ON THE NUMERICAL SOLUTION OF A NONLINEAR EIGENVALUE PROBLEM FOR THE MONGE-AMPÈRE OPERATOR

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Dédié à François Murat pour ses contributions à l'analyse mathématique des équations elliptiques non linéaires dont

$$-\nabla^2 u = \lambda e^u$$

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0. MOTIVATION

In the early 2000, while looking for ***PDE's from Geometry***, I came across an article of ***Alice Chang*** discussing the ***Laplace-Beltrami-Bratu-Gelfand*** problem

(LBBG)
$$\Delta_\Sigma u + \lambda e^u = 0 \text{ on } \Sigma$$

where Σ is a surface of $\mathbf{R}^d$ and Δ_Σ the associated ***Laplace-Beltrami operator.*** At the time I was more interested in the ***Monge-Ampère equation***

$$\det \mathbf{D}^2 u = f(>0) \text{ in } \Omega$$

where Ω is a ***bounded convex domain*** of $\mathbf{R}^d$. For some strange reason, **(LBBG)** became in my mind

$$\det \mathbf{D}^2 u = \lambda e^u \text{ in } \Omega$$

explaining what's follows (when I realized my mistake it was too late).

1. FORMULATION OF THE EIGENVALUE PROBLEM

Assuming that Ω is a ***bounded convex*** domain of $\mathbf{R}^2$, our goal is to solve ***numerically*** the following ***nonlinear eigenvalue problem***: (**Monge-Ampère-Bratu-Gelfand** problem)

(MABG)

$$\begin{cases} \text{Find } u \ \textbf{\textit{convex}} \text{ and } \lambda > 0 \text{ such that} \\ \begin{cases} \det \mathbf{D}^2 u = \lambda e^{-u} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \\ \int_\Omega (e^{-u} - 1) d\mathbf{x} = C(> 0). \end{cases}$$

Above $d\mathbf{x} = dx_1 dx_2$.

REMARK 1. The ***convexity*** of Ω and ***u***, and the condition $u = 0$ **on** $\partial\Omega \Rightarrow u < 0$ **in** Ω.

REMARK 2. **P.L. Lions**, ***Annali di Matematica Pura ed Applicada,*** **142**(1), 1985 contains mathematical results associated with a closely related nonlinear eigenvalue problem (λu^2 instead of λe^{-u}).

REMARK 3. Suppose that Ω is the ***unit disk*** centered at **(0, 0)**. Looking for ***radial solutions*** to **(MABG)** we solve

(1)
$$\begin{cases} u \le 0, \lambda \ge 0, \\ u'u'' = \lambda r e^{-u} \quad \text{on} \quad (0,1), \\ u(1) = 0, u'(0) = 0, \end{cases}$$

(by a ***shooting method*** for example). The related ***bifurcation diagram*** has been visualized on **Figure 1**, below, showing a ***turning point*** at $\lambda \approx 3.7617$, the associated function ***u*** taking its ***minimal value*** at **(0, 0)** with $u(0,0) \approx -2.5950$.

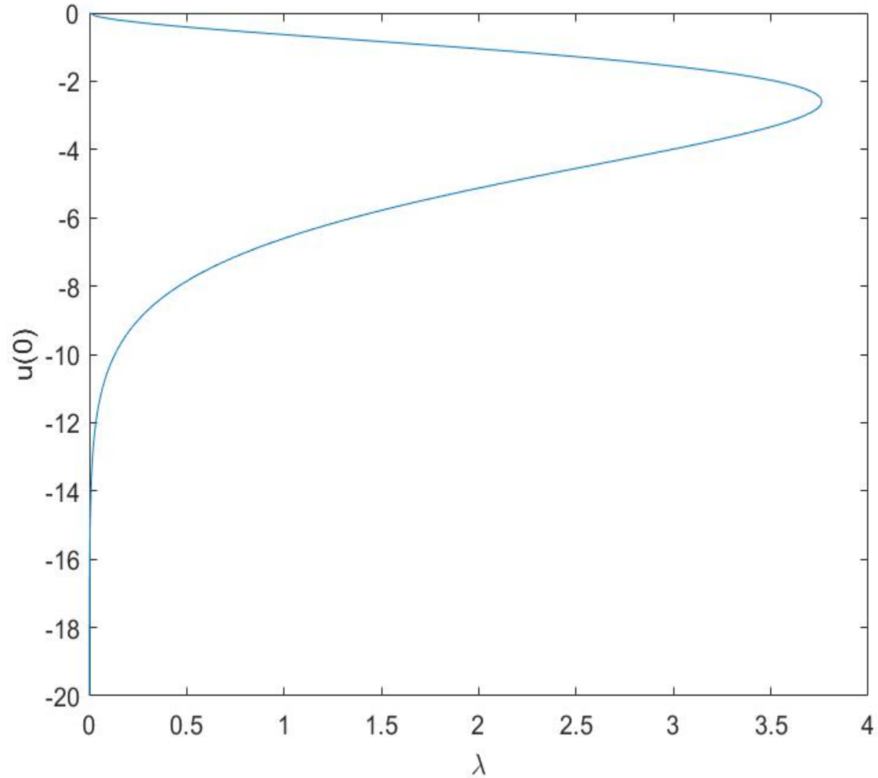


Figure 1: **MABG problem on the unit disk: Exact solution bifurcation diagram**

2. DIVERGENCE FORMULATIONS OF (MABG)

An alternative formulation to **(MABG)** is given by

(MABG.DIV.1)

$$\begin{cases} \text{Find } u \ \textbf{\textit{convex}} \text{ and } \lambda > 0 \text{ such that} \\ \begin{cases} -\nabla.(\text{cof } \mathbf{D}^2 u)\nabla u = -2\lambda e^{-u} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \\ \int_\Omega (e^{-u} - 1)dx = C. \end{cases}$$

In order to take advantage (via ***operator-splitting***) of the methodology developed in

R. G., H. Liu, S. Leung & J. Qian, ***J. Scient. Comp.***, **2019**, for the numerical solution of the ***canonical elliptic Monge-Ampère equation***

(E-MA) $\det \mathbf{D}^2 u = f\ (> 0) \text{ in } \Omega,\ u = g \text{ on } \partial\Omega,$

we reformulate **(MABG.DIV1)** as

$$
\textbf{(MABG.DIV.2)} \quad \begin{cases} \text{Find } u \textbf{\textit{ convex}}, \mathbf{p} \textbf{\textit{ symmetric positive semi-definite}} \text{ and } \lambda > 0 \text{ s.t.} \\ \begin{cases} -\nabla.(\operatorname{cof}\mathbf{p})\nabla u = -2\lambda e^{-u} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \\ \mathbf{p} - \mathbf{D}^2 u = \mathbf{0}, \\ \int_\Omega (e^{-u} - 1)\,d\mathbf{x} = C. \end{cases}
$$

3. AN ASSOCIATED INITIAL VALUE PROBLEM

We are going to associate with **(MABG.DIV.2)** an ***initial value problem*** whose ***steady state solutions*** solve **(MABG).** After ***time-discretization***, the resulting algorithm can be viewed as a nonlinear variant of the ***inverse power method with shift*** (the shift being here associated with the operator $\mathbf{I}/\tau$, τ being a ***time-discretization step***).

(MABG.DIV.2) ⟶ (MABG.IVP)

Find $u(t) \leq 0$, $\mathbf{p}(t)$ **SPSD** (point wise), $\lambda(t) > 0$ so that :

(MABG.IVP)

$$
\begin{cases}
\begin{cases}
\dfrac{\partial u}{\partial t} - \nabla . [\varepsilon \mathbf{I} + \operatorname{cof} \mathbf{p}] \nabla u = -2\lambda e^{-u} \text{ in } \Omega \times (0,+\infty), \\
u = 0 \text{ on } \partial\Omega \times (0,+\infty),
\end{cases} \\
\dfrac{\partial \mathbf{p}}{\partial t} + \gamma(\mathbf{p} - \mathbf{D}^2 u) = \mathbf{0} \text{ in } \Omega \times (0,+\infty), \\
\int_\Omega (e^{-u} - 1) d\mathbf{x} = 1, \forall t > 0, \\
(u(0), \mathbf{p}(0)) = (u_0, \mathbf{p}_0).
\end{cases}
$$

Above: **(i)** $\varphi(t): x \to \varphi(x, t)$. **(ii)** $\varepsilon > 0$ **(**$\varepsilon \approx h^2$ **in practice)**. **(iii)** $u_0 \leq 0$. **(iv)** $\gamma > 0$. **(v)** $\mathbf{p}_0$ **SPSD (pointwise)**.

4. TIME-DISCRETIZATION BY OPERATOR-SPLITTING OF (MABG.IVP)

With $\tau\,(>0)$ a ***time-discretization step*** (***fixed*** for simplicity) we approximate **(MABG.IVP)** by:

(0) $$(u^0, \mathbf{p}^0) = (u_0, \mathbf{p}_0).$$

For $n \geq 0$, $(u^n, \mathrm{p}^n) \rightarrow u^{n+1/3} \rightarrow (u^{n+2/3}, \mathrm{p}^{n+1}) \rightarrow u^{n+1}$ as follows:

First Step: **Solve the following (*well-posed*) *linear elliptic problem***

(1) $$\begin{cases} u^{n+1/3} - \tau\nabla.\left[\varepsilon\mathbf{I} + \operatorname{cof}\mathbf{p}^n\right]\nabla u^{n+1/3} = u^n \text{ in } \Omega, \\ u^{n+1/3} = 0 \text{ on } \partial\Omega. \end{cases}$$

Second Step:

(2)$_1$ $$\mathbf{p}^{n+1}(\mathbf{x}) = P_+\left[e^{-\gamma\tau}\mathbf{p}^n(\mathbf{x}) + (1 - e^{-\gamma\tau})\mathbf{D}^2 u^{n+1/3}(\mathbf{x})\right], \text{a.e. } \mathbf{x} \in \Omega,$$

(2)$_2$ $$\begin{cases} u^{n+2/3} - u^{n+1/3} = -2\tau\lambda^{n+1} e^{-u^{n+2/3}}, \\ \int_\Omega (e^{-u^{n+2/3}} - 1)\,d\mathbf{x} = C (\Leftrightarrow u^{n+2/3} \in S_C = \{\varphi \text{ measurable}, \int_\Omega (e^{-\varphi} - 1)\,d\mathbf{x} = C\}). \end{cases}$$

Third Step:

(3) $$u^{n+1} = \inf(0, u^{n+2/3}).$$

Above:

♦ Problem **(1)** is a very classical ***linear elliptic problem***. It has the following property:

$$u^n \leq 0 \Rightarrow u^{n+1/3} \leq 0.$$

♦ P_+ is an operator, mapping the space of the **2 × 2** ***real symmetric matrices*** onto the ***closed convex cone of the real SPSD*** **2 × 2** ***matrices*** (if **q** is a **2 × 2** ***real symmetric matrix*** with ***eigenvalues*** μ_1 and μ_2, **∃ S ∈ *O*(2)** s.t. $\mathbf{q} = \mathbf{S}\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}\mathbf{S}^{-1};$ then, operator P_+ is defined by

$$P_+(\mathbf{q}) = \mathbf{S}\begin{pmatrix} \max(0,\mu_1) & 0 \\ 0 & \max(0,\mu_2) \end{pmatrix}\mathbf{S}^{-1}).$$

♦ We consider system **(2)**$_2$ as an ***optimality system*** associated with the following ***minimization problem***

(MIN)
$$u^{n+2/3} = \arg\min_{v \in S_C} \left[\frac{1}{2} \int_\Omega |v|^2 d\mathbf{x} - \int_\Omega u^{n+1/3} v \, d\mathbf{x} \right] .$$

It follows from **(MIN)** that $u^{n+2/3}$ is the ***L^2 - projection*** of $u^{n+1/3}$ onto S_C. Unless $u^{n+1/3} \in S_C$ problem **(MINP)** may have no solution since S_C ***is not weakly closed*** in $L^2(\Omega)$. ***Fortunately its***

discrete analogues have solutions.

♦**Algorithm (0) – (3)** has clearly the flavor of an ***inverse power method*** (with ***truncation***).

♦ ***Step 3*** has been included to be ***on the safe side***. Numerical experiments suggest that if ***algorithm* (0) – (3)** ***is properly initialized,*** **(3)** ***is useless***.

Two important remarks are in order:

Remark 4: ***Algorithm*** **(0) – (3)** 'enjoys' a ***splitting error*** forcing us to use a ***small time – discretization step*** $\boldsymbol{\tau}$.

Remark 5: There is no need to compute $\boldsymbol{\lambda^{n+1}}$ at each time step. Indeed,

$$\frac{u^{n+2/3}-u^{n+1/3}}{\tau} = -2\lambda^{n+1}e^{-u^{n+2/3}} \Rightarrow (\text{since } u^{n+2/3}\in S_C)\lambda^{n+1} = \frac{\int_\Omega (u^{n+1/3}-u^{n+2/3})d\mathbf{x}}{2\tau(C+|\Omega|)},$$

i.e., $\boldsymbol{\lambda^{n+1}}$ is obtained by the ***ratio of two small numbers***. It is safer **(?)** to proceed as follows:

Denote by $(\boldsymbol{u_\tau}, \mathbf{p}_\tau)$ the limit of $(\boldsymbol{u^{n+1/3}}, \mathbf{p}^n)_n$. It makes sense to approximate the (nonlinear) eigenvalue $\boldsymbol{\lambda}$ by

$$\lambda_\tau = -\frac{\int_\Omega (\varepsilon\mathbf{I}+\operatorname{cof}\mathbf{p}_\tau)\nabla u_\tau \cdot \nabla u_\tau d\mathbf{x}}{2\int_\Omega u_\tau e^{-u_\tau} d\mathbf{x}}$$

a (kind of) ***generalized Rayleigh quotient***.

5 . FINITE ELEMENT APPROXIMATION OF (MABG)

Assuming that Ω is a ***bounded convex polygonal domain*** of $\mathbf{R}^2$ (or has been approximated by a family of such domains) we introduce a family $(\mathcal{T}_h)_h$ of triangulations of Ω like those in **Figure 2** (*h* is, typically, the ***length of the largest edge(s)*** of $\mathcal{T}_h$). Next, we approximate the functional spaces $H^1(\Omega)$ and $H_0^1(\Omega)$ by

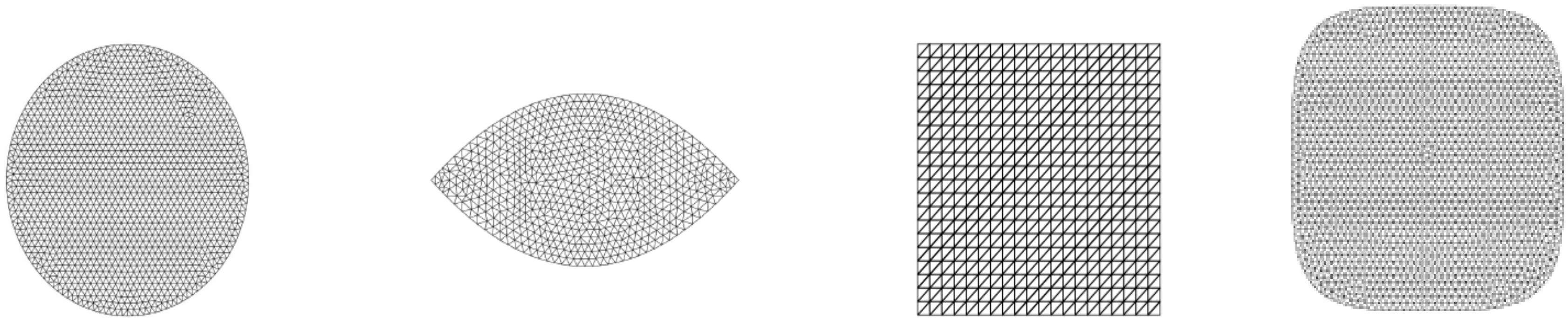
$$V_h = \{\varphi \in C^0(\overline{\Omega}), \varphi|_T \in P_1, \forall T \in \mathcal{T}_h\},$$

and

$$V_{0h} = \{\varphi \in V_h, \varphi|_{\partial\Omega} = 0\} (= V_h \cap H_0^1(\Omega)),$$

respectively, P_1 being the space of the ***polynomial functions of two variables*** of **degree ≤ 1**.

Figure 2 ***Some triangulations***



(a) (b) (c)

Let us denote by Σ_h (resp., Σ_{0h}) the set of the ***vertices*** of $\mathcal{T}_h$ (resp., the set $\Sigma_h \setminus \Sigma_h \cap \partial\Omega$).
We have then

$$\mathbf{dim}\ V_h = \mathbf{Card}\ \Sigma_h\ (: = N_h) \text{ and } \mathbf{dim}\ V_{0h} = \mathbf{Card}\ \Sigma_{0h}\ (: = N_{0h}).$$

We assume that the vertices of $\mathcal{T}_h$ have been numbered so that $\Sigma_{0h} = \{Q_k\}_{k=1}^{N_{0h}}$.
For $k = 1, \ldots, N_{0h}$, we define ω_k as the union of those triangles of $\mathcal{T}_h$ which have Q_k as a common vertex. We denote by $|\omega_k|$ the ***measure*** (**area**) of ω_k.

Approximating the ***Monge-Ampère part*** of the ***splitting scheme*** **(0) – (3)** is a (boring) and time

consuming repetition of **RG-HL-TL & JQ, *J. Scient. Comp.*, 2019.** Focusing on the ***eigenvalue*** part, we approximate S_C by S_{Ch} defined (***trapezoidal rule***) by:

$$S_{Ch} = \{\varphi \in V_{0h}, \sum_{k=1}^{N_{0h}} |\omega_k| (e^{-\varphi(Q_k)} - 1) = 3C\}.$$

The ***discrete analogue*** of the ***L^2 – projection*** onto ***S_C*** is done by a ***SQP*** method very easy to implement, showing (as expected) fast convergence properties.

Remark 6: We can take advantage of the fact that τ is ***small*** by replacing **(MIN)** by **(MIN.LIN)** obtained by ***linearization***

(MIN.LIN) $$u^{n+2/3} = \arg\min_{v \in DS_C^{n+1/3}} \left[\frac{1}{2} \int_\Omega |v|^2 d\mathbf{x} - \int_\Omega u^{n+1/3} v \, d\mathbf{x} \right],$$

where

$$DS_C^{n+1/3} = \{ v \,|\, \int_\Omega e^{-u^{n+1/3}} (1 + u^{n+1/3} - v) d\mathbf{x} = C + |\Omega| \}.$$

The ***closed form solution*** of **(MJN.LIN)** is given by

$$u^{n+2/3} = u^{n+1/3} + \frac{\int_\Omega e^{-u^{n+1/3}} d\mathbf{x} - (C + |\Omega|)}{\int_\Omega e^{-2u^{n+1/3}} d\mathbf{x}} e^{-u^{n+1/3}}$$

The above function coincides with the **1**$^{\text{st}}$ ***iterate*** of the ***SQP method*** initialized

by $u^{n+1/3}$.

6. NUMERICAL RESULTS

6.1. TEST PROBLEM FOR THE UNIT DISK

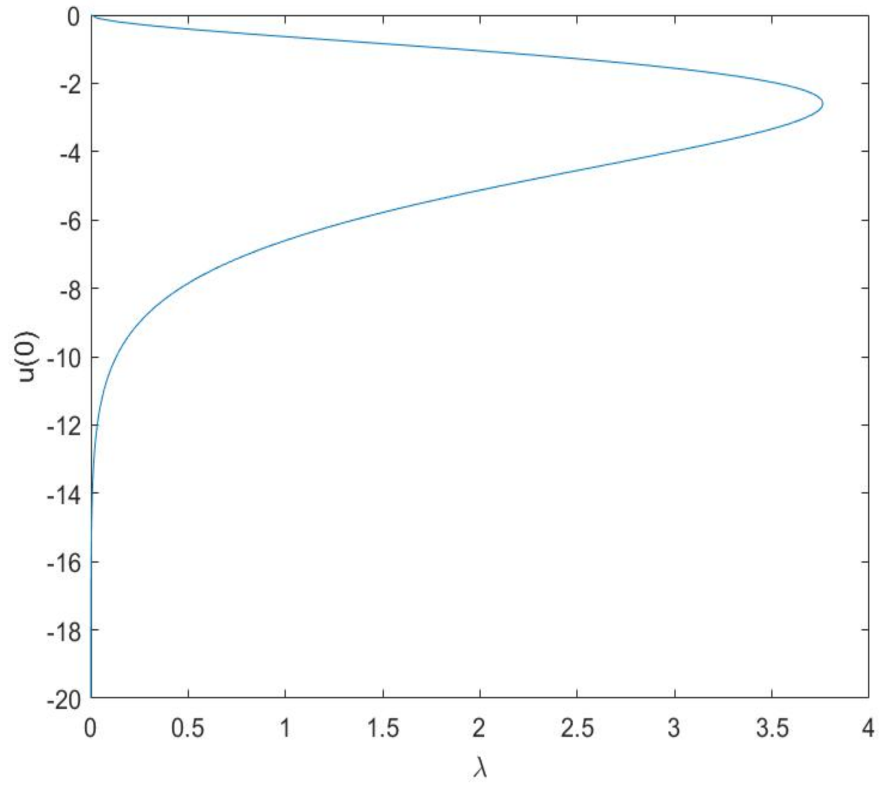


Figure 3: **MABG problem on the unit disk**: **Exact solution bifurcation diagram**

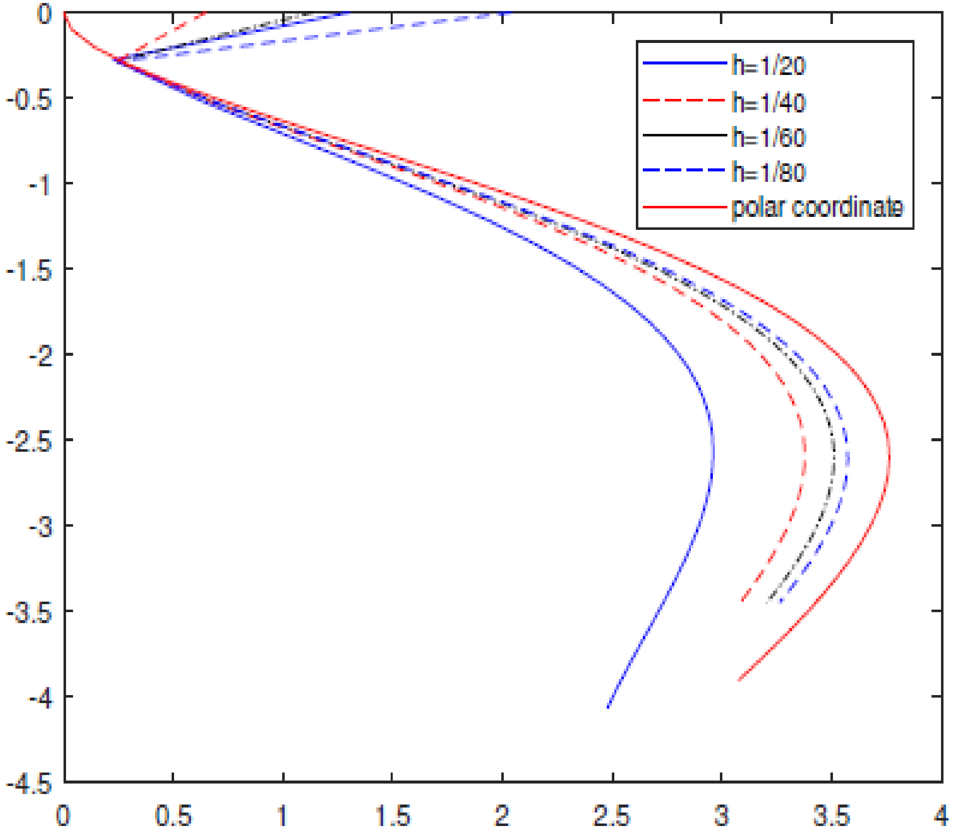


Figure 4: ***Bifurcation diagram*** **comparison for the** ***unit disk***

The ***discrepancy*** shown on **Figure 4** for ***small values*** of ***C*** is the result of an ***erroneous initialization***. ***This mistake has been corrected***. The reason we exhibit these partially wrong results is to show the ***robustness*** of our methodology. Indeed, without human intervention, our method ***'returns quickly by itself'*** on the ***correct*** approximate bifurcation diagrams as ***C*** increases.

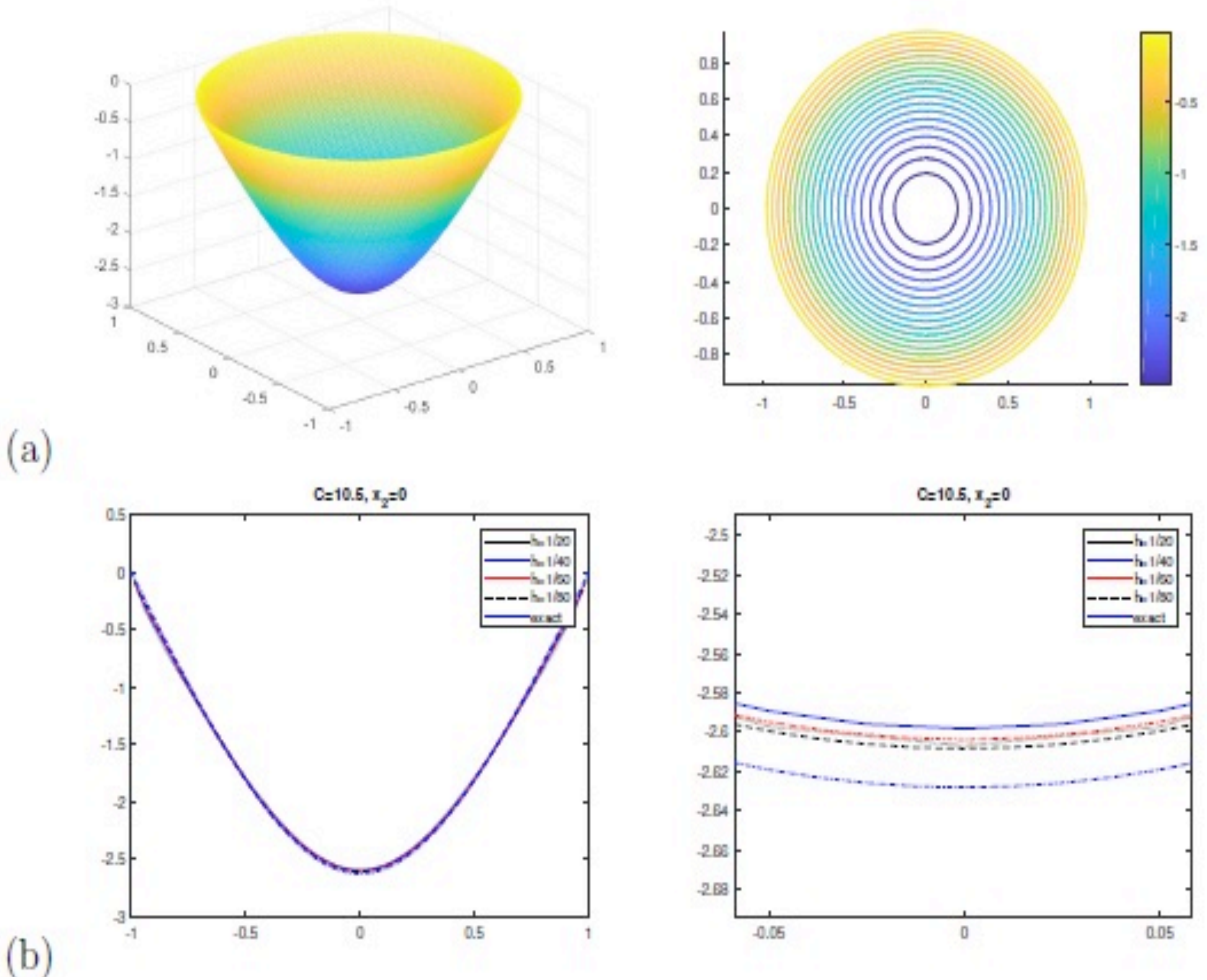


Figure 5.

Results at the *exact turning point* (*C* = 10.5)

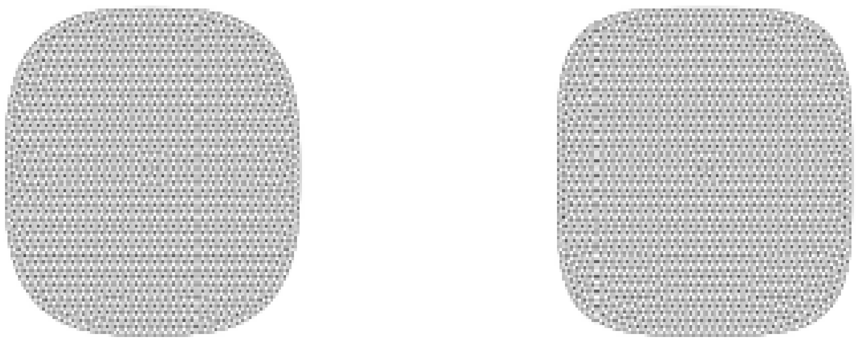
h	$u(0, 0)$	λ
1/20	– 2.5411	2.95
1/40	– 2.5983	3.3773
1/60	– 2.6038	3.5102
1/80	– 2.6084	3.5723
0([1])	– 2.5950	3.7617

Table 3. *MABG problem*: *Variation with h of the computed turning points*

([1]) Exact solution

$\lambda_h - \lambda_0$ = ***quasi-textbook*** **O(*h*)**

6.2. TEST PROBLEMS FOR REGULARIZED SQUARES



(a) (b)

Mesh for $|(x_1 - 1/2)^q| + |(x_2 - 1/2)^q| \leq (1/2)^q$ with $h = 1/40$: (a) q=3. (b) ***q* = 4**

FIGURE 6

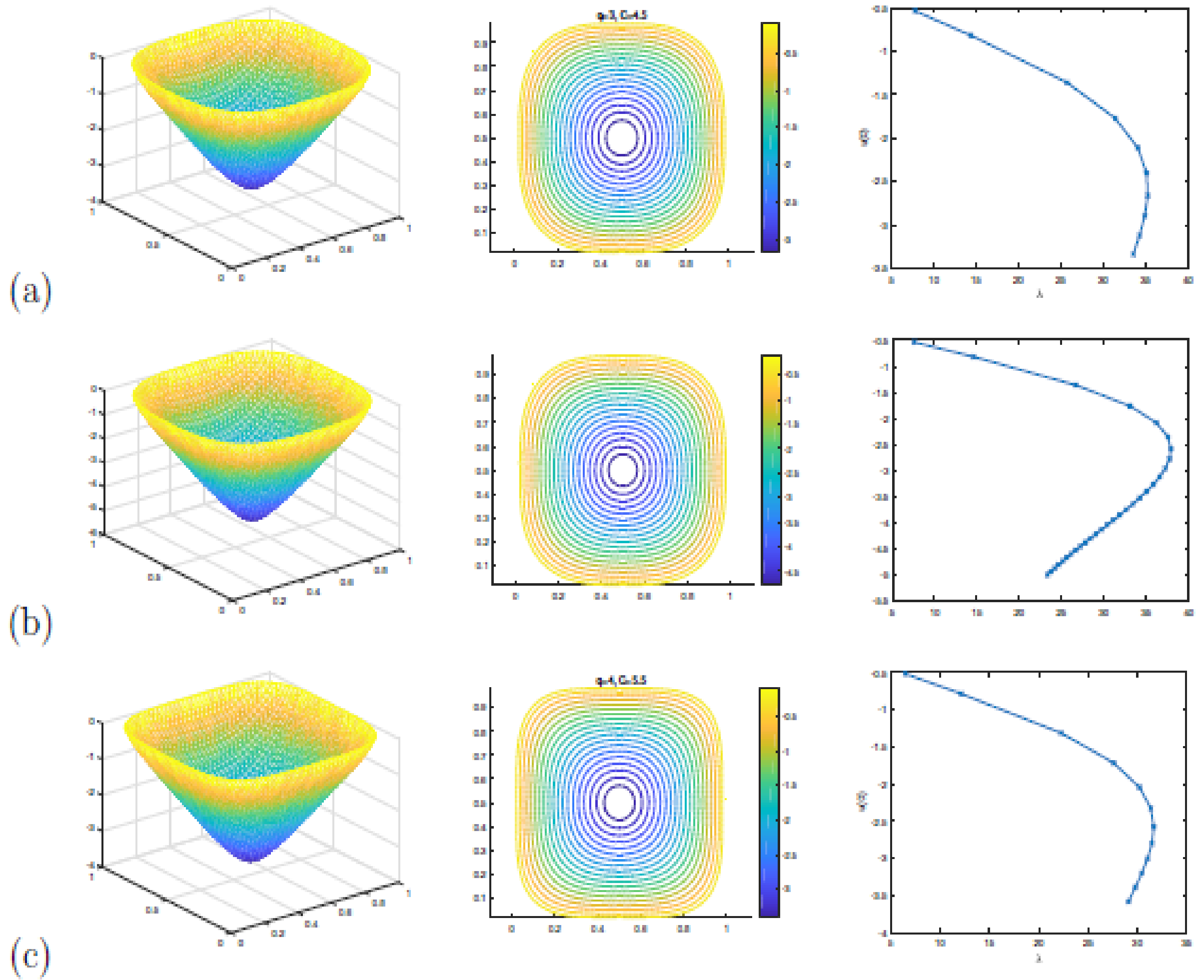


Figure 5: Graph of solution, contour of solution and bifurcation diagram. (a) $q = 3, \Delta t = h^2/2$, with $C = 4.5$. (b) $q = 3, \Delta t = h^2/8$ with $C = 15$.(c) $q = 4, \Delta t = h^2/2$ with $C = 5.5$.

« il faut essayer, vas-y »(1)

CEDRIC VILLANI

DEA d' ANALYSE NUMERIQUE (Université P. & M. Curie)

Fields Medal 2010

MERCI POUR VOTRE ATTENTION

(1) " One has to try, let's go"