# Problème inverse dans une équation de Schrödinger

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en collaboration avec Alberto Mercado, Chili.

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- Introduction
  - Usual Schrödinger equation
  - Non-linear inverse problem
  - Uniqueness and Stability
  - Transmission Schrödinger equation
- Method to prove stability for some linear inverse problem
- Carleman estimates
  - Weight function
  - Proof scheme of the usual case
  - Transmission case
- Stability Theorem

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# Schrödinger equation

$T > 0$ , $\Omega \subset \mathbb{R}^2$ bounded domain

Smooth boundary $\partial\Omega$

potential $q = q(x) \in L^\infty(\Omega)$

$$y = y(x,t)$$
$$y' = \frac{\partial y}{\partial t}$$

$$\begin{cases} iy'(x,t) + \operatorname{div}(a(x)\nabla y(x,t)) + q(x)y(x,t) = 0, & x \in \Omega,\ t \in (0,T) \\ y(x,t) = h(x,t), & x \in \partial\Omega,\ t \in (0,T) \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases}$$

If $a = 1$, $y_0 \in L^2(\Omega)$, $h \in L^2(\partial\Omega \times (0,T))$, $\exists!$ solution

$$y = y(q) \in C([0,T], H^{-1}(\Omega)) \cap H^{-1}(0,T; L^2(\Omega))$$

such that $\frac{\partial y}{\partial \nu} \in H^{-2}([0,T], H^{-\frac{3}{2}}(\partial\Omega))$.

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# Nonlinear inverse problem :

$$\begin{cases} iy' + \operatorname{div}(a(x)\nabla y) + qy = 0, & \Omega \times (0,T) \\ y = h, & \partial\Omega \times (0,T) \\ y(0) = y_0, & \Omega. \end{cases}$$

*Is it possible to retrieve the potential* $q = q(x),\ x \in \Omega$ *from measurement of the normal derivative* $\left.\dfrac{\partial y}{\partial \nu}\right|_{\partial\Omega \times (0,T)}$ *?*

- $q \mapsto \dfrac{\partial y(q)}{\partial \nu}$ is non linear
- Local result ( $q$ known, $p$ unknown, close to $q$ )
- This is a one-measurement inverse problem.

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# Uniqueness and stability

- Uniqueness :

$$\left(\frac{\partial y(p)}{\partial \nu}\Big|_{\partial\Omega\times(0,T)} = \frac{\partial y(q)}{\partial \nu}\Big|_{\partial\Omega\times(0,T)}\right) \quad \Rightarrow \quad (p = q \text{ sur } \Omega)\ ?$$

- Stability : It is possible to estimate $(q-p)|_\Omega$ by

$$\frac{\partial y(q)}{\partial \nu} - \frac{\partial y(p)}{\partial \nu}\Big|_{\partial\Omega\times(0,T)}$$

in suitable norms ?

# Schrödinger transmission equation

$\Omega_1$, smooth boundary $\Gamma_1$ *st* $\overline{\Omega}_1 \subset \Omega$ and $\Omega_0 = \Omega \setminus \overline{\Omega}_1$.

We set $a(x) = \begin{cases} a_1 & x \in \Omega_1 \\ a_0 & x \in \Omega_0 \end{cases}$ with $a_j > 0$ for $j = 0, 1$

$$\begin{cases} iy' - \mathrm{div}(a(x)\nabla y) + q(x)y = 0 & \Omega \times (0,T) \\ y = 0 & \partial\Omega \times (0,T) \\ y(0) = y_0, & \Omega \end{cases}$$

Inverse problem well-posed provided that :

- $\Omega_1$ is strongly convex
  *ie* $\Omega_1$ is convex and $\Gamma_1$ has a non-vanishing curvature
- $a_0 < a_1$

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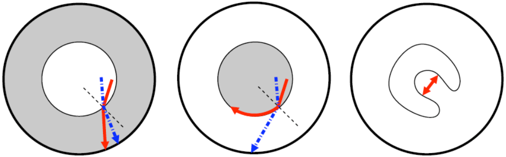

$a_1 > a_0$ (good) $\quad a_1 < a_0$ (bad) $\quad \Omega_1$ non convex (bad)

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# References

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# Method

Takes its roots in the works of Bukhgeim, Klibanov and Malinski.

- Equation in $y(q)$
- Equation in $u = y(q) - y(p)$
- Equation in $z = u'$
- use of Carleman Estimate

$$\begin{cases} iy' - \text{div}(a\nabla y) + qy = 0, & \Omega \times (0,T) \\ y = h, & \partial\Omega \times (0,T) \\ y(0) = y_0, & \Omega \end{cases} \quad \begin{cases} iu' - \text{div}(a\nabla u) + (f+q)u = fR \\ u = 0 \\ u(0) = 0. \end{cases}$$

$$\left| \begin{array}{l} u = y(q) - y(p) \\ f = p - q \\ R = y(q) \end{array} \right. \quad \Rightarrow \quad \text{Linearized version of the inverse problem :}$$

*Can we determine* $f$ *in* $\Omega$ *from* $\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_0 \times (0,T)}$ ?

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Extension of the equation on $(-T,T)$

- Since $R(0) \in \mathbb{R}$, we extend $v$ and $R$ on $\Omega \times (-T,0)$ by $v(x,t) = -\overline{v}(x,-t)$ and $R(x,t) = \overline{R}(x,-t)$.
  $z(x,t) = -\bar{z}(x,-t)$ , $R(x,t) = \bar{R}(x,-t)$ on $(-T,0)$
- If we assume $R(0) \in i\mathbb{R}$, an other extension can be defined.

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- If we assume $R(0) \in i\mathbb{R}$, an other extension can be defined.

We take $q \in L^\infty(\Omega)$ and $u_0 \in H_0^1(\Omega)$ such that

- $R = y(q) \in H^1(0, T; L^\infty(\Omega))$
- $0 < r < |R(x, 0)|$ a.e. on $\Omega$

and we will use an estimate like

$$C_1(s) \int_\Omega |z'(0)|^2 \, dx \leq C_2(s) \int_{-T}^{T} \int_{\partial\Omega} \left| \frac{\partial z}{\partial \nu} \right|^2 d\sigma dt + C_3(s) \int_{-T}^{T} \int_\Omega |\text{rhs}|^2 \, dx$$

with $\dfrac{C_3(s)}{C_1(s)} \to 0$ when $s \to \infty$.

$$\begin{cases} iz' - \text{div}(a\nabla z) + qz = fR', \\ z = 0, \\ z(0) = -ifR(0). \end{cases}$$

# Carleman Estimate for the Schrödinger equation

$Lv = iv' + \Delta v + qv$ we suppose that $||q||_{L^\infty(\Omega)} \leq m$.

For $\lambda$ and $s$ large enough, $\exists\, M = M(\Omega, T, m) > 0$ such that :

$$s\lambda \int_{-T}^{T} \int_\Omega |\nabla v|^2 \varphi e^{-2s\varphi}\, dxdt + s^3\lambda^4 \int_{-T}^{T} \int_\Omega |v|^2 \varphi^3 e^{-2s\varphi}\, dxdt$$
$$\leq M \int_{-T}^{T} \int_\Omega |Lv|^2 \varphi^3 e^{-2s\varphi}\, dxdt + M s\lambda \int_{-T}^{T} \int_{\partial\Omega} \left|\frac{\partial v}{\partial \nu}\right|^2 d\sigma dt$$

$\forall\, v \in L^2(-T, T; H_0^1(\Omega))$ such that $Lv \in L^2(\Omega \times (-T, T))$ and $\dfrac{\partial v}{\partial \nu} \in L^2(\Omega \times (-T, T))$.

# Schrödinger weight function

$$\varphi(x,t) = \frac{\alpha - e^{\lambda\psi(x)}}{(T-t)(T+t)}$$

- $\lambda > 0$, $\|e^{\lambda\psi}\|_{L^\infty} \leq \alpha$
- $\psi > 0$ regular on $\Omega$
- $|\nabla\psi| \geq \beta > 0$ in $\Omega$
- $\exists\, \varepsilon > 0$ such that $\forall\, \xi \in \mathbb{R}^N$, $\lambda|\nabla\psi.\, \xi|^2 + D^2\psi\left(\xi,\overline{\xi}\right) \geq \varepsilon\, |\xi|^2$
- $\nabla\psi \cdot \nu < 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma_0$

ex : $\psi(x) = |x - x_0|^2,\ x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$

We set $w = e^{-s\varphi} v$ and we calculate $Pw = e^{-s\varphi} L(e^{s\varphi} w)$.

We get $Pw = P_1 w + P_2 w + qw$ with

$$\begin{cases} P_1 w = iw' + \Delta w + s^2 |\nabla\varphi|^2 w \\ P_2 w = is\varphi' w + 2s\nabla\varphi \cdot \nabla w + s\Delta\varphi w. \end{cases}$$

$$\int_{-T}^{T}\int_{\Omega} |Pw - qw|^2 \, dxdt$$

$$= \int_{-T}^{T}\int_{\Omega} \left(|P_1 w|^2 + |P_2 w|^2\right) \, dxdt + 2 \, Re \int_{-T}^{T}\int_{\Omega} P_1 w \overline{P_2 w} \, dxdt.$$

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The point is to obtain in the calculous :

- some “dominant” terms with the right sign

$$s\lambda \int_{-T}^{T}\int_{\Omega} |\nabla w|^2 \, dxdt, \quad s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} |w|^2 \, dxdt$$

- and the “measurement” term

$$s\lambda \int_{-T}^{T}\int_{\Gamma_0} \left|\frac{\partial w}{\partial \nu}\right|^2 \nabla\psi \cdot \nu \, d\sigma dt$$

# Discontinuous main coefficient

$$a(x) = \begin{cases} a_1 & x \in \Omega_1 \\ a_0 & x \in \Omega_0 \end{cases} \quad \text{with } a_j > 0 \text{ for } j = 0, 1.$$

Then for each $f \in L^2(\Omega \times (0,T))$, $y$ satisfies the equation

$$iy' - \text{div}(a\nabla y) + qy = f \quad \text{in } \Omega \times (0,T)$$

iff $j \in \{0,1\}$, $y_j = y|_{\Omega_j}$ satisfies

$$iy_j' - a_j \Delta y_j + qy_j = f|_{Q_j} \quad \text{in } \Omega_j \times (0,T)$$

with the *transmission conditions* on the interface $\Gamma_1 = \partial\Omega_1$

$$\begin{cases} y_0 = y_1 \\ a_0 \dfrac{\partial y_0}{\partial \nu_0} + a_1 \dfrac{\partial y_1}{\partial \nu_1} = 0 \end{cases} \quad \text{on } \Gamma_1 \times (0,T).$$

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# Carleman estimate - discontinuous $a$

We use a weight-function $\varphi(x,t) = \dfrac{\alpha - e^{\lambda\psi(x)}}{(T-t)(T+t)}$ such that :

- **Condition 1 :** $\varphi|_{\Omega_j}$ = appropriate weight-function in $\Omega_j$.
- **Condition 2 :** We can control the "interface" terms on $\partial\Omega_1$.
- The choice of $\psi$ is crutial :

$$\left\{ \begin{array}{l} |\nabla\psi| \geq \beta > 0 \\ 2D_a^2\psi(\xi,\bar{\xi}) + 2a^2\lambda|\nabla\psi\cdot\xi|^2 - a\nabla a\cdot\nabla\psi|\xi|^2 \geq \epsilon|\xi|^2, \quad \forall\xi\in\mathbb{C}^n \end{array} \right. \quad \text{in } \Omega_0\cup\Omega_1$$

where $D_a^2\psi \left(a\frac{\partial}{\partial x_i}\left(a\frac{\partial\psi}{\partial x_j}\right)\right)_{1\leq i,j\leq N}$

$$\left\{ \begin{array}{l} \psi_0 = \psi_1 = cst \\ a_0\dfrac{\partial\psi_0}{\partial\nu_0} + a_1\dfrac{\partial\psi_1}{\partial\nu_1} = 0 \text{ and } \dfrac{\partial\psi_0}{\partial\nu_0} + \dfrac{\partial\psi_1}{\partial\nu_1} < 0 \end{array} \right. \quad \text{on } \Gamma_1$$

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- Condition 2 : We can control the "interface" terms on $\partial\Omega_1$.
- The choice of $\psi$ is crutial :

$$\begin{cases} |\nabla\psi| \geq \beta > 0 \\ 2D_a^2\psi(\xi,\bar{\xi}) + 2a^2\lambda|\nabla\psi\cdot\xi|^2 - a\nabla a\cdot\nabla\psi|\xi|^2 \geq \epsilon|\xi|^2, \quad \forall\xi\in\mathbb{C}^n \end{cases} \text{ in } \Omega_0\cup\Omega_1$$

where $D_a^2\psi \left(a\frac{\partial}{\partial x_i}\left(a\frac{\partial\psi}{\partial x_j}\right)\right)_{1\leq i,j\leq N}$

$$\begin{cases} \psi_0 = \psi_1 = cst \\ a_0\dfrac{\partial\psi_0}{\partial\nu_0} + a_1\dfrac{\partial\psi_1}{\partial\nu_1} = 0 \text{ and } \dfrac{\partial\psi_0}{\partial\nu_0} + \dfrac{\partial\psi_1}{\partial\nu_1} < 0 \end{cases} \text{ on } \Gamma_1$$

# Explicit weight function

We will actually define an $\varepsilon$-pair of transmission weight functions $(\psi^1, \psi^2) \in C^4(\Omega_0 \cup \Omega_1)^2$ such that for each $j, k \in \{1, 2\}$ with $j \neq k$,

$$\psi^j - \psi^k \geq \delta > 0 \qquad \text{in } B_\varepsilon(x_k)$$

and

$$\begin{cases} |\nabla \psi^j| \geq \beta > 0 \\ \forall \xi \in \mathbb{C}^n, \\ 2D_a^2\psi^j(\xi, \bar{\xi}) + 2a^2\lambda|\nabla\psi^j \cdot \xi|^2 - a\nabla a \cdot \nabla\psi^j|\xi|^2 \geq \epsilon|\xi|^2 \end{cases} \quad \text{in } \Omega_0 \cup \Omega_1 \setminus B_\varepsilon(x_j)$$

$$\psi^j(x) = \eta(x)\frac{a_k}{\rho(x)^2}|x - x_j|^2 + M_j, \; x_j \in \Omega_1$$

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# Intermediate step

$$Z = \left\{ v \in L^2(-T,T;H_0^1(\Omega)) : Lv \in L^2(Q), \frac{\partial v}{\partial \nu} \in L^2(\Sigma), v \text{ satisfies } (Tr) \right\},$$

$$\|w\|_{s,\lambda,\psi} = s^3\lambda^4 \int_{-T}^{T}\int_\Omega \theta^3|w|^2 dxdt + s\lambda \int_{-T}^{T}\int_\Omega \theta|\nabla w|^2 dxdt$$

and $\|\cdot\|_{s,\lambda,\psi,B}$ corresponds to the above terms defined in $B \subset \Omega$.

We prove first that

$$\|P_1(w)\|_{L^2}^2 + \|P_2(w)\|_{L^2}^2 + \|w\|_{s,\lambda,\psi}^2 \leq C\|P(w)\|_{L^2}^2 + C\|w\|_{s,\lambda,\psi,B_\varepsilon(x_j)}^2 + s\lambda C \iint_{\partial\Omega} \left| a\frac{\partial w}{\partial \nu} \right|^2.$$

# Carleman estimate

Suppose there exists for some $\varepsilon > 0$ an $\varepsilon$-pair of transmission weight functions $(\psi^1, \psi^2)$ belonging to $C^4(\Omega_0 \cup \Omega_1)$.

$\forall q \in L^\infty(\Omega)$ with $\|q\|_{L^\infty(\Omega)} \leq m$, $\exists\, C = C(\Omega, T, m) > 0$ such that

$$\sum_{k=1}^{2} \left( s\lambda \int_{-T}^{T}\int_{\Omega} |\nabla v|^2 e^{-2s\varphi^k}\, dxdt + s^3\lambda^4 \int_{-T}^{T}\int_{\Omega} |v|^2 e^{-2s\varphi^k}\, dxdt \right) \leq C \sum_{k=1}^{2} \left( \int_{-T}^{T}\int_{\Omega} |Lv|^2 e^{-2s\varphi^k}\, dxdt + s\lambda \int_{0}^{T}\int_{\partial\Omega} \left| a_1 \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt \right)$$

$\forall\, v \in L^2(-T, T; H_0^1(\Omega))$ satisfying the transmission conditions... and for all $\lambda$ and $s$ large enough.

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$\forall\, v \in L^2(-T, T; H_0^1(\Omega))$ satisfying the transmission conditions... and for all $\lambda$ and $s$ large enough.

# Stability theorem

$$\begin{cases} iy' + \operatorname{div}(a(x)\nabla y) + qy = 0 \\ y = h \\ y(0) = y_0 \end{cases}$$

Let $\mathcal{U}$ be a bounded subset of $L^\infty(\Omega)$, $q \in L^\infty(\Omega)$, $T_0 > 0$ and $r > 0$.
If $y_0 \in H_0^1(\Omega)$ and $y_0$ takes its values in $\mathbb{R}$ and if

$$|y_0(x)| \geq r, \quad \text{a.e. in } \Omega,$$

$$y(q) \in H^1(0, T; L^\infty(\Omega)),$$

then $\exists\, C = C(\Omega_0, \Omega_1, T, a_0, a_1, \|q\|_{L^\infty}, y_0, h, \mathcal{U}, r) > 0$ such that $\forall p \in \mathcal{U}$, we have

$$\|q - p\|_{L^2(\Omega)} \leq C \left\| a_1 \frac{\partial y(q)}{\partial \nu} - a_1 \frac{\partial y(p)}{\partial \nu} \right\|_{H^1(0,T;L^2(\Gamma_0))}.$$

$$\begin{cases} iz' - \text{div}(a\nabla z) + qz = fR' \\ z = 0 \\ z(0) = -ifR(0) \end{cases}$$

We set $w = e^{s\varphi}z$ and define

$$I = Im \int_{-T}^{0} \int_{\Omega} P_1 w \, \overline{w} \, dxdt.$$

where $P_1 w = iw' + \Delta w + s^2|\nabla\varphi|^2 w$.

- Calculation :

$$\begin{aligned} I &= Im \int_{-T}^{0} \int_{\Omega} (iw' + \Delta w + s^2|\nabla\varphi|^2 w) \quad \overline{w} \, dxdt \\ &= \frac{1}{2} \int_{\Omega} |w(x,0)|^2 \, dx \\ &= \frac{1}{2} \int_{\Omega} |R(x,0)|^2 e^{-2s\varphi(0)} |f|^2 \, dx \end{aligned}$$

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- Carleman :

$$\begin{aligned} I &\leq \left(\int_{-T}^{T}\int_{\Omega}|P_1 w|^2\,dxdt\right)^{\frac{1}{2}}\left(\int_{-T}^{T}\int_{\Omega}|w|^2\,dxdt\right)^{\frac{1}{2}} \\ &\leq Ms^{-\frac{3}{2}}\left(\int_{0}^{T}\int_{\Omega}|Pw|^2\,dxdt + s\int_{0}^{T}\int_{\Gamma_0}\left|\frac{\partial w}{\partial \nu}\right|^2\,d\sigma dt\right) \\ &\leq Ms^{-\frac{3}{2}}\left(\int_{0}^{T}\int_{\Omega}|fR'|^2 e^{-2s\varphi}\,dxdt + s\int_{0}^{T}\int_{\Gamma_0}\left|\frac{\partial v}{\partial \nu}\right|^2\,d\sigma dt\right) \end{aligned}$$

Since $|R(x,0)| \geq r > 0$, then

$\exists\, g_0 \in L^2(0,T),\ |R'(x,t)| \leq g_0(t)|R(x,0)|$ so that

$$\begin{aligned} \int_{\Omega}|f|^2|R(0)|^2 e^{-2s\varphi(0)}\,dx &\leq Ms^{-\frac{3}{2}}\int_{0}^{T}\int_{\Omega}|f|^2|g_0|^2|R(0)|^2 e^{-2s\varphi(0)}\,dxdt \\ &+ Ms^{-\frac{1}{2}}\int_{0}^{T}\int_{\Gamma_0}\left|\frac{\partial v}{\partial \nu}\right|^2\,d\sigma dt. \end{aligned}$$

- Carleman :

$$I \leq \left(\int_{-T}^{T}\int_{\Omega} |P_1 w|^2 \, dx dt\right)^{\frac{1}{2}} \left(\int_{-T}^{T}\int_{\Omega} |w|^2 \, dx dt\right)^{\frac{1}{2}}$$

$$\leq M s^{-\frac{3}{2}} \left(\int_{0}^{T}\int_{\Omega} |Pw|^2 \, dx dt + s\int_{0}^{T}\int_{\Gamma_0} \left|\frac{\partial w}{\partial \nu}\right|^2 \, d\sigma dt\right)$$

$$\leq M s^{-\frac{3}{2}} \left(\int_{0}^{T}\int_{\Omega} |f R'|^2 e^{-2s\varphi} \, dx dt + s\int_{0}^{T}\int_{\Gamma_0} \left|\frac{\partial v}{\partial \nu}\right|^2 \, d\sigma dt\right)$$

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We finally obtain

$$(1 - Cs^{-\frac{3}{2}}) \int_\Omega |f(x)|^2 \, dx \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma dt,$$

and for $s$ large enough, we prove the stability of the linear inverse problem :

$$||f||_{L^2(\Omega)} \leq C \left|\left| \frac{\partial u}{\partial \nu} \right|\right|_{H^1(0,T;L^2(\Gamma_0))} .$$

# Theorem - Linear inverse problem

Let $\|q\|_{L^\infty} \leq m$ and $r > 0$. We assume that $a(x) = \begin{cases} a_1 & x \in \Omega_1 \\ a_0 & x \in \Omega_0 \end{cases}$ with $a_1 > a_0 > 0$ and

$$R(0) \in \mathbb{R}$$
$$|R(x,0)| \geq r \quad \text{a.e. in } \Omega,$$
$$R \in H^1(0,T;L^\infty(\Omega)).$$

Then $\exists\, C = C(\Omega_0, \Omega_1, T, \Gamma_0, m, \|R\|) > 0$ such that $\forall f \in L^2(\Omega)$, the solution $u$ of

$$\begin{cases} iu' + \mathrm{div}(a\nabla u) + qu = fR, & \Omega \times (0,T) \\ u = 0, & \partial\Omega \times (0,T) \\ u(0) = 0, & \Omega, \end{cases}$$

satifies

$$||f||_{L^2(\Omega)} \leq C \left\| a_1 \frac{\partial u}{\partial \nu} \right\|_{H^1(0,T;L^2(\partial\Omega))}.$$

# Further problems

- Recovering the main coefficient.
- Observations in a part of the boundary.
- Logarithmic stability (without geometric hypothesis).
- Neumann boundary data and Dirichlet measurement.

Thank you for your attention !

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