k_2 < \dots < k_j \to \infty$



$\rho(k_\ell) \sim k_\ell^2$



$\rho(k_m) \sim e^{c k_m}$

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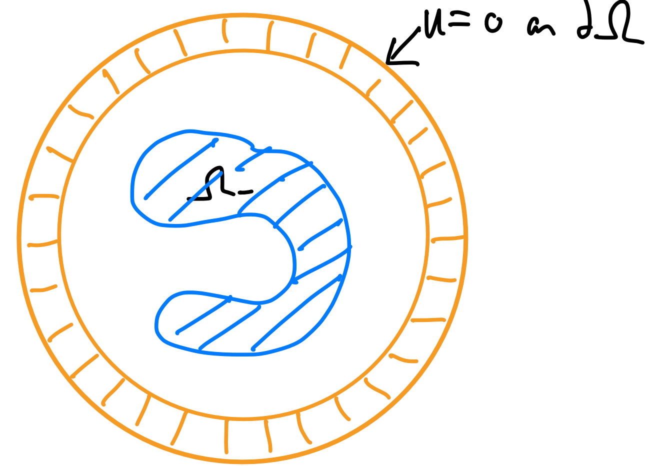
in general, $\rho(k) \leq C k^M$ for "most" k [Lafontaine, Spence, Wunsch 2021]

PML truncation, variational formulation, FEM

truncate Ω_+ and approximate radiation condition by a radial PML



find $u \in H^1_0(\Omega)$ s.t. $a(u,v) = F(v)$ $\forall\, v \in H^1_0(\Omega)$

where $a(u,v) = \displaystyle\int_\Omega k^{-2}(A\nabla u)\cdot\overline{\nabla v} - n u \bar{v}$

$$F(v) = \int_\Omega f\,\bar{v}$$

h-version of FEM: $\{V_h\}_{h>0}$ p-wise polys, fixed degree p, meshwidth h, $V_h \subset H^1_0(\Omega)$

find $u_h \in V_h$ s.t. $a(u_h, v_h) = F(v_h)$ $\forall\, v_h \in V_h$

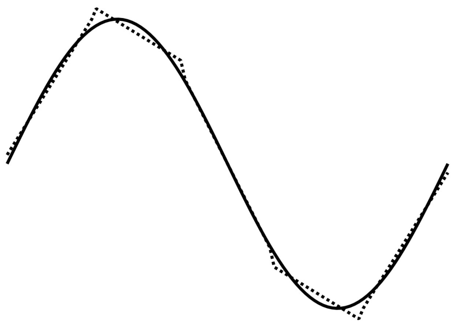
How quickly must h decrease as $k \to \infty$ to maintain accuracy?

How small must h be to accurately approximate a Helmholtz solution?

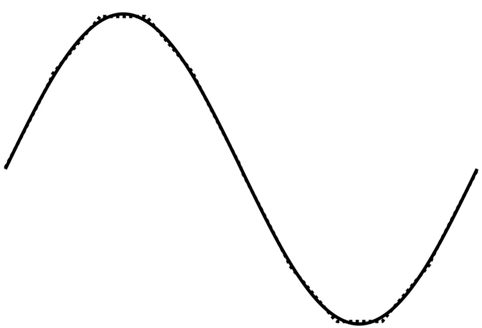
Wavelength of oscillations is $2\pi k^{-1}$

$\therefore$ functions are close to linear when $h \ll k^{-1}$

$hk = \frac{2\pi}{5}$, $p = 1$



$hk = \frac{2\pi}{10}$



classic polynomial approximation result

$$\min_{v_h} \| u - v_h \|_{H^1_k} \leq C (hk)^p \| u \|_{H^{p+1}_k}$$

where

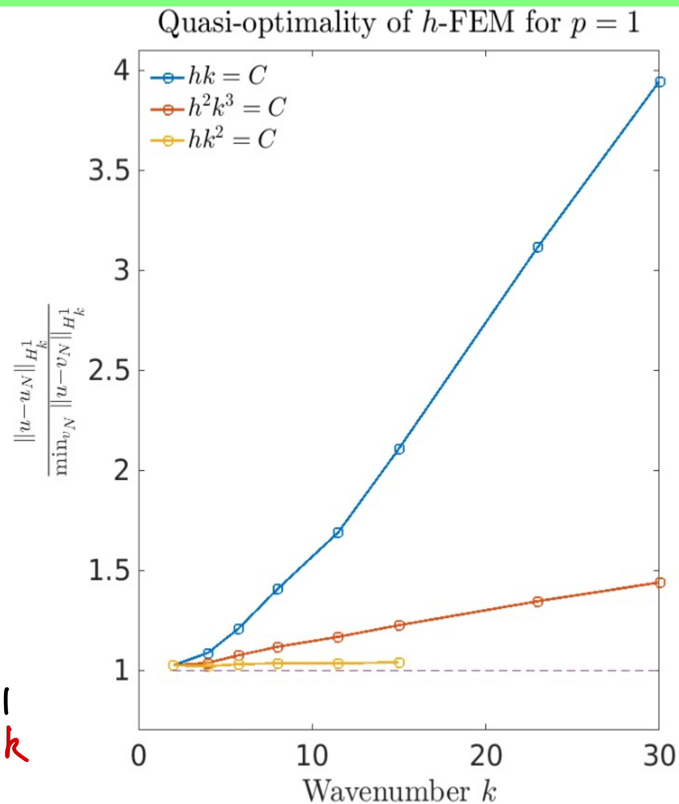
$$\| u \|^2_{H^m_k} := \sum_{|\alpha| \leq m} \| k^{-|\alpha|} \partial^\alpha u \|_{L^2}$$

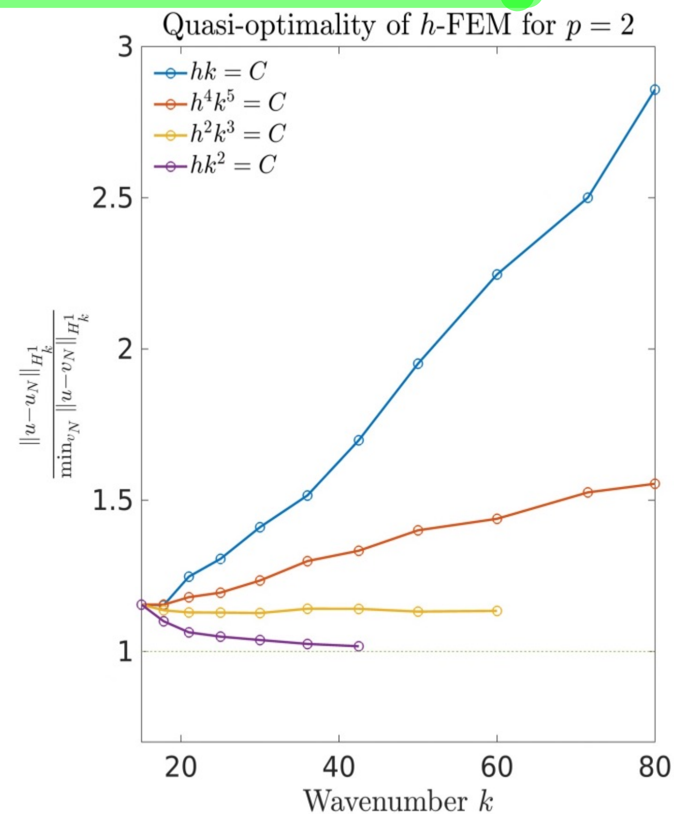
How quickly must h decrease as $k \to \infty$ to maintain accuracy of h-FEM?

"pollution effect": hk small not sufficient for accuracy as $k \to \infty$!

(term coined in 1995 by Babuška, Ihlenburg, Sauter)

$$\frac{\|u-u_h\|_{H^1_k}}{\min_{v_h} \|u-v_h\|_{H^1_k}}$$





How quickly must h decrease as $k\to\infty$ to maintain accuracy of h-FEM?

$$\boxed{(hk)^p\rho(k)\leq\varepsilon \quad\Rightarrow\quad \|u-u_h\|_{H^1_k}\leq C\min_{v_h\in V_h}\|u-v_h\|_{H^1_k}}$$

"asymptotic regime" k independent quasi-optimality

$$\boxed{\begin{array}{l}(hk)^{2p}\rho(k)\leq\varepsilon \\ \text{and data } k\text{-oscillatory}\end{array} \quad\Rightarrow\quad \frac{\|u-u_h\|_{H^1_k}}{\|u\|_{H^1_k}}\leq C\left(\sqrt{\frac{\varepsilon}{\rho(k)}}+\varepsilon\right)}$$

relative error controllably small

$$(hk)^{2p}\rho(k)\leq\varepsilon \quad\Rightarrow\quad \boxed{\|u-u_h\|_{H^1_k}\leq C\left[1+(hk)^p\rho(k)\right]\min_{v_h\in V_h}\|u-v_h\|_{H^1_k}}$$

$$\leq C\left[1+(hk)^p\rho(k)\right](hk)^p\underbrace{\|u\|_{H^{p+1}_k}}_{\leq C\|u\|_{H^1_k}\text{ if data }k\text{-oscillatory}}$$

short history of work obtaining these mesh thresholds

- [Aziz, Kellogg, Stephens, 1988], $d=1$ $\quad hk^2 \leq \varepsilon$ for Q.O.
  - [Melenk, 1995], $d=2$ $\quad$ ''
- [Ihlenburg, Babuška, 1995, 1997], $d=1$ $\quad (hk)^{2p} k \leq \varepsilon$ for relative error controlled

$\Big\}$ nontrapping problems

Q.O. if $(hk)^p \rho(k) \leq \varepsilon$

- [Melenk, Sauter, 2010, 2011]
  p-explicit (constant coeff problems)
- [Chaumont-Frelet, Nicaise 2020]
  general Helmholtz problems

relative error controlled if $(hk)^{2p} \rho(k) \leq \varepsilon$

- [Du, Wu, 2015] nontrapping problem truncated with impedance boundary condition
- [Galkowski, Spence, 2023]
  general Helmholtz problems

Two questions

1. For a (uniform) h satisfying either $(hk)^p \rho(k) \leq C$ or $(hk)^{2p}\rho(k) \leq C$, can one prove that FEM error is smaller away from trapping?

2. Can one choose a non-uniform (k-dependent) h to achieve different goals, e.g., control error in trapping
control error away from trapping?

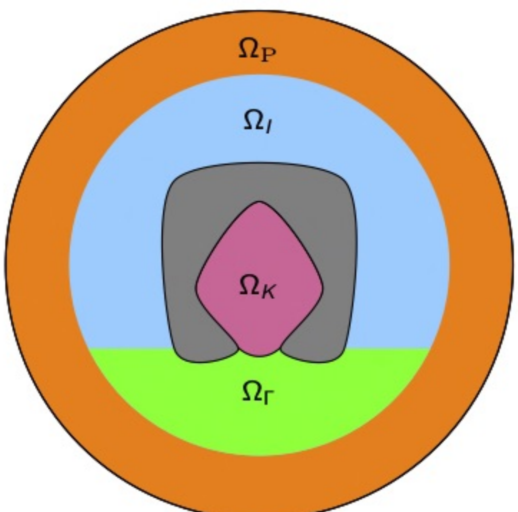
Four regions

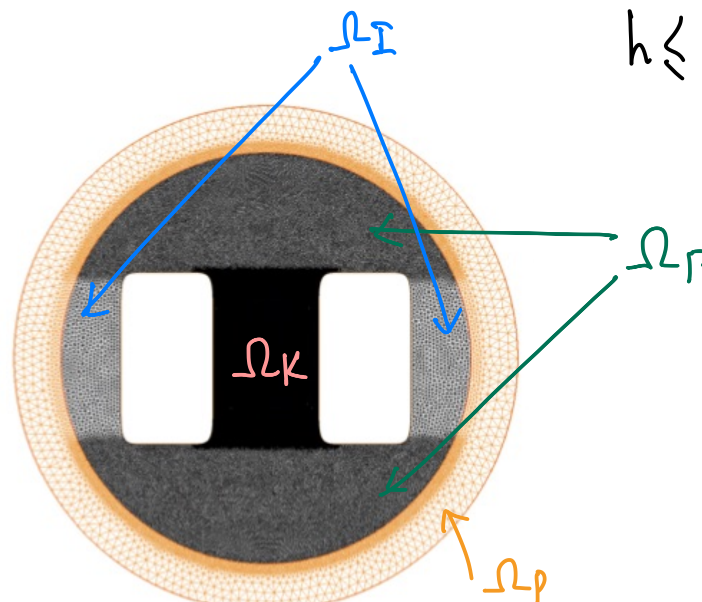
Ω_K : the cavity – all forward and backward trapped rays $\qquad h \lesssim h_K$

Ω_Γ : the visible set – all rays that are trapped (either forward or backward) $\qquad h \lesssim h_\Gamma$

Ω_P : a set contained in the PML region $\qquad h \lesssim h_P$

Ω_I : the invisible set – everything else $\qquad h \lesssim h_I$





Structure of the main result

given $k_0 > 0$

if h_K, h_Γ, h_I, h_p all suff. small in different k-dependent ways
(before $(hk)^{2p}\rho(k) \leq \varepsilon$)

Then Galerkin sol$^{\underline{n}}$ exists, is unique, and $\exists\, C > 0$ s.t. $\forall k \geq k_0$, $\forall v_h \in V_h$

$$\begin{pmatrix} \|u-u_h\|_{H^1_k(\Omega_K)} \\ \|u-u_h\|_{H^1_k(\Omega_\Gamma)} \\ \|u-u_h\|_{H^1_k(\Omega_I)} \\ \|u-u_h\|_{H^1_k(\Omega_p)} \end{pmatrix} \leq C \underbrace{\left(I + \begin{pmatrix} \text{matrix} \\ \text{depending on} \\ h_K, h_\Gamma, h_I, h_p \\ \text{and } k \end{pmatrix} \right)}_{\left(\text{before } 1 + (hk)^p \rho(k)\right)} \begin{pmatrix} \|u-v_h\|_{H^1_k(\Omega_K)} \\ \|u-v_h\|_{H^1_k(\Omega_\Gamma)} \\ \|u-v_h\|_{H^1_k(\Omega_I)} \\ \|u-v_h\|_{H^1_k(\Omega_p)} \end{pmatrix}$$

Ingredients of the main result

"communication matrix"

$$\mathcal{C}(k) := \begin{array}{c} \\ \Omega_K \\ \Omega_\Gamma \\ \Omega_I \\ \Omega_P \end{array} \begin{array}{c} \begin{array}{cccc} \Omega_K & \Omega_\Gamma & \Omega_I & \Omega_P \end{array} \\ \begin{pmatrix} \rho(k) & \sqrt{k\rho(k)} & 0 & 0 \\ \sqrt{k\rho(k)} & k & k & 1 \\ 0 & k & k & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

where $0 \to O(k^{-\infty})$

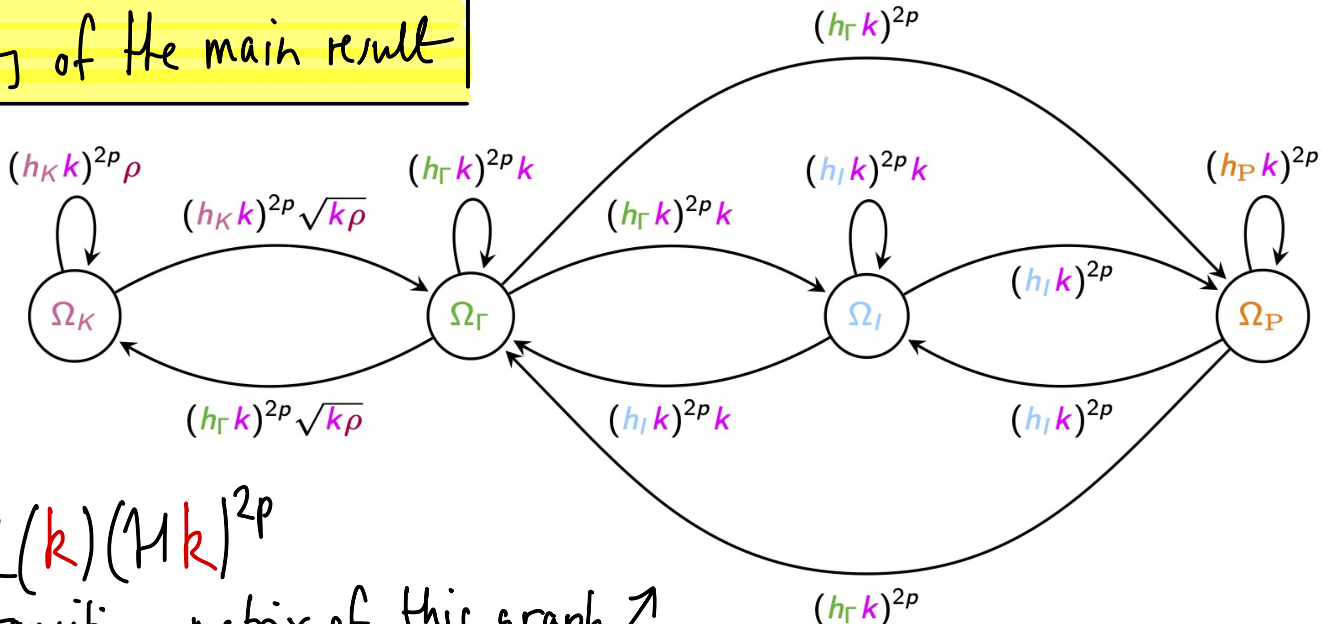
mesh-width matrix

$$\mathcal{H} := \begin{array}{c} \\ \Omega_K \\ \Omega_\Gamma \\ \Omega_I \\ \Omega_P \end{array} \begin{array}{c} \begin{array}{cccc} \Omega_K & \Omega_\Gamma & \Omega_I & \Omega_P \end{array} \\ \begin{pmatrix} h_K & 0 & 0 & 0 \\ 0 & h_\Gamma & 0 & 0 \\ 0 & 0 & h_I & 0 \\ 0 & 0 & 0 & h_P \end{pmatrix} \end{array}$$

$$\mathcal{C}_{ij}(k) = \left\| 1_{\Omega_i} (\text{sol}^{\underline{n}} \text{ operator}) 1_{\Omega_j} \right\|_{L^2 \to L^2}$$

[Burq 2002], [Cardoso, Vodev, 2002], [Datchev, Vasy, 2012]

Ingredients of the main result



let $T := \mathcal{C}(k)(Hk)^{2p}$

= transition matrix of this graph ↗

given $C' > 0$ let

$$\boxed{\nu(C') := (I - C'T)^{-1} = \sum_{n=0}^{\infty} (C'T)^n} \Big\} \; ij\text{th entry} = \sum_{\substack{\text{paths from} \\ j \text{ to } i}} (C')^{(\text{length of path})} \begin{pmatrix}\text{product of} \\ \text{weights along} \\ \text{path}\end{pmatrix}$$

where edge $(l,m) \leftrightarrow$ weight T_{ml}

The main result

$\forall k_0 > 0 \ \exists \ \varepsilon, C^*, C$ s.t. $\forall k \geq k_0$

if $\boxed{(h_K k)^{2p} \rho(k) + (h_\Gamma k)^{2p} k + (h_I k)^{2p} k + h_p k \leq \varepsilon}$ (before $(hk)^{2p} \rho(k) \leq \varepsilon$)

then Galerkin sol$^{\underline{n}}$ exists, is unique, and $\forall v_h \in V_h$

$$\begin{pmatrix} \|u-u_h\|_{H^1_k(\Omega_K)} \\ \|u-u_h\|_{H^1_k(\Omega_\Gamma)} \\ \|u-u_h\|_{H^1_k(\Omega_I)} \\ \|u-u_h\|_{H^1_k(\Omega_p)} \end{pmatrix} \leq C \left(I + \underbrace{\nu(C^*)\,\rho(k)}\underbrace{(\mathcal{H}k)^p} \right) \begin{pmatrix} \|u-v_h\|_{H^1_k(\Omega_K)} \\ \|u-v_h\|_{H^1_k(\Omega_\Gamma)} \\ \|u-v_h\|_{H^1_k(\Omega_I)} \\ \|u-v_h\|_{H^1_k(\Omega_p)} \end{pmatrix}$$

(before $1 + \rho(k)(hk)^p$)

The main result: implications

mesh threshold	asymptotic cost	guarantee
$h_K = h_F = h_I = h_p = h$, $(hk)^p \rho(k) \lesssim \varepsilon$	$\mathrm{vol}(\Omega)\, k^d \rho(k)^{\frac{d}{p}}$	k-independent Q.O.
$h_K = h_F = h_I = h_p = h$, $(hk)^{2p} \rho(k) \lesssim \varepsilon$	$\mathrm{vol}(\Omega)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled

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$(h_K k)^p \rho + (h_\Gamma k)^p \sqrt{k\rho} + (h_I k)^p k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{p}}$	k-independent Q.O.
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For a uniform h, the error is smaller away from trapping

$$h_K = h_P = h_I = h, \quad (hk)^p \rho(k) \le \varepsilon$$

$$\begin{pmatrix} \|u-u_h\|_{H^1_k(\Omega_K)} \\ \|u-u_h\|_{H^1_k(\Omega_P)} \\ \|u-u_h\|_{H^1_k(\Omega_I)} \end{pmatrix} \lesssim C \begin{pmatrix} 1 & \sqrt{\frac{k}{\rho}} & \left(\frac{k}{\rho}\right)^{\frac{3}{2}}\frac{1}{\rho} \\ \sqrt{\frac{k}{\rho}} & 1 & \frac{k}{\rho} \\ \left(\frac{k}{\rho}\right)^{\frac{3}{2}}\frac{1}{\rho} & \frac{k}{\rho} & 1 \end{pmatrix} \begin{pmatrix} \|u-v_h\|_{H^1_k(\Omega_K)} \\ \|u-v_h\|_{H^1_k(\Omega_P)} \\ \|u-v_h\|_{H^1_k(\Omega_I)} \end{pmatrix} \quad \forall v_h \in V_h$$

$$(h_K k)^p \rho + (h_P k)^p \sqrt{k\rho} + (h_I k)^p k \le \varepsilon$$

$$\begin{pmatrix} \\ \text{''} \\ \\ \end{pmatrix} \lesssim C \begin{pmatrix} 1 & 1 & \frac{1}{\sqrt{k\rho}} \\ \sqrt{\frac{k}{\rho}} & 1 & 1 \\ \frac{1}{\rho}\sqrt{\frac{k}{\rho}} & \sqrt{\frac{k}{\rho}} & 1 \end{pmatrix} \begin{pmatrix} \\ \text{''} \\ \\ \end{pmatrix}$$

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$(h_K k)^p \rho + (h_\Gamma k)^p \sqrt{k\rho} + (h_I k)^p k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{p}}$	k-independent Q.O.
$h_K = h_\Gamma = h_I = h_p = h,\ (hk)^{2p} \rho(k) \lesssim \varepsilon$	$\mathrm{vol}(\Omega)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled
$(h_K k)^{2p} \rho + (h_\Gamma k)^{2p} \sqrt{k\rho} + (h_I k)^{2p} k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled

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$(h_K k)^p \rho + (h_\Gamma k)^p \sqrt{k\rho} + (h_I k)^p k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{p}}$	k-independent Q.O.
$(h_K k)^p \sqrt{k\rho} + (h_\Gamma k)^p k + (h_I k)^p k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^{d+\frac{1}{2p}} \rho(k)^{\frac{d}{2p}}$	k-independent Q.O. away from Ω_K
$h_K = h_\Gamma = h_I = h_p = h,\ (hk)^{2p} \rho(k) \lesssim \varepsilon$	$\mathrm{vol}(\Omega)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled
$(h_K k)^{2p} \rho + (h_\Gamma k)^{2p} \sqrt{k\rho} + (h_I k)^{2p} k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled

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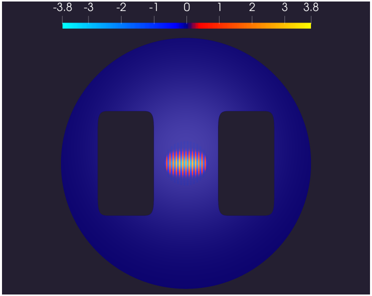
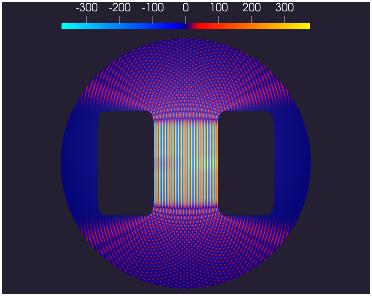
mesh threshold	asymptotic cost	guarantee
$h_K = h_\Gamma = h_I = h_p = h,\ (hk)^p \rho(k) \lesssim \varepsilon$	$\mathrm{vol}(\Omega)\, k^d \rho(k)^{\frac{d}{p}}$	k-independent Q.O.
$(h_K k)^p \rho + (h_\Gamma k)^p \sqrt{k\rho} + (h_I k)^p k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{p}}$	k-independent Q.O.
$(h_K k)^p \sqrt{k\rho} + (h_\Gamma k)^p k + (h_I k)^p k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^{d+\frac{1}{2p}} \rho(k)^{\frac{d}{2p}}$	k-independent Q.O. away from Ω_K
$h_K = h_\Gamma = h_I = h_p = h,\ (hk)^{2p} \rho(k) \lesssim \varepsilon$	$\mathrm{vol}(\Omega)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled
$(h_K k)^{2p} \rho + (h_\Gamma k)^{2p} \sqrt{k\rho} + (h_I k)^{2p} k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled
$(h_K k)^{2p} \rho + (h_\Gamma k)^{2p} k + (h_I k)^{2p} k + h_p k \lesssim \varepsilon$	$\mathrm{vol}(\Omega_K)\, k^d \rho(k)^{\frac{d}{2p}}$	rel. error controlled away from Ω_K

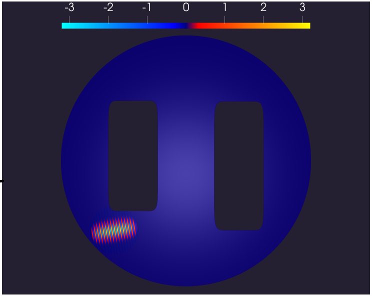
Data and solutions for numerical experiments

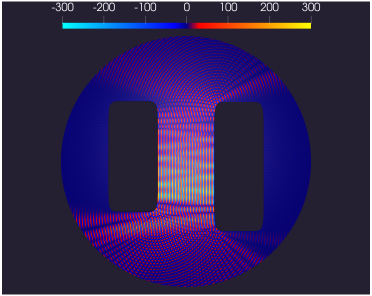
beam inside





beam outside

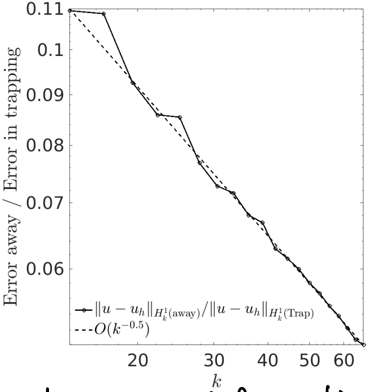




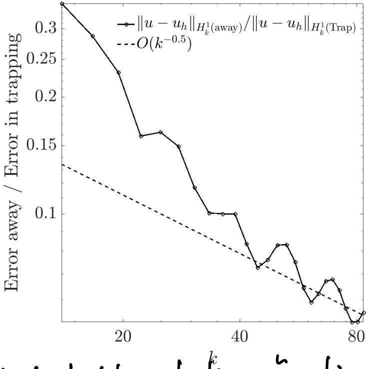
Uniform h for Q.O., error smaller away from trappings

$h_K = h_\Gamma = h_I = h_p = h$, $(hk)^p \rho(k) \lesssim \varepsilon$, predict $\boxed{\dfrac{\|u-u_h\|_{H^1_k(\Omega_\Gamma)}}{\|u-u_h\|_{H^1_k(\Omega_K)}} \sim k^{-\frac{1}{2}}}$

beam inside



beam outside

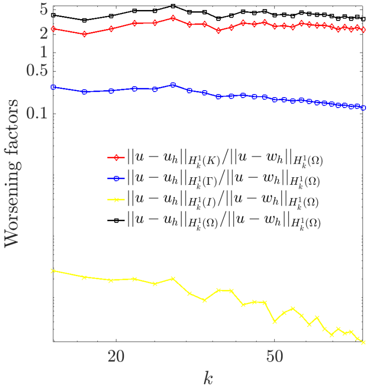


N.B. have used information that both solutions "activate" the trappings to a sufficient extent

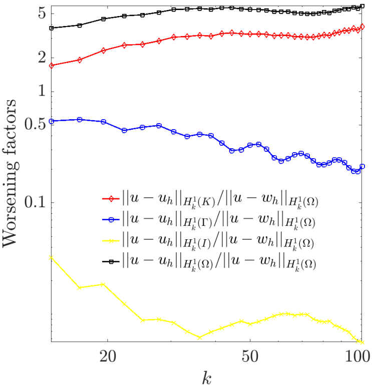
k-independent Q.O. via non-uniform mesh

$$(h_K k)^p \rho(k) + (h_\Gamma k)^p \sqrt{k\rho(k)} + (h_I k)^p k + h_p k \lesssim \varepsilon$$

beam inside



beam outside



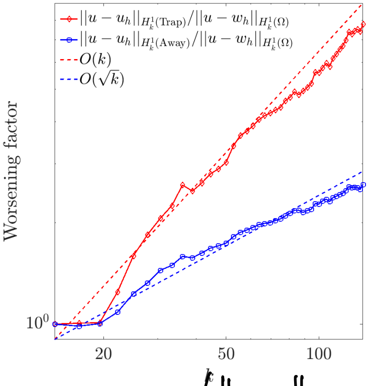
plotting $\dfrac{\|u-u_h\|_{H^1_k(\Omega_i)}}{\min_{v_h}\|u-v_h\|_{H^1_k(\Omega)}}$, theory predicts all $\lesssim C$

k-dependent Q.O. via coarsest mesh

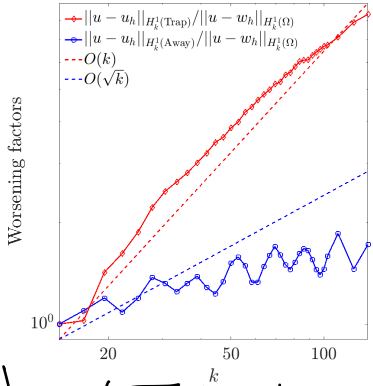
$$(h_K k)^{2p}\rho(k) + (h_\Gamma k)^p k + (h_I k)^p k + h_p k \lesssim \varepsilon$$

N.B. $\rho(k) \sim k^2$ here

beam inside



beam outside



theory predicts

$$\begin{pmatrix} \|u-u_h\|_{H^1_k(\Omega_K)} \\ \|u-u_h\|_{H^1_k(\Omega_\Gamma)} \end{pmatrix} \lesssim C \begin{pmatrix} \sqrt{\rho(k)}\,\|u-v_h\|_{H^1_k(\Omega)} \\ \sqrt{k}\,\|u-v_h\|_{H^1_k(\Omega)} \end{pmatrix} \quad \forall v_h \in V_h$$

Take-home messages

1) even for nontrapping problems, only need $hk \leq \varepsilon$ (i.e. fixed number of points per wavelength) strictly inside PML region

2) for trapping problems, to achieve k-indep. Q.O. / rel. error controlled, only need to respect existing mesh thresholds inside the trapping region Ω_K; can use larger meshwidths ($\Rightarrow$ fewer DOF) elsewhere