Carleman estimates, results on control and stabilization for partial differential equations

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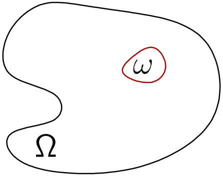
NULL CONTROL FOR HEAT EQUATION

Let Ω a bounded domain with smooth boundary.
Let ω subset open in Ω.
Let $T > 0$.
The problem is the following, for all $u_0 \in L^2(\Omega)$, does exist f supported on $[0, T] \times \omega$ such that the solution of

$$\begin{cases} (\partial_t - \Delta)u = f \text{ in } [0, T] \times \Omega, \\ u(0, \cdot) = u_0, \text{ in } \Omega, \\ u_{[0,T]\times\partial\Omega} = 0, \end{cases}$$

satisfied $u(T, .) = 0$.



ANSWER IS YES. (Fursikov-Imanuvilov, Lebeau-R)

STABILIZATION FOR WAVE EQUATIONS

Let $\omega \subset \Omega$ be an open and $a \in \mathscr{C}_0^\infty(\omega)$ where $a(x) \geq 0$ for all $x \in \omega$. Let $(u_0, u_1) \in H_0^1(\Omega) \oplus L^2(\Omega)$, and u be the solution of

$$\begin{cases} \partial_{tt}^2 u(t,x) - \Delta u(t,x) + a(x)\partial_t u(t,x) = 0 \text{ in } (0,+\infty) \times \Omega, \\ (u(0,\cdot), \partial_t u(0,\cdot)) = (u_0, u_1), \\ u_{|(0,+\infty)\times\partial\Omega} = 0. \end{cases}$$

Energy: $E(u)(t) = \int_\Omega \left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\right) dx$.

$E(u)(T) - E(u)(0) = -2\int_0^T \int_\omega a(x)|\partial_t u(t,x)|^2 dxdt$.

Question: how fast the energy decreases?

- Under geometrical condition we have $E(u)(t) \leq C\|(u_0,u_1)\|^2_{H^1(\Omega)\oplus L^2(\Omega)} e^{-t/C}$.
- If $a > 0$ on an open subset we have $(E(u)(t))^{1/2} \leq \frac{C\|(u_0,u_1)\|_{\mathcal{D}(A^k)}}{\log(2+t)^k}$.

CARLEMAN ESTIMATES

WHAT ARE CARLEMAN ESTIMATES?

For P a partial differential operator,

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \text{ where } \alpha \in \mathbb{N}^n, \ D_j = -i\partial_{x_j}.$$

Carleman estimates have the following form,

$$\|e^{\tau\phi(x)} v(x)\| \leq C \|e^{\tau\phi(x)} P v(x)\|,$$

where

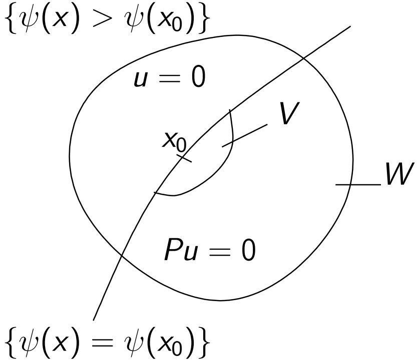
- $\tau \geq \tau_0 > 0$,
- C does not depend on τ,
- ϕ, the weight, plays an important role.
- v compactly supported.

APPLICATION : UNIQUE CONTINUATION

UNIQUENESS

Assume $Pu = 0$ in W a neighborhood of x_0 and $u = 0$ in $\{x \in W,\ \psi(x) > \psi(x_0)\}$, $(d\psi(x_0) \neq 0)$.

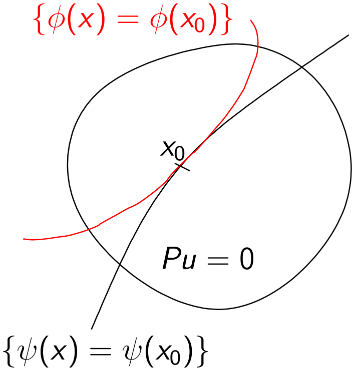
QUESTION: Does there exist V, a neighborhood of x_0, such that $u = 0$ on V.



CARLEMAN ESTIMATE

$$\|e^{\tau\phi(x)}v(x)\| \leq C\|e^{\tau\phi(x)}Pv(x)\|,$$

for v supported in W.



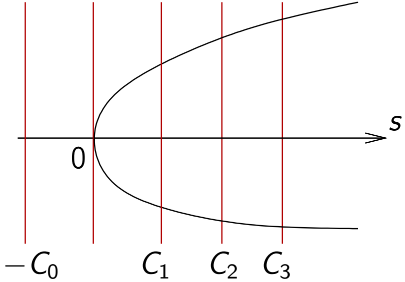
PROOF OF UNIQUE CONTINUATION I

Changing local coordinates, we can take $\phi(x) = -s$.

Carleman estimate is $\|e^{-\tau s} v\| \leq C \|e^{-\tau s} P v\|$,

$v(x) = \chi(s) u(x)$, where $\chi(s) = 1$ for $s \leq C_2$, and χ supported in $s < C_3$.

$Pu = 0$ in $s < C_3$.
$Pv = [P, \chi] u$, where $[P, \chi] u$ supported in $C_2 \leq s \leq C_3$.



PROOF OF UNIQUE CONTINUATION II

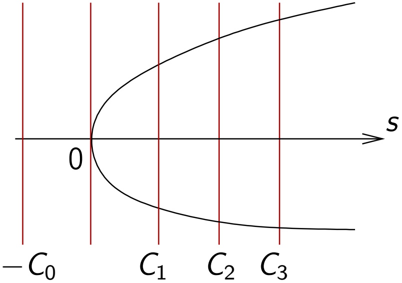
Carleman estimate give

$$\|e^{-\tau s}v\| \leq C\|e^{-\tau s}Pv\|$$

$$\|e^{-\tau s}u\|_{L^2(0\leq s\leq C_1)} \leq C\|e^{-\tau s}u\|_{H^{m-1}(C_2\leq s\leq C_3)}$$

$$e^{-\tau C_1}\|u\|_{L^2(0\leq s\leq C_1)} \leq Ce^{-\tau C_2}\|u\|_{H^{m-1}(C_2\leq s\leq C_3)}$$

$$\|u\|_{L^2(0\leq s\leq C_1)} \leq Ce^{-\tau(C_2-C_1)}\|u\|_{H^{m-1}(C_2\leq s\leq C_3)} \to 0 \text{ when } \tau\to\infty$$



INTERPOLATION

If $Pu \neq 0$ and $u \neq 0$ in $\{x \in W,\ \psi(x) > \psi(x_0)\}$. Same computations give

$$\|u\|_{L^2(0 \leq s \leq C_1)} \leq Ce^{-\tau(C_2 - C_1)} \|u\|_{H^{m-1}(C_2 \leq s \leq C_3)} + Ce^{C_0 \tau} (\|u\|_{H^{m-1}(\psi(x) > \psi(x_0))} + \|Pu\|_{L^2(V)}),$$

where $C_0 > 0$.

By optimisation on τ, we obtain

$$\|u\|_{L^2(0 \leq s \leq C_1)} \lesssim \|u\|^{1-\delta}_{H^{m-1}(-C_0 \leq s \leq C_3)} (\|u\|_{H^{m-1}(\psi(x) > \psi(x_0))} + \|Pu\|_{L^2(V)})^{\delta}.$$

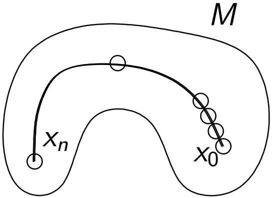
In general $\delta \in (0, 1)$, and $\delta = 1$ only for P hyperbolic.

PROPAGATION AND INTERPOLATION

Advantage of interpolation estimate: We can propagate the estimates.
Assume

$$\|u\|_{H^{m-1}(B(x,3R))} \leq C_R \|u\|^{1-\delta}_{H^{m-1}(B(x,4R))} (\|u\|_{H^{m-1}(B(x,R))} + \|Pu\|_{L^2(M)})^{\delta},$$

for all x.



We obtain

$$\|u\|_{H^{m-1}(M_\varepsilon))} \leq C_\varepsilon \|u\|^{1-\delta}_{H^{m-1}(M)} (\|u\|_{H^{m-1}(B(x_0,R))} + \|Pu\|_{L^2(M)})^{\delta},$$

where $M_\varepsilon = \{x \in M,\ d(x, \partial M) > \varepsilon\}$.

APPLICATION: EIGENFUNCTIONS

Let Ω be a Riemannian manifold without boundary, let ω be an open subset of Ω and $-\Delta\varphi_j = \lambda_j\varphi_j$, where λ_j are the eigenvalues and φ_j the associated eigenfunctions.

$$\|\varphi_j\|_{L^2(\Omega)} \leq Ce^{C\sqrt{\lambda_j}}\|\varphi_j\|_{L^2(\omega)}.$$

PROOF: Let $u(s,x) = e^{s\sqrt{\lambda_j}}\varphi_j(x)$, we have $-\partial_s^2 u - \Delta u = 0$. We have the following interpolation estimate

$$\|u\|_{L^2(\Omega_1)} \leq C\|u\|^{\delta}_{L^2(\omega_1)}\|u\|^{1-\delta}_{L^2(\Omega_2)},$$

where $\Omega_S = (-S,S)\times\Omega$ and $\omega_1 = (-1,1)\times\omega$. As

$$e^{-\sqrt{\lambda_j}}\|\varphi_j\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega_1)}$$

$$\|u\|_{L^2(\Omega_2)} \leq e^{2\sqrt{\lambda_j}}\|\varphi_j\|_{L^2(\Omega)}$$

$$\|u\|_{L^2(\omega_1)} \leq e^{\sqrt{\lambda_j}}\|\varphi_j\|_{L^2(\omega)}.$$

We have

$$e^{-\sqrt{\lambda_j}}\|\varphi_j\|_{L^2(\Omega)} \leq Ce^{2(1-\delta)\sqrt{\lambda_j}}\|\varphi_j\|^{1-\delta}_{L^2(\Omega)}e^{\delta\sqrt{\lambda_j}}\|\varphi_j\|^{\delta}_{L^2(\omega)}.$$

BOUNDARY CARLEMAN ESTIMATES

WHY: to follow the previous strategy for Dirichlet boundary eigenfunctions.

Can be apply for all boundary value problem where we use Carleman estimate.

TWO KINDS OF CARLEMAN ESTIMATES AT THE BOUNDARY

- If we know the two boundary conditions.
- If we have only one boundary condition as Dirichlet or Neumann boundary conditions.

NOTATIONS AND ASSUMPTIONS

$$P = \sum_{|\alpha|\leq 2} a_\alpha(x) D^\alpha.$$

$$p(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq C|\xi|^2,$$

where $a_\alpha(x)$ smooth functions, real valued, $\alpha \in \mathbb{N}^n$, $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$, $D^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$ and $D_{x_j} = -i\partial_{x_j}$.

SUB-ELLIPTICITY CONDITION

$$\forall \xi \in \mathbb{R}^n \setminus \{0\},\ \forall \tau > 0,\ p(x_0, \xi + i\tau d\phi(x_0)) = 0,$$
$$\Rightarrow \mathrm{Im}\{p(x,\xi - i\tau d\phi(x), p(x, \xi + i\tau d\phi(x)\} > 0 \text{ at } (x,\xi) = (x_0,\xi).$$

This condition can be satisfied by $\phi(x) = e^{\lambda\psi(x)}$ if $|d\psi(x)| \neq 0$ and λ large enough.

CARLEMAN ESTIMATE: TWO BOUNDARY CONDITIONS

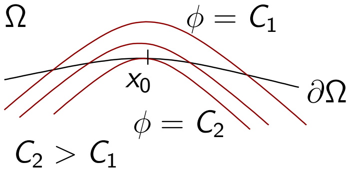
Theorem

Let $x_0 \in \partial\Omega$ and P be an elliptic operator of order 2 with real coefficients and $\phi \in \mathscr{C}^\infty$ satisfying sub-ellipticity condition at x_0 and $d\phi(x_0) \neq 0$. Then there exist W a neighborhood of x_0 in $\mathbb{R}^n$, $C > 0$ and $\tau_0 > 0$ such that for all $v \in \mathscr{C}_0^\infty(\overline{V})$, where $V = W \cap \Omega$, and all $\tau \geq \tau_0$,

$$\sum_{|\alpha|\leq 1} \tau^{3-2|\alpha|} \|e^{\tau\phi(x)} D^\alpha v(x)\|^2_{L^2(V)} \leq C\|e^{\tau\phi(x)} P v(x)\|^2_{L^2(V)}$$

$$+C \sum_{|\alpha|\leq 1} \tau^{3-2|\alpha|} |(e^{\tau\phi(x)} D^\alpha v)_{|\partial\Omega}|^2_{L^2(\partial V)},$$

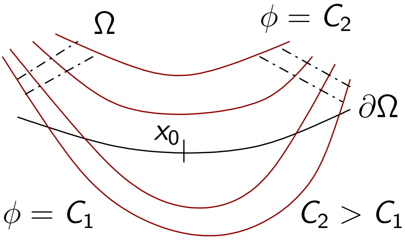
where $\partial V = \overline{V} \cap \partial\Omega$.



CARLEMAN ESTIMATE: DIRICHLET BOUNDARY CONDITION

Theorem

Let $x_0 \in \partial\Omega$ and P be an elliptic operator of order 2 with real coefficients and $\phi \in \mathscr{C}^\infty$ satisfying sub-ellipticity condition at x_0 and $\partial_\nu \phi(x_0) < 0$, where ∂_ν is the exterior normal derivative at ∂V. Then there exist W a neighborhood of x_0 in $\mathbb{R}^n$, $C > 0$ and $\tau_0 > 0$ such that for all $v \in \mathscr{C}_0^\infty(\overline{V})$, where $V = W \cap \Omega$, satisfying $v_{|\partial V} = 0$ and all $\tau \geq \tau_0$,



$$\sum_{|\alpha| \leq 1} \tau^{3-2|\alpha|} \| e^{\tau\phi(x)} D^\alpha v(x) \|^2_{L^2(V)} + \tau |(e^{\tau\phi(x)} \partial_\nu v)_{|\partial\Omega}|^2_{L^2(\partial V)} \leq C \| e^{\tau\phi(x)} P v(x) \|^2_{L^2(V)}.$$

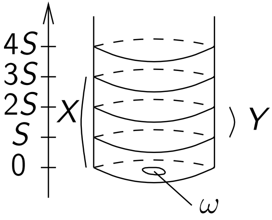
A GLOBAL INTERPOLATION ESTIMATE

We introduce an other variable denote by s and by $Q = D_s^2 + P$. Let $S > 0$, $\omega \subset \{0\} \times \Omega$ be an open, $X = (0, 3S) \times \Omega$ and $Y = (S, 2S) \times \Omega$.

Theorem

There exist $C > 0$ and $\delta \in (0,1)$ such that for all $u \in \mathscr{C}^\infty(\overline{X})$ satisfying $u = 0$ on $(0, 4S) \times \partial\Omega$ and on $\{0\} \times \Omega$, we have

$$\|u\|_{H^1(Y)} \leq C \left(\|Qu\|_{L^2(X)} + |\partial_s u_{|\omega}|_{L^2(\omega)} \right)^\delta \|u\|_{H^1(X)}^{1-\delta}.$$



APPLICATION: SUM OF EIGENFUNCTIONS

Here P is assumed self-adjoint.
The Dirichlet eigenfunctions are φ_j associated with eigenvalue λ_j, i.e. $P\varphi_j = \lambda_j \varphi_j$, $(\varphi_j)_{|\partial\Omega} = 0$.

Theorem

There exists $C > 0$ such that for all $\mu > 0$ and $w = \sum_{\lambda_j \leq \mu^2} a_j \varphi_j$, we have

$$\|w\|_{L^2(\Omega)} \leq Ce^{C\mu} \|w\|_{L^2(\omega)}.$$

This estimate is optimal (Jerison-Lebeau).

SUM OF EIGENFUNCTIONS (PROOF)

Let $u(s,\cdot) = \sum_{\lambda_j \le \mu^2} (a_j/\sqrt{\lambda_j}) \sinh(\sqrt{\lambda_j} s) \varphi_j$.
$Qu = 0$ and $\partial_\nu u = -\partial_s u = -\sum_{\lambda_j \le \mu^2} a_j \varphi_j$ on $s = 0$.

$$\|u\|_{H^1(Y)} \le C \left(\|Qu\|_{L^2(X)} + |\partial_\nu u_{|\omega}|_{L^2(\{0\}\times\omega)} \right)^{\delta} \|u\|_{H^1(X)}^{1-\delta}.$$

$$\|u\|_{H^1(X)} \le Ce^{C\mu} \|w\|_{H^1(\Omega)},$$

$$|\partial_\nu u_{|\omega}|_{L^2(\{0\}\times\omega)} \le C\|w\|_{L^2(\omega)},$$

$$C_0\|w\|_{L^2(\Omega)} \le \|u\|_{L^2(Y)} \le \|u\|_{H^1(Y)}.$$

We obtain

$$C_0\|w\|_{L^2(\Omega)} \le C\|w\|_{L^2(\omega)}^{\delta} \left(e^{C\mu} \|w\|_{H^1(\Omega)} \right)^{1-\delta} \le Ce^{C\mu} \|w\|_{L^2(\omega)}^{\delta} \|w\|_{L^2(\Omega)}^{1-\delta},$$

as $\|w\|_{H^1(\Omega)} \le C\mu\|w\|_{L^2(\Omega)}$.
This gives

$$\|w\|_{L^2(\Omega)} \le Ce^{C\mu} \|w\|_{L^2(\omega)}.$$

CONTROL FOR HEAT EQUATION

Previous result yields a first control result.
Let $u_0 \in L^2(\Omega)$, there exists f supported on $[0, T] \times \omega$ such that

$$\begin{cases} (\partial_t + P)u = f \text{ in } [0, T] \times \Omega, \\ u(0, \cdot) = u_0, \text{ in } \Omega, \\ u_{[0,T]\times\partial\Omega} = 0. \end{cases}$$

such that $u(T, \cdot)$ is orthogonal to φ_j if $\lambda_j \leq \mu^2$ and

$$\|f\|_{L^2([0,T]\times\omega)} \leq C_T e^{C\mu} \|u_0\|_{L^2(\Omega)},$$
$$\|u(T, \cdot)\|_{L^2(\Omega)} \leq C_T e^{C\mu} \|u_0\|_{L^2(\Omega)}.$$

As heat equation is dissipative, if we do not apply control between T and $2T$ we have (if $T\mu >> 1$)

$$\|u(2T, \cdot)\|_{L^2(\Omega)} \leq C_T e^{C\mu - KT\mu^2} \|u_0\|_{L^2(\Omega)} \leq C_T e^{-C_1 T\mu^2} \|u_0\|_{L^2(\Omega)}.$$

Let T_j a sequence, well chosen, such that $T_j \to T$, we can repeat the previous procedure in $[T_j, T_{j+1}]$ for frequencies less than $\mu_j \to +\infty$. We can prove that the procedure converges.

STABILIZATION FOR WAVE EQUATIONS

Let $\omega \subset \Omega$ be an open and $a \in \mathscr{C}_0^\infty(\omega)$ where $a(x) \geq 0$ for all $x \in \omega$. Let $(u_0, u_1) \in H_0^1(\Omega) \oplus L^2(\Omega)$, and u be the solution of

$$\begin{cases} \partial_{tt}^2 u(t,x) - \Delta u(t,x) + a(x)\partial_t u(t,x) = 0 \text{ in } (0,+\infty) \times \Omega, \\ (u(0,\cdot), \partial_t u(0,\cdot)) = (u_0, u_1), \\ u_{|(0,+\infty)\times\partial\Omega} = 0. \end{cases}$$

Energy: $E(u)(t) = \int_\Omega \left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\right) dx$.

$E(u)(T) - E(u)(0) = -2\int_0^T \int_\omega a(x)|\partial_t u(t,x)|^2 dxdt$.

Question: how fast the energy decreases?

If $a > 0$ on an open subset we have $(E(u)(t))^{1/2} \leq \frac{C\|(u_0,u_1)\|_{\mathcal{D}(A^k)}}{\log(2+t)^k}$.

STABILIZATION AND RESOLVENT

General result on C_0-semigroup e^{tA}, $t \geq 0$, with $\sup_{t\geq 0} \|e^{tA}\| < \infty$. If there exist $C > 0$ and $\delta > 0$ such that for all $\lambda \in \mathbb{R}$,

$$\|(i\lambda - A)^{-1}\| \leq Ce^{\delta|\lambda|},$$

where $\|\cdot\|$ is the norm operator between $H \to H$. Then

$$\|e^{tA}v\| \leq C\frac{\|A^k v\|}{\log(2+t)^k}.$$

For damped wave equation

$$A = \begin{pmatrix} 0 & Id \\ \Delta & -a(x) \end{pmatrix},$$

and $H = H_0^1(\Omega) \oplus L^2(\Omega)$.

RESOLVENT ESTIMATE I

Equation

$$(i\lambda - A)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \Rightarrow \Delta u_0 + \lambda^2 u_0 + ia\lambda u_0 = g,$$

where g depends on f_0 and f_1. Estimate on resolvent equivalent to

$$\|u_0\|_{L^2(\Omega)} \leq Ce^{C\lambda}\|g\|_{L^2(\Omega)}.$$

Let $v(s,x) = e^{s\lambda}u_0$, v satisfies

$$(\partial_s^2 + \Delta)v + ia\partial_s v = e^{s\lambda}g.$$

Let $\omega = \{x \in \Omega,\ a(x) \geq \mu > 0\}$ where μ small enough such that $\omega \neq \emptyset$.

Let $\Omega_S = (-S,S) \times \Omega$, $\omega_1 = (-1,1) \times \omega$.

We use the following interpolation estimate

$$\|v\|_{L^2(\Omega_1)} \leq C\left(\|v\|_{L^2(\omega_1)} + \|(\partial_s^2 + \Delta)v + ia\partial_s v\|_{L^2(\Omega_2)}\right)^{\delta}\|v\|_{L^2(\Omega_2)}^{1-\delta}.$$

RESOLVENT ESTIMATE II

Recall interpolation estimate and $v(s,x) = e^{s\lambda}u_0$

$$\|v\|_{L^2(\Omega_1)} \leq C\left(\|v\|_{L^2(\omega_1)} + \|(\partial_s^2+\Delta)v + ia\partial_s v\|_{L^2(\Omega_2)}\right)^{\delta}\|v\|_{L^2(\Omega_2)}^{1-\delta}.$$

As

$$\begin{aligned}
&e^{-|\lambda|}\|u_0\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega_1)},\\
&\|v\|_{L^2(\omega_1)} \leq e^{|\lambda|}\|u_0\|_{L^2(\omega)},\\
&\|(\partial_s^2+\Delta)v + ia\partial_s v\|_{L^2(\Omega_2)} \leq Ce^{2|\lambda|}\|g\|_{L^2(\Omega)},\\
&\|u\|_{L^2(\Omega_2)} \leq Ce^{2|\lambda|}\|u_0\|_{L^2(\Omega)}.
\end{aligned}$$

We have

$$\|u_0\|_{L^2(\Omega)} \leq Ce^{C\lambda}(\|u_0\|_{L^2(\omega)} + \|g\|_{L^2(\Omega)}).$$

RESOLVENT ESTIMATE III

From

$$\Delta u_0 + \lambda^2 u_0 + ia\lambda u_0 = g,$$

multiplying by $\overline{u}_0$, integrating on Ω and taking the imaginary part, we obtain

$$\lambda \int_\Omega a|u_0|^2 dx = \mathrm{Im} \int_\Omega g\overline{u}_0 dx.$$

This implies

$$\|u_0\|^2_{L^2(\omega)} \leq C\|g\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)}.$$

With the previous estimate

$$\|u_0\|_{L^2(\Omega)} \leq Ce^{C\lambda}(\|u_0\|_{L^2(\omega)} + \|g\|_{L^2(\Omega)}),$$

we have

$$\|u_0\|_{L^2(\Omega)} \leq Ce^{C\lambda}\|g\|_{L^2(\Omega)}.$$

OTHER APPLICATIONS OF CARLEMAN ESTIMATES

- Carleman estimates for heat equation by Fursikov-Imanuvilov method. Non linear heat equations.
- Energy decay for wave equation with boundary damping and for different boundary conditions.
- Local energy decay for wave equation in exterior domain.
- Discrete Carleman estimates.
- Control bang-bang.
- Stochastic heat equations.
- Control for degenerate heat equations.
- Inverse problems.