Analysis of the Blade Element Momentum Theory, application to river current power extraction

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50 ans du LJLL - Roscoff, 4.3.19

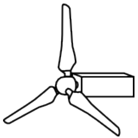
Evaluation and design of propeller, Seminal models :

- **1865 :** "Momentum Theory" : William J. M. Rankine [1], also Greenhill and Froude.
- **1878 :** "Blade Element Theory" : William Froude [2], also Taylor and Drzewiecki.
- **1919 :** The 1D-model of Betz and the Betz limit

$$C_{p,Betz} = 16/27 \approx 0.5926$$

- **1926 :** "Blade Element Momentum Theory" : Glauert's breakthrough.

  1. Combine "Momentum Theory" and "Blade Element Theory",
  2. Take into account the wake momentum.



1. W. J. M. Rankine. On the mechanical principles of the action of propellers. Transactions, Institute of Naval Architects, 6 :13-30, 1865.
2. W. Froude. On the elementary relation between pitch, slip and propulsive efficiency. Trans. Roy. Inst. Naval Arch., 19(47) :47-57, 1878.

Outline

Hermann Glauert, 1892-1934.

"The Elements of Aerofoil and Airscrew Theory" - 1926

Ideas :

- Decompose the blade into elements, considered to be **independent**.
- Coupling of two models :
    1. Local 2D model, describing the **lift and drag forces on a 2D profile**
    2. Macroscopic model, describing the **evolution of a fluid ring crossing the propeler**

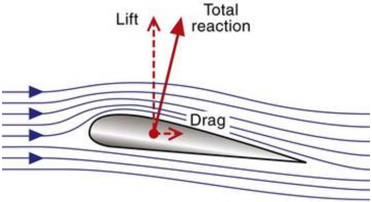
Local 2D model :

Using windtunnel, or Computational Fluid Mechanics, one use a 2D prototype or model to assess the Drag and Lift forces on a profile, assuming they are on the form :

$$dL = C_L(\alpha)\frac{1}{2}\rho W^2 c(r)dr$$
$$dD = C_D(\alpha)\frac{1}{2}\rho W^2 c(r)dr.$$

with :

- α = angle of incidence,
- W = macroscopic velocity in $x = -\infty$,
- c = is the chord distribution.



http://www.pilotwings.org/

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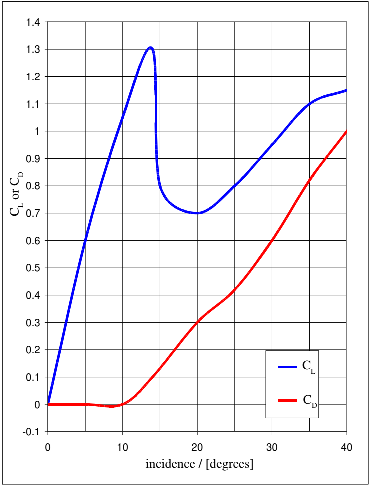
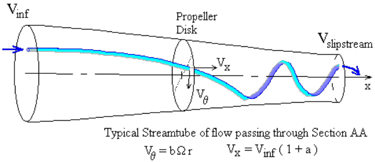


"Wind Turbine Blade Analysis using the Blade Element Momentum Method", Notes by G. Ingram.

Macroscopic model :
axial and rotational interference factors

$$a = \frac{U_{-\infty} - U_{x=0}}{U_{-\infty}}$$

$$a' = \frac{\omega_{x=0^+}}{2\Omega}$$



"Aerodynamics for students",

http://www-mdp.eng.cam.ac.uk

Macroscopic model :
axial and rotational interference factors

$$a = \frac{U_{-\infty} - U_{x=0}}{U_{-\infty}}$$

$$a' = \frac{\omega_{x=0^+}}{2\Omega}$$

occur. This consideration will be modified in the next section. A helicopter, in going from vertical ascent to autorotational descent can pass through the various states illustrated in Figure 4-6. Glauert (10) used experimental results to quantify the turbulent windmill and vortex ring states of a rotor.

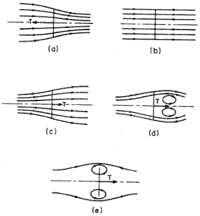


Figure 4.6 Working States of a Rotor: (a) propeller; (b) zero-thrust; (c) windmill; (d) turbulent windmill; (e) vortex ring

Wilson, Lissaman, "Applied Aerodynamics of wind power machines", 1974, p.51.

The Glauert's model

Local/Macro decomposition

Macroscopic model :
Angular relations

$\theta =$ Blade angle

$\alpha =$ Incidence angle

$\varphi =$ Relative angle deviation

"The element will work at

$$\alpha = \theta - \varphi."$$

16·2. Consider next the aerodynamic forces experienced by the blade element at radial distance r. The blade element

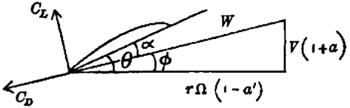


Fig. 110.

is subject to an axial velocity $V(1+a)$ and a rotational velocity $r\Omega(1-a')$, so that the resultant velocity W is inclined at angle ϕ to the plane of rotation, where

$$\tan\phi = \frac{V}{r\Omega}\cdot\frac{1+a}{1-a'}.$$

If θ is the blade angle, the element will work at an angle of incidence $\alpha = \theta - \phi$ and will give the corresponding lift and drag coefficients, C_L and C_D, appropriate to the aerofoil section in two-dimensional motion. The components of these force coefficients, resolved in the direction of the thrust and torque, are respectively

$$\lambda_1 = C_L\cos\phi - C_D\sin\phi,$$
$$\lambda_2 = C_L\sin\phi + C_D\cos\phi,$$

and the elements of thrust and torque given by the blade element of area $c\,dr$ are

$$dT = \lambda_1\,\tfrac{1}{2}\rho W^2 c\,dr,$$
$$dQ = \lambda_2\,\tfrac{1}{2}\rho W^2 cr\,dr.$$

"The Elements of Aerofoil and Airscrew Theory", 1926, p.212.

Macroscopic model :
Angular relations
Glauert motivation : aeronautics

$$\Rightarrow a_{Glauert} \to -a,$$
$$a'_{Glauert} \to -a'$$
$$\theta \to \gamma_\lambda.$$

$$\tan^{-1}\varphi = \lambda\frac{1+a'}{1-a}$$
$$\lambda = \frac{r\Omega}{U_{-\infty}}$$

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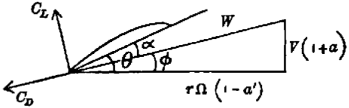

Fig. 110.

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"The Elements of Aerofoil and Airscrew Theory", 1926, p.212.

Using Bernouilli's relation, one can find the elementary force and torque :

$$dF_x = \rho U_{x=0}^2 (4a(1-a))\pi r dr,$$
$$dT = 4a'(1-a)\rho U_{x=0} r^3 \Omega \pi dr.$$

But the lift and drag coefficients definitions imply

$$dF_x = \sigma(r)\pi\rho \frac{U_{x=0}^2 (1-a)^2}{\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\cos\varphi + C_D(\varphi-\gamma_\lambda)\sin\varphi) r dr,$$
$$dT = \sigma(r)\pi\rho \frac{U_{x=0}^2 (1-a)^2}{\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\sin\varphi - C_D(\varphi-\gamma_\lambda)\cos\varphi) r^2 dr,$$

where we have introduced the *local solidity*, defined by :

$$\sigma(r) = \frac{Bc(r)}{2\pi r}.$$

We end up with the **Glauert's system** :

$$\tan^{-1}\varphi = \lambda\frac{1+a'}{1-a},$$
$$\frac{a}{1-a} = \frac{\sigma(r)}{4\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\cos\varphi + C_D(\varphi-\gamma_\lambda)\sin\varphi),$$
$$\frac{a'}{1-a} = \frac{\sigma(r)}{4\lambda\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\sin\varphi - C_D(\varphi-\gamma_\lambda)\cos\varphi).$$

Outline

Simplification : we assume $C_D = 0 \rightarrow$ fits with the practical cases [3]

$$\tan^{-1}\varphi = \lambda\frac{1+a'}{1-a},$$
$$\frac{a}{1-a} = \frac{\sigma(r)}{4\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\cos\varphi),$$
$$\frac{a'}{1-a} = \frac{\sigma(r)}{4\lambda\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\sin\varphi).$$

3. *"In the calculation of induction factors,[...] accepted practice is to set C_D equal to zero [...]. For airfoils with low drag coefficients, this simplification introduces negligible errors."*, Manwell et al. p.125.

$$
\begin{aligned}
\tan^{-1}\varphi &= \lambda\frac{1+a'}{1-a},\\
\frac{a}{1-a} &= \frac{\sigma(r)}{4\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\cos\varphi),\\
\frac{a'}{1-a} &= \frac{\sigma(r)}{4\lambda\sin^2\varphi}(C_L(\varphi-\gamma_\lambda)\sin\varphi).
\end{aligned}
$$

Remarks :

- Also for practical cases, we are interested in solution such that $C_L(\varphi - \gamma_\lambda) > 0 \Leftrightarrow \gamma_\lambda > \varphi$,
- $(a, a', \varphi) = (1, a', \frac{\pi}{2})$ is always a (non-interesting) solution of this system.

Set $\mu_{C_L} = \mu_{C_L}(\varphi) := \dfrac{\sigma(r) C_L(\varphi - \gamma_\lambda)}{4}$:

$$\begin{aligned} \tan^{-1}\varphi &= \lambda \frac{1+a'}{1-a}, \\ \frac{a}{1-a} &= \frac{\sigma(r)}{4\sin^2\varphi}\left(C_L(\varphi - \gamma_\lambda)\cos\varphi\right), \\ \frac{a'}{1-a} &= \frac{\sigma(r)}{4\lambda\sin^2\varphi}\left(C_L(\varphi - \gamma_\lambda)\sin\varphi\right). \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \tan^{-1}\varphi &= \lambda \frac{1+a'}{1-a}, \\ \frac{a}{1-a} &= \frac{\mu_{C_L}}{\sin^2\varphi}\cos\varphi, \\ \frac{a'}{1-a} &= \frac{\mu_{C_L}}{\lambda\sin\varphi}. \end{aligned}$$

$$\tan^{-1}\varphi = \lambda\frac{1+a'}{1-a},$$
$$\frac{a}{1-a} = \frac{\mu_{C_L}}{\sin^2\varphi}\cos\varphi,$$
$$\frac{a'}{1-a} = \frac{\mu_{C_L}}{\lambda\sin\varphi}.$$

$$\Updownarrow$$

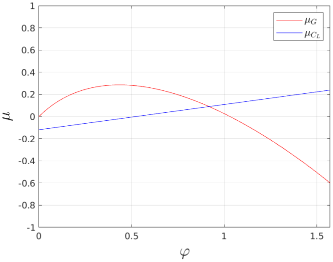
$$\tan^{-1}\varphi = \lambda\left(1+\frac{\mu_{C_L}}{\sin^2\varphi}\cos\varphi\right)+\frac{\mu_{C_L}}{\sin\varphi}.$$

To study the solution(s) of Glauert's system, we rewrite it :

$$\tan^{-1}\varphi = \lambda\left(1 + \frac{\mu_{C_L}}{\sin^2\varphi}\cos\varphi\right) + \frac{\mu_{C_L}}{\sin\varphi}$$
$$\Leftrightarrow \mu_{C_L} = \sin\varphi\frac{\cos\varphi - \lambda\sin\varphi}{\sin\varphi + \lambda\cos\varphi}$$
$$\Leftrightarrow \mu_{C_L} = \sin\varphi\tan(\theta_\lambda - \varphi) =: \mu_G.$$

Solving Glauert's approach consists in solving :

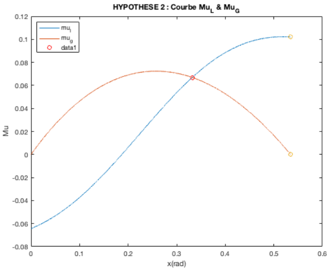
$$\frac{\sigma(r)C_L(\varphi - \gamma_\lambda)}{4} = \sin\varphi\tan(\theta_\lambda - \varphi)$$
$$\Updownarrow$$
$$\mu_{C_L}(\varphi) = \mu_G(\varphi)$$

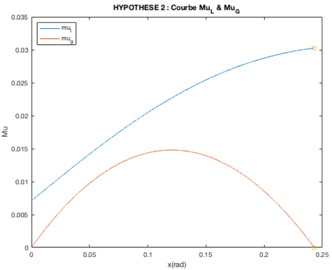


Example : river current power, 'Hydrolienne H3'

HYDROTUBE ENERGIE

$\rightarrow$ *A.N.R HyFloEFlu*


HYPOTHESE 2 : Courbe Mu_L & Mu_G
mu_l
mu_g
data1
Mu
x(rad)
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
0
0.1
0.2
0.3
0.4
0.5
0.6


HYPOTHESE 2 : Courbe Mu_L & Mu_G
mu_l
mu_g
Mu
x(rad)
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
0.05
0.1
0.15
0.2
0.25

Outline

Recall that :

$$dF_x = 4a(1-a)U_{-\infty}^2 \rho \pi r dr.$$

The quantity

$$C_T = \frac{dF_x}{\frac{1}{2}U_{-\infty}^2 \rho 2\pi r dr},$$

is called *local thrust coefficient.*

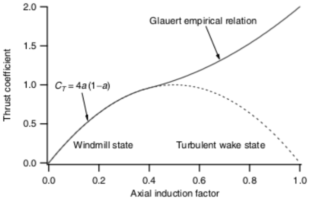


Figure 3.29 Fits to measured wind turbine thrust coefficients

Manwell et al, "Wind Energy Explained Theory, Design and Application", 2nd Ed., p.130

$$dF_x = 4a(1-a)U_{-\infty}^2 \rho \pi r dr$$
$$\Downarrow$$
$$dF_x = 4\chi(a, a_c)U_{-\infty}^2 \rho \pi r dr$$
$$= 4\left(a(1-a) + \psi\left((a-a_c)_+\right)\right)U_{-\infty}^2 \rho \pi r dr$$

Order	Author	a_c	$\psi\left((a-a_c)_+\right)$
3	Glauert	1/3	$\frac{(a-a_c)_+}{4}\left(\frac{(a-a_c)_+^2}{a_c} + 2(a-a_c)_+ + a_c\right)$
2	Glauert emp.	2/5	$a_c(1-a_c) + \frac{(a-a_c)_+[F_\lambda(\varphi)(a-a_c)_+ + 2F_\lambda(\varphi)a_c - 0.286]}{2.5708} F_\lambda(\varphi)$
2	Buhl	2/5	$\frac{1}{2F_\lambda(\varphi)}\left(\frac{(a-a_c)_+}{1-a_c}\right)^2$
1	Wilson et al.	1/3	$(a-a_c)_+^2$

Lemma

Suppose that (φ, a, a') satisfies Glauert's model.

1. *There exists $\tau : \varphi \mapsto a = \tau(\varphi)$ satisfying :*

$$\frac{a}{1-a} + \left(1 - \frac{\cos\theta_\lambda \cos\varphi}{\cos(\theta_\lambda - \varphi)}\right) \frac{\psi\left((a-a_c)_+\right)}{(1-a)^2} = g(\varphi),$$

with

$$g(\varphi) := \tan^{-1}\varphi \tan(\theta_\lambda - \varphi) + \frac{\mu_{C_D}}{\sin\varphi}\left(1 + \tan^{-1}\varphi \tan(\theta_\lambda - \varphi)\right),$$

2. *The unknown φ satisfies*

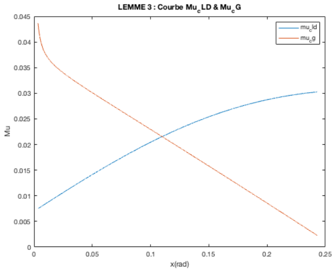
$$\mu_{C_L}(\varphi) - \tan(\theta_\lambda - \varphi)\mu_{C_D}(\varphi) = \mu^c_G(\varphi),$$

where

$$\mu^c_G(\varphi) := \mu_G(\varphi) + \frac{\cos\theta_\lambda \sin^2\varphi}{\cos(\theta_\lambda - \varphi)} \frac{\psi\left((\tau(\varphi) - a_c)_+\right)}{(1-\tau(\varphi))^2}.$$

Expanding further, one finds

$$a = 1 - \sqrt{\frac{\psi(1-a_c)}{\mu_{C_D}(0)}}\varphi^{3/2}$$



Lemma

The function μ_G^c satisfies

$$\mu_G^c(\varphi) \approx_{\varphi=0^+} \frac{\mu_{C_D}(0)}{\varphi}.$$

Outline

Outline

Usual way to solve this system :

airfoil characteristics. Thus, the total system of equation, which is both nonlinear and implicit, needs to be solved either by employing a nonlinear solution technique for the full system of equations or by using a simple iterative updating technique. For several reasons, the latter is the most convenient method to be used. A solution procedure may proceed as follows:

1. Divide the rotor blade into a number of spanwise elements (typically 20–30) and start an iterative procedure for each element.
2. Guess a and a'. The guess may either be based on the values obtained at the previous element or, e.g., by putting $a = 1/3$ and $a' = 0$.
3. Compute the flow angle from the expression: $\phi = \tan^{-1}\left(\frac{1-a}{\lambda x(1+a')}\right)$, where $\lambda = \frac{\Omega R}{U_0}$ is the tip speed ratio and $x = r/R$.
4. Compute the angle of attack, $\alpha = \phi - \gamma$, and based on this, determine the airfoil characteristics, $C_l = C_l(\alpha)$ and $C_d = C_d(\alpha)$.
5. Compute C_n and C_t.
6. Update a and a' and continue the process until convergence.

"General Momentum Theory for Horizontal Axis Wind Turbines", J. N. Sorensen

Usual way to solve this system :

$$\tan^{-1}\varphi^{k+1} = \lambda\frac{1+a'^{k}}{1-a^{k}},$$
$$\frac{a^{k}}{1-a^{k}} = \frac{\sigma(r)}{4\sin^2\varphi^{k}}(C_L(\varphi^{k}-\gamma_\lambda)\cos\varphi^{k} + C_D(\varphi^{k}-\gamma_\lambda)\sin\varphi^{k}),$$
$$\frac{a'^{k}}{1-a^{k}} = \frac{\sigma(r)}{4\lambda\sin^2\varphi^{k}}(C_L(\varphi^{k}-\gamma_\lambda)\sin\varphi^{k} - C_D(\varphi^{k}-\gamma_\lambda)\cos\varphi^{k}).$$

Theorem

Let

$$\mu_{C_L}(\theta_\lambda) \leq \mu_G(\gamma_\lambda).$$

and

$$\frac{\|\mu'_{C_L}\|_\infty h(\gamma_\lambda)}{1+\lambda^2} \leq 1$$
$$\frac{\|\mu_{C_L}\|_\infty |h'(\gamma_\lambda)|}{1+\lambda^2} \leq 1.$$

Then, the sequence $(\varphi^k)_{k\in\mathbb{N}}$ *defined by the classical solver converges to a solution of Glauert's model.*

Outline

All the variant of the model may give rise to multiple solutions. More precisely :

1. With the simplification $C_D = 0$ and C_L approximately linear around 0 : two solutions.
2. Stall : possible other solution after the critical angle.
3. Corrected model : μ_G^c may change of concavity.

$\rightarrow$ possible problem of convergence..
$\Rightarrow$ Bisection method will always work...

Outline

Quantity to maximize :

$$C_P = \frac{8}{\lambda_R^2} \int_{\lambda_r}^{\lambda_R} \lambda^3 a'(1-a) \left(1 - \frac{C_D(\varphi - \gamma_\lambda)}{C_L(\varphi - \gamma_\lambda)} \tan\varphi \right) d\lambda.$$

Design parameters : $c(r), \gamma_\lambda(r) \Rightarrow (\mu_{C_L}, \mu_{C_D})$.

Indeed :

$$\mu_{C_L} = \sigma(r) \frac{C_L(\gamma_\lambda(r) - \varphi)}{4}$$

$$\mu_{C_D} = \sigma(r) \frac{C_D(\gamma_\lambda(r) - \varphi)}{4}$$

$$\sigma(r) = \frac{Bc(r)}{2\pi r}.$$

Mathematical formulation, for fixed λ :

$$\min J(\mu_{C_L}, \mu_{C_D}) = a'(1-a)\left(1 - \frac{\mu_{C_D}}{\mu_{C_L}} \tan^{-1}\varphi\right),$$

under the constraints

$$\begin{aligned}
\tan^{-1}\varphi &= \lambda \frac{1+a'}{1-a}, \\
\frac{a}{1-a} &= \frac{1}{\sin^2\varphi}\left(\mu_{C_L}\cos\varphi + \mu_{C_D}\sin\varphi\right), \\
\frac{a'}{1-a} &= \frac{1}{\lambda\sin^2\varphi}\left(\mu_{C_L}\sin\varphi - \mu_{C_D}\cos\varphi\right).
\end{aligned}$$

and with

$$\mu_{C_L} = \frac{\sigma(r) C_L(\gamma_\lambda - \varphi)}{4} \qquad \mu_{C_D} = \frac{\sigma(r) C_D(\gamma_\lambda - \varphi)}{4}.$$

If the correction on a is not included, the following procedure is used :

3.10.1.2 Define the Blade Shape

5. Obtain and examine the empirical curves for the aerodynamic properties of the airfoil at each section (the airfoil may vary from the root to the tip), i.e. C_l vs. α, C_d vs. α. Choose the design aerodynamic conditions, $C_{l,design}$ and α_{design}, such that $C_{d,design}/C_{l,design}$ is at a minimum for each blade section.

"Wind Energy Explained", Manwell et al. 2nd Ed.

$$J(\mu_{C_L}, \mu_{C_D}) = a'(1-a)\left(1 - \frac{C_D(\varphi - \gamma_\lambda)}{C_L(\varphi - \gamma_\lambda)} \tan^{-1}\varphi\right)$$

$$\Downarrow \quad C_D \approx 0$$

$$J(\mu_{C_L}) = a'(1-a).$$

After this step $\alpha^\star = \varphi - \gamma_\lambda$ is fixed !

Consider then :

$$J(\mu_{C_L}) = a'(1-a).$$

Taking into account that $\mu_{C_L}(\varphi) = \mu_G(\varphi) := \sin\varphi\tan(\theta_\lambda - \varphi)$, one can rewrite J only in term of φ.

$$J(\varphi) = \frac{1}{2\lambda}\frac{\sin^2\varphi}{\sin\theta_\lambda}\sin(2(\varphi - \theta_\lambda))$$

whose optimum is obtained for :

$$\varphi^\star = \frac{2}{3}\theta_\lambda$$

End of the design procedure :

Recall that $\alpha^\star = \varphi - \gamma_\lambda$ and $\mu_{C_L} = \dfrac{\sigma(r) C_L(\varphi - \gamma_\lambda)}{4}$, then

$$\gamma_\lambda^\star = \alpha^\star + \varphi^\star$$
$$c^\star = \frac{8\pi r}{B C_L(\alpha^\star)} \mu_G(\varphi^\star)$$

Summary :

$$\min J(\mu_{C_L}, \mu_{C_D}, \varphi) \; s.t. \; Eq(\mu_{C_L}, \mu_{C_D}, \varphi) = 0$$

$$\Downarrow (1)$$

$$\min J(\mu_{C_L}, \varphi) \; s.t. \; Eq(\mu_{C_L}, \varphi) = 0$$

$$\Downarrow (2)$$

$$\min J(f(\varphi), \varphi)$$

$$\Downarrow$$

Explicit solution !

Outline

Using the expansion :

$$a = 1 - \sqrt{\frac{\psi(1-a_c)}{\mu_{C_D}(0)}}\varphi^{3/2},$$

we obtain

$$J(\mu_{C_L}, \mu_{C_D}) \approx \frac{\psi(1-a_c)\tan\theta_\lambda}{\lambda}\left(1-\mu_{C_D}(0)\right)\varphi^2.$$

Theorem

There exists $a_c^0 < 1$ such that for $a_c \leq a_c^0$, the optimal solution does not activate the thrust correction.

Outline

The BEM is a $0D \times 2D$ coupled model.

- Condition on γ_λ and C_L to guarantee existence of solution of interest
- Possible multiple solutions
- Optimization : existence, definition of a research interval

Possible extension to $1D \times 2D$?

Pub

Une conférence Maths/Énergies Marines :
https://emrsim2019.sciencesconf.org/
4-7 Juillet

$\Rightarrow$ À Roscoff ! !