Espaces grossiers adaptatifs pour les méthodes de décomposition de domaines à deux niveaux

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Outline

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1. Introduction
   - Classical coarse space method
2. An abstract 2-level Schwarz: the GenEO algorithm
3. Conclusion

Motivation

Large discretized system of PDEs
strongly heterogeneous coefficients
(high contrast, nonlinear, multiscale)

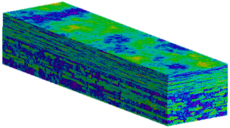
E.g. Darcy pressure equation,
P^1-finite elements:

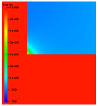
$$A\mathbf{U} = \mathbf{F}$$

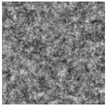
$$\text{cond}(A) \sim \frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} h^{-2}$$

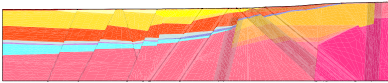
Goal:
iterative solvers
robust in size and heterogeneities

Applications:
flow in heterogeneous / stochastic / layered media
structural mechanics
electromagnetics
etc.







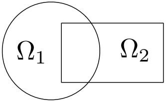


Linear Algebra from the End User point of view

Direct	DDM	Iterative
Cons: Memory Difficult to $\|\|$ Pros: Robustness	Pro: Flexible Naturally $\|\|$	Pros: Memory Easy to $\|\|$ Cons: Robustness
solve(MAT,RHS,SOL)	Few black box routines Few implementations of efficient DDM	solve(MAT,RHS,SOL)

Multigrid methods: very efficient but may lack robustness, not always applicable (Helmholtz type problems, complex systems) and difficult to parallelize.

The original Schwarz Method (H.A. Schwarz, 1870)



$$-\Delta(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Schwarz Method : $(u_1^n, u_2^n) \to (u_1^{n+1}, u_2^{n+1})$ with

$$\begin{array}{ll} -\Delta(u_1^{n+1}) = f \quad \text{in } \Omega_1 & -\Delta(u_2^{n+1}) = f \quad \text{in } \Omega_2 \\ u_1^{n+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega & u_2^{n+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \\ u_1^{n+1} = u_2^n \quad \text{on } \partial\Omega_1 \cap \overline{\Omega_2}. & u_2^{n+1} = u_1^{n+1} \quad \text{on } \partial\Omega_2 \cap \overline{\Omega_1}. \end{array}$$

Parallel algorithm, converges but very slowly, overlapping subdomains only.
The parallel version is called **Jacobi Schwarz method (JSM)**.

Strong and Weak scalability

How to evaluate the efficiency of a domain decomposition?

Strong scalability (Amdahl)

"How the solution time varies with the number of processors for a fixed *total* problem size"

Weak scalability (Gustafson)

"How the solution time varies with the number of processors for a fixed problem size *per processor*."

Not achieved with the one level method

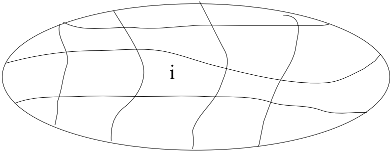
Number of subdomains	8	16	32	64
ASM	18	35	66	128

The iteration number increases linearly with the number of subdomains in one direction.

How to achieve scalability

Stagnation corresponds to a few very low eigenvalues in the spectrum of the preconditioned problem. They are due to the lack of a global exchange of information in the preconditioner.

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$



The mean value of the solution in domain i depends on the value of f on all subdomains.

A classical remedy consists in the introduction of a coarse problem that couples all subdomains. This is closely related to deflation technique classical in linear algebra (see Nabben and Vuik's papers in SIAM J. Sci. Comp, 200X).

We add a coarse space correction (*aka* second level)
Let V_H be the coarse space and Z be a basis, $V_H = \mathrm{span}\, Z$, writing $R_0 = Z^T$ we define the two level preconditioner as:

$$M_{ASM,2}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i .$$

The Nicolaides approach is to use the kernel of the operator as a coarse space, this is the constant vectors, in local form this writes:

$$Z := (R_i^T D_i R_i \mathbf{1})_{1 \le i \le N}$$

where D_i are chosen so that we have a partition of unity:

$$\sum_{i=1}^{N} R_i^T D_i R_i = Id .$$

Theoretical convergence result

Theorem (Widlund, Dryija)

Let $M_{ASM,2}^{-1}$ be the two-level additive Schwarz method:

$$\kappa(M_{ASM,2}^{-1}A) \leq C\left(1+\frac{H}{\delta}\right)$$

where δ is the size of the overlap between the subdomains and H the subdomain size.

This does indeed work very well

Number of subdomains	8	16	32	64
ASM	18	35	66	128
ASM + Nicolaides	20	27	28	27

$$
\begin{aligned}
-\nabla \cdot (\alpha(x,y)\nabla u) &= 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \\
u &= 0 \quad \text{on} \quad \partial\Omega_D, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega_N.
\end{aligned}
$$

Decomposition

$\alpha(x,y)$

Jump	1	10	10^2	10^3	10^4
ASM	39	45	60	72	73
ASM + Nicolaides	30	36	50	61	65

Our approach

Fix the problem by an optimal and proven choice of a coarse space Z.

Outline

Objectives

Strategy

Define an appropriate coarse space $V_{H2} = \mathrm{span}(Z_2)$ and use the framework previously introduced, writing $R_0 = Z_2^T$ the two level preconditioner is:

$$P_{ASM2}^{-1} := R_0^T \left(R_0 A R_0^T\right)^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i .$$

The coarse space must be

- Local (calculated on each subdomain) $\rightarrow$ parallel
- Adaptive (calculated automatically)
- Easy and cheap to compute
- Robust (must lead to an algorithm whose convergence is proven not to depend on the partition nor the jumps in coefficients)

Abstract eigenvalue problem

Gen.EVP per subdomain:

Find $p_{j,k} \in V_{h|\Omega_j}$ and $\lambda_{j,k} \geq 0$:

$$a_{\Omega_j}(p_{j,k},\, v) \;=\; \lambda_{j,k}\, a_{\Omega_j^\circ}(\Xi_j p_{j,k},\, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

$$A_j \mathbf{p}_{j,k} \;=\; \lambda_{j,k}\, \mathbf{X}_j A_j^\circ \mathbf{X}_j\, \mathbf{p}_{j,k} \qquad (\mathbf{X}_j \ldots \text{diagonal})$$

$a_D \ldots$ restriction of a to D

In the two-level ASM:

Choose first m_j eigenvectors per subdomain:

$$V_0 \;=\; \operatorname{span}\{\Xi_j p_{j,k}\}_{k=1,\ldots,m_j}^{j=1,\ldots,N}$$

This automatically includes Zero Energy Modes.

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Comparison with existing works

Galvis & Efendiev (SIAM 2010):

$$\int_{\Omega_j} \boldsymbol{\kappa} \nabla p_{j,k} \cdot \nabla v \, dx \; = \; \lambda_{j,k} \int_{\Omega_j} \boldsymbol{\kappa} \, p_{j,k} \, v \, dx \qquad \forall v \in V_{h|\Omega_j}$$

Efendiev, Galvis, Lazarov & Willems (submitted):

$$a_{\Omega_j}(p_{j,k}, \, v) \; = \; \lambda_{j,k} \sum_{i \in \text{neighb}(j)} a_{\Omega_j}(\xi_j \, \xi_i \, p_{j,k}, \, \xi_j \, \xi_i \, v) \qquad \forall v \in V_{|\Omega_j}$$

$\xi_j \ldots$ partition of unity, calculated adaptively (MS)

Our gen.EVP:

$$a_{\Omega_j}(p_{j,k}, \, v) \; = \; \lambda_{j,k} \, a_{\Omega_j^\circ}(\Xi_j p_{j,k}, \, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

both matrices typically singular $\implies \lambda_{j,k} \in [0, \, \infty]$

Two technical assumptions.

Theorem (Spillane, Dolean, Hauret, N., Pechstein, Scheichl)

If for all j: $0 < \lambda_{j,m_{j+1}} < \infty$*:*

$$\kappa(M^{-1}_{ASM,2}A) \leq (1+k_0)\left[2 + k_0\,(2k_0+1)\,\max_{j=1}^{N}\left(1 + \frac{1}{\lambda_{j,m_j+1}}\right)\right]$$

Possible criterion for picking m_j: (used in our Numerics)

$$\lambda_{j,m_j+1} < \frac{\delta_j}{H_j}$$

H_j ... subdomain diameter, δ_j ... overlap

Numerical results via a Domain Specific Language

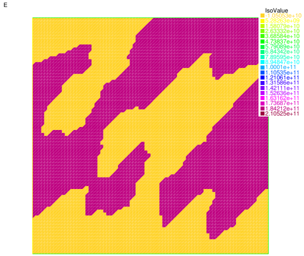
FreeFem++ (http://www.freefem.org/ff++), with:

- Metis Karypis and Kumar 1998
- SCOTCH Chevalier and Pellegrini 2008
- UMFPACK Davis 2004
- ARPACK Lehoucq et al. 1998
- MPI Snir et al. 1995
- Intel MKL
- PARDISO Schenk et al. 2004
- MUMPS Amestoy et al. 1998
- PaStiX Hénon et al. 2005

Why use a DS(E)L instead of C/C++/Fortran/.. ?

- performances close to low-level language implementation,
- hard to beat something as simple as:

$$\text{varf a(u, v)} = \textbf{int3d(mesh)}([\textbf{dx}(\text{u}), \textbf{dy}(\text{u}), \textbf{dz}(\text{u})]' * [\textbf{dx}(\text{v}), \textbf{dy}(\text{v}), \textbf{dz}(\text{v})]) \\ + \textbf{int3d(mesh)}(\text{f} * \text{v}) + \textbf{on}(\text{boundary\_mesh})(\text{u} = 0)$$



$$E_1 = 2 \cdot 10^{11}$$
$$\nu_1 = 0.3$$

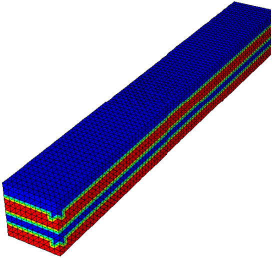
$$E_2 = 2 \cdot 10^{7}$$
$$\nu_2 = 0.45$$

METIS partitions with 2 layers added

subd.	dofs	AS-1	AS-ZEM	(V_H)	GENEO	(V_H)
4	13122	93	134	*(12)*	42	*(42)*
16	13122	164	165	*(48)*	45	*(159)*
25	13122	211	229	*(75)*	47	*(238)*
64	13122	279	167	*(192)*	45	*(519)*

Numerics – 3D Elasticity

Iterations (CG) vs. number of subdomains



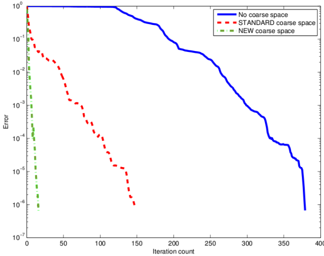
$E_1 = 2 \cdot 10^{11}$

$\nu_1 = 0.3$

$E_2 = 2 \cdot 10^{7}$

$\nu_2 = 0.45$

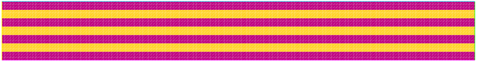
Relative error vs. iterations
16 regular subdomains



subd.	dofs	AS-1	AS-ZEM	(V_H)	GENEO	(V_H)
4	1452	79	54	*(24)*	16	*(46)*
8	29040	177	87	*(48)*	16	*(102)*
16	58080	378	145	*(96)*	16	*(214)*

AS-ZEM (Rigid body motions): $m_j = 6$

Numerical results – Optimality



Layers of hard and soft elastic materials

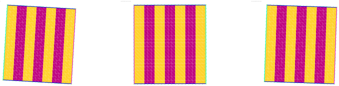
m_i is given automatically by the strategy.

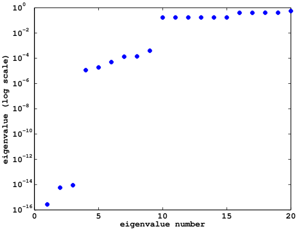
$\#Z$ per subd.	one level	ZEM	GenEO
$\max(m_i - 1, 3)$			2600 (93)
m_i	$5.1\,e5$ (184)	$1.4\,e4$ (208)	53 (35)
$m_i + 1$			45 (25)

condition number (iteration count) for one and two level ASMs

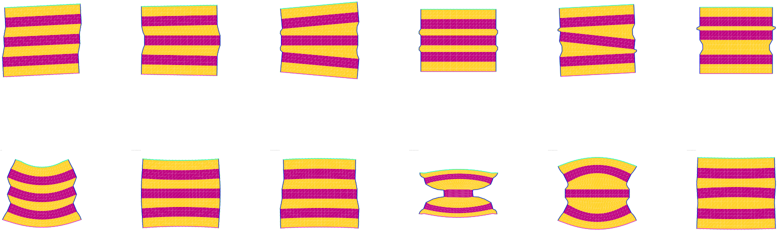
- Taking one fewer eigenvalue has a huge influence on the iteration count
- Taking one more has only a small influence

Eigenvalues and eigenvectors





Logarithmic scale



Darcy pressure equation

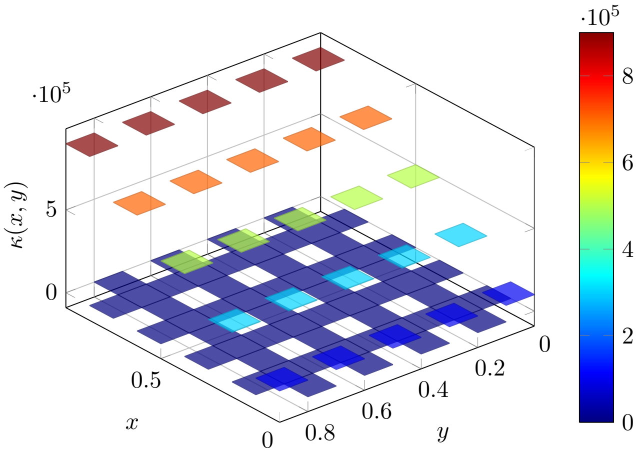
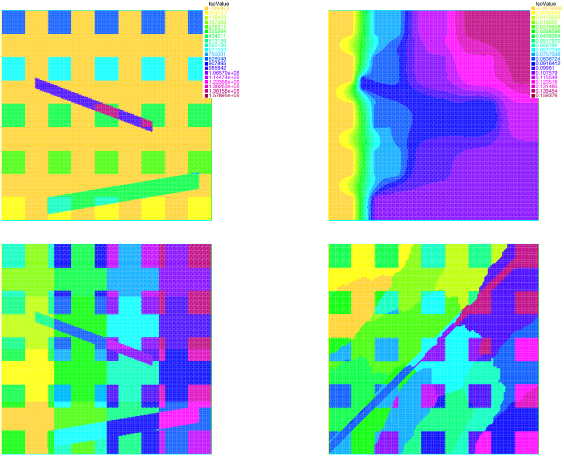


Figure : Two dimensional diffusivity κ

Channels and inclusion



Channels and inclusions: $1 \leq \alpha \leq 1.5 \times 10^6$, the solution and partitionings (Metis or not)

Parallel implementation

PhD of Pierre Jolivet.

Since version 1.16, bundled with the `Message Passing Interface`. `FreeFem++` is working on the following parallel architectures (among others):

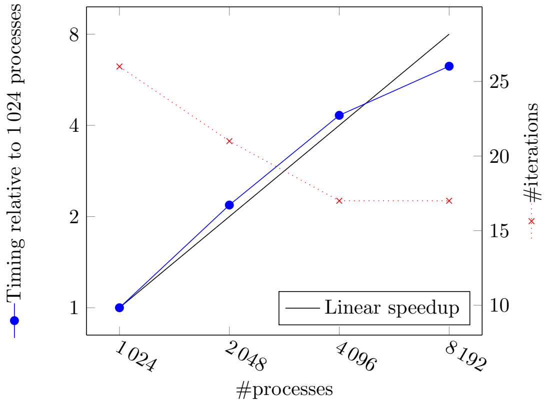
	N° of cores	Memory	Peak perf
hpc1@LJLL	160@2.00 Ghz	640 Go	~ 10 TFLOP/s
titane@CEA	12192@2.93 Ghz	37 To	140 TFLOP/s
babel@IDRIS	40960@850 Mhz	20 To	139 TFLOP/s
curie@CEA	92000@2.93 Ghz	315 To	1.6 PFLOP/s

`http://www-hpc.cea.fr`, Bruyères-le-Châtel, France.
`http://www.idris.fr`, Orsay, France.

`http://www.prace-project.eu`.

Strong scalability in two dimensions heterogeneous elasticity

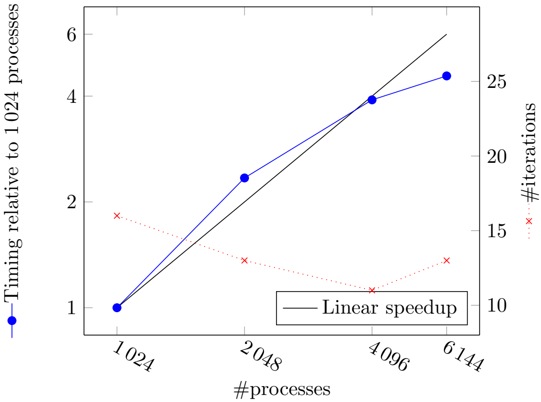
Elasticity problem with heterogeneous coefficients



Speed-up for a 1.2 billion unknowns 2D problem. Direct solvers in the subdomains. Peak performance wall-clock time: 26s.

Strong scalability in three dimensions heterogeneous elasticity

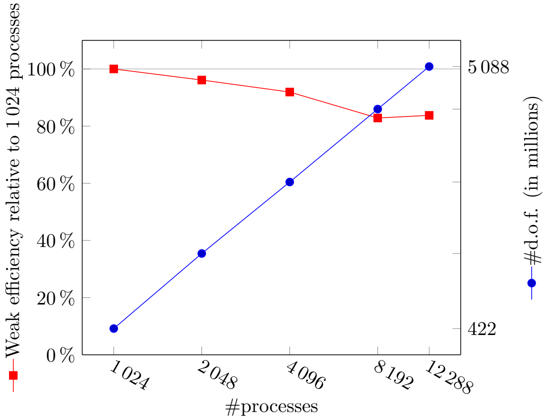
Elasticity problem with heterogeneous coefficients



Speed-up for a 160 million unknowns 3D problem. Direct solvers in subdomains. Peak performance wall-clock time: 36s.

Weak scalability in two dimensions

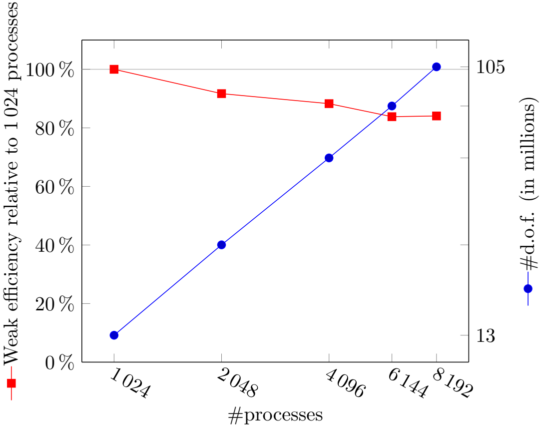
Darcy problems with heterogeneous coefficients



Efficiency for a 2D problem. Direct solvers in the subdomains.
Final size: 22 billion unknowns. Wall-clock time: $\simeq$ 200s.

Weak scalability in three dimensions

Darcy problems with heterogeneous coefficients



Efficiency for a 3D problem. Direct solvers in the subdomains.
Final size: 2 billion unknowns. Wall-clock time: $\simeq$ 200s.

Outline

Conclusion

GenEO for ASM

- Implementation requires only element stiffness matrices + connectivity
- FreeFem++ MPI implementation

Not shown here

- GenEO for BNN/FETI methods

Outlook

- Nonlinear time dependent problem (Reuse of the coarse space)
- Coarse problem satisfies assembling property
  $\mapsto$ multilevel method — link to σAMGe ?
- Other discretizations: finite volume, finite difference, ...

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Bibliography Schwarz: DtN and GenEO

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V. Dolean, F. Nataf, P. Jolivet: *”Domain decomposition methods. Theory and parallel implementation with FreeFEM++”*, Lecture notes (coming soon).

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