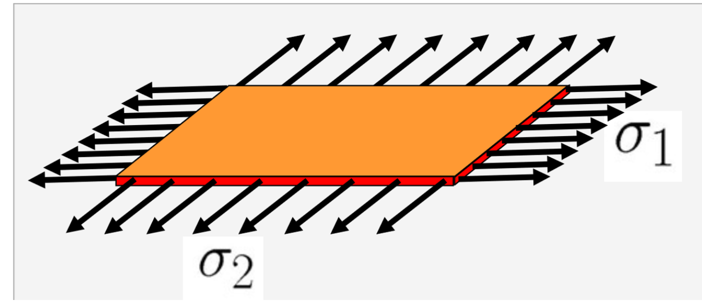
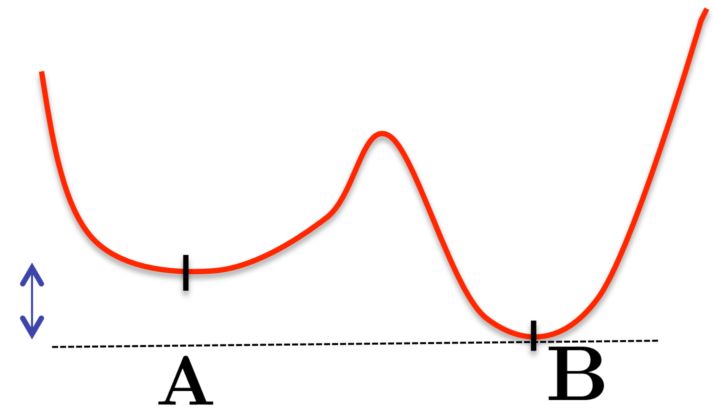
# Incompatibility-induced metastability

## Example 1




Special case of JB/James 2014 designed to explain hysteresis in the bi-axial experiments of Chu & James on CuAlNi single crystals, in which a transformation occurs under load between two martensitic variants.




Consider the integral $W(\mathbf{A}) - W(\mathbf{B})$

$$I(\mathbf{y}) = \int_\Omega W(D\mathbf{y})\, d\mathbf{x},$$

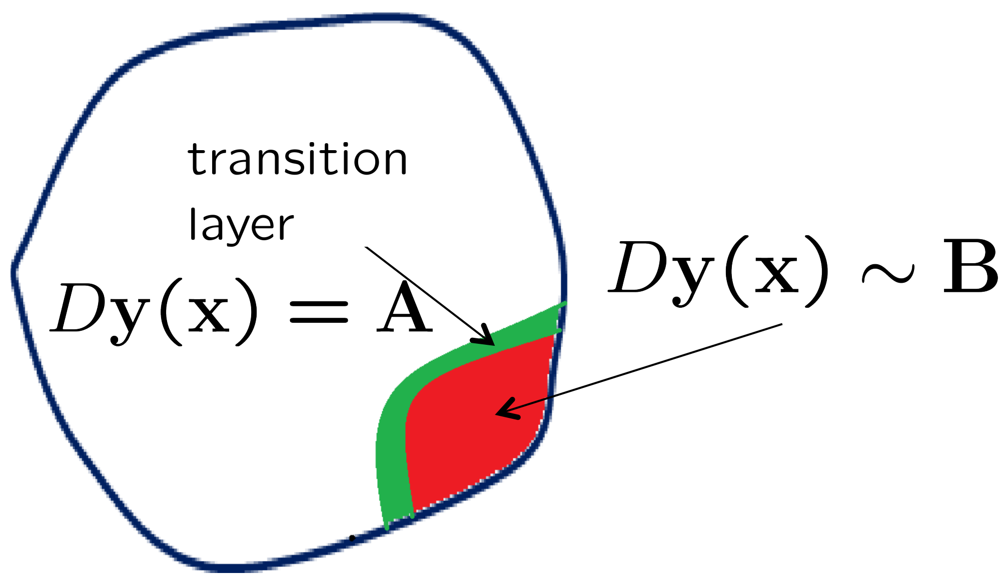
$$W(\mathbf{A}) = \psi(\mathbf{A}, \theta) - \mathbf{T} \cdot \mathbf{A}$$

where $W : GL^+(3, \mathbb{R}) \to \mathbb{R}$ and $W$ has two local minimizers at $\mathbf{A}, \mathbf{B}$ with rank$\,(\mathbf{A} - \mathbf{B}) > 1$ and $W(\mathbf{A}) - W(\mathbf{B}) > 0$ sufficiently small.

Claim. Under suitable growth hypotheses on $W$, $\bar{\mathbf{y}}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{c}$ is a local minimizer of $I$ in $L^1(\Omega;\mathbb{R}^3)$, i.e. there exists $\varepsilon > 0$ such that $I(\mathbf{y}) \geq I(\bar{\mathbf{y}})$ if $\int_\Omega |\mathbf{y} - \bar{\mathbf{y}}|\, d\mathbf{x} < \varepsilon$.

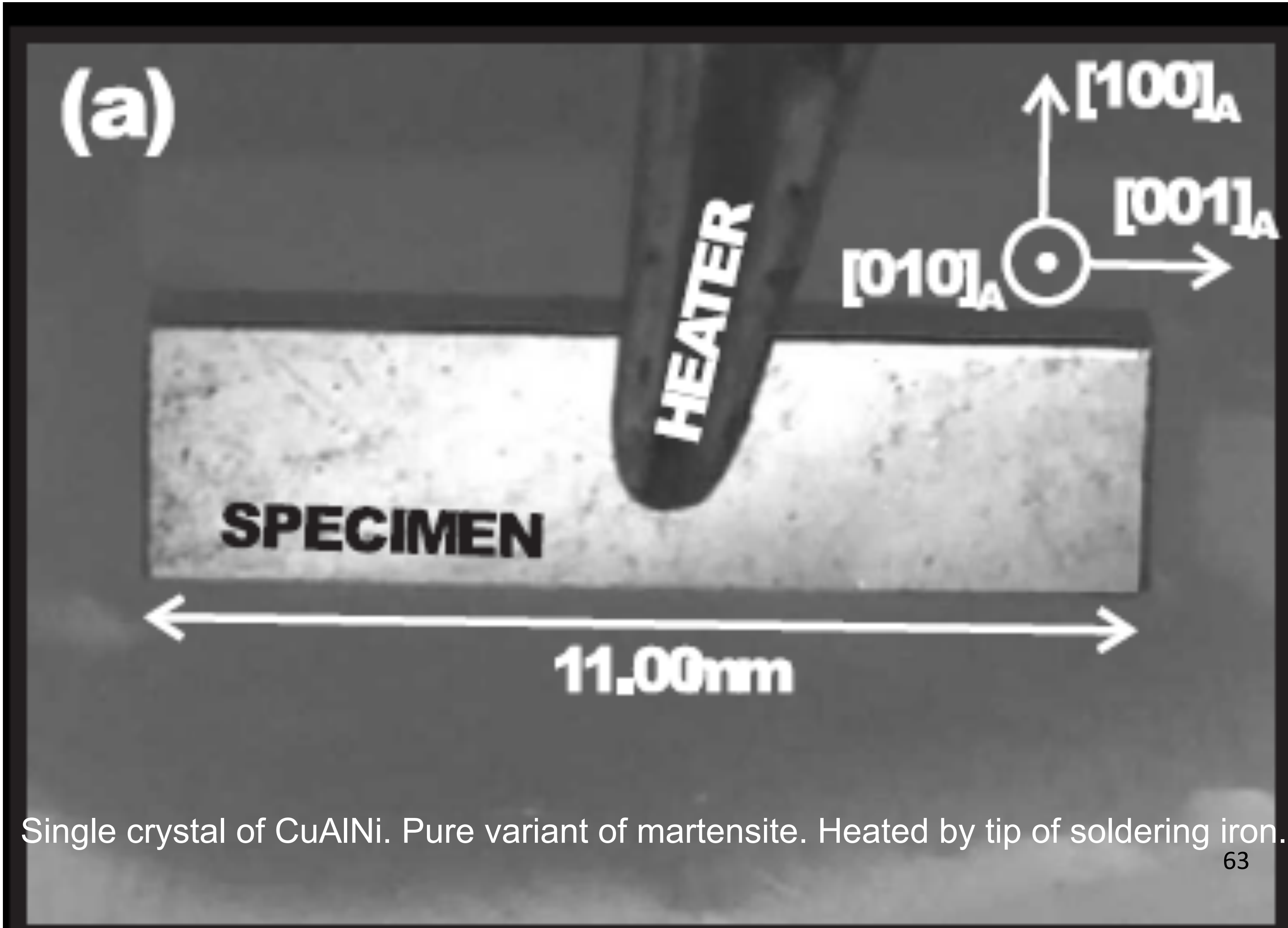
Idea: since $\mathbf{A}$ and $\mathbf{B}$ are incompatible, if we nucleate a region in which $D\mathbf{y}(\mathbf{x}) \sim \mathbf{B}$ there must be a transition layer in which the increase of energy is greater than the decrease of energy in the nucleus.

Related work:
Kohn & Sternberg 1989,
Grabovsky & Mengesha 2009

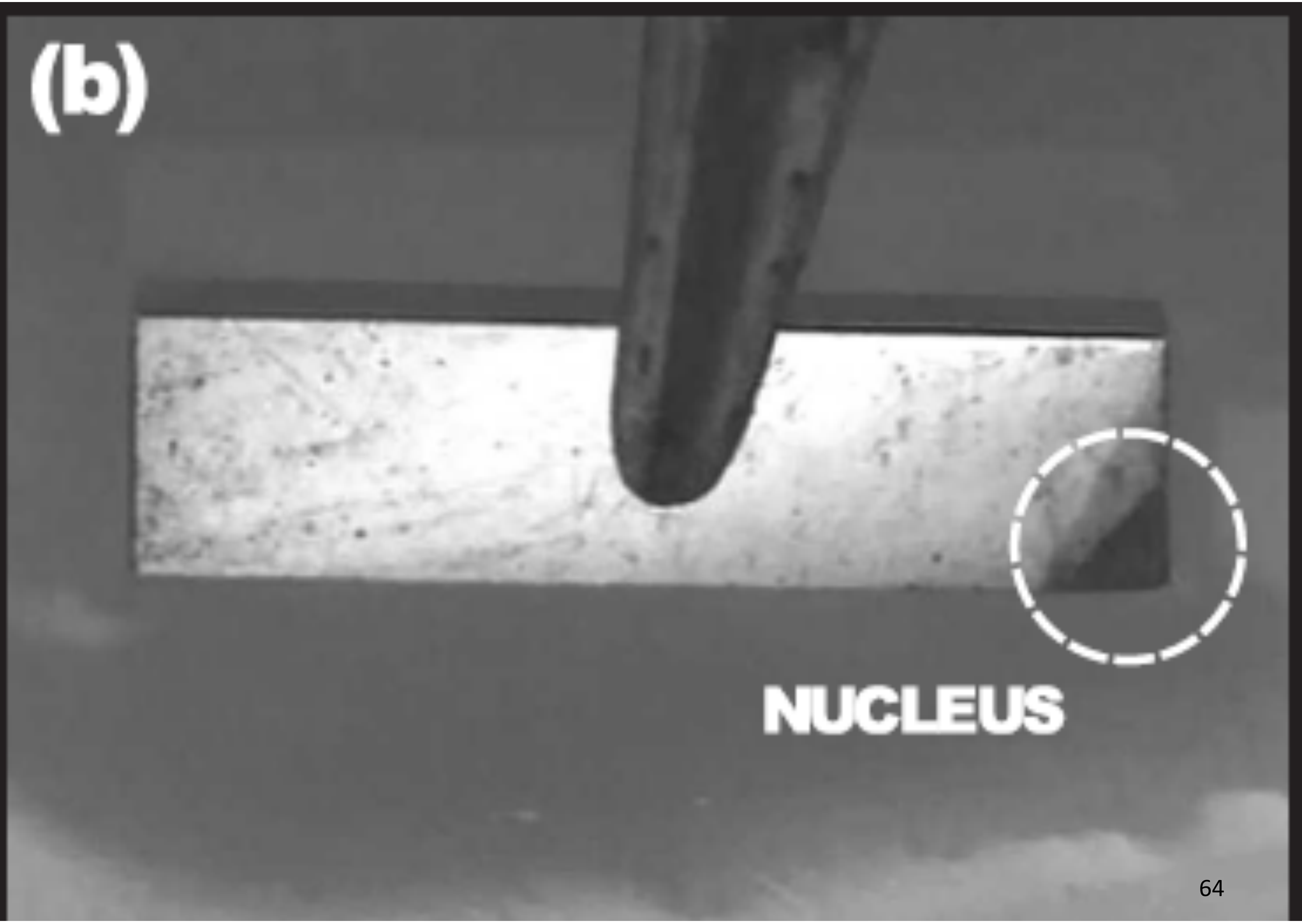


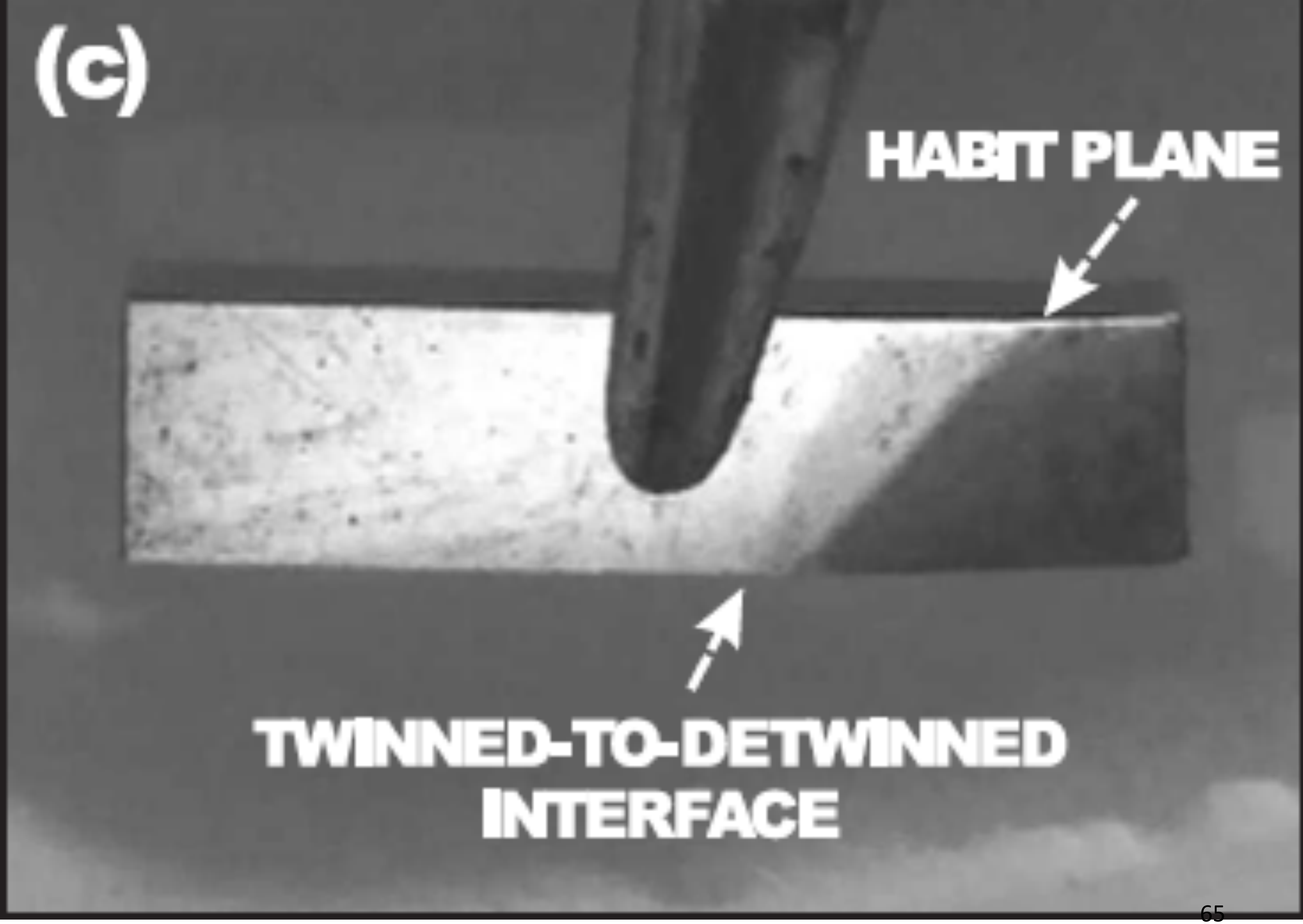
# Example 2. Nucleation of austenite in martensite

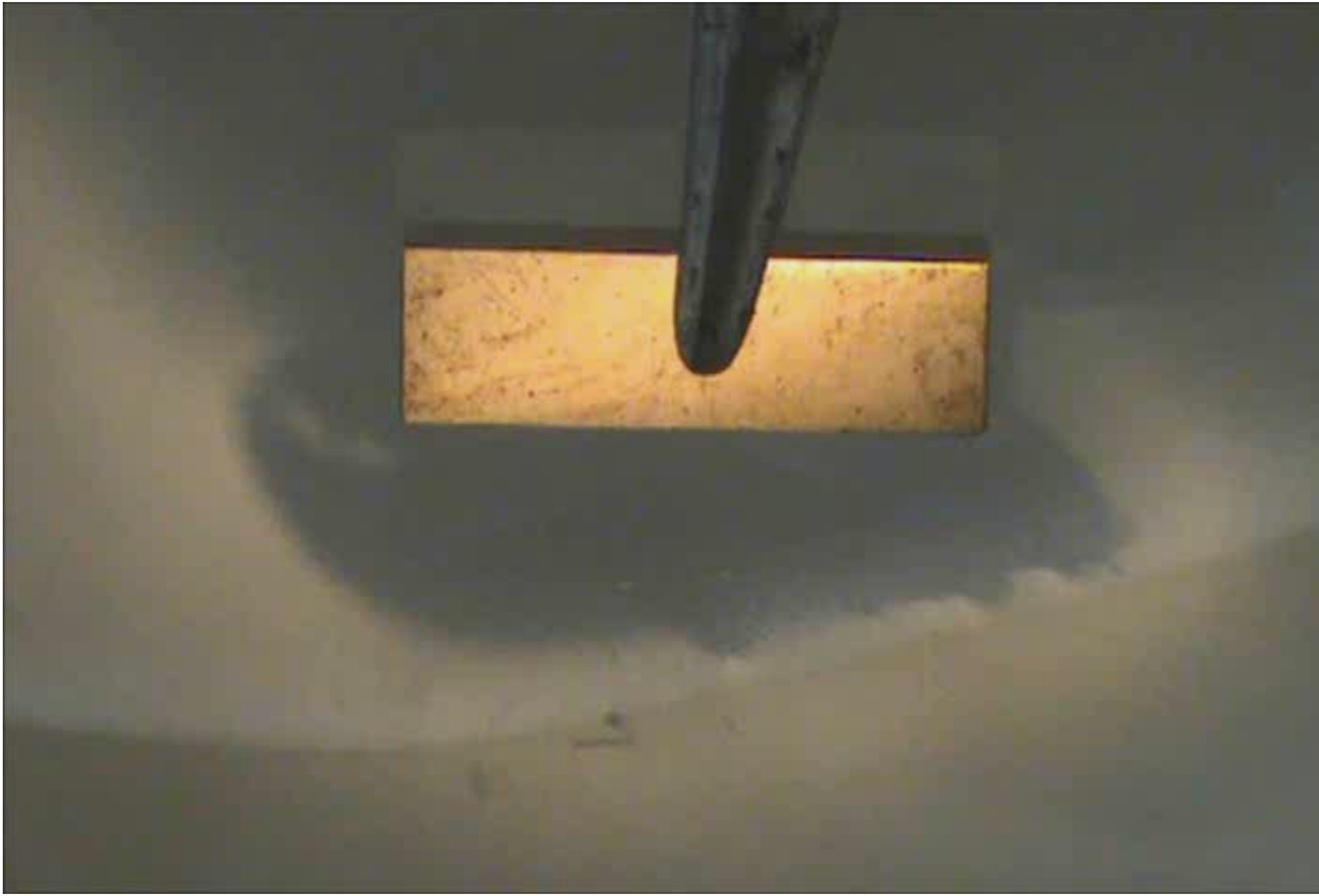
(JB/K. Koumatos/H. Seiner 2013,2014)




Single crystal of CuAlNi. Pure variant of martensite. Heated by tip of soldering iron.


(b)
NUCLEUS


(c)
HABIT PLANE
TWINNED-TO-DETWINNED
INTERFACE



# Twinning and slip in Bravais lattices

Consider a Bravais lattice $\mathbf{B}$. What are the rank-one connections between $SO(3)$ and $SO(3)\mathbf{M}$, where $\mathbf{M} = \mathbf{B}\boldsymbol{\mu}\mathbf{B}^{-1} \notin P(\mathbf{B})$?

We try $\boldsymbol{\mu} = -\mathbf{1} + \mathbf{p} \otimes \mathbf{q}$ with $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^3$ and $\mathbf{p} \cdot \mathbf{q} = 2$, when

$$\mathbf{B}\boldsymbol{\mu}\mathbf{B}^{-1} = -\mathbf{1} + \mathbf{B}\mathbf{p} \otimes \mathbf{B}^{-T}\mathbf{q},$$

$$\begin{aligned}
\mathbf{M}^T\mathbf{M} - \mathbf{1} &= (-\mathbf{1} + \mathbf{B}^{-T}\mathbf{q} \otimes \mathbf{B}\mathbf{p})(-\mathbf{1} + \mathbf{B}\mathbf{p} \otimes \mathbf{B}^{-T}\mathbf{q}) - \mathbf{1} \\
&= -\mathbf{B}^{-T}\mathbf{q} \otimes \mathbf{B}\mathbf{p} - \mathbf{B}\mathbf{p} \otimes \mathbf{B}^{-T}\mathbf{q} + |\mathbf{B}\mathbf{p}|^2\mathbf{B}^{-T}\mathbf{q} \otimes \mathbf{B}^{-T}\mathbf{q} \\
&= (-\mathbf{B}\mathbf{p} + \frac{1}{2}|\mathbf{B}\mathbf{p}|^2\mathbf{B}^{-T}\mathbf{q}) \otimes \mathbf{B}^{-T}\mathbf{q} \\
&\qquad + \mathbf{B}^{-T}\mathbf{q} \otimes (-\mathbf{B}\mathbf{p} + \frac{1}{2}|\mathbf{B}\mathbf{p}|^2\mathbf{B}^{-T}\mathbf{q}).
\end{aligned}$$

Hence $SO(3)$ and $SO(3)\mathbf{M}$ are rank-one connected, with normals parallel to $\mathbf{B}^{-T}\mathbf{q}$ and $-\mathbf{Bp}+\frac{1}{2}|\mathbf{Bp}|^2\mathbf{B}^{-T}\mathbf{q}$.

Note also that if $\mathbf{1}+\mathbf{a}\otimes\mathbf{n}=\mathbf{QM}$ then

$$\begin{aligned}\mathrm{tr}\,\mathbf{M}^T\mathbf{M}-3 &= \mathrm{tr}\,(\mathbf{1}+\mathbf{n}\otimes\mathbf{a})(\mathbf{1}+\mathbf{a}\otimes\mathbf{n})-3\\ &= |\mathbf{Bp}|^2|\mathbf{B}^{-T}\mathbf{q}|^2-4,\end{aligned}$$

so that $|\mathbf{a}|^2=|\mathbf{Bp}|^2|\mathbf{B}^{-T}\mathbf{q}|^2-4$.

For a bcc lattice we can take

$$\mathbf{B} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \ \mathbf{B}^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then the first case with $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ gives the normals $\begin{pmatrix} \pm 1 \\ \pm 1 \\ 2 \end{pmatrix}$ and $|\mathbf{a}|^2 = \frac{1}{2}$.

These are the most commonly observed normals for bcc metals and alloys, and work of Bevis & Crocker (1968,1969), Jaswon & Dove (1956,1957,1960) probably shows that they minimize $|\mathbf{a}|$.

For fcc we can take

$$\mathbf{B} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ \mathbf{B}^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then with $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ we get the commonly observed normals $\begin{pmatrix} \pm 1 \\ \pm 1 \\ 1 \end{pmatrix}$ and $|\mathbf{a}|^2 = \frac{1}{2}$.