Control and Pattern Formation in Collective Dynamics

Nastassia Pouradier Duteil

INRIA Paris, Team MAMBA
Laboratoire Jacques-Louis Lions, Sorbonne University

Roscoff
March 7, 2019

Collective-dynamics models

First-order models: Hegselmann-Krause

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(x_j - x_i), \quad i \in \{1, \dots, N\} \tag{1}$$

Second-order models: Cucker-Smale

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i), \quad i \in \{1, \dots, N\} \end{cases} \tag{2}$$

Kinetic formulation:

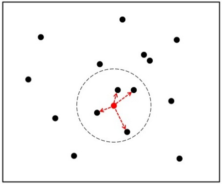
$$\partial_t \mu + \nabla \cdot (V[\mu]\mu) = 0$$

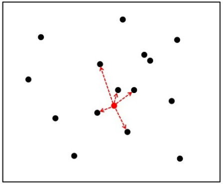
- 1st order: $\mu = \mu(t,x)$; $V[\mu](x) = \int a(\|x-y\|)(y-x)d\mu(y)$
- 2nd order: $\mu = \mu(t,x,v)$; $V[\mu](x,v) = \begin{pmatrix} v \\ \int\int a(\|x-y\|)(w-v)d\mu(y,w) \end{pmatrix}$.

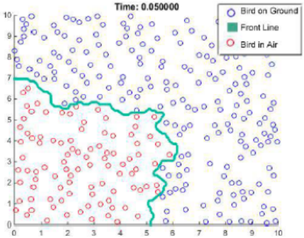
I - Example of application

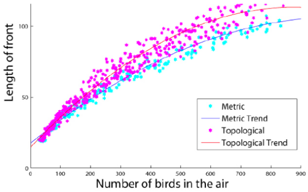
Video credits: Dr. W. Saidel, CCIB, Rutgers University

$$\dot{x}_i = \frac{1}{N} \sum_{j \in \mathcal{N}_i} a(\|x_i - x_j\|)(x_j - x_i)$$









Metric or topological interaction?

II - Optimal control of a collective migration model

Collective Migration Model

$$\dot{v}_i = (1-\alpha_i)\frac{1}{N}\sum_{j=1}^{N}(v_j - v_i) + \alpha_i(V - v_i), \quad i \in \{1,\ldots,N\} \tag{3}$$

where:

- $v_i \in \mathbb{R}^d$ is the state variable (velocity)
- $V \in \mathbb{R}^d$ is the target velocity
- $\alpha \in \mathcal{U} := \{\alpha : [0,T] \to [0,1]^N \mid \alpha$ measurable, s.t. $\sum_{i=1}^{N} \alpha_i \leq 1\}$

Two forces guide the system (balanced by the parameters α_i):

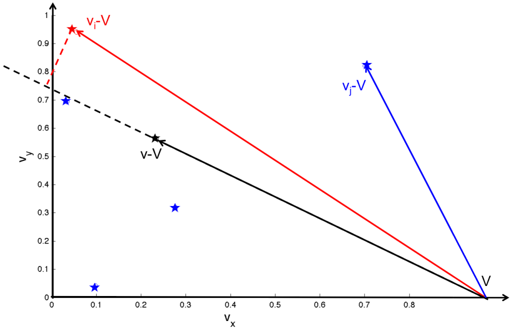
- the consensus dynamics
- the attraction towards a target velocity V

Given $T > 0$, minimize $\mathbb{V}(T) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\|v_i(T) - V\|^2$

Theorem: Full control optimal strategy

Let $T > 0$. Let $\xi_i := \langle v_i - V, \frac{\bar{v}-V}{\|\bar{v}-V\|}\rangle$. There exist $\{t_1, \cdots, t_N\}$ depending only on $\xi(0)$ such that if $T \geq t_N$, then $\sum_{i=1}^N \alpha_i \equiv 1$ and $\xi_i(T) = \xi_j(T)$ for all $i, j \in \{1, \ldots, N\}$.
In particular the following strategy is optimal:

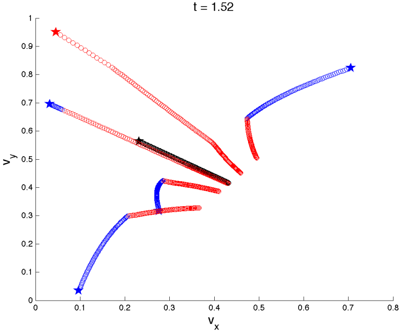
$$\forall t \in [t_k, t_{k+1}), \alpha_i(t) = \begin{cases} \frac{1}{k} & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}$$



Theorem: Full control optimal strategy

Let $T > 0$. Let $\xi_i := \langle v_i - V, \frac{\bar{v}-V}{\|\bar{v}-V\|}\rangle$. There exist $\{t_1, \cdots, t_N\}$ depending only on $\xi(0)$ such that if $T \geq t_N$, then $\sum_{i=1}^N \alpha_i \equiv 1$ and $\xi_i(T) = \xi_j(T)$ for all $i, j \in \{1, \ldots, N\}$.
In particular the following strategy is optimal:

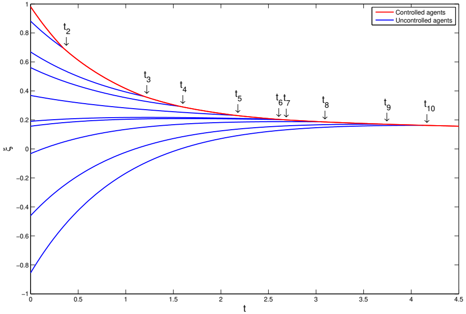
$$\forall t \in [t_k, t_{k+1}), \alpha_i(t) = \begin{cases} \frac{1}{k} & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}$$



Theorem: Full control optimal strategy

Let $T > 0$. Let $\xi_i := \langle v_i - V, \frac{\bar{v}-V}{\|\bar{v}-V\|}\rangle$. There exist $\{t_1, \cdots, t_N\}$ depending only on $\xi(0)$ such that if $T \geq t_N$, then $\sum_{i=1}^N \alpha_i \equiv 1$ and $\xi_i(T) = \xi_j(T)$ for all $i, j \in \{1, \ldots, N\}$. In particular the following strategy is optimal:

$$\forall t \in [t_k, t_{k+1}), \alpha_i(t) = \begin{cases} \frac{1}{k} & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}$$

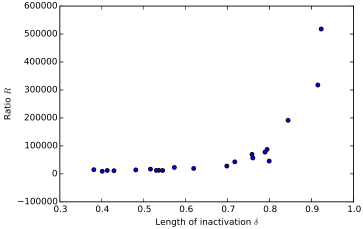


Theorem: Inactivation principle

If $T < t_N$, then there exists $\delta \in [0, T]$ such that the optimal constrol strategy is *inactive* on $[0, \delta]$ and acts with full control on $[\delta, T]$, i.e.

$$\sum_{i=1}^{N} \alpha_i^{\text{opt}}(t) = \begin{cases} 0 & \text{if } t < \delta \\ 1 & \text{if } t \geq \delta \end{cases}$$

$$\mathbb{V} = \frac{1}{N}\sum_{i=1}^{N} \|v_i(T) - V\|^2 = \underbrace{\|\bar{v} - V\|^2}_{\text{Drive system to } V} + \underbrace{\frac{1}{N}\sum_{i=1}^{N} \|v_i - \bar{v}\|^2}_{\text{Reach consensus}}$$



Relation between time of inactivation and $R := \frac{1}{N}\sum_{i=1}^{N} \|v_i - \bar{v}\|^2 / \|\bar{v} - V\|^2$

Thank you for your attention!