Mean Field Games: a survey

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March, 2018 — Roscoff

Outline

The Nash system for $N-$player differential games

N identical agents whose dynamics are

$$dX_{i,t} = \sqrt{2}\, dW_t^i + \gamma_{i,t} dt$$

- (W_t^i): N independent Brownian motions in $\mathbb{R}^d$
- A *common noise* may be added
- Global information: the control of the agent i, i.e. $\gamma_{i,t}$ is a bounded process adapted to $(W_t^1, \dots, W_t^N)$
- Models with partial information can be considered as well.

Symmetry assumption: the cost of the agent i at time t is

$$J_i(t) = \mathbb{E}\left\{ \int_t^T \mathcal{L}\left(X_{i,s}, \gamma_{i,s}; m_s^i\right) ds + G\left(X_{i,T}; m_T^i\right) \right\} \quad \text{where} \quad m_s^i = \frac{1}{N-1} \sum_{j \neq i} \delta_{X_{j,s}}.$$

We will sometimes focus on the simpler case (separate dependency + periodicity) when

$$\mathcal{L}\left(X_{i,s}, \gamma_{i,s}; m_s^i\right) = L\left(X_{i,s}, \gamma_{i,s}\right) + F\left(X_{i,s}; m_s^i\right),$$
$$L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}, \qquad F, G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}.$$

Nash Equilibria

Nash equilibrium: $\bar\gamma_i = \underset{\gamma_i}{\operatorname{argmin}}\, J_i(t, \bar\gamma_1, \dots, \bar\gamma_{i-1}, \gamma_i, \bar\gamma_{i+1}, \dots, \bar\gamma_N), \qquad \forall i$

Hamiltonian: $H(x,p) = \sup_{\gamma \in \mathbb{R}^d} \left(-p\cdot\gamma - L(x,\gamma)\right)$

The $N-$agents differential game is described by the Nash system of N coupled equations in $[0,T]\times(\mathbb{T}^d)^N$:

$$\begin{cases} -\partial_t v^{N,i}(t,X) - \sum_{j=1}^N \Delta_{x_j} v^{N,i}(t,X) + H(x_i, D_{x_i} v^{N,i}(t,X)) \\ \qquad + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,X)) \cdot D_{x_j} v^{N,i}(t,X) = F(x_i, m_X^{N,i}) \\ \qquad\qquad \text{in } [0,T]\times(\mathbb{T}^d)^N, \\ v^{N,i}(T,X) = G(x_i; m_X^{N,i}) \qquad \text{in } (\mathbb{T}^d)^N, \end{cases}$$

where $X = (x_1, \dots, x_N)$ and

$$m_X^{N,i} = \frac{1}{N-1} \sum_{j\neq i} \delta_{x_j}.$$

Feedback law: $\bar\gamma_i(s) = -D_p H\left(X_{i,s}, D_{x_i} v^{N,i}\left(s, X_{1,s}, \dots, X_{N,s}\right)\right).$

Important observation

Because of symmetry, all the functions $v^{N,i}$ may be expressed in the same form,

$$v^{N,i}(t,X) = U^N\left(t, x_i; m_X^{N,i}\right),$$

where

- $$m_X^{N,i} = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}$$
- U^N is a continuous real valued function defined on $[0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)$
- $\mathcal{P}(\mathbb{T}^d)$ is endowed with the **Monge-Kantorovich distance:**

$$\mathbf{d}_1(m,m') = \sup_{|\phi|_{\mathrm{Lip}}\le 1}\int_{\mathbb{T}^d}\phi(y)\ (dm(y) - dm'(y)).$$

$N \to \infty$: a system of PDEs

- **A system of 2 PDEs.** Under suitable assumptions on H, F and G, Lasry-Lions proved that there exists a unique solution to the system of PDEs in $\mathbb{T}^d \times [0,T]$:

$$(*)\begin{cases} \dfrac{\partial u}{\partial t} + \Delta u - H(x, \nabla u) & = & F(x; m_t), \\ \dfrac{\partial m}{\partial t} - \Delta m - \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) & = & 0, \end{cases}$$

with the terminal and initial conditions

$$u(t=T, x) = G(x; m_T), \quad \text{and} \quad m(0,x) = m_0(x).$$

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$$u(t = T, x) = G(x; m_T), \quad \text{and} \quad m(0, x) = m_0(x).$$

- **Convergence.** In 2015, under much stronger assumptions (discussed later), Cardaliaguet, Delarue, Lasry and Lions proved that for any $m_0 \in \mathcal{P}(\mathbb{T}^d)$, for any $t \in [0,T]$, for any $i \le N$,

$$\left\| \int_{(\mathbb{T}^d)^{N-1}} v^{N,i}(t, X) \prod_{j \neq i} dm_t(x_j) - u(t, x_i) \right\|_{L^1_{m_t}} \le \begin{cases} C N^{-\frac{1}{d}} & \text{if } d \ge 3, \\ C N^{-\frac{1}{2}} \log(N) & \text{if } d = 2, \end{cases}$$

where $X = (x_1, \dots, x_N)$ and C does not depend on N, m_0, t, i.

The main idea: a transport equation posed in a set of measures

- System $(\star)$ yields the characteristics of a nonlinear transport equation posed in the set of probability measures, that Lasry-Lions call the master equation: find $U : [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ s.t.

$$\left\{\begin{array}{l} -\partial_t U - \Delta_x U + \displaystyle\int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U(t,x,m,y)\right] \, dm(y) + H(x, D_x U) \\ + \displaystyle\int_{\mathbb{T}^d} D_m U(t,x,m,y) \cdot D_p H(y, D_x U) \, dm(y) = F(x;m), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(T,x,m) = G(x;m), \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \end{array}\right. \tag{M}$$

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$$\left\{\begin{array}{l} -\partial_t U - \Delta_x U + \displaystyle\int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U(t,x,m,y)\right] \, dm(y) + H(x, D_x U) \\ + \displaystyle\int_{\mathbb{T}^d} D_m U(t,x,m,y) \cdot D_p H(y, D_x U) \, dm(y) = F(x;m), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(T,x,m) = G(x;m), \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \end{array}\right. \tag{M}$$

- Recall that the Nash eq. system is: for $i = 1, \dots, N$,

$$\left\{\begin{array}{l} -\partial_t v^{N,i}(t,X) - \displaystyle\sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(t,X) + H(x_i, D_{x_i} v^{N,i}(t,X)) \\ \qquad + \displaystyle\sum_{j \neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,X)) \cdot D_{x_j} v^{N,i}(t,X) = F(x_i, m_X^{N,i}), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times (\mathbb{T}^d)^N, \\ v^{N,i}(T,X) = G(x_i; m_X^{N,i}) \qquad \text{in } (\mathbb{T}^d)^N, \end{array}\right.$$

with $v^{N,i}(t,X) = U^N(t, x_i, m_X^{N,i})$.

Differential calculus in the space of measures (1/2)

- **Directional derivatives:** $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous map $\frac{\delta U}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\lim_{s \to 0^+} \frac{U((1-s)m + sm') - U(m)}{s} = \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m,y)(dm'(y) - dm(y)).$$

- **Intrinsic derivative:** If $\frac{\delta U}{\delta m}$ is of class $\mathcal{C}^1$ with respect to y, the intrinsic derivative $D_m U$ is defined by

$$D_m U : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}^d, \qquad D_m U(m,y) = D_y \frac{\delta U}{\delta m}(m,y).$$

Then, for a regular vector field Φ and a small $h \in \mathbb{R}$:

$$U((I + h\Phi)\sharp m) - U(m) \sim h \int D_m U(m,y) \cdot \Phi(y) dm(y).$$

- **Ex.** $U(m) = \int_{\mathbb{T}^d} g(x) dm(x) \Rightarrow \frac{\delta U}{\delta m}(m,y) = g(y)$ and $D_m U(m,y) = Dg(y)$.

Differential calculus in the space of measures (2/2)

- **Second order derivatives:** when possible,

$$\lim_{s\to 0^+} \frac{\frac{\delta U}{\delta m}((1-s)m+sm',y)-\frac{\delta U}{\delta m}(m,y)}{s} = \int_{\mathbb{T}^d} \frac{\delta^2 U}{\delta m^2}(m,y,z)d(m'-m)(z),$$

and

$$D^2_{m,m}U(m,y,z) = D_{y,z}\frac{\delta^2 U}{\delta m^2}(m,y,z).$$

- **Differential calculus for symmetric functions of N variables in $\mathbb{T}^d$:**

If $u^N(X) = U^N(m_X^N)$, with $m_X^N = \frac{1}{N}\sum_j \delta_{x_j}$, then

$$\begin{aligned} D_{x_j}u^N(X) &= \frac{1}{N}D_mU^N(m_X^N,x_j) \\ D^2_{x_j,x_j}u^N(X) &= \frac{1}{N}D_x[D_mU^N](m_X^N,x_j) + \frac{1}{N^2}D^2_{m,m}U^N(m_X^N,x_j,x_j), \\ \Delta_{x_j}u^N(X) &= \frac{1}{N}\mathrm{div}_x[D_mU^N](m_X^N,x_j) + \frac{1}{N^2}\mathrm{Tr}\left(D^2_{m,m}U^N(m_X^N,x_j,x_j)\right). \end{aligned}$$

Typical assumptions for proving the asymptotic behavior

- $\mathcal{H}$ depends separately on m and p: $\mathcal{H}(x,m,p) = H(x,p) - F(x,m)$
- H smooth, globally Lipschitz continuous, with $D^2_{pp}H > 0$,
- F and G "smooth" and monotone:
  - Typical smoothness assumptions: for some $n \geq 1$, $\alpha \in (0,1)$,

$$\sup_{m\in\mathcal{P}(\mathbb{T}^d)} \left\{ \|F(\cdot,m)\|_{n+\alpha} + \left\| \frac{\delta F}{\delta m}(\cdot,m,\cdot) \right\|_{n+\alpha,n+\alpha} \right\} < \infty,$$

$$\sup_{m1\neq m_2} \frac{1}{d_1(m_1,m_2)} \left\{ \left\| \frac{\delta F}{\delta m}(\cdot,m_1,\cdot) - \frac{\delta F}{\delta m}(\cdot,m_2,\cdot) \right\|_{n+\alpha,n+\alpha} \right\} < \infty.$$

  - Monotonicity

$$\int_{\mathbb{T}^d} (F[m_1](x) - F[m_2](x))(dm_2(x) - dm_1(x)) \leq 0 \qquad \Rightarrow \qquad m_1 = m_2.$$

Important extension: common noise

$$dX_{i,t} = \sqrt{2}\, dW_t^i + \sqrt{2\beta}\, dB_t + \gamma_{i,t} dt$$

The master equation can be written. Then $(\star)$ becomes a system of stochastic PDEs. Much more difficult.

Outline

Special structure

$$(*)\left\{\begin{array}{rcl}\dfrac{\partial u}{\partial t}+\nu\Delta u-\mathcal{H}(x,\nabla u;m) & = & 0,\\ \dfrac{\partial m}{\partial t}-\nu\Delta m-\mathrm{div}\left(m\dfrac{\partial\mathcal{H}}{\partial p}(x,\nabla u;m)\right) & = & 0,\end{array}\right.$$

with the terminal and initial conditions

$$u(t=T,x)=G(x;m_T),\quad \text{and}\quad m(0,x)=m_0(x).$$

- forward/backward
- the operator in the Fokker-Planck equation is the adjoint of the linearized version of the operator in the Bellman equation
- the PDEs are coupled and there is also a possible coupling via the terminal condition on u.

Uniqueness for $(\star)$ when $\mathcal{H}(x,p;m) = H(x,p) - F(x;m)$

Theorem (Lasry-Lions) If F and G are monotone, i.e.

$$\int_{\mathbb{T}^d} (F(\cdot;m) - F(\cdot;\tilde{m}))(m - \tilde{m}) \leq 0 \Rightarrow F(\cdot;m) = F(\cdot;\tilde{m}),$$
$$\int_{\mathbb{T}^d} (G(\cdot;m) - G(\cdot;\tilde{m}))(m - \tilde{m}) \leq 0 \Rightarrow G(\cdot;m) = G(\cdot;\tilde{m}),$$

then

$(\star)$ has at most a classical solution.

Remark The assumption on F has a clear economical meaning if F is local: crowd aversion.

Proof

Consider two solutions of $(\star)$: (u_1, m_1) and (u_2, m_2):

- multiply $\text{HJB}_1 - \text{HJB}_2$ by $m_1 - m_2$:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} (u_1 - u_2)(\partial_t m_1 - \partial_t m_2) + \nu \nabla(u_1 - u_2) \cdot \nabla(m_1 - m_2) \\
&\quad + \int_0^T \int_{\mathbb{T}^d} \Big(H(x, \nabla u_1) - H(x, \nabla u_2) \Big) (m_1 - m_2) \\
= &\int_0^T \int_{\mathbb{T}^d} (F(\cdot; m_1) - F(\cdot; m_2))(m_1 - m_2) \\
&\quad + \int_{\mathbb{T}^d} \Big(G(\cdot; m_{1|t=T}) - G(\cdot; m_{2|t=T})(m_{1|t=T} - m_{2|t=T} \Big).
\end{aligned}$$

- multiply $\text{FP}_1 - \text{FP}_2$ by $u_1 - u_2$:

$$\begin{aligned}
0 = &\int_0^T \int_{\mathbb{T}^d} (u_1 - u_2)(\partial_t m_1 - \partial_t m_2) + \nu \nabla(u_1 - u_2) \cdot \nabla(m_1 - m_2) \\
&+ \int_0^T \int_{\mathbb{T}^d} \left(m_1 \frac{\partial H}{\partial p}(x, \nabla u_1) - m_2 \frac{\partial H}{\partial p}(x, \nabla u_2) \right) \cdot \nabla(u_1 - u_2).
\end{aligned}$$

- subtract:

$$0 = \begin{cases} \displaystyle\int_0^T \int_{\mathbb{T}^d} m_1 \left(H(x, \nabla u_2) - H(x, \nabla u_1) - \frac{\partial H}{\partial p}(x, \nabla u_1) \cdot \nabla(u_2 - u_1) \right) \\ \displaystyle + \int_0^T \int_{\mathbb{T}^d} m_2 \left(H(x, \nabla u_1) - H(x, \nabla u_2) - \frac{\partial H}{\partial p}(x, \nabla u_2) \cdot \nabla(u_1 - u_2) \right) \\ \displaystyle + \int_0^T \int_{\mathbb{T}^d} (F(\cdot; m_1) - F(\cdot; m_2))(m_1 - m_2) \\ \displaystyle + \int_{\mathbb{T}^d} (\cdot; G(m_{1|t=T}) - G(\cdot; m_{2|t=T}))(m_{1|t=T} - m_{2,t=T}) \end{cases}$$

- Since H is convex, F and G are monotone, the 4 terms vanish.
- Then, the strict monotonicity of F and G implies that $F(\cdot; m_1) = F(\cdot; m_2)$ and $u_1(t = T) = u_2(t = T)$.
- $u_1 = u_2$ is obtained from the uniqueness for the Bellman equation.
- $m_1 = m_2$ is obtained from the uniqueness for the Fokker-Planck equation.

Classical solutions with local couplings

$$\begin{cases} -\partial_t u - \nu\Delta u + \mathcal{H}(x, Du; m) = 0, & \text{in } (0,T)\times\mathbb{T}^d, \\ \partial_t m - \nu\Delta m - \operatorname{div}\left(m\mathcal{H}_p(x, Du; m)\right) = 0, & \text{in } (0,T)\times\mathbb{T}^d, \\ u(T,x) = G(x, m(T)), \qquad m(t_0,\cdot) = m_0 & \text{in } \mathbb{T}^d. \end{cases} \qquad \text{(MFG)}$$

If

- the Hamiltonian $\mathcal{H}(x,p;m)$ depends separately on p and m, and locally on m, i.e.
$$\mathcal{H}(x,p;m) = H(x,p) - F(x, m(x))$$
- $G(x,m) = G(x, m(x))$

then existence of classical solutions was proved in several cases:

1. H is globally Lipschitz
2. Growth conditions on H, F and if $G = G(x)$ (Lions 2012, Gomes-Pimentel-Sanchez 2013)
3. $H(x,p) = |p|^2$, $F \geq 0$, $G = G(x)$ (Cardaliaguet-Lasry-Lions-Porretta 2012)

Weak solutions

A pair $(u,m) \in L^1(Q_T) \times L^1(Q_T)_+$ is a weak solution of (MFG) if

(i) $m \in C^0([0,T]; L^1(\mathbb{T}^d))$, $G(x, m(T))$, $m(T)G(x; m(T)) \in L^1(\mathbb{T}^d)$.

(ii) $\mathcal{H}(x, Du; m) \in L^1(Q_T)$, $m\mathcal{H}(x, Du; m) \in L^1(Q_T)$ and

$$m\,|\mathcal{H}_p(x, Du; m)|^2 \in L^1(Q_T),$$

(iii) the PDEs hold in the sense of distributions:

$$\int_0^T \int_{\mathbb{T}^d} u\,\varphi_t\,dxdt - \int_0^T \int_{\mathbb{T}^d} u\,\Delta\varphi\,dxdt + \int_0^T \int_{\mathbb{T}^d} \mathcal{H}(x, Du; m)\varphi\,dxdt = \int_{\mathbb{T}^d} G(x; m(T))\varphi(T)\,dx$$

for every $\varphi \in C_c^\infty((0,T] \times \mathbb{T}^d)$,

$$\int_0^T \int_{\mathbb{T}^d} m\left\{-\varphi_t - \Delta\varphi + \mathcal{H}_p(x, Du; m)D\varphi\right\}\,dxdt = \int_{\mathbb{T}^d} m_0\,\varphi(0)\,dx$$

for every $\varphi \in C_c^\infty([0,T) \times \mathbb{T}^d)$.

Weak solutions

Weak solutions are useful in many aspects:

- More general assumptions on H, F and G (Lasry-Lions 2007, Porretta 2014)
- Degenerate diffusion (Cardaliaguet-Graber-Porretta-Tonon 2013)
- Congestion, for example $\mathcal{H}(x, p; m) = \frac{|p|^\beta}{(\mu+m)^\alpha} - F(x, m(x))$, with $1 < \beta \le 2$, $\mu \ge 0$ and $0 < \alpha < 4/\beta'$. (Y.A-Laurière 2015, Y.A.-Porretta 2016).
- Proof of convergence of numerical schemes, (Y.A.-Porretta 2015).

Difficulties:

- Uniqueness may fail for weak solutions of HJB equations (in Sobolev spaces)
- Definition of the Fokker-Planck equation: as shown in [Porretta 2014], a typical condition to ensure the uniqueness of weak solutions of the Fokker-Planck equation is

$$m|D_p\mathcal{H}(x, Du; m)|^2 \in L^1$$

- Uniqueness of weak solutions of (MFG) is more difficult to prove. It still stems from the structure of the system and monotonicity conditions.

Outline

Finite difference scheme [YA-I Capuzzo Dolcetta (2010)]

Take $d = 1$ and revert time:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = F(m) & \text{in } [0,T) \times \mathbb{T} \\ \dfrac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) = 0 & \text{in } (0,T] \times \mathbb{T} \\ u(t=0) = u_\circ \\ m(t=T) = m_\circ \end{cases} \qquad \text{(MFG)}$$

- Let $\mathbb{T}_h$ be a uniform grid on the torus with mesh step h, and x_i be a generic point in $\mathbb{T}_h$
- Uniform time grid: $\Delta t = T/N_T$, $t_n = n\Delta t$
- The values of u and m at (x_i, t_n) are approximated by u_i^n and m_i^n
- The discrete Laplace operator: $(\Delta_h w)_i = \frac{1}{h^2}(w_{i+1} - 2w_i + w_{i-1})$

Finite difference scheme

Notation The collection of the right and left sided first order finite difference formulas at x_i is noted

$$[\nabla_h w]_i = \left\{ \frac{w_{i+1} - w_i}{h}, \frac{w_i - w_{i-1}}{h} \right\} \in \mathbb{R}^2$$

For the Bellman equation, a semi-implicit monotone scheme

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = F(m)$$

$$\downarrow$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \nu(\Delta_h u^{n+1})_i + g(x_i, [\nabla_h u^{n+1}]_i) = -F(m_i^n)$$

The numerical Hamiltonian $g(x, q) = g(x, q_1, q_2)$ is $\searrow$ w.r.t. q_1 and $\nearrow$ w.r.t. q_2 , consistent with H, $\mathcal{C}^1$ regular, convex in q.

For example, if the Hamiltonian is of the form $H(x, \nabla u) = \psi(x, |\nabla u|)$, a possible choice is the **upwind scheme**:

$$g(x, q_1, q_2) = \psi\left(x, \sqrt{(q_1^-)^2 + (q_2^+)^2}\right).$$

Discrete Fokker-Planck equation

Discretize $$-\int_{\mathbb{T}} \operatorname{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w$$

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Discretize $$-\int_{\mathbb{T}} \operatorname{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w = \int_{\mathbb{T}} m\frac{\partial H}{\partial p}(x,\nabla u)\cdot\nabla w$$

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by $$h\sum_i m_i \nabla_q g(x_i,[\nabla_h u]_i)\cdot[\nabla_h w]_i$$

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by $$-h\sum_{i,j}\mathcal{T}_i(u,m)w_i \equiv h\sum_i m_i\nabla_q g(x_i,[\nabla_h u]_i)\cdot[\nabla_h w]_i$$

Discrete version of $\operatorname{div}(mH_p(x,\nabla u))$:

$$\begin{aligned}&\mathcal{T}_i(u,m)\\ =&\frac{1}{h}\left(\begin{array}{l} m_i\dfrac{\partial g}{\partial q_1}(x_i,[\nabla_h u]_i) - m_{i-1}\dfrac{\partial g}{\partial q_1}(x_{i-1},[\nabla_h u]_{i-1}) \\ +m_{i+1}\dfrac{\partial g}{\partial q_2}(x_{i+1},[\nabla_h u]_{i+1}) - m_i\dfrac{\partial g}{\partial q_2}(x_i,[\nabla_h u]_i)\end{array}\right)\end{aligned}$$

Discrete Fokker-Planck equation

Discretize $$-\int_{\mathbb{T}} \mathrm{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w = \int_{\mathbb{T}} m\frac{\partial H}{\partial p}(x,\nabla u)\cdot\nabla w$$

by $$-h\sum_{i,j}\mathcal{T}_i(u,m)w_i \equiv h\sum_i m_i\nabla_q g(x_i,[\nabla_h u]_i)\cdot[\nabla_h w]_i$$

Discrete version of $\mathrm{div}(mH_p(x,\nabla u))$:

$$\mathcal{T}_i(u,m) = \frac{1}{h}\left(\begin{array}{l} m_i\frac{\partial g}{\partial q_1}(x_i,[\nabla_h u]_i) - m_{i-1}\frac{\partial g}{\partial q_1}(x_{i-1},[\nabla_h u]_{i-1}) \\ +m_{i+1}\frac{\partial g}{\partial q_2}(x_{i+1},[\nabla_h u]_{i+1}) - m_i\frac{\partial g}{\partial q_2}(x_i,[\nabla_h u]_i) \end{array}\right)$$

Semi-implicit finite difference scheme for Fokker-Planck equation:

$$\frac{m_i^{n+1}-m_i^n}{\Delta t} + \nu(\Delta_h m^n)_i + \mathcal{T}_i(u^{n+1},m^n) = 0$$

The discrete MFG system has the same structure as the continuous one

$$\begin{cases} \dfrac{u_i^{n+1}-u_i^n}{\Delta t} - \nu(\Delta_h u^{n+1})_i + g(x_i, [\nabla_h u^{n+1}]_i) = F\left(m_i^n\right) \\ \dfrac{m_i^{n+1}-m_i^n}{\Delta t} + \nu(\Delta_h m^n)_i + \mathcal{T}_i(u^{n+1}, m^n) = 0 \end{cases}$$

Convergence to classical sol. if it exists: [YA- ICD 2010] [YA-F.Camilli-ICD 2013]

Assumptions

- *growth conditions*: there exist positive constants c_1, c_2, c_3, c_4 such that

$$\begin{aligned} g_q(x,q)\cdot q - g(x,q) &\geq c_1|g_q(x,q)|^2 - c_2, \\ |g_q(x,q)| &\leq c_3|q| + c_4. \end{aligned}$$

- F is continuous and bounded from below
- u_0 is continuous
- m_0 is a bounded nonnegative function such that $\int_{\mathbb{T}^d} m_0 = 1$

Convergence result

Theorem (Y.A.-Porretta)
Let $u_{h,\Delta t}$, $m_{h,\Delta t}$ be the piecewise constant functions which take the values $u_{i,j}^{n+1}$ and $m_{i,j}^{n}$, respectively, in $(t_n, t_{n+1}) \times (ih - h/2, ih + h/2) \times (jh - h/2, jh + h/2)$.

There exist functions $\tilde{u}$, $\tilde{m}$ such that

1. *after the extraction of a subsequence, $u_{h,\Delta t} \to \tilde{u}$ and $m_{h,\Delta t} \to \tilde{m}$ in $L^\beta(Q)$ for all $\beta \in [1, \frac{d+2}{d})$*
2. *$\tilde{u}$ and $\tilde{m}$ belong to $L^\alpha(0, T; W^{1,\alpha}(\mathbb{T}^2))$ for any $\alpha \in [1, \frac{d+2}{d+1})$*
3. *$(\tilde{u}, \tilde{m})$ is a weak solution to the system (MFG) in the following sense:*
   1. $$H(\cdot, D\tilde{u}) \in L^1(Q), \qquad \tilde{m}F(\tilde{m}) \in L^1(Q),$$
      $$\tilde{m}\Big(H_p(\cdot, D\tilde{u}) \cdot D\tilde{u} - H(\cdot, D\tilde{u})\Big) \in L^1(Q)$$
   2. *$(\tilde{u}, \tilde{m})$ satisfies the system (MFG) in the sense of distributions*
   3. *$\tilde{u}, \tilde{m} \in C^0([0, T]; L^1(\mathbb{T}^2))$ and $\tilde{u}|_{t=0} = u_0$, $\tilde{m}|_{t=T} = m_T$.*

Outline

Modeling congestion

$$
\begin{aligned}
-\partial_t u - \nu\Delta u + H(t,x,m,Du) &= F(t,x,m)\,, && (t,x)\in(0,T)\times\mathbb{T}^d\\
\partial_t m - \nu\Delta m - \operatorname{div}(mH_p(t,x,m,Du)) &= 0\,, && (t,x)\in(0,T)\times\mathbb{T}^d\\
m(0,x)=m_0(x)\,,\ u(T,x) &= G(x,m(T))\,, && x\in\mathbb{T}^d
\end{aligned}
$$

Typical Hamiltonian: $H(t,x,m,p) = \frac{|p|^\beta}{(m+\mu)^\alpha}$ for some $1<\beta\le 2$ and $0<\alpha\le\frac{4(\beta-1)}{\beta}$ and $\mu\ge 0$.

- The related cost is $L(\gamma,m)\sim(m+\mu)^{\frac{\alpha}{\beta-1}}|\gamma|^{\beta'}$
- $0\le\alpha\le\frac{4(\beta-1)}{\beta}$ was found by P-L. Lions as a sufficient condition for the uniqueness of a classical solution: it implies that

$$
\begin{pmatrix} 2H_{p,p}(t,x,m,p) & H_{m,p}(t,x,m,p)\\ H_{m,p}(t,x,m,p) & -\frac{2}{m}H_m(t,x,m,p)\end{pmatrix} > 0
$$

- Possible singularity : note that if $\mu=0$, then the Hamiltonian is $+\infty$ as $m=0$.

Main assumptions on the coupling cost F: F is measurable w.r.t. $(t,x) \in Q_T$ and C^0 w.r.t. m;

$$\exists c \in \mathbb{R}: \quad F(t,x,m) \geq c$$

$$\forall L > 0, \quad \sup_{m \in [0,L]} |F(t,x,m)| \in L^1(Q_T).$$

Theorem (stated here only for non singular congestion: $\mu > 0$)
There exists a weak solution (unique if F is increasing), i.e. a pair $(u,m) \in L^1(Q_T) \times (L^1(Q_T))_+$ such that

1. $m \in C^0([0,T]; L^1(\mathbb{T}^d))$, $m(T)G(x,m(T)) \in L^1(\mathbb{T}^d)$, $mF(t,x,m) \in L^1(Q_T)$
2. $$\frac{|Du|^\beta}{(m+\mu)^\alpha} \in L^1(Q_T), \qquad m\frac{|Du|^\beta}{(m+\mu)^\alpha} \in L^1(Q_T), \qquad m^{1+\frac{\alpha}{\beta-1}} \in L^1(Q_T)$$
3. the HJB and FP equations hold in the sense of distributions.

Weak solutions when $\mu = 0$: $(u,m) \in L^1(Q_T) \times L^1(Q_T)_+$ s.t.

(i) $m \in C^0([0,T]; L^1(\mathbb{T}^d))$, $m(T)G(x,m(T)) \in L^1(\mathbb{T}^d)$ and $mF(t,x,m) \in L^1(Q_T)$

(ii)

$$m^{-\alpha}|Du|^\beta \, \mathbb{1}_{\{m>0\}} \in L^1(Q_T), \qquad m^{1-\alpha}|Du|^\beta \, \mathbb{1}_{\{m>0\}} \in L^1(Q_T)$$

$$Du = 0 \quad \text{a.e. in } \{m=0\}, \qquad m^{1+\frac{\alpha}{\beta-1}} \in L^1(Q_T)$$

(iii) u is a subsolution of the HJB equation: for every $0 \le \varphi \in C_c^\infty((0,T] \times \mathbb{T}^d)$,

$$\int_0^T \int_{\mathbb{T}^d} u\,\varphi_t \,dxdt - \int_0^T \int_{\mathbb{T}^d} u\,\Delta\varphi \,dxdt + \int_0^T \int_{\mathbb{T}^d} H(t,x,m,Du) \mathbb{1}_{\{m>0\}} \varphi \,dxdt$$
$$\le \int_0^T \int_{\mathbb{T}^d} F(t,x,m)\varphi \,dxdt + \int_{\mathbb{T}^d} G(x,m(T))\varphi(T)\,dx$$

(iv) m is a solution of the Kolmogorov equation: for every $\varphi \in C_c^\infty([0,T) \times \mathbb{T}^d)$,

$$\int_0^T \int_{\mathbb{T}^d} m \left\{ -\varphi_t - \Delta\varphi + H_p(t,x,m,Du) \mathbb{1}_{\{m>0\}} D\varphi \right\} dxdt = \int_{\mathbb{T}^d} m_0\,\varphi(0)\,dx$$

(v) Energy identity:

$$\int_{\mathbb{T}^d} m_0\,u(0)\,dx = \int_{\mathbb{T}^d} G(m(T))\,m(T)\,dx + \int_0^T \int_{\mathbb{T}^d} F(t,x,m)m \,dxdt$$
$$+ \int_0^T \int_{\mathbb{T}^d} \left[m\,H_p(t,x,m,Du)\cdot Du - H(t,x,m,Du) \right] \mathbb{1}_{\{m>0\}} dxdt$$

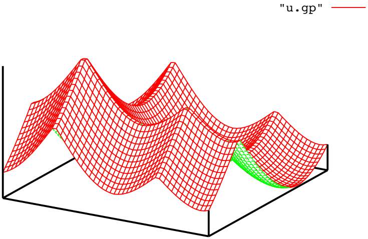
Outline

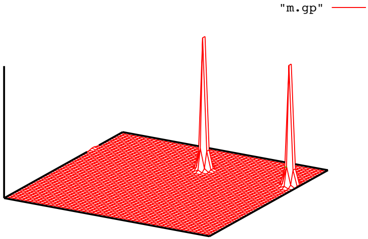
Robustness in the deterministic limit: an example with an infinite horizon

$$(*) \begin{cases} \rho - \nu\Delta u + H(x, \nabla u) = F(x; m), \\ -\nu\Delta m - \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u)\right) = 0, \end{cases}$$

with

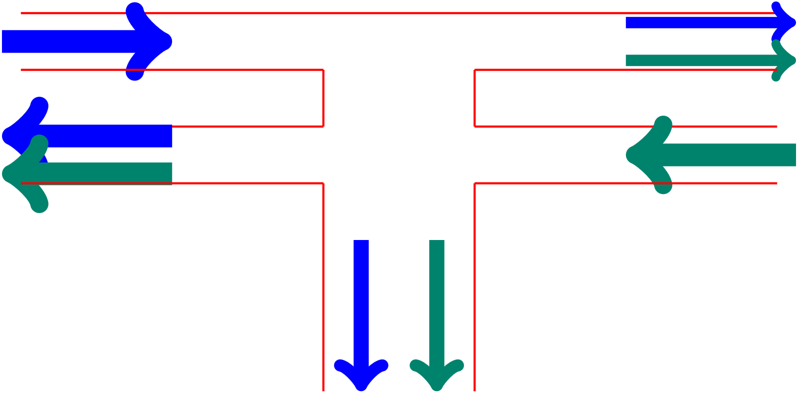
$$\nu = 0.001, \qquad H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^{3/2},$$
$$F(x;m) = \left((1-\Delta)^{-1}(1-\Delta)^{-1} m\right)(x)$$





MFG with 2 populations and congestion effects

Purpose: try a MFG model for a crossroad with two main flows of vehicles.



The system of PDEs

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + \nu \Delta u_1 - H_1(x, \nabla u_1; m_1, m_2) &= -\Phi_1(m_1, m_2) \\
\frac{\partial m_1}{\partial t} - \nu \Delta m_1 - \operatorname{div}\left(m_1 \frac{\partial H_1}{\partial p}(x, \nabla u_1; m_1, m_2) \right) &= 0 \\
\frac{\partial u_2}{\partial t} + \nu \Delta u_2 - H_2(x, \nabla u_2; m_2, m_1) &= -\Phi_2(m_2, m_1) \\
\frac{\partial m_2}{\partial t} - \nu \Delta m_2 - \operatorname{div}\left(m_2 \frac{\partial H_2}{\partial p}(x; \nabla u_2; m_2, m_1) \right) &= 0
\end{aligned}
$$

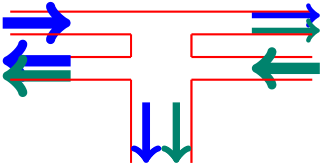
- The Hamiltonian for the population labeled i is

$$
H_i(x, p; m_i, m_j) = \frac{|p|^2}{1 + m_i + 5 m_j}
$$

- The coupling cost for the population labeled i is

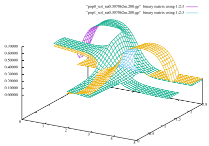
$$
\Phi_i(x, m_i, m_j) = 0.5 + 0.5 \left(\frac{m_i}{m_i + m_j + \epsilon} - 0.5 \right)_- + (m_i + m_j - 4)_+
$$

Boundary conditions

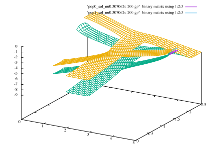


- Exit costs for the population labeled 0:
  1. North-West and South-East exits: 0
  2. South-West exit : -8.5
  3. North-East exit : -4
  4. South exit : -7
- Exit costs for the population labeled 1:
  1. North-West and South-East exits: 0
  2. South-West exit : -7
  3. North-East and South exit : -4
- Entry fluxes
  1. Population 0: at the North-West exit, the entry flux is 1
  2. Population 1: at the South-East exit, the entry flux is 1

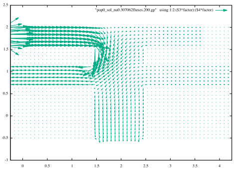
Stationary equilibrium for $\nu \sim 0.3$



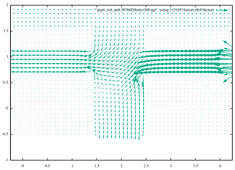
((a)) Distributions of the two populations



((b)) Value functions of the two populations



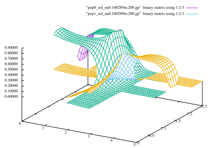
((c)) Fluxes for population 0



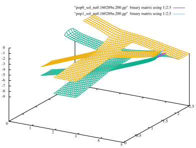
((d)) Fluxes for population 1

Figure : Numerical Solution to Stationary Equilibrium $\nu \sim 0.31$

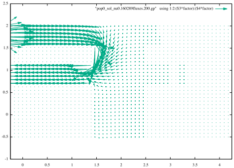
Stationary equilibrium for $\nu \sim 0.15$



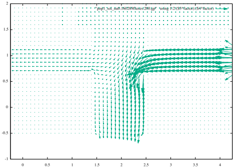
((a)) Distributions of the two populations



((b)) Value functions of the two populations



((c)) Fluxes for population 0



((d)) Fluxes for population 1

Figure : Numerical Solution to Stationary Equilibrium $\nu \sim 0.16$

MFGs of control (with Z. Kobeissi)

The agents interact via the distribution of states **and controls** : $\mu(t) = \mathcal{L}(X_t, \gamma_t)$

- Let $V(t,x)$ be an average drift at $x \in \Omega$ and $t \in [0,T]$:

$$V(t,x) = \frac{1}{Z(t,x)} \int_{\Omega \times \mathbb{R}^d} \gamma \, K(x,y) \, d\mu(t,y,\gamma), \text{ with } \mu(t) = \mathcal{L}(X_t, \gamma_t),$$

  where K is a kernel and $Z(t,x)$ is a normalization factor.

- The cost to be minimized by a given agent is

$$J(\gamma) = \mathbb{E}\Big\{ G(X_T; m(T)) + \int_0^T \frac{\alpha}{2} |\gamma_t - \lambda V(t,X_t)|^2 + \frac{1-\alpha}{2} |\gamma_t|^2 + F(X_t; m(t))\, dt \Big\},$$

  with $\lambda < 1, 0 \leq \alpha \leq 1, m(t) = \mathcal{L}(X_t)$.

- The Hamiltonian associated to this control problem is

$$H(x,p,V) = \frac{1}{2} |p - \lambda \alpha V|^2 - \frac{\lambda^2 \alpha}{2} |V|^2.$$

The forward-backward system

The system of PDEs is

$$\begin{cases} -\partial_t u(t,x) - \nu\Delta u(t,x) + \frac{1}{2}\left|\nabla_x u(t,x) - \lambda\alpha V(t,x)\right|^2 - \frac{\lambda^2\alpha}{2}\left|V(t,x)\right|^2 = F(x;m(t)), \\ \partial_t m_t(t,x) - \nu\Delta m(t,x) - \operatorname{div}\Big(\big(\nabla_x u - \lambda\alpha V(t,x)\big)m(t,x)\Big) = 0, \\ V(t,x) = \int_\Omega \big(-\nabla_x u(t,y) + \lambda\alpha V(t,y)\big)\frac{K(x,y)}{Z(t,x)}dm(t,y), \\ Z(t,x) = \int_\Omega K(x,y)dm(t,y), \end{cases}$$

with boundary conditions

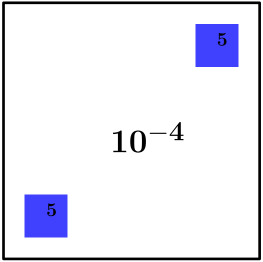
$$\begin{cases} u(T,x) = G(x;m(T)), & m(0) = m_\circ, \\ \frac{\partial u}{\partial n} - \lambda\alpha V\cdot n = 0, & \frac{\partial m}{\partial n} = 0, \text{ on } \partial\Omega. \end{cases}$$

- Existence of solutions (Z. Kobeissi)
- Uniqueness: no, in general

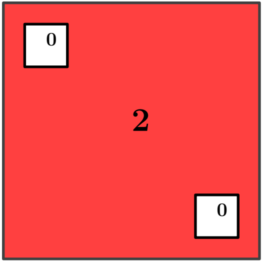
Parameters of the simulations

- $\Omega = [-0.5, 0.5]^2$.
- V is a piecewise linear interpolation of averages of the control in 3×3 equal subdomains.
- $T = 1, \lambda = 0.9, \alpha = 1$ and $\nu = 10^{-4}$.
- No coupling cost.
- The initial mass is distributed in two symmetric corners of the domain and the terminal cost pushes the agents to go towards the other two corners of the domains.

Initial distribution



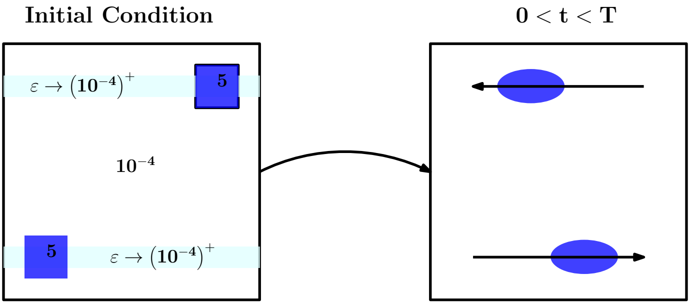
Terminal cost



A first symmetrical solution

Non-symmetrical solutions

An evanescent part is added to the initial distribution, and a continuation method is used



Non-symmetrical solutions

With another cost which models crowd aversion and a more local formula for V