VARIATIONS ON A LEMMA OF JACQUES-LOUIS LIONS

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Table of Contents

2. AN EQUIVALENCE THEOREM

2.1 Classical J.L. Lions lemma $\Rightarrow$ J. Nečas inequality

2.2 J. Nečas inequality $\Rightarrow$ ***grad*** has closed range

2.3 ***grad*** has closed range $\Rightarrow$ de Rham theorem in $\boldsymbol{H}^1(\Omega)$

2.4 de Rham theorem in $\boldsymbol{H}^1(\Omega)$ $\Rightarrow$ div is onto

2.5 div is onto $\Rightarrow$ "approximation lemma"

2.6 "approximation lemma" $\Rightarrow$ J.L. Lions lemma

2.7 An equivalence theorem

2.8 Two proofs of J.L. Lions lemma

3. TWO FURTHER EQUIVALENCES

3.1 J.L. Lions lemma $\Leftrightarrow$ weak Poincaré lemma

3.2 J.L. Lions lemma $\Leftrightarrow$ de Rham theorem in $\boldsymbol{H}^{-1}(\Omega)$ $\Leftrightarrow$ "refined" de Rham theorem in $\boldsymbol{H}^{-1}(\Omega)$

4. EXTENSIONS

4.1 Vector version of J.L. Lions lemma

4.2 J.L. Lions lemma in $W^{-m,p}(\Omega)$

1. J.L. LIONS LEMMA

1.1 THE CLASSICAL J.L. LIONS LEMMA

LEMMA Ω: *open in* $\mathbb{R}^N$; $H^{-1}(\Omega)$: *dual of* $H_0^1(\Omega)$

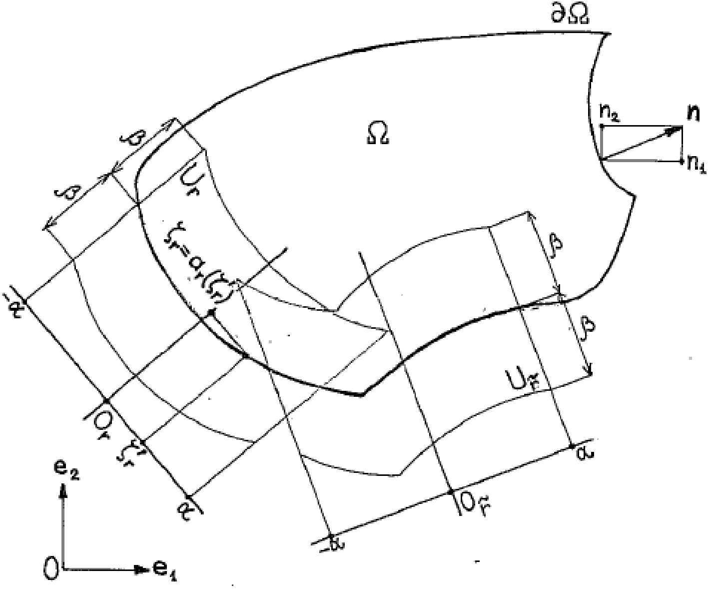
$$f \in L^2(\Omega) \Rightarrow f \in H^{-1}(\Omega) \text{ and } \textbf{grad}\, f := (\partial_i f)_{i=1}^N \in \boldsymbol{H}^{-1}(\Omega).$$

Proof $\langle T, \varphi \rangle := {}_{\mathcal{D}'(\Omega)}\langle T, \varphi \rangle_{\mathcal{D}(\Omega)}$ for any $T \in \mathcal{D}'(\Omega)$ and any $\varphi \in \mathcal{D}(\Omega)$
$f \in L^2(\Omega) \Rightarrow \langle f, \varphi \rangle = \int_\Omega f \varphi \mathrm{dx}$ and $\langle \partial_i f, \varphi \rangle = -\langle f, \partial_i \varphi \rangle = -\int_\Omega f \partial_i \varphi \mathrm{dx}$ for any $\varphi \in \mathcal{D}(\Omega)$. Therefore, for any $\varphi \in \mathcal{D}(\Omega)$,

$$|\langle f, \varphi \rangle| \leq \|f\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)},$$
$$|\langle \partial_i f, \varphi \rangle| \leq \|f\|_{L^2(\Omega)} \|\partial_i \varphi\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)}. \quad \square$$

Ω: *domain* in $\mathbb{R}^N$: bounded, connected, open subset of $\mathbb{R}^N$ such that $\partial\Omega$ is Lipschitz-continuous and Ω is locally on the same side of $\partial\Omega$.

The *classical J.L. Lions lemma* asserts that $\Leftarrow$ holds if Ω is a *domain*.


∂Ω
Ω
n
n₂
n₁
β
β
β
β
U_r
ζ_r=a_r(ζ_r')
−α
O_r
ζ_r'
α
U_r̃
α
O_r̃
−α
e₂
O
e₁

CLASSICAL J.L. LIONS LEMMA Ω: *domain in* $\mathbb{R}^N$

$$f \in H^{-1}(\Omega) \text{ and } \textbf{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega).$$

E. Magenes & G. Stampacchia [1958]: Footnote[27]

G. Duvaut & J.L. Lions [1972]: English translation: *Inequalities in Mechanics and Physics*, Springer, 1976: First published proof for domains with *smooth* boundaries.

L. Tartar [1978]: Another proof, again for domains with *smooth* boundaries.

Annali della Scuola Normale Superiore di Pisa
Vol. 12 (1958), 247-357

I PROBLEMI AL CONTORNO PER LE EQUAZIONI DIFFERENZIALI DI TIPO ELLITTICO

di ENRICO MAGENES e GUIDO STAMPACCHIA (Genova)

Lo studio dei problemi al contorno per le equazioni differenziali lineari di tipo ellittico di ordine qualunque ha avuto negli ultimi anni uno sviluppo notevole e ancora attualmente sono in corso interessanti ricerche.

Il presente lavoro è un'esposizione, che riteniamo abbastanza completa e generale, delle diverse teorie relative ai problemi in questioni, sviluppata in una serie di seminari all'Istituto matematico dell'Università di Genova.

Abbiamo ritenuto utile pubblicare questo lavoro, sia perchè un'esposizione generale non ci sembra ancora fatta — anche le monografie esistenti, quale ad esempio quella di C. MIRANDA [3] (*), sono quasi esclusivamente dedicate alle equazioni del secondo ordine o a equazioni particolari — sia perchè abbiamo cercato di portare, in alcuni punti, qualche contributo nuovo.

Ci sono state utili le conversazioni avute con i proff. MIRANDA, FICHERA, PRODI; in modo particolare desideriamo ringraziare il prof. LIONS oltre che per i suoi consigli anche per averci dato in visione manoscritti non ancora pubblicati. E siamo anche grati al prof. ARUFFO e ai dott. CAMPANATO e GAGLIARDO per la loro collaborazione.

Genova, Giugno 1958.

(*) I numeri tra [ ] si riferiscono alla bibliografia finale.

ove

$$|\xi'|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \dots + \xi_{n-1}^2 .$$

Possiamo allora concludere che se $u \in C^\infty(\overline{\pi}) \cap K(\pi)$ segue $Pu \in H^1(R^n)$.

Il lemma di LIONS segue immediatamente osservando che se $u \in K(\omega)$, φu con $\varphi \in \mathfrak{D}(R^n, \omega)$ si può considerare come una funzione di $K(\pi)$ e pertanto $\in K(R^n)$. Ciò implica che $u \in H^1(\sigma_r)$ con $r < R$. c. v. d. (27)

c) Prima di terminare questo numero osserviamo che i ragionamenti fin qui svolti ci permettono di assicurare che le formule di maggiorazione trovate alla fine del n. 10 si estendono in modo ovvio.

Ci interessa per il seguito segnalare che la formula (10.27) sussiste anche, in virtù dei risultati ottenuti in questo numero, quando $t \equiv (t_1, t_2, \dots, t_{n-1}, t_n)$ con $|t| \leq m$. Si ha cioè

$$\| \varphi u \|_{2m,\omega} \leq c \left(\| f \|_{0,\omega} + \| u \|_{m,\omega} + ||| g ||| \right) \tag{11.18}$$

ove $||| g |||$ è data dalla (10.28) o dalla (10.30) nelle stesse ipotesi su g ivi fatte e con c indipendente da u.

Si osservi poi che la (10.26) quando sia $\mathcal{V} \equiv H_0^m(\omega)$ vale anche se $s \equiv (s_1, s_2, \dots, s_n)$ con $|s| \leq k$.

(27) È interessante osservare, aprendo una breve parentesi, che il tipo di dimostrazione ora dato permette di risolvere una questione relativa a certi spazi di distribuzioni che si pone abbastanza naturalmente a proposito delle ipotesi F_k) e F_k^*) da noi introdotte nel n. 10: *dato un insieme Ω aperto e limitato di R^n, ogni distribuzione T tale che $D^p T \in H^{-m}(\Omega)$ per $|p| \leq k$, verifica anche la $T \in H^{-m+k}(\Omega)$*? (la reciproca è vera come si è visto nel n. 10.). J. L. Lions ha ottenuto in proposito i seguenti risultati che ci ha gentilmente comunicati: si possono dare esempi di aperti Ω per cui la risposta a tale questione è negativa; essa è però affermativa se la frontiera di Ω è *una varietà di classe $C^{(m+k)}$ cioè $m+k$ — volte differenziabile con continuità*. La dimostrazione di questo risultato si ottiene riportando il problema, mediante una opportuna trasformazione di coordinate, al seguente teorema: Sia π il semispazio di R^n con $x_n > 0$ e sia T una distribuzione su π tale che

$$D_{x'}^s T \in H^{-m}(\pi), \ |s| \leq k, \ \frac{\partial^k T}{\partial x_n^k} \in H^{-m}(\pi)$$

allora $T \in H^{-m+k}(\pi)$. E questo teorema si ottiene proprio con una dimostrazione del tipo di quella ora data per il lemma 11.2.

G. Geymonat & P. Suquet [1986]: First proof for *general domains*; point of departure:

NEČAS INEQUALITY Ω: *domain in* $\mathbb{R}^N$. *There exists* $C_0(\Omega)$ *such that*

$$\|f\|_{L^2(\Omega)} \leq C_0(\Omega) \left(\|f\|_{H^{-1}(\Omega)} + \|\textbf{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)} \right)$$

$$\text{for all } \; f \in L^2(\Omega)$$

J. Nečas [1965]: *Equations aux Dérivées Partielles*, Université de Montréal

1.2 THE "GENERAL" J.L. LIONS LEMMA

J.L. LIONS LEMMA Ω: *domain in* $\mathbb{R}^N$

$$f \in \mathcal{D}'(\Omega) \text{ and } \textbf{grad}\, f = \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega).$$

W. Borchers & H. Sohr [1990]; point of departure:

SURJECTIVITY OF div

Ω: *domain in* $\mathbb{R}^N$

$$\boldsymbol{H}_0^1(\Omega) = \left\{\boldsymbol{v} = (v_i)_{i=1}^N;\ v_i \in H_0^1(\Omega)\right\}.$$

The operator

$$\text{div} : \boldsymbol{H}_0^1(\Omega) \rightarrow L_0^2(\Omega) := \left\{f \in L^2(\Omega); \int_\Omega f\,dx = 0\right\}$$

is onto

O.A. LADYZHENSKAYA [1969]: Surjectivity of div already implicit there, for domains in $\mathbb{R}^3$ with smooth boundaries

M.E. BOGOVSKII [1979]: Constructive proof (see Sect. 2.8)

B. DACOROGNA [2002]: Constructive proof for domains with a smooth boundary

Different proof: C. AMROUCHE & V. GIRAULT [1994]; point of departure: Nečas inequality

Extension to $\boldsymbol{W}^{-1,p}(\Omega)$; $1 < p < \infty$: GEYMONAT & SUQUET [1986]

Extension to $\boldsymbol{W}^{-m,p}(\Omega)$; $m \geq 1$, $1 < p < \infty$: W. BORCHERS & H. SOHR [1990]; C. AMROUCHE & V. GIRAULT [1994]

1.3 A FIRST APPLICATION: KORN'S INEQUALITY

Ω: open subset of $\mathbb{R}^N$

Given $\boldsymbol{v} = (v_i)_{i=1}^N \in \boldsymbol{H}^1(\Omega)$ (e.g., a displacement field with $N = 3$ in elasticity theory), let ($\mathbb{S}^N$: space of $N \times N$ symmetric matrices)

$$\boldsymbol{\nabla}_s \boldsymbol{v} := \frac{1}{2}\left(\boldsymbol{\nabla} \boldsymbol{v}^T + \boldsymbol{\nabla} \boldsymbol{v}\right) = \left(\frac{1}{2}(\partial_i v_j + \partial_j v_i)\right) \in \mathbb{L}^2(\Omega) := L^2(\Omega; \mathbb{S}^N)$$

denote the corresponding **linearized strain tensor**. So:

$$\boldsymbol{\nabla}_s : \boldsymbol{H}^1(\Omega) \to \mathbb{L}^2(\Omega)$$

Then ($\mathbb{A}^N$: space of $N \times N$ antisymmetric matrices)

$$\textbf{\textit{Ker}}\, \boldsymbol{\nabla}_s = \Big\{ \boldsymbol{v} : x \in \Omega \to \boldsymbol{v}(x) = \boldsymbol{b} + \boldsymbol{B} x \in \mathbb{R}^N \text{ for some } \boldsymbol{b} \in \mathbb{R}^N \text{ and } \boldsymbol{B} \in \mathbb{A}^N \Big\}$$

THEOREM: KORN'S INEQUALITY: Ω: *domain in* $\mathbb{R}^n$.

There exists a constant C *such that, for all* $\boldsymbol{v} = (v_i) \in \boldsymbol{H}^1(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} := \Big(\sum_i \|v_i\|^2_{L^2(\Omega)} + \sum_{i,j} \|\partial_j v_i\|^2_{L^2(\Omega)} \Big)^{1/2}$$

$$\leq C \Big(\sum_i \|v_i\|^2_{L^2(\Omega)} + \sum_{i,j} \Big\| \frac{1}{2}(\partial_j v_i + \partial_i v_j) \Big\|^2_{L^2(\Omega)} \Big)^{1/2}$$

So: The $L^2(\Omega)$-norms of $\frac{n(n+1)}{2}$ linear combinations

$$e_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) = (\boldsymbol{\nabla}_s \boldsymbol{v})_{ij}$$

control the $L^2(\Omega)$-norms of n^2 partial derivatives $\partial_j v_i$.

Proof (i) Define

$$\boldsymbol{K}(\Omega) := \{\boldsymbol{v} = (v_i);\; v_i \in L^2(\Omega), e_{ij}(\boldsymbol{v}) \in L^2(\Omega)\} \supset \boldsymbol{H}^1(\Omega)$$

Also, $\boldsymbol{K}(\Omega) \subset \boldsymbol{H}^1(\Omega)$ (again, $\frac{n(n+1)}{2}$ vs. n^2):

$$\boldsymbol{v} = (v_i) \in \boldsymbol{K}(\Omega) \Rightarrow \begin{cases} \partial_k v_i \in H^{-1}(\Omega) \\ \partial_j(\partial_k v_i) = (\partial_j e_{ik}(\boldsymbol{v}) + \partial_k e_{ij}(\boldsymbol{v}) - \partial_i e_{jk}(\boldsymbol{v})) \in H^{-1}(\Omega) \end{cases}$$

Classical J.L. Lions lemma: $\partial_k v_i \in H^{-1}(\Omega)$ and $\partial_j(\partial_k v_i) \in H^{-1}(\Omega) \Rightarrow \partial_k v_i \in L^2(\Omega)$

Therefore $\boxed{\boldsymbol{K}(\Omega) = \boldsymbol{H}^1(\Omega).}$

(ii) Apply *Banach open mapping theorem* to $\boldsymbol{id} : \boldsymbol{H}^1(\Omega) \to \boldsymbol{K}(\Omega) = \boldsymbol{H}^1(\Omega)$. $\square$

Remarks: **(1)** There exist *different* proofs, i.e., that *do not* use J.L. Lions lemma, of the Korn inequality on a domain in $\mathbb{R}^N$:

J. Gobert [1962]: Proof uses *Calderón-Zygmund singular integrals*

P.P. Mosolov & V.P. Mjasnikov [1971]: Proof uses *Cesàro-Volterra path integral formula* and *Calderón-Zygmund singular integrals*

V.A. Kondrat'ev & O.A. Oleinik [1988]: Proof uses integral inequalities with $(\text{dist}(\cdot, \partial\Omega))^2$ as a weight and *hypoellipticity of* Δ.

(2) Using *J.L. Lions lemma* as in the proof of the Korn inequality on a domain in $\mathbb{R}^N$, one can establish a *Korn inequality on a surface* or, more generally, on a *Riemannian manifold*:

M. Bernadou, P.G. Ciarlet & B. Miara [1994]: Surface in $\mathbb{R}^3$ with boundary

S. Mardare [2003]: Compact surface in $\mathbb{R}^3$ without boundary

W. Chen & J. Jost [2002]: Riemannian manifold

1.4 A SECOND APPLICATION: STOKES EQUATIONS

THEOREM Ω: *domain in* $\mathbb{R}^N$; *viscosity* $\nu > 0$. *Given* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, *there exists a unique solution* $(\boldsymbol{u}, \lambda) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ *to the* **Stokes equations**

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \textbf{grad}\, \lambda &= \boldsymbol{h} \text{ in } \boldsymbol{H}^{-1}(\Omega) \\ \operatorname{div} \boldsymbol{u} &= 0 \text{ in } \Omega \\ \boldsymbol{u} &= \boldsymbol{0} \text{ on } \partial\Omega \end{aligned}$$

Principle of proof We will see later (cf. Part 2) that:

Classical J.L. Lions lemma $\Rightarrow$ J. Nečas inequality $\Rightarrow$ $\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ is onto

Then:

$$\left.\begin{array}{l} \operatorname{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega) \text{ is onto} \\ \text{Babuška-Brezzi inf-sup condition} \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \text{existence and uniqueness} \\ \text{for the Stokes equations} \end{array}\right. \qquad \square$$

R. Temam [1977]: *Navier-Stokes Equations*, North-Holland, Amsterdam

V. Girault & P.A. Raviart [1986]: *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin

F. Brezzi & M. Fortin [1991]: *Mixed and Hybrid Finite Element Methods*, Springer, New York

P.G. Ciarlet [2013]: *Linear and Nonlinear Functional Analysis with Applications*, SIAM, Philadelphia

1.5 A THIRD APPLICATION: WEAK POINCARÉ LEMMA

WEAK POINCARÉ LEMMA (P.G. Ciarlet & P. Ciarlet, Jr. [2005]; then simpler proof by S. Kesavan [2005])

Ω: *simply-connected domain in* $\mathbb{R}^N$. *Let* $\boldsymbol{h} = (h_i) \in \boldsymbol{H}^{-1}(\Omega)$ *be such that*

$$\partial_i h_j = \partial_j h_i \text{ in } H^{-2}(\Omega) \Leftrightarrow \textbf{curl}\, \boldsymbol{h} = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-2}(\Omega)$$

Then there exists $p \in L^2(\Omega)$, *unique up to the addition of constants,such that*

$$\partial_i p = h_i \text{ in } H^{-1}(\Omega) \Leftrightarrow \textbf{grad}\, p = \boldsymbol{h} \text{ in } \boldsymbol{H}^{-1}(\Omega)$$

Proof There exists $(\boldsymbol{u}, \lambda) \in \boldsymbol{H}_0^1(\Omega) \times L^2(\Omega)$ such that (*Stokes equations*; cf. Sect. 1.4)

$$-\boldsymbol{\Delta u} + \boldsymbol{grad}\,\lambda = \boldsymbol{h} \text{ in } \boldsymbol{H}^{-1}(\Omega) \text{ and } \operatorname{div} \boldsymbol{u} = 0 \text{ in } L^2(\Omega).$$

Then $\boldsymbol{curl}\, \boldsymbol{h} = \boldsymbol{0} \Rightarrow \Delta(\boldsymbol{curl}\, \boldsymbol{u}) = \boldsymbol{0} \Rightarrow \boldsymbol{curl}\, \boldsymbol{u} \in \mathcal{C}^\infty(\Omega; \mathbb{R}^n)$ (*hypo-ellipticity of* Δ)

$$\Rightarrow \partial_j(\partial_j u_i - \partial_i u_j) = \Delta u_i - \partial_i(\operatorname{div} \boldsymbol{u}) = \Delta u_i \in \mathcal{C}^\infty(\Omega).$$

Consequently,

$$\boldsymbol{\Delta u} \in \mathcal{C}^\infty(\Omega; \mathbb{R}^N) \text{ and } \boldsymbol{curl}\, \boldsymbol{\Delta u} = \Delta(\boldsymbol{curl}\, \boldsymbol{u}) = \boldsymbol{0}$$

Hence there exists $\widetilde{\lambda} \in \mathcal{C}^\infty(\Omega) \subset \mathcal{D}'(\Omega)$ such that

$$\boldsymbol{grad}\, \widetilde{\lambda} = \boldsymbol{\Delta u} = \boldsymbol{grad}\, \lambda - \boldsymbol{h}$$

by the *classical Poincaré lemma* (this is where the assumption that Ω is simply-connected is used). Then

$$p := \lambda - \widetilde{\lambda} \in \mathcal{D}'(\Omega) \text{ satisfies } \boldsymbol{grad}\, p = \boldsymbol{grad}\, \lambda - \boldsymbol{grad}\, \widetilde{\lambda} = \boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega),$$

and *J.L. Lions lemma* implies that $p \in L^2(\Omega)$. □

1.6 A FOURTH APPLICATION: WEAK SAINT-VENANT LEMMA

WEAK SAINT-VENANT LEMMA (P.G. Ciarlet & P. Ciarlet, Jr., *M3AS* [2005])

Ω: *simply-connected domain in* $\mathbb{R}^N$. *Let* $(e_{ij}) \in \mathbb{L}^2(\Omega) = L^2(\Omega; \mathbb{S}^N)$ *be such that the following* **SAINT-VENANT COMPATIBILITY CONDITIONS** *are satisfied*:

$$\partial_{\ell j} e_{ik} + \partial_{ki} e_{j\ell} - \partial_{\ell i} e_{jk} - \partial_{kj} e_{i\ell} = 0 \text{ in } H^{-2}(\Omega).$$

Then there exists $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, *unique up to the addition of a vector field in* $\boldsymbol{Ker}\, \boldsymbol{\nabla}_s$ (*equivalently, there exists a unique* $\dot{\boldsymbol{v}} \in \dot{\boldsymbol{H}}^1(\Omega) = \boldsymbol{H}^1(\Omega)/\boldsymbol{Ker}\,\boldsymbol{\nabla}_s$), *such that*

$$(\boldsymbol{\nabla}_s \boldsymbol{v})_{ij} := \frac{1}{2}(\partial_j v_i + \partial_i v_j) = e_{ij} \text{ in } L^2(\Omega)$$

Proof Same as for the "classical" Saint-Venant lemma:

$$(e_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^N) \Rightarrow \boldsymbol{v} \in \mathcal{C}^3(\Omega; \mathbb{R}^N),$$

but with the "classical" Poincaré lemma replaced by the weak Poincaré lemma. □

2. AN EQUIVALENCE THEOREM

C. Amrouche, P.G. Ciarlet & C. Mardare: *JMPA* **104** (2015), 207–226.

Ω: *domain in* $\mathbb{R}^N$

$C(\Omega), C_0(\Omega), C_1(\Omega), \ldots$ designate various constants only dependent on Ω

Proofs of (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e): see also P.G. Ciarlet [2013]: *Linear and Nonlinear Functional Analysis with Applications*, SIAM.

2.1 CLASSICAL J.L. LIONS LEMMA $\Rightarrow$ J. NEČAS INEQUALITY

(a) **Classical J.L. Lions lemma**:
$f \in H^{-1}(\Omega)$ and $\textbf{\textit{grad}}\, f \in \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega)$

implies

(b) **J. Nečas inequality**:
$\|f\|_{L^2(\Omega)} \leq C_0(\Omega) \left(\|f\|_{H^{-1}(\Omega)} + \|\textbf{\textit{grad}}\, f\|_{\boldsymbol{H}^{-1}(\Omega)} \right)$ for all $f \in L^2(\Omega)$

Sketch of proof The space

$$V(\Omega) := \left\{ f \in H^{-1}(\Omega);\ \textbf{\textit{grad}}\, f \in \boldsymbol{H}^{-1}(\Omega) \right\},$$

equipped with the norm

$$f \in V(\Omega) \to \left(\|f\|_{H^{-1}(\Omega)} + \|\textbf{\textit{grad}}\, f\|_{\boldsymbol{H}^{-1}(\Omega)} \right),$$

is *complete*. The canonical injection

$$\iota : L^2(\Omega) \to V(\Omega)$$

is one-to-one, (clearly) continuous, and *onto* by the *classical J.L. Lions lemma*. Therefore, by *Banach open mapping theorem*, ι^{-1} is also continuous. There thus exists a constant $C_0(\Omega)$ such that *J. Nečas inequality holds*:

$$\|f\|_{L^2(\Omega)} \leq C_0(\Omega) \left(\|f\|_{H^{-1}(\Omega)} + \|\textbf{\textit{grad}}\, f\|_{\boldsymbol{H}^{-1}(\Omega)} \right) \text{ for all } f \in L^2(\Omega) \qquad \square$$

2.2 J. NEČAS INEQUALITY $\Rightarrow$ *grad* HAS CLOSED RANGE

(b) **J. Nečas inequality**:

$$\|f\|_{L^2(\Omega)} \leq C_0(\Omega)\left(\|f\|_{H^{-1}(\Omega)} + \|\boldsymbol{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)}\right) \text{ for all } f \in L^2(\Omega)$$

implies

(c) $\boldsymbol{grad} : L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ has closed range

Sketch of proof To show that $\textbf{\textit{grad}} : L_0^2(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ *has closed range*, it suffices to show that

$$\|f\|_{L^2(\Omega)} \leq C(\Omega)\, \|\textbf{\textit{grad}}\, f\|_{\boldsymbol{H}^{-1}(\Omega)} \text{ for all } f \in L_0^2(\Omega).$$

If not, there exists $(f_k)_{k=1}^{\infty}$ with $f_k \in L_0^2(\Omega)$ such that

$$\|f_k\|_{L^2(\Omega)} = 1 \text{ for all } k, \text{ and } \|\textbf{\textit{grad}}\, f_k\|_{\boldsymbol{H}^{-1}(\Omega)} \to 0 \text{ as } k \to \infty.$$

Hence a subsequence $(f_\ell)_{\ell=1}^{\infty}$ converges in $H^{-1}(\Omega)$ (the canonical injection from $L^2(\Omega)$ into $H^{-1}(\Omega)$ is compact) and thus $(f_\ell)_{\ell=1}^{\infty}$ is a Cauchy sequence for the norm

$$f \in L^2(\Omega) \to \|f\|_{H^{-1}(\Omega)} + \|\textbf{\textit{grad}}\, f\|_{\boldsymbol{H}^{-1}(\Omega)}.$$

By *Nečas inequality*, $(f_\ell)_{\ell=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$. So

$$f_\ell \to f \text{ in } L^2(\Omega) \text{ as } \ell \to \infty.$$

Since the mapping $f \in L^2(\Omega) \to \textbf{\textit{grad}}\, f \in \boldsymbol{H}^{-1}(\Omega)$ is continuous,

$$\textbf{\textit{grad}}\, f_\ell \to \textbf{\textit{grad}}\, f = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega) \text{ as } \ell \to \infty.$$

So $f = 0$ since $f \in L_0^2(\Omega)$, in contradiction with $\|f_\ell\|_{L^2(\Omega)} = 1$ for all ℓ. □

2.3 *grad* HAS CLOSED RANGE $\Rightarrow$ de RHAM THEOREM IN $H^{-1}(\Omega)$

(c) $\textbf{\textit{grad}} : L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ has closed range

implies

(d) **de Rham theorem in $H^{-1}(\Omega)$**: *Given* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, *there exists* $p \in L^2_0(\Omega)$ *such that* $\textbf{\textit{grad}}\, p = \boldsymbol{h}$ *in* $\boldsymbol{H}^{-1}(\Omega)$ *if* (and clearly only if) ${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{H}^1_0(\Omega)} = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$ that satisfy $\operatorname{div} \boldsymbol{v} = 0$ in Ω

Proof By definition of $\textbf{\textit{grad}}\, f$ for $f \in L^2_0(\Omega)$,

$${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \textbf{\textit{grad}}\, f, \boldsymbol{v}\rangle_{\boldsymbol{H}^1_0(\Omega)} = -\int_\Omega f \operatorname{div} \boldsymbol{v} dx \text{ for all } f \in L^2_0(\Omega) \text{ and all } \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$$

Hence $\textbf{\textit{grad}} : L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ *is the dual of* $-\operatorname{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega)$.

Therefore, by *Banach closed range theorem*:

$$\operatorname{Im} \textbf{\textit{grad}} = (\operatorname{Ker}(-\operatorname{div}))^0 = \Big\{\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega);\ {}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{H}^1_0(\Omega)} = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega) \text{ that satisfy } \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\Big\}. \qquad \square$$

2.4 de RHAM THEOREM IN $H^{-1}(\Omega) \Rightarrow$ div IS ONTO

(d) **de Rham theorem in $H^{-1}(\Omega)$**: *Given* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, *there exists* $p \in L^2_0(\Omega)$ *such that* $\textbf{\textit{grad}}\, p = \boldsymbol{h}$ *in* $\boldsymbol{H}^{-1}(\Omega)$ *if* (and clearly only if) ${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{H}^1_0(\Omega)} = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$ that satisfy $\operatorname{div} \boldsymbol{v} = 0$ in Ω

implies

(e) $\operatorname{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega)$ is onto

Consequently, for each $f \in L^2_0(\Omega)$, *there exists a unique* $\boldsymbol{u}_f \in (\textbf{\textit{Ker}}\operatorname{div})^\perp \subset \boldsymbol{H}^1_0(\Omega)$ *such that*

$$\operatorname{div} \boldsymbol{u}_f = f,$$

and, *by Banach open mapping theorem*,

$$\|\boldsymbol{u}_f\|_{\boldsymbol{H}^1(\Omega)} \leq C_1(\Omega)\, \|f\|_{L^2(\Omega)} \text{ for all } f \in L^2_0(\Omega)$$

Proof Again by *Banach closed range theorem*,

$$\operatorname{Im} \operatorname{div} = (\operatorname{Ker} \boldsymbol{grad})^0$$

and $\operatorname{Ker} \boldsymbol{grad} = \{0\}$ since $\boldsymbol{grad}\, f = \mathbf{0}$ and $f \in L^2_0(\Omega)$ implies $f = 0$. Therefore

$$\operatorname{Im} \operatorname{div} = L^2_0(\Omega).$$

□

2.5 div IS ONTO $\Rightarrow$ "APPROXIMATION LEMMA"

A domain Ω is *starlike with respect to a ball* $B(x;r)$ if, for each $z \in \Omega$,

$$\text{co}(\{z\} \cup B(x;r)) \subset \Omega.$$

(e) $\text{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$ is onto

implies

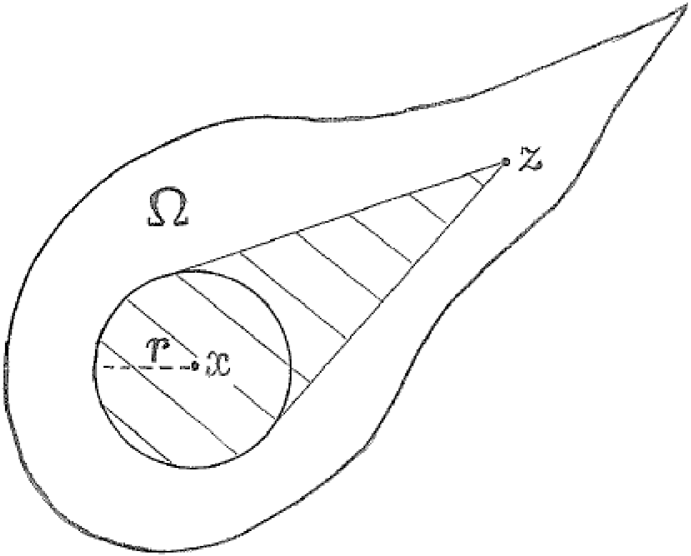
(f) **Approximation lemma**: *Assume the domain* Ω *is* **starlike with respect to a ball**. *Then, given any*

$$\varphi \in \mathcal{D}(\Omega) \text{ such that } \int_\Omega \varphi dx = 0,$$

there exist $\boldsymbol{v}_n = \boldsymbol{v}_n(\varphi) \in \boldsymbol{\mathcal{D}}(\Omega)$, $n \geq 1$, *such that*

$$\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \leq C_2(\Omega) \|\varphi\|_{L^2(\Omega)} \text{ for all } n \geq 1, \text{ and}$$

$$\text{div}\, \boldsymbol{v}_n \to \varphi \text{ in } \mathcal{D}(\Omega) \text{ as } n \to \infty$$



Ω
z
r
x

Sketch of proof Without loss of generality, assume Ω is starlike with respect to a ball $B(0; r)$ centered at the *origin*. Let

$$\mathcal{D}_0(\Omega) := \left\{\varphi \in \mathcal{D}(\Omega); \int_\Omega \varphi \mathrm{d}x = 0\right\} \subset L_0^2(\Omega),$$

and *let* $\varphi \in \mathcal{D}_0(\Omega)$ *be given*.

(i) **Definition of auxiliary fields** $\boldsymbol{u}_n = \boldsymbol{u}_n(\varphi)$. By assumption, there exists a unique $\boldsymbol{u} = \boldsymbol{u}(\varphi) \in (\boldsymbol{Ker}\,\mathrm{div})^\perp \subset \boldsymbol{H}_0^1(\Omega)$ such that

$$\mathrm{div}\,\boldsymbol{u} = \varphi \text{ in } \Omega \text{ and } \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C_1(\Omega)\,\|\varphi\|_{L^2(\Omega)}.$$

Let $\boldsymbol{w} = \boldsymbol{w}(\varphi) := \boldsymbol{u}$ in Ω and $\boldsymbol{w} := \boldsymbol{0}$ in $\mathbb{R}^N - \Omega$, so that

$$\boldsymbol{w} \in \boldsymbol{H}^1(\mathbb{R}^N),\ \|\boldsymbol{w}\|_{\boldsymbol{H}^1(\mathbb{R}^N)} = \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C_1(\Omega)\,\|\varphi\|_{L^2(\Omega)}, \text{ and}$$
$$\mathrm{div}\,\boldsymbol{w} = \varphi \text{ in } \Omega \text{ and } \mathrm{div}\,\boldsymbol{w} = 0 \text{ in } \mathbb{R}^N - \Omega$$

Let $n_0 \geq 1$ be such that $n_0 > \dfrac{2}{r}$, and let, for each $n \geq n_0$,

$$\boxed{\lambda_n := 1 - \frac{2}{nr} \text{ and } \Omega_n := \left\{\lambda_n x \in \mathbb{R}^N;\ x \in \Omega\right\} \subset \Omega.}$$

Because Ω *is starlike with respect to* $B(0; r)$, *Thales theorem gives*:

$$\boxed{\text{for each } n \geq n_0,\ \operatorname{dist}(x, \partial\Omega) > \frac{2}{n} \text{ for all } x \in \Omega_n}$$

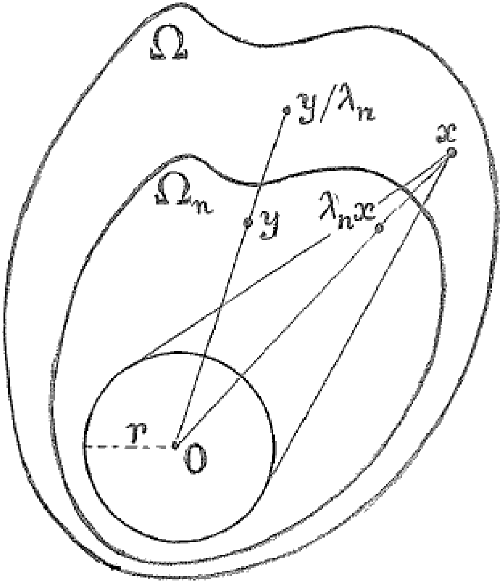
For each $n \geq n_0$, let

$$\boxed{\boldsymbol{u}_n : y \in \mathbb{R}^N \to \boldsymbol{u}_n(y) := \lambda_n \boldsymbol{w}\left(\frac{y}{\lambda_n}\right),}$$

so that, for each $n \geq n_0$,

$$\boxed{\boldsymbol{u}_n \in \boldsymbol{H}^1(\mathbb{R}^N),\ \boldsymbol{u}_n = \boldsymbol{0} \text{ in } \mathbb{R}^N - \Omega_n \text{ and } \operatorname{div} \boldsymbol{u}_n = \varphi\left(\frac{\cdot}{\lambda_n}\right) \text{ in } \mathbb{R}^N}$$

where the same notation φ designates the extension of φ by 0 in $\mathbb{R}^N - \Omega$.


Ω
y/λn
x
Ωn
y
λnx
r
0

(ii) *Definition of the fields* $\boldsymbol{v}_n = \boldsymbol{v}_n(\varphi) \in \boldsymbol{\mathcal{D}}(\Omega)$. Let $(\rho_n)_{n=1}^{\infty}$ be a family of *mollifiers*:

$$\rho_n \in \mathcal{C}^\infty(\mathbb{R}^N),\ \operatorname{supp}\rho_n \subset \overline{B\left(0; \frac{1}{n}\right)}, \rho_n \geq 0, \text{ and } \int_{\mathbb{R}^N} \rho_n(x)\mathrm{d}x = 1,$$

and let, for each $n \geq n_0$,

$$\boldsymbol{w}_n := \boldsymbol{u}_n * \rho_n, \text{ i.e., } \boldsymbol{w}_n(x) := \int_{B\left(x;\frac{1}{n}\right)} \rho_n(x-y)\boldsymbol{u}_n(y)\mathrm{d}y,\ x \in \mathbb{R}^N$$

Then

$$\operatorname{supp}\boldsymbol{w}_n \subset \overline{\left\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\right\}},$$

and thus

$$\boldsymbol{v}_n := \boldsymbol{w}_n|_\Omega \in \boldsymbol{\mathcal{D}}(\Omega).$$

Besides, by a well-known property of convolution operators,

$$\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} = \|\boldsymbol{w}_n\|_{\boldsymbol{H}^1(\mathbb{R}^N)} = \|\boldsymbol{u}_n * \rho_n\|_{\boldsymbol{H}^1(\mathbb{R}^N)} \leq \|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\mathbb{R}^N)},\ n \geq n_0$$

(iii) *The vector fields* $\boldsymbol{v}_n \in \boldsymbol{\mathcal{D}}(\Omega)$ *satisfy*

$$\boxed{\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \leq C_1(\Omega)\,\|\varphi\|_{L^2(\Omega)} \ \text{ for all } n \geq n_0.}$$

Taking $y := \dfrac{x}{\lambda_n}$ as the new variable in the integrals below shows that

$$\begin{aligned}
\|\boldsymbol{u}_n\|^2_{\boldsymbol{H}^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left|\lambda_n \boldsymbol{w}\left(\frac{x}{\lambda_n}\right)\right|^2 \mathrm{d}x + \sum_{i,j} \int_{\mathbb{R}^N} \left|\partial_i w_j\left(\frac{x}{\lambda_n}\right)\right|^2 \mathrm{d}x \\
&= \lambda_n^{N+2} \int_{\mathbb{R}^N} |\boldsymbol{w}(y)|^2 \,\mathrm{d}y + \sum_{i,j} \lambda_n^N \int_{\mathbb{R}^N} |\partial_i w_j(y)|^2 \,\mathrm{d}y \\
&\leq \|\boldsymbol{w}\|^2_{\boldsymbol{H}^1(\mathbb{R}^N)} = \|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)},
\end{aligned}$$

so that, by (i) and (ii),

$$\boxed{\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \leq \|\boldsymbol{u}_n\|_{\boldsymbol{H}^1(\mathbb{R}^N)} \leq \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C_1(\Omega)\,\|\varphi\|_{L^2(\Omega)} \ \text{ for all } n \geq n_0.}$$

(iv) *The vector fields* $\boldsymbol{v}_n \in \boldsymbol{\mathcal{D}}(\Omega)$, $n \geq n_0$, *satisfy*

$$\operatorname{div} \boldsymbol{v}_n \to \varphi \text{ in } \mathcal{D}(\Omega) \text{ as } n \to \infty.$$

By definition of the convergence in $\mathcal{D}(\Omega)$, we have to find a compact subset K of Ω such that

$$\operatorname{supp} \varphi \subset K \text{ and } \operatorname{supp}(\operatorname{div} \boldsymbol{v}_n) \subset K \text{ for all } n \text{ large enough, and for } \textit{each} \text{ multi-index } \boldsymbol{\alpha},\ \sup_{x \in K} |\partial^{\boldsymbol{\alpha}}(\operatorname{div} \boldsymbol{v}_n)(x) - \partial^{\boldsymbol{\alpha}} \varphi(x)| \to 0 \text{ as } n \to \infty.$$

Since $\boldsymbol{u}_n = \boldsymbol{0}$ in $\mathbb{R}^N - \Omega_n$, $\boldsymbol{w}_n = \boldsymbol{u}_n * \rho_n$ with $\operatorname{supp} \rho_n \subset \overline{B\left(0; \frac{1}{n}\right)}$, and $\boldsymbol{v}_n = \boldsymbol{w}_n|_\Omega$, there exists $\beta > 0$ and $n_1 \geq n_0$ such that

$$\operatorname{supp}(\operatorname{div} \boldsymbol{v}_n) \cup \operatorname{supp} \varphi \subset K := \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \geq \beta\} \text{ for all } n \geq n_1$$

That $\sup_{x\in K}|\partial^{\boldsymbol{\alpha}}(\operatorname{div}\boldsymbol{v}_n)(x)-\partial^{\boldsymbol{\alpha}}\varphi(x)|\to 0$ as $n\to\infty$ follows by noting that

$$\begin{aligned}\partial^{\boldsymbol{\alpha}}(\operatorname{div}\boldsymbol{v}_n) &= \partial^{\boldsymbol{\alpha}}\operatorname{div}(\boldsymbol{w}_n) = \partial^{\boldsymbol{\alpha}}\left(\operatorname{div}(\boldsymbol{u}_n * \rho_n)\right)\\ &= (\partial^{\boldsymbol{\alpha}}(\operatorname{div}\boldsymbol{u}_n)) * \rho_n = \left(\partial^{\boldsymbol{\alpha}}\varphi\left(\frac{\cdot}{\lambda_n}\right)\right) * \rho_n \text{ in } \Omega,\end{aligned}$$

so that, for each $n \geq n_1$,

$$\begin{aligned}\partial^{\boldsymbol{\alpha}}(\operatorname{div}\boldsymbol{v}_n)(x) - \partial^{\boldsymbol{\alpha}}\varphi(x) &= \left(\partial^{\boldsymbol{\alpha}}\varphi\left(\frac{\cdot}{\lambda_n}\right)\right) * \rho_n(x) - \partial^{\boldsymbol{\alpha}}\varphi(x)\\ &= \int_{\mathbb{R}^N}\left(\frac{1}{\lambda_n^{|\boldsymbol{\alpha}|}}\partial^{\boldsymbol{\alpha}}\varphi\left(\frac{x-y}{\lambda_n}\right) - \partial^{\boldsymbol{\alpha}}\varphi(x)\right)\rho_n(y)\mathrm{d}y \text{ at each } x\in\Omega,\end{aligned}$$

which in turn implies that

$$\sup_{x\in K} |\partial^{\boldsymbol{\alpha}}(\operatorname{div} \boldsymbol{v}_n)(x) - \partial^{\boldsymbol{\alpha}}\varphi(x)|$$

$$= \sup_{x\in K} \left| \int_{\mathbb{R}^N} \left[\left(\frac{1}{\lambda_n^{|\boldsymbol{\alpha}|}} - 1 \right) \partial^{\boldsymbol{\alpha}}\varphi\left(\frac{x-y}{\lambda_n}\right) \rho_n(y) + \left(\partial^{\boldsymbol{\alpha}}\varphi\left(\frac{x-y}{\lambda_n}\right) - \partial^{\boldsymbol{\alpha}}\varphi(x) \right) \rho_n(y) \right] \mathrm{d}y \right|$$

$$\leq \sup_{z\in\mathbb{R}^N} |\partial^{\boldsymbol{\alpha}}\varphi(z)| \left(\frac{1}{\lambda_n^{|\boldsymbol{\alpha}|}} - 1 \right) + \sup_{x\in K} \left| \int_{B(0;\frac{1}{n})} \Big(\partial^{\boldsymbol{\alpha}}\varphi(x+\delta_n(x,y)) - \partial^{\boldsymbol{\alpha}}\varphi(x) \Big) \rho_n(y) \mathrm{d}y \right|,$$

where $\delta_n(x,y) := \left(\frac{1-\lambda_n}{\lambda_n}\right)x - \frac{y}{\lambda_n}$. Since then $\sup_{x\in K} \sup_{y\in B(0;\frac{1}{n})} |\delta_n(x,y)|$ can be made arbitrarily small if n is large enough, it follows that, for each multi-index $\boldsymbol{\alpha}$,

$$\boxed{\sup_{x\in K} |\partial^{\boldsymbol{\alpha}}(\operatorname{div} \boldsymbol{v}_n)(x) - \partial^{\boldsymbol{\alpha}}\varphi(x)| \to 0 \text{ as } n \to \infty,}$$

since the function $\partial^{\boldsymbol{\alpha}}\varphi$ *is uniformly continuous* and *bounded*. □

2.6 “APPROXIMATION LEMMA” $\Rightarrow$ J.L. LIONS LEMMA

(f) **Approximation lemma**: *Assume the domain* Ω *is* **starlike with respect to a ball**. *Then, given any*

$$\varphi \in \mathcal{D}(\Omega) \text{ such that } \int_\Omega \varphi dx = 0,$$

there exist $\boldsymbol{v}_n = \boldsymbol{v}_n(\varphi) \in \mathcal{D}(\Omega)$, $n \geq 1$, *such that*

$$\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \leq C_2(\Omega)\,\|\varphi\|_{L^2(\Omega)} \text{ for all } n \geq 1, \text{ and}$$
$$\operatorname{div} \boldsymbol{v}_n \to \varphi \text{ in } \mathcal{D}(\Omega) \text{ as } n \to \infty$$

implies

(g) **J.L. Lions lemma** : Ω : *domain in* $\mathbb{R}^N$

$$f \in \mathcal{D}'(\Omega) \text{ and } \boldsymbol{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega)$$

Sketch of proof (i) Assume first that Ω is *starlike with respect to an open ball*, and let $f \in \mathcal{D}'(\Omega)$ be such that $\textbf{\textit{grad}}\, f \in \boldsymbol{H}^{-1}(\Omega)$. To show that $f \in L^2(\Omega)$, it suffices to show that there exists a constant $C_0(f, \Omega)$ such that

$$\left|{}_{\mathcal{D}'(\Omega)}\langle f, \varphi\rangle_{\mathcal{D}(\Omega)}\right| \leq C_0(f, \Omega) \left\|\varphi\right\|_{L^2(\Omega)} \text{ for all } \varphi \in \mathcal{D}(\Omega)$$

Let $\varphi_1 \in \mathcal{D}(\Omega)$ such that $\int_\Omega \varphi_1 \mathrm{d}x = 1$. Given any $\varphi \in \mathcal{D}(\Omega)$,

$$\varphi_0 = \varphi_0(\varphi) := \varphi - \left(\int_\Omega \varphi \mathrm{d}x\right) \varphi_1 \in \mathcal{D}_0(\Omega)$$

and there exists a constant $C(\Omega, \varphi_1)$ independent of φ such that

$$\left\|\varphi_0\right\|_{L^2(\Omega)} \leq C(\Omega, \varphi_1) \left\|\varphi\right\|_{L^2(\Omega)} .$$

By assumption,

$$\left|{}_{\mathcal{D}'(\Omega)}\langle f, \operatorname{div} \boldsymbol{\psi}\rangle_{\mathcal{D}(\Omega)}\right| = \left|{}_{\boldsymbol{\mathcal{D}}'(\Omega)}\langle \textbf{\textit{grad}}\, f, \boldsymbol{\psi}\rangle_{\boldsymbol{\mathcal{D}}(\Omega)}\right| \leq C_1(f, \Omega) \left\|\boldsymbol{\psi}\right\|_{\boldsymbol{H}^1(\Omega)} \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega)$$

By the *approximation lemma*, there exist vector fields $\boldsymbol{v}_n = \boldsymbol{v}_n(\varphi_0) = \boldsymbol{v}_n(\varphi) \in \boldsymbol{\mathcal{D}}(\Omega)$, $n \geq 1$, such that

$$\operatorname{div} \boldsymbol{v}_n \to \varphi_0 \text{ in } \mathcal{D}(\Omega) \text{ as } n \to \infty \text{ and } \|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \leq C_2(\Omega) \|\varphi_0\|_{L^2(\Omega)} \text{ for all } n \geq 1.$$

The relations

$$\begin{aligned}
{}_{\mathcal{D}'(\Omega)}\langle f, \varphi\rangle_{\mathcal{D}(\Omega)} &= {}_{\mathcal{D}'(\Omega)}\langle f, \varphi_0\rangle_{\mathcal{D}(\Omega)} + \left(\int_\Omega \varphi \mathrm{d}x\right) {}_{\mathcal{D}'(\Omega)}\langle f, \varphi_1\rangle_{\mathcal{D}(\Omega)}, \\
{}_{\mathcal{D}'(\Omega)}\langle f, \varphi_0\rangle_{\mathcal{D}(\Omega)} &= \lim_{n\to\infty} {}_{\mathcal{D}'(\Omega)}\langle f, \operatorname{div} \boldsymbol{v}_n\rangle_{\mathcal{D}(\Omega)}, \\
\left|{}_{\mathcal{D}'(\Omega)}\langle f, \operatorname{div} \boldsymbol{v}_n\rangle_{\mathcal{D}(\Omega)}\right| &\leq C_1(f, \Omega) \|\boldsymbol{v}_n\|_{\boldsymbol{H}^1(\Omega)} \\
&\leq C_1(f, \Omega) C_2(\Omega) \|\varphi_0\|_{L^2(\Omega)} \text{ for all } n \geq 1,
\end{aligned}$$

together imply that

$$\left|{}_{\mathcal{D}'(\Omega)}\langle f, \varphi\rangle_{\mathcal{D}(\Omega)}\right| \leq C_0(f, \Omega) \|\varphi\|_{L^2(\Omega)} \text{ for all } \varphi \in \mathcal{D}(\Omega), \text{where}$$
$$C_0(f, \Omega) := C(\Omega, \varphi_1) C_1(f, \Omega) C_2(\Omega) + (\operatorname{meas} \Omega)^{1/2} \left|{}_{\mathcal{D}'(\Omega)}\langle f, \varphi_1\rangle_{\mathcal{D}(\Omega)}\right|$$

(ii) Assume next that Ω *is a general domain*. Then, *there exists a finite number of domains* Ω_i, $i \in I$, *each one contained in* Ω *and starlike with respect to an open ball, such that* (use ideas from V. MAZ'YA [1985] or M. COSTABEL & MCINTOSH [2010])

$$\Omega = \bigcup_{i \in I} \Omega_i .$$

Given any $\varphi \in \mathcal{D}(\Omega)$, let $(\alpha_i)_{i \in I}$ be a *partition of unity* associated with the open cover $\bigcup_{i \in I} \Omega_i$ of the compact set

$$K := \operatorname{supp} \varphi ,$$

i.e., $\alpha_i \in \mathcal{D}(\Omega)$, $\operatorname{supp} \alpha_i \subset \Omega_i$, and $\sum_{i \in I} \alpha_i(x) = 1$ for all $x \in K$. Then *J.L.Lions lemma on* Ω follows from the application of *J.L. Lions lemma on each* Ω_i, $i \in I$. $\square$

2.7 AN EQUIVALENCE THEOREM

Clearly, (g): *J.L. Lions lemma* $\Rightarrow$ (a): *classical J.L. Lions lemma*. So:

EQUIVALENCE THEOREM Ω : *domain in* $\mathbb{R}^N$. *The following statements are equivalent*:

(a) *Classical J.L. Lions lemma*:
$f \in H^{-1}(\Omega)$ and $\boldsymbol{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega)$

(b) *J. Nečas inequality*:
$\|f\|_{L^2(\Omega)} \leq C_0(\Omega)(\|f\|_{H^{-1}(\Omega)} + \|\boldsymbol{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)})$ for all $f \in L^2(\Omega)$

(c) $\boldsymbol{grad} : L^2_0(\Omega) \to \boldsymbol{H}^{-1}(\Omega)$ *has closed range*

(d) *de Rham theorem in* $\boldsymbol{H}^{-1}(\Omega)$

(e) $\mathrm{div} : \boldsymbol{H}^1_0(\Omega) \to L^2_0(\Omega)$ *is onto*

(f) *"Approximation lemma"*

(g) *J.L. Lions lemma:* $f \in \mathcal{D}'(\Omega)$ and $\boldsymbol{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega)$

Conclusion: *Any "independent" proof of* (a), *or* (b), *or* (c), *or* (d), *or* (e), *or* (f), *provides a proof of J.L. Lions lemma.*

2.8 TWO PROOFS OF J.L. LIONS LEMMA

One proof of **J.L. Lions lemma** follows from the **equivalence theorem** *together with*:

THEOREM Ω: *domain in* $\mathbb{R}^N$. *Then the operator*

$$\operatorname{div} : \boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega) := \left\{ f \in L^2(\Omega); \int_\Omega f \, dx = 0 \right\}.$$

is onto.

M.E. Bogovskii; *Soviet Math. Dokl.* **20** (1979), 1094–1098.

Note that this surjectivity holds as well for the operator
$\operatorname{div} : W_0^{1,p}(\Omega) \to L_0^p(\Omega) = \{ f \in L^p(\Omega); \int_\Omega f \, dx = 0 \}$, $1 < p < \infty$.

Brief idea of the proof: *One shows that, given any* $f \in L_0^2(\Omega)$, *there exist a vector field* $\boldsymbol{u}_f = \boldsymbol{R} f \in \boldsymbol{H}_0^1(\Omega)$ *with* $\boldsymbol{R} : L_0^2(\Omega) \to \boldsymbol{H}_0^1(\Omega)$ *linear and a constant* $C(\Omega)$ *independent of* f *such that*

$$\operatorname{div} \boldsymbol{u}_f = f \text{ in } \Omega \text{ and } \|\boldsymbol{u}_f\|_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega) \, \|f\|_{L^2(\Omega)}$$

Assume that Ω is starlike with respect to a ball B and let $\theta \in \mathcal{D}(\mathbb{R}^N)$ be such that $0 \leq \theta \leq 1$, $\operatorname{supp}\theta \subset B$, and $\int_B \theta \mathrm{d}x = 1$. Then M.E. Bogovskii gives a remarkable *explicit formula* for such a vector field $\boldsymbol{u}_f$ in this case in the following form:

$$\begin{aligned} \boldsymbol{u}_f(x) &:= \int_\Omega f(y)\boldsymbol{K}(x,y)\mathrm{d}y, \; x \in \Omega, \text{ where} \\ \boldsymbol{K}(x,y) &:= \left(\int_1^\infty t^{N-1}\theta(y + t(x-y))\mathrm{d}t\right)(\boldsymbol{x} - \boldsymbol{y}) \end{aligned}$$

Note that $\boldsymbol{K}(x,y)$ *is not defined if* $x = y$ and that establishing the estimate $\|\boldsymbol{u}_f\|_{\boldsymbol{H}^1(\Omega)} \leq C(\Omega)\|f\|_{L^2(\Omega)}$ is delicate, as it relies in particular on the theory of *Calderón-Zygmund singular integrals*.

Assume next that Ω is a general domain. The rest of the proof then follows like that of the implication (f) $\Rightarrow$ (g) in the equivalence theorem. $\square$

Remark: As already noted, W. Borchers & H. Sohr [1990] showed that J.L. Lions lemma can be established as a consequence of the surjectivity of div : $\boldsymbol{H}_0^1(\Omega) \to L_0^2(\Omega)$. However, our proof is shorter and simpler, thanks to the *approximation lemma*.

Another proof of **J.L. Lions lemma** follows from the **equivalence theorem** *together with*:

THEOREM: J. NEČAS INEQUALITY: Ω: *domain in* $\mathbb{R}^n$.
There exists a constant $C_0(\Omega)$ *such that*

$$\|f\|_{L^2(\Omega)} \leq C_0(\Omega)\left(\|f\|_{H^{-1}(\Omega)} + \|\mathbf{grad}\, f\|_{\boldsymbol{H}^{-1}(\Omega)}\right) \text{ for all } f \in L^2(\Omega)$$

See J. Nečas [1967], *op.cit.*, or
J.H. Bramble, *Math. Models Appl. Sci.* **13** (2003), 361–371.

Remark: As already noted, G. GEYMONAT & P. SUQUET [1986] showed that the classical J.L. Lions lemma can be established as a consequence of J. Nečas inequality. However, our proof is shorter and simpler, again thanks to the *approximation lemma*.

3. TWO FURTHER EQUIVALENCES

3.1 J.L. LIONS LEMMA $\Leftrightarrow$ WEAK POINCARÉ LEMMA

We saw in Sect. 1.5 that *the* **classical J.L. Lions lemma** *implies the* **weak Poincaré lemma**. Conversely:

THEOREM

Weak Poincaré lemma: Ω : *simply-connected domain in* $\mathbb{R}^N$.
Given $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$ *such that*

$$\boldsymbol{curl}\, \boldsymbol{h} = \boldsymbol{0} \text{ in } \boldsymbol{H}^{-2}(\Omega),$$

there exists $p \in L^2(\Omega)$ *such that*

$$\boldsymbol{grad}\, p = \boldsymbol{h} \text{ in } H^{-1}(\Omega)$$

implies

J.L. Lions lemma

$$f \in \mathcal{D}'(\Omega) \text{ and } \boldsymbol{grad}\, f \in \boldsymbol{H}^{-1}(\Omega) \Rightarrow f \in L^2(\Omega)$$

Proof (i) Assume first that the domain Ω is *simply-connected*, and let $f \in \mathcal{D}'(\Omega)$ be such that $\boldsymbol{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)$. Since then

$$\boldsymbol{curl}\,\boldsymbol{grad}\, f = 0,$$

the **weak Poincaré lemma** implies that there exists $p \in L^2(\Omega)$ such that

$$\boldsymbol{grad}\, p = \boldsymbol{grad}\, f.$$

Hence there exists a constant C such that

$$f = p + C \text{ in } \mathcal{D}'(\Omega),$$

which shows that $f \in L^2(\Omega)$, i.e., that **J.L. Lions lemma** *holds in this case*.

(ii) Assume next that Ω is a general domain, and let $f \in \mathcal{D}'(\Omega)$ be such that $\boldsymbol{grad}\, f \in \boldsymbol{H}^{-1}(\Omega)$. There exist a finite number of *simply-connected* domains Ω_i, $i \in I$, such that $\Omega = \bigcup_{i\in I} \Omega_i$. Then, given any $\varphi \in \mathcal{D}(\Omega)$, use a *partition of unity* associated with the open cover $\bigcup_{i\in I} \Omega_i$ of the compact set $\operatorname{supp}\varphi$ and use *J.L. Lions lemma on each* Ω_i, $i \in I$. $\square$

3.2 J.L. LIONS LEMMA $\Leftrightarrow$ de RHAM THEOREM IN $H^{-1}(\Omega)$ $\Leftrightarrow$ "REFINED" de RHAM THEOREM IN $H^{-1}(\Omega)$

We saw in the **equivalence theorem** that **J.L. Lions lemma** *is equivalent to*

(d) **"Coarse" de Rham theorem**: *Given* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, *there exists* $p \in L^2_0(\Omega)$ *such that* $\textbf{grad}\, p = \boldsymbol{h}$ *in* $\boldsymbol{H}^{-1}(\Omega)$ *if* (and clearly only if) ${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{H}^1_0(\Omega)} = 0$ for all $\boldsymbol{v} \in \boldsymbol{H}^1_0(\Omega)$ that satisfy $\operatorname{div} \boldsymbol{v} = 0$ in Ω

THEOREM: **"Coarse" de Rham theorem** *implies*

"Refined" de Rham theorem: *Given* $\boldsymbol{h} \in \boldsymbol{H}^{-1}(\Omega)$, *there exists* $p \in L^2_0(\Omega)$ *such that* $\textbf{grad}\, p = \boldsymbol{h}$ in $\boldsymbol{H}^{-1}(\Omega)$ *if* (and clearly only if) ${}_{\boldsymbol{H}^{-1}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^1_0(\Omega)} = 0$ for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\Omega)$ that satisfy $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω

Sketch of proof Follows same idea as in V. GIRAULT & P.A. RAVIART [1986] (see also F. BOYER & P. FABRIE [2013]), with a significant simplification because the "general" **J.L. Lions lemma** can be used. □

4. EXTENSIONS

4.1 VECTOR VERSION DE J.L. LIONS LEMMA

Capital Roman letters denote spaces of symmetric $N \times N$ matrix fields.

Ω: *domain in* $\mathbb{R}^N$

$$\boldsymbol{v} \in \mathcal{D}'(\Omega) \text{ and } \boldsymbol{\nabla}_s \boldsymbol{v} := \frac{1}{2}\left(\boldsymbol{\nabla} \boldsymbol{v}^T + \boldsymbol{\nabla} \boldsymbol{v}\right) \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$$

C. Amrouche, P.G. Ciarlet, L. Gratie & S. Kesavan [2006]. Proof based on the "scalar" version of J.L. Lions lemma (cf. *supra*).

The following "equivalence theorem" is due to P.G. Ciarlet, M. Malin & C. Mardare [2018]. For brevity, the corresponding "approximation lemma" is not mentioned.

EQUIVALENCE THEOREM Ω : *domain in* $\mathbb{R}^N$. *The following statements are equivalent*:

(a) $\boldsymbol{v} \in \boldsymbol{H}^{-1}(\Omega)$ and $\boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$

(b) *Vector version of J. Nečas inequality*:
$\|\boldsymbol{v}\|_{L^2(\Omega)} \leq C_0(\Omega)(\|\boldsymbol{v}\|_{H^{-1}(\Omega)} + \|\boldsymbol{\nabla}_s \boldsymbol{v}\|_{\mathbb{H}^{-1}(\Omega)})$ for all $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$

(c) $\boldsymbol{\nabla}_s : \boldsymbol{L}_0^2(\Omega) \to \mathbb{H}^{-1}(\Omega)$ *has closed range, where*

$\boldsymbol{L}_0^2(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ \int_\Omega \boldsymbol{v} \cdot \boldsymbol{r}\, dx = 0\}$ for all $\boldsymbol{r} \in \operatorname{Ker} \boldsymbol{\nabla}_s$

(d) *Donati compatibility conditions:*
Given $\boldsymbol{e} \in \mathbb{H}^{-1}(\Omega)$, *there exists* $\boldsymbol{v} \in \boldsymbol{L}_0^2(\Omega)$ *such that*
$\boldsymbol{\nabla}_s \boldsymbol{v} = \boldsymbol{e}$ *if* (and clearly only if) ${}_{\mathbb{H}^{-1}(\Omega)}\langle \boldsymbol{e}, \boldsymbol{s} \rangle_{\mathbb{H}_0^1(\Omega)} = 0$
for all $\boldsymbol{s} \in \mathbb{H}_0^1(\Omega)$ that satisfy $\mathbf{div}\, \boldsymbol{s} = \mathbf{0}$ in Ω

(e) $\mathbf{div} : \mathbb{H}_0^1(\Omega) \to \boldsymbol{L}_0^2(\Omega)$ *is onto.*

(f) *Vector version of J.L. Lions lemma:*
$\boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$ and $\boldsymbol{\nabla}_s \boldsymbol{v} \in \mathbb{H}^{-1}(\Omega) \Rightarrow \boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.

4.2 J.L. LIONS LEMMA IN $W^{-m,p}(\Omega)$

EQUIVALENCE THEOREM Ω : *domain in* $\mathbb{R}^N$. *Let* $m \geq 1$, $1 < p < \infty$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *The following statements are equivalent*:

(a) $f \in W^{-m,p}(\Omega)$ and $\textbf{grad}\, f \in \boldsymbol{W}^{-m,p}(\Omega) \Rightarrow f \in W^{-m+1,p}(\Omega)$

(b) *J. Nečas inequality in* $W^{-m,p}(\Omega)$:

$$\|f\|_{W^{-m+1,p}(\Omega)} \leq C_0(\Omega, m, p)(\|f\|_{W^{-m,p}(\Omega)} + \|\textbf{grad}\, f\|_{\boldsymbol{W}^{-m,p}(\Omega)})$$

for all $f \in W^{-m+1,p}(\Omega)$

(c) $\textbf{grad} : W^{-m+1,p}(\Omega) \to \boldsymbol{W}^{-m,p}(\Omega)$ *has closed range*

(d) *de Rham theorem in* $W^{-m,p}(\Omega)$ *: Given* $\boldsymbol{h} \in \boldsymbol{W}^{-m,p}(\Omega)$, *there exists* $p \in W^{-m+1,p}(\Omega)$ *such that* $\textbf{grad}\, p = \boldsymbol{h}$ *if* (and clearly only if) ${}_{\boldsymbol{W}^{-m,p}(\Omega)}\langle \boldsymbol{h}, \boldsymbol{v} \rangle_{\boldsymbol{W}_0^{m,q}(\Omega)} = 0$ for all $\boldsymbol{v} \in \boldsymbol{W}_0^{m,q}(\Omega)$ that satisfy $\operatorname{div} \boldsymbol{v} = 0$ in Ω

(e) $\operatorname{div} : \boldsymbol{W}_0^{m,q}(\Omega) \to \left\{ f \in W_0^{m-1,q}(\Omega);\ \int_\Omega f\, dx = 0 \right\}$ *is onto*

(f) *J.L. Lions lemma in* $W^{-m,p}(\Omega)$:

$f \in \mathcal{D}'(\Omega)$ and $\textbf{grad}\, f \in \boldsymbol{W}^{-m,p}(\Omega) \Rightarrow f \in W^{-m+1,p}(\Omega)$