

# Stabilization of the Korteweg-de Vries equation on a periodic domain

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# Korteweg-de Vries equation

- Korteweg-de Vries equation (1895):

$$\partial_t u + \partial_x^3 u + u \partial_x u = 0$$

- $x \in I$  ( $I = \mathbb{R}, \mathbb{T}, (0, L)$ ),  $t \in \mathbb{R}$ ,  $u = u(x, t) \in \mathbb{R}$ .
- Classical model for propagation of small amplitude long waves in nonlinear dispersive media

# Well-posedness

- $x \in I = \mathbb{R}$   
 Temam [1969], Saut-Temam [1976], Bona-Scott [1976], Kato [1983], Kenig-Ponce-Vega [1991,1996], Bourgain [1993]  
**Colliander-Keel-Staffilani-Takaoka-Tao [2003]: GWP in  $H^s(\mathbb{R})$ ,  $s > -3/4$ .**
- $x \in I = \mathbb{T}$   
 Bourgain [1993,1996]  
**Kappeler-Topalov [2006]: GWP in  $H^{-1}(\mathbb{T})$**
- $x \in I = (0, L) + 3 \text{ b.c.}$   
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# Basic issues in Control Theory

$$(S) \quad \frac{du}{dt} = f(u, h), \quad u(0) = u_0$$

- **Exact Controllability**

Given  $T > 0$ ,  $u_0$  and  $u_1$  in some space, can we find a control input  $h = h(t)$  driving system (S) from  $u_0$  at  $t = 0$  to  $u_1$  at  $t = T$ ?

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Can we find a closed-loop control  $h = h(u)$  such that the origin is asymptotically (or exponentially) stable for (S)?

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# Tools in Control Theory

- Nonharmonic Fourier analysis (1D PDEs, Schrödinger)
- Multipliers (wave, plates)
- Carleman estimates (heat, Navier-Stokes)
- Microlocal analysis (wave)

# Control properties of KdV: bounded domain



$$\partial_t u + \partial_x^3 u + \partial_x u + u \partial_x u = 0$$

$$u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \quad u_x(L, t) = h_3(t)$$

- **Controllability:** Rosier [1997,2000,2004], Zhang [1999], Coron-Crépeau [2004], Cerpa [2007], Glass-Guerrero [2008], Cerpa-Crépeau [2009], Chapouly [preprint]
- **Stabilization:** Perla Menzala-Vasconcellos-Zuazua [2002], Pazoto [2005], Rosier-Zhang [2006], Linares-Pazoto [2007,2009]
- **Main tools:** Compactness argument based upon Kato smoothing effect: assume  $h_i = 0$  ( $i = 1, 2, 3$ ).

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$$\partial_t u + \partial_x^3 u + u \partial_x u = f(x, t), \quad x \in \mathbb{T} = \mathbb{R}/(2\pi)\mathbb{Z}$$

Forcing term  $f$  supported in some given open set  $\omega \subset \mathbb{T}$ .

- To keep the mass  $\int_{\mathbb{T}} u(x, t) dx$  conserved, we impose the condition  $[f] := (2\pi)^{-1} \int_{\mathbb{T}} f dx = 0$ . We shall assume that

$$f(x, t) = [Gh](x, t) = g(x) \left( h(x, t) - \int_{\mathbb{T}} g(y) h(y, t) dy \right)$$

$g$  being a fixed nonnegative, smooth function supported in  $\omega$  with  $\int_{\mathbb{T}} g(x) dx = 1$ .

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Zhang [1990], Komornik-Russell-Zhang [1991], Russell-Zhang [1993, 1996]



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# Ideas of the proof

- Kato smoothing effect no longer valid on the torus. Fortunately, Bourgain [1993] discovered a more subtle smoothing effect thanks to which he proved the GWP in  $L^2(\mathbb{T})$ .
- Tools for the proofs: contraction mapping principle in Bourgain spaces
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# Bourgain spaces

- For  $b, s \in \mathbb{R}$  and  $u(x, t) = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \hat{u}(k, \tau) e^{i(kx + \tau t)} d\tau$ , let

$$\|u\|_{X_{b,s}}^2 = \sum_k \int_{\mathbb{R}} \langle k \rangle^{2s} \langle \tau - k^3 \rangle^{2b} |\hat{u}(k, \tau)|^2 d\tau,$$

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- $X_{b,s}$  (resp.  $Y_{b,s}$ ) completion of  $\mathcal{S}(\mathbb{T} \times \mathbb{R})$  for the norm  $\|\cdot\|_{X_{b,s}}$  (resp.  $\|\cdot\|_{Y_{b,s}}$ ). Finally  $Z_{b,s} = X_{b,s} \cap Y_{b-\frac{1}{2},s}$ . Let  $X_{b,s}^T, Z_{b,s}^T$  be the restriction spaces to  $(0, T)$ .
- Bilinear estimate**

$$\|(uv)_x\|_{Z_{-\frac{1}{2},s}^T} \leq CT^\theta \|u\|_{X_{\frac{1}{2},s}^T} \|v\|_{X_{\frac{1}{2},s}^T}.$$

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# The results

*Joint work with*

**Camille Laurent**, *Université Paris-Sud (France)*, and

**Bing-Yu Zhang**, *University of Cincinnati*

Consider

$$\partial_t u + u \partial_x u + \partial_x^3 u = Gh = g(x)(h(x, t) - \int_{\mathbb{T}} g(y)h(y, t) dy)$$

$$u(., 0) = u_0$$

To simplify the exposition, assume  $[u(., t)] = [u_0] = 0$ .

# Global exact controllability

**Thm 1:** Assume given  $s \geq 0$ ,  $R > 0$ . There exists  $T > 0$  s.t. for

$$u_0, u_1 \in H^s(\mathbb{T}), \quad [u_0] = [u_1] = 0, \quad \|u_0\|_s + \|u_1\|_s \leq R$$

one can find  $h \in L^2(0, T, H^s(\mathbb{T}))$  driving the system from  $u_0$  at  $t = 0$  to  $u_1$  at  $t = T$



# Global exponential stabilization

**Thm 2:**  $s \geq 0$  given. There exists a constant  $\mu > 0$  such that for  $u_0 \in H^s(\mathbb{T})$  with  $[u_0] = 0$ , we have for  $t \geq 0$

$$\|u(\cdot, t)\|_s \leq \alpha(\|u_0\|_s) e^{-\mu t} \|u_0\|_s$$

$\alpha$  is a nondecreasing function depending on  $s$ .

# Ideas of the proofs

- Thm 1 follows from Thm 2 and Russell-Zhang (local) controllability
- To prove Thm 2 for  $s = 0$ , apply Zuazua's compactness - uniqueness strategy: using the Energy Identity ( $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{T})}$ ):

$$\|u(t)\|^2 = \|u_0\|^2 - \int_0^t \|Gu(\tau)\|^2 d\tau$$

it is sufficient to prove the **Observability Inequality**:

$$\|u_0\|^2 \leq \text{const} \int_0^T \|Gu(\tau)\|^2 d\tau$$

This is done by contradiction, using a propagation of compactness / regularity as in Dehman-Lebeau-Gerard [2006] and Laurent [2009] for NLS.

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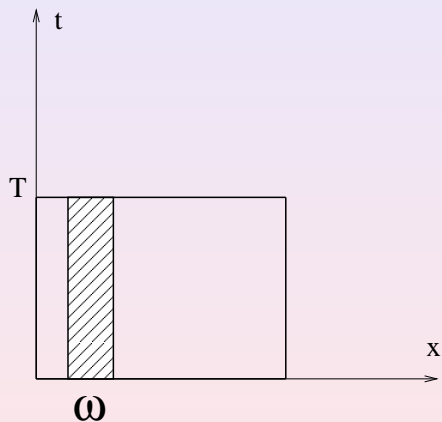
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# Propagation of compactness/regularity



# Propagation of compactness

Assume  $T > 0$ ,  $\omega \subset \mathbb{T}$ ,  $0 \leq b' \leq b \leq 1$  and  $u_n \in X_{b,0}^T$ ,  $f_n \in X_{-b,-2+2b}^T$  satisfy

$$\partial_t u_n + \partial_x^3 u_n = f_n, \quad n = 1, 2, \dots$$

Assume further that  $\|u_n\|_{X_{b,0}^T} \leq \text{const}$  and

$$\|u_n\|_{X_{-b,-2+2b}^T} + \|f_n\|_{X_{-b,-2+2b}^T} + \|u_n\|_{X_{-b',-1+2b'}^T} \rightarrow 0.$$

If  $u_n \rightarrow 0$  in  $L^2(0, T, L^2(\omega))$ , then

$$u_n \rightarrow 0 \text{ in } L_{loc}^2(0, T, L^2(\mathbb{T})).$$

# Propagation of regularity

Assume  $T > 0$ ,  $\omega \subset \mathbb{T}$ ,  $0 \leq b < 1$ ,  $r \in \mathbb{R}$  and  $f \in X_{-b,r}^T$ . Let  $u \in X_{b,r}^T$  solve

$$\partial_t u + \partial_x^3 u = f.$$

If  $u \in L_{loc}^2(0, T, H^{r+\rho}(\omega))$  for some  $\rho$  with

$$0 < \rho \leq \min\left\{1 - b, \frac{1}{2}\right\}$$

Then

$$u \in L_{loc}^2(0, T, H^{r+\rho}(\mathbb{T})).$$

**Corollary:** Let  $u \in X_{\frac{1}{2},0}^T$  solves  $u_t + u_{xxx} + uu_x = 0$ . Then

$$u \in C^\infty(\omega \times (0, T)) \quad \Rightarrow \quad u \in C^\infty(\mathbb{T} \times (0, T))$$

# Rapid stabilization

**Thm 3.** Let  $\lambda > 0$  and  $s \geq 0$ . There exists  $\delta > 0$  and  $K_\lambda \in \mathcal{L}(H^s(\mathbb{T}), H^s(\mathbb{T}))$  such that for  $\|u_0\|_s \leq \delta$  and  $[u_0] = 0$ , the solution  $u$  of

$$\partial_t u + u \partial_x u + \partial_x^3 u = -GK_\lambda u, \quad u(\cdot, 0) = u_0,$$

satisfies

$$\|u(\cdot, t)\|_s \leq Ce^{-\lambda t} \|u_0\|_s.$$

$h = -K_\lambda u$  is the feedback law given by Slemrod [1974] for the linearized system.

# Time-varying feedback law

- The feedback law  $h = -Gu$  (resp.  $h = -K_\lambda u$ ) yields a global (resp. local) exponential stabilization with a given (resp. arbitrary) decay rate.
- Aim: combine both feedback laws to obtain a **global** stabilization with an **arbitrary** decay rate.
- Idea: use the feedback law  $h = -Gu$  far from 0 to get the global stabilization, and the feedback law  $h = -K_\lambda u$  close to 0 to get a large decay rate.
- In practice: avoid discontinuous feedback laws (difficulty to define a solution!), and use a smooth (periodic) time-varying feedback law, coinciding successively on half periods with  $-Gu$  and with  $-K_\lambda u$  (at least close to 0).



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## Time-varying feedback law (2)

- Pick a 2-periodic smooth function  $\theta(t)$  with  $\theta(t) = 1$  on  $[\delta, 1 - \delta]$ ,  $\theta(t) = 0$  on  $[1, 2]$ , and a smooth function  $\rho(r)$  with  $\rho(r) = 1$  for  $0 \leq r \leq r_0 < 1$ ,  $\rho(r) = 0$  for  $r \geq 1$
- Set

$$-K(u, t) := \rho(\|u\|_s)[\theta(t/T)K_\lambda u + \theta((t-T)/T)Gu] + (1 - \rho(\|u\|_s))Gu$$

- **Thm 4:** Let  $\lambda > 0$  be given. There exists  $T_0 > 0$  such that for  $T > T_0$ , for each pair  $(t_0, u_0)$  with  $[u_0] = 0$ , the solution of

$$u_t + uu_x + u_{xxx} = GK(u, t), \quad u(., t_0) = u_0$$

satisfies

$$\|u(., t)\|_s \leq \alpha(\|u_0\|_s)e^{-\frac{1}{2}\lambda(t-t_0)}\|u_0\|_s, \quad \text{for } t \geq t_0.$$

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# Future directions of research

- **Duration** of the control process: can two states  $u_0$  and  $u_1$  be connected by a trajectory of KdV in arbitrarily small time  $T$ ? If not, how  $T$  is related to the magnitude of  $\|u_0\|_s$  and  $\|u_1\|_s$ ? Same question for NLS.
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**Thank you for your attention!**