Convergence of Domain Decomposition Methods via Semi-Classical Calculus

Frédéric Nataf* and Francis Nier**

* CMAP, Ecole Polytechnique
URA-CNRS 756
91128 Palaiseau Cedex, France
E-mail address: nataf@cmapx.polytechnique.fr
** CMAT, Ecole Polytechnique
URA-CNRS 169
91128 Palaiseau Cedex, France
E-mail address: nier@orphee.polytechnique.fr

In this paper, we study the convergence of domain decomposition methods for the solving of advection-diffusion equations

$$(B\nabla - h\ \mathrm{div}\, C\nabla)(u) = f, \tag{0.1}$$

where B is a given vector field ($B_x > 0$), h is the viscosity and C is a positive definite symmetric matrix. Equation (0.1) models the transport of a quantity u (e.g. dyer, temperature, energy, ...) by a vector field B and its diffusion scaled by a usually small viscosity coefficient h. It arises in many different areas like environmental flows, semi-conductors, fluid dynamics,... . It is also involved in the numerical computation of Navier-Stokes solutions by successive linearizations techniques (see e.g. [9]). Domain decomposition methods are well-fitted to the solving of (0.1) for very large scale problems on parallel computers. Roughly speaking, the idea is to solve a boundary value problem by decomposing the domain into overlapping or nonoverlapping subdomains. The equation is satisfied in each subdomain. In order to enforce continuity of the solution and of its derivatives, the interface conditions are imposed in an iterative manner. The main computational interest lies in the saving of memory which enables to treat very large scale problems (see [13] and references herein). It is also of mathematical interest since it is related to the factorization of elliptic operators and to the study of Dirichlet-to-Neumann operator.

We consider three iterative domain decomposition methods in a simple geometry. The whole space is decomposed into N vertical strips with possible overlaps. The three methods differ by the updating of the solution in every in subdomain. The first algorithm is the additive Schwarz method (ASM) which consists in updating the solution at the same time in every subdomain. In the second algorithm denoted by DSA, the updating is made by double sweeps over the domain. In the third method called FDA, the solution is updated by flow directed sweeps over the domain (flow directed Gauss-Seidel method, see [12]). In the overlapping case for which few results are available up to now, we obtain geometric convergence. In [8] and [15], convergence was proved for specific interface conditions, derived from Dirichlet/Fourier boundary conditions, by a blend of energy estimates and maximum principles. Here, we follow a different approach and extend for general boundary conditions the results obtained with constant coefficients in [16]. In the nonoverlapping case, our convergence analysis is based on energy estimates similar in principle to those of [6],[14] and [17]. Moreover, we check that the convergence is faster in some sense for a judicious choice of the interface conditions, derived from absorbing boundary conditions.

The clue is a careful study of the Dirichlet-to-Neumann operator associated with the convection-diffusion operator (0.1) on the half-space. Owing to the presence of the viscosity h as a small parameter, we develop a semi-classical analysis which extends naturally the Fourier analysis of [16] for constant coefficients. Hence, all the results, which we mentionned and which are detailed further, hold for small enough values of h. This analysis is developed in the framework of Weyl-Hörmander calculus allowed by our flat geometry. Indeed this calculus takes into account low frequencies and gives at once global norm estimates, which is crucial while proving convergence.

Contents

1 Semi-Classical Second Order Elliptic Half-Space Problems

In this section, we first consider the Dirichlet-to-Neumann operator associated with semi-classical elliptic half-space problems. By Beals criterion (see Appendix A) we check that it is an h-pseudo-differential operator. Then by using the factorization of Appendix B, we construct an approximation of this operator which provides additional information. Our final aim is to write exact solutions of semi-classical second order elliptic half-space problems in terms of parabolic evolution systems.

1.1 Dirichlet-to-Neumann operators

We consider second order differential operators on $\mathbb{R}^{1+d} = \mathbb{R}_x \times \mathbb{R}_y^d$ which depend on a small parameter $h \in (0, h_0)$,

$$\mathcal{L}^h = a + {}^tB(h\partial) - {}^t(h\partial)C(h\partial) \tag{1.1}$$

$$\text{with} \quad \partial = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \quad B = \begin{pmatrix} b_x \\ B_y \end{pmatrix} \quad \text{and } C = {}^tC = \begin{pmatrix} 1 & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}.$$

Here and in the sequel we use small letters for scalar coefficients and capital letters for matrices.

Hypothesis: a) The coefficients a, b_x, b_{y_i}, c_{x,y_i}, $c_{y_i,x}$ and c_{y_i,y_j}, $i, j = 1 \dots d$, are real-valued functions of $(x, y; h) \in \mathbb{R}^{1+d} \times (0, h_0)$ supposed to be uniformly bounded with respect to $h \in (0, h_0)$ in $S(1, dx^2 + dy^2)$.
b) Moreover we assume

$$a(x, y; h) \geq \alpha^2, \quad \text{and } C(x, y; h) \geq \gamma^2 Id, \quad \forall (x, y; h) \in \mathbb{R}^{1+d} \times (0, h_0),$$

where α and γ are two positive constants.

Remark 1.1 *a) If the matrix C satisfies all the above assumptions except $c_{xx}=1$, we go back to the described situation by taking suitable constants α, γ and h_0 for the operator*

$$\frac{1}{c_{xx}}\mathcal{L}^h = a' + {}^tB'(h\partial) - {}^t(h\partial)C'(h\partial)$$

where $a'=\frac{a}{c_{xx}}$, $C'=\frac{1}{c_{xx}}C$ and $B'=\frac{1}{c_{xx}}\begin{pmatrix} b_x - h\partial_x c_{xx} - \frac{h}{c_{xx}}({}^t\partial_y c_{xx})C_{yx} \\ B_y - h\frac{\partial_x c_{xx}}{c_{xx}}C_{xy} - \frac{h}{c_{xx}}(\partial_y c_{xx})C_{yy}\end{pmatrix}$.

b) The drift-diffusion operators (0.1) do not exactly correspond to these hypotheses. Meanwhile if we assume $b_x \geq 2\beta$ and $a\geq 0$ in (0.1), conjugating with $e^{-\frac{\beta x}{h}}$ brings back to our assumptions. Indeed we have

$$e^{-\frac{\beta x}{h}}\mathcal{L}^h e^{\frac{\beta x}{h}} = a_\beta + {}^tB_\beta(h\partial) - {}^t(h\partial)C(h\partial)$$

with $B_\beta = \begin{pmatrix} b_x - 2\beta \\ B_y - 2\beta C_{x,y}\end{pmatrix}$ and $a_\beta = \beta b_x - \beta^2 - h\,{}^t\partial_y C_{y,x}\beta \geq \dfrac{\beta^2}{2}$

for h small enough.
c) In the sequel we may have to restrict the domain of the small parameter h. The upper bound h_0 will generically denote a constant which is determined by the operator $\mathcal{L}^h$ (more precisely by a finite number of semi-norms in $S(1,dx^2+dy^2)$ of its coefficients)

Definition 1.2 *The Dirichlet-to-Neumann operator denoted by $\Lambda^{\mp,h}(x_0)$ is defined on $H^{s+1/2,h}(\mathbb{R}^d)$, $s\geq 0$, $h\in(0,h_0)$, by $\Lambda^{\mp,h}(x_0)u_0 = h\partial_x u\big|_{x=x_0}$ where u is the variational solution of*

$$\begin{cases} \mathcal{L}^h u = 0, & x \gtrless x_0 \\ u\big|_{x=x_0} = u_0. \end{cases} \tag{1.2}$$

The precise framework of this definition will be briefly reviewed in the next lemmas. Our aim is to prove

Theorem 1.3 *The operator $\Lambda^{\pm,h}(x_0)$ equals $\lambda^\pm(x_0,y,hD_y;h)$, where $\lambda^\pm$ belongs to the symbol class $\mathbb{P}^0S^{h_0}(\langle\eta\rangle,g_\eta)$ defined in Appendix B.1.*

In order to write accurate estimates uniform with respect to $h\in(0,h_0)$, we will use Sobolev spaces $H^{s,h}(\Omega)$ which depend on h via their norm. When $\Omega=\mathbb{R}^n$, we take $\|u\|_{H^{s,h}(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}$, according to the notations of Appendix A. The definition of $H^{s,h}(\Omega)$ and $H_0^{s,h}(\Omega)$ for any "smooth" open set Ω follow as usual. Notice $\|u\|_{H^{s,h}(\Omega)} = \|D_h^{-1}u\|_{H^s(\Omega)}$ with $D_hu(x) = h^{-\frac{n}{2}}u(\frac{x}{h})$, when Ω is a convex cone of $\mathbb{R}^n$ with vertex 0, while $\|u\|^2_{H^{s,h}(\Omega)} = \sum_{|\alpha|\leq s}\|(hD)^\alpha u\|^2_{L^2(\Omega)}$ for a general Ω and $s\in\mathbb{N}$. We also need h-dependent norms on some partial Sobolev spaces (see [11]-Appendix B for the case $h=1$). We will consider the Hilbert-spaces $H^{(m,s),h}(I\times\mathbb{R}^d_y)$, where I is an open interval of $\mathbb{R}_x$ and $(m,s,h)\in\mathbb{N}\times\mathbb{R}\times(0,h_0)$, endowed with the norm

$$\|u\|_{H^{(m,s),h}(I\times\mathbb{R}^d)} = \left(\sum_{j=1\ldots m}\|(h\partial_x)^j u\|^2_{L^2(I,H^{m+s-j,h}(\mathbb{R}^d))}\right)^{1/2}.$$

Among other properties we have to mention the

Lemma 1.4 *a) For any $x_0 \in \overline{I}$, the trace $(h\partial_x)^j u\big|_{x=x_0}$, $j \in \mathbb{N}$, defines a continuous operator from $H^{(m,s),h}(I \times \mathbb{R}^d)$ into $H^{m+s-j-1/2,h}(\mathbb{R}^d)$ as soon as $m > j + 1/2$, with the uniform estimates*

$$\|(h\partial_x)^j u\big|_{x=x_0}\|_{H^{m+s-j-1/2,h}(\mathbb{R}^d)} \leq C_{m,s,j} h^{-1/2} \|u\|_{H^{(m,s),h}(\mathbb{R}^{d+1})}. \tag{1.3}$$

b) Reciprocally, the lifting map E^h_ϱ, defined after a partial Fourier transform in the y-direction by

$$\widehat{E^h_\varrho u}(x,\eta) = \varrho\left(\frac{x\langle h\eta\rangle}{h}\right)\hat{u}(\eta) \tag{1.4}$$

with $\varrho \in \mathcal{S}(\mathbb{R})$, $\varrho(0) = 1$, is continuous from $H^{m+s-1/2,h}(\mathbb{R}^d)$ into $H^{(m,s),h}(\mathbb{R}^{d+1})$ for $m \in \mathbb{N}$, $s \in \mathbb{R}$. Moreover the estimates

$$\|E^h_\varrho u\|_{H^{(m,s),h}(\mathbb{R}^{d+1})} \leq C_{m,s} h^{1/2} \|u\|_{H^{m+s-1/2,h}(\mathbb{R}^d)}$$

hold with constants $C_{m,s}$ independent of $h \in (0, h_0)$.

Proof : For a general open interval I, $H^{(m,s),h}(I \times \mathbb{R}^d_y)$ is the space of restrictions of elements of $H^{(m,s),h}(\mathbb{R}^{1+d})$. Hence, we consider as usual the case $I = \mathbb{R}$. We can also take $x_0 = 0$ and set $\gamma^h_j u = h\partial^j_x u\big|_{x=0}$. Assertions a) and b) comes at once from standard results for $h = 1$ because $\gamma^h_j u = h^{-1/2} D_h \gamma^1_j D_h^{-1} u$ and $E^h_\varrho u = h^{1/2} D_h^{-1} E^1_\varrho D_h u$. Note that the $h^{1/2} = h^{\frac{d+1}{2}}/h^{\frac{d}{2}}$ factor results from the normalization factor depending on the dimension of the unitary dilation D_h. $\square$

Lemma 1.5 *a) There exists a constant $h_0 > 0$ so that the boundary value problem*

$$\begin{cases} \mathcal{L}^h u = f, \ x > 0 \\ \quad u\big|_{x=0} = u_0 \end{cases} \tag{1.5}$$

admits a unique solution in $H^{1,h}(\mathbb{R}^{d+1}_+)$ for all $h \in (0, h_0)$ as soon as $u_0 \in H^{1/2,h}(\mathbb{R}^d)$ and $f \in H^{-1,h}(\mathbb{R}^{d+1}_+)$. The estimate

$$\|u\|_{H^{1,h}} \leq C_0 \left(h^{1/2}\|u_0\|_{H^{1/2,h}} + \|f\|_{H^{-1,h}}\right)$$

holds uniformly with respect to $h \in (0, h_0)$. Moreover when $f \in L^2(\mathbb{R}^d)$, the second trace $\gamma^h_1 u = h\partial_x u\big|_{x=0}$ can be extended from its usual definition as an element of $H^{-1/2,h}(\mathbb{R}^d)$ with

$$\|\gamma^h_1 u\|_{H^{-1/2,h}} \leq C_0 \left(\|u_0\|_{H^{1/2,h}} + h^{-1/2}\|f\|_{L^2}\right).$$

b) Whenever $u_0 \in H^{m+s+3/2,h}(\mathbb{R}^d)$ and $f \in H^{(m,s),h}(\mathbb{R}^{d+1}_+)$ with $m \in \mathbb{N}$ and $s \geq 0$, the solution u of (1.5) belongs to $H^{(m+2,s),h}(\mathbb{R}^{d+1}_+)$. Moreover the estimates

$$\|u\|_{H^{(m+2,s),h}} \leq C_{m,s}\left(h^{1/2}\|u_0\|_{H^{m+s+3/2,h}} + \|f\|_{H^{(m,s),h}}\right)$$

$$\text{and} \qquad \|\gamma^h_1 u\|_{H^{m+s+1/2,h}} \leq C'_{m,s}\left(\|u_0\|_{H^{m+s+3/2,h}} + h^{-1/2}\|f\|_{H^{(m,s),h}}\right).$$

hold with constants $C_{m,s}$ and $C'_{m,s}$ independent of $h \in (0, h_0)$.

Proof : Like in Lemma 1.4 we refer to standard results by using the dilations D_h. Here are some details in order to check the uniform control of the constants. We set $U_0 = D_h^{-1}u_0$, $F = D_h^{-1}f$ and $U = D_h^{-1}u$. Then we have $\|u_0\|_{H^{s,h}} = \|U_0\|_{H^s}$ with similar identities for F and U. Equation (1.5) writes

$$\begin{cases} D_h^{-1}\mathcal{L}^h D_h U = F, \\ U\big|_{x=0} = h^{1/2}U_0 \end{cases} \tag{1.6}$$

with $D_h^{-1}\mathcal{L}^h D_h = a(hx, hy; h) + {}^t B(hx, hy; h)\partial + {}^t\partial C(hx, hy; h)\partial$.

a) Let E_ϱ^1 be the continuous lifting map: $H^{1/2}(\mathbb{R}^d) \to H^1(\mathbb{R}_+^{d+1})$ defined by (1.4) with $h = 1$. If U is replaced by $V = U - h^{1/2}E_\varrho^1 U_0$, equation (1.6) writes

$$\begin{cases} D_h^{-1}\mathcal{L}^h D_h V = F - h^{1/2}D_h^{-1}\mathcal{L}^h D_h E_\varrho^1 U_0, \\ \qquad\qquad U\big|_{x=0} = 0 \end{cases}$$

where the H^{-1}-norm of the right-hand side is bounded by $C_0(\|F\|_{H^{-1}} + h^{1/2}\|U_0\|_{H^{1/2}})$. The bilinear form $(V, W) \to (V, D_h^{-1}\mathcal{L}^h D_h W)_{L^2}$ is continuous on $H_0^1(\mathbb{R}_+^{d+1})$ and integrating by part the first order term gives

$$\mathbb{R}\mathrm{e}(V, D_h^{-1}\mathcal{L}^h D_h V)_{L^2} \geq \min(\alpha, \gamma)\|V\|_{H^1}^2 - h\|{}^t\partial B\|_{L^\infty}\|V\|_{L^2}^2, \quad \forall V \in H_0^1(\Omega_0).$$

Lax-Milgram theorem applies if one takes h_0 small enough so that $h\|{}^t\partial B\|_{L^\infty} \leq \frac{1}{2}\min(\alpha, \gamma)$ for all $h \in (0, h_0)$ and leads to uniform estimates.

For the second trace, we recall $\gamma_1^h u = h^{-1/2}D_h\partial_x U\big|_{x=0}$ and we still work with the dilated equation (1.6). Integration by part with $U \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^{d+1}})$ and $V \in H^{1/2}(\mathbb{R}^d)$ gives

$$\int_{\mathbb{R}^d} \overline{V}[\partial_x U] = \int_{\mathbb{R}_+^{d+1}} \overline{\left[{}^t\partial E_\varrho^1 V\right] C}\,[\partial U] + \int_{\mathbb{R}_+^{d+1}} \overline{E_\varrho^1 V}\left[{}^t\partial C\partial U\right] + \int_{\mathbb{R}^d} \left[{}^t\partial_y C_{yx}\overline{V}\right] U,$$

where we omit the arguments $(hx, hy; h)$ and $(0, hy; h)$. The right-hand side extends to any $U \in H^1(\mathbb{R}_+^{d+1})$ such that $[{}^t\partial C\partial U] \in L^2(\mathbb{R}_+^{d+1})$ and especially to the solution of (1.6), if we assume $F \in L^2(\mathbb{R}_+^{d+1})$. Moreover the L^2-norm of $[{}^t\partial C\partial U]$ is then bounded by $C_0(\|F\|_{L^2} + h^{1/2}\|U_0\|_{H^{1/2}})$.

b) We write (1.6) in the form

$$\begin{cases} (a(hx, hy; h) + {}^t\partial C(hx, hy; h)\partial)\, V = F - h^{1/2}D_h^{-1}\mathcal{L}^h D_h E_\varrho^1 U_0 - {}^t B(hx, hy; h)\partial U, \\ \qquad\qquad U\big|_{x=0} = 0 \end{cases}$$

where Lemma 1.4-b) with $h = 1$ implies

$$\|D_h^{-1}\mathcal{L}^h D_h E_\varrho^1 U_0\|_{H^{(m,s)}} \leq C_{m,s}\|U_0\|_{H^{m+s+3/2}}.$$

We first prove $U \in H^{(m+2,s)}(\Omega_0)$ for $m, s \in \mathbb{N}$ by induction on $m + s$ with Nirenberg's method of differential quotients. We get uniform estimates because all the derivatives of the coefficients $a(hx,hy;h)$, $B(hx,hy;h)$ and $C(hx,hy;h)$ are bounded uniformly with respect to $h\in(0,h_0)$. Finally the result for general $s\geq 0$ comes from interpolation. $\square$

This lemma gives a meaning to Definition 1.2 for any $s\geq 0$. Next we work with $s=1$, for which the second trace γ_1^h is naturally defined. One makes sure that the constant $h_0>0$ does not depend on x_0 by writing (1.2) in the form

$$\begin{cases} \mathcal{L}^h_{x_0}u=0,\ x\gtrless 0\\ \quad u\big|_{x=0}=u_0. \end{cases} \tag{1.7}$$

with $\mathcal{L}^h_{x_0}=\tau_{-x_0}\mathcal{L}^h\tau_{x_0}$, $[\tau_{x_0}u](x)=u(x-x_0)$. In the sequel we focus on $\Lambda^{-,h}(x_0)$ and the properties of $\Lambda^{+,h}(x_0)$ follow by symmetry. Lemma 1.5 applied with homogeneous boundary conditions ensures that for any $m\in\mathbb{N}$, $s\geq 0$ and $h\in(0,h_0)$, $\mathcal{L}^h_{x_0}$ defines an isomorphism from $H^1_0(\mathbb{R}^{d+1}_+)\cap H^{(m+2,s),h}(\mathbb{R}^{d+1}_+)$ onto $H^{(m,s),h}(\mathbb{R}^{d+1}_+)$ which will be denoted by $\mathcal{L}^h_{D,x_0}$. With such an operator and the lifting map E^h_ϱ, $\Lambda^{-,h}(x_0)$ writes explicitly as

$$\Lambda^{-,h}(x_0)u_0=\gamma_1^h\left[\mathcal{L}^{h\ -1}_{D,x_0}\left(-\mathcal{L}^h_{x_0}E^h_\varrho u_0\right)+E^h_\varrho u_0\right]. \tag{1.8}$$

Proof of Theorem 1.3 : The regularity with respect to x_0 of $\mathcal{L}^h_{D,x_0}$, $\mathcal{L}^{h\ -1}_{D,x_0}$ and $\Lambda^{-,h}(x_0)$ is induced by our assumptions on the coefficients $a(x,y;h)$, $B(x,y;h)$ and $C(x,y;h)$. As a continuous operator $S(\mathbb{R}^d)\to S'(\mathbb{R}^d)$, $\Lambda^{-,h}(x_0)$ writes $\lambda^-(x_0,y,hD_y;h)$ with $\lambda^-(x_0;h)\in S'(T^*\mathbb{R}^d)$. We will get that $\lambda^-(x_0,h)$ is bounded in $S(\langle\eta\rangle,g_\eta)$ uniformly with respect to $(x_0,h)\in\mathbb{R}\times(0,h_0)$ by Beals criterion (Proposition A.11). We have to verify the estimates

$$\|\left(\mathrm{ad}^\beta_y\,\mathrm{ad}^\alpha_{hD_y}\,\Lambda^{-,h}(x_0)\right)u\|_{H^{|\beta|+1/2,h}}\leq C_{\alpha,\beta}h^{|\alpha|+|\beta|}\|u\|_{H^{3/2,h}},\quad \forall u\in S(\mathbb{R}^d),$$

with constants $C_{\alpha,\beta}$ independent of $(x_0,h)\in\mathbb{R}\times(0,h_0)$. One easily checks from (1.4) that E^h_ϱ is continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{1+d}_+)=\{u\big|_{x>0},\ u\in\mathcal{S}(\mathbb{R}^{1+d})\}$, endowed with its natural quotient topology. Meanwhile, we have

$$\Lambda^{-,h}(x_0)u=F^{-,h}(x_0)E^h_\varrho u,\quad \forall u\in\mathcal{S}(\mathbb{R}^d),$$

where $F^{-,h}(x_0)=\gamma_1^h\left[-\mathcal{L}^{h\ -1}_{D,x_0}\mathcal{L}^h_{x_0}+Id\right]$ is continuous from $\mathcal{S}(\mathbb{R}^{1+d}_+)$ into $\mathcal{S}'(\mathbb{R}^d)$. In this framework, Leibnitz formula

$$\mathrm{ad}^\beta_y\,\mathrm{ad}^\alpha_{hD_y}\,\Lambda^{-,h}(x_0)=\sum_{(\alpha',\beta')\leq(\alpha,\beta)}C_{\alpha',\beta'}\left(\mathrm{ad}^{\beta'}_y\,\mathrm{ad}^{\alpha'}_{hD_y}\,F^{-,h}(x_0)\right)\left(\mathrm{ad}^{\beta-\beta'}_y\,\mathrm{ad}^{\alpha-\alpha'}_{hD_y}\,E^h_\varrho\right) \tag{1.9}$$

makes sense for it involves only bounded operators. The second factor will be treated directly. In order to avoid questions about domains while looking at the first one, we introduce the mollified commutators

$$\mathrm{ad}^\beta_{y,\varepsilon}\,\mathrm{ad}^\alpha_{hD_y,\varepsilon}=\prod_{1\leq i\leq d}\mathrm{ad}^{\beta_i}_{\chi(\varepsilon y)y_i}\prod_{1\leq i'\leq d}\mathrm{ad}^{\alpha_{i'}}_{\chi(\varepsilon hD_y)hD_{y_{i'}}} \tag{1.10}$$

with $\chi \in S(\mathbb{R}^d)$, $\chi(0) = 1$, and $\varepsilon \in (0,1)$. We note

$$(1.11) \qquad \lim_{\varepsilon \to 0} \left(\mathrm{ad}_{y,\varepsilon}^{\beta'} \, \mathrm{ad}_{hD_y,\varepsilon}^{\alpha'} \, F^{-,h}(x_0) \right) u = \left(\mathrm{ad}_{y}^{\beta'} \, \mathrm{ad}_{hD_y}^{\alpha'} \, F^{-,h}(x_0) \right) u \quad \text{in } S'(\mathbb{R}^d)$$

for all $u \in S(\mathbb{R}_+^{1+d})$, while the boundedness of $\chi(\varepsilon y) y_i$ and $\chi(\varepsilon h D_y) h D_{y_{i'}}$ on the functional spaces $H^{s+1/2,h}(\mathbb{R}^d)$, $H^{(0,s),h}(\mathbb{R}_+^{d+1})$, $H^{(2,s),h}(\mathbb{R}_+^{d+1})$ and $H_0^1(\mathbb{R}_+^{d+1}) \cap H^{(2,s),h}(\mathbb{R}_+^{d+1})$, $s \geq 0$, allows

$$(1.12) \qquad \mathrm{ad}_{y,\varepsilon}^{\beta'} \, \mathrm{ad}_{hD_y,\varepsilon}^{\alpha'} \, F^{-,h}(x_0) = -\gamma_1^h \left[\sum_{(\alpha'',\beta'') \leq (\alpha',\beta')} C_{\alpha'',\beta''} \left(\mathrm{ad}_{y,\varepsilon}^{\beta''} \, \mathrm{ad}_{hD_y,\varepsilon}^{\alpha''} \, \mathcal{L}_{D,x_0}^{h\ -1} \right) \left(\mathrm{ad}_{y,\varepsilon}^{\beta'-\beta''} \, \mathrm{ad}_{hD_y,\varepsilon}^{\alpha'-\alpha''} \, \mathcal{L}_{x_0}^{h} \right) \right]$$

for $(\alpha', \beta') > (0,0)$. We next consider each factor by itself.

a)$\underline{\mathrm{ad}_y^\beta \, \mathrm{ad}_{hD_y}^\alpha \, E_\varrho^h}$: Since $hD_{y_{i'}}$ commutes with E_ϱ^h, we only consider the case $\alpha = 0$. For $v \in S(\mathbb{R}^d)$ and $j \in \{0,1,2\}$, we have

$$(h\partial_x)^j \widehat{\mathrm{ad}_{y,\varepsilon}^{\beta} \, E_\varrho^h} v(x,\eta) = \left[(h\partial_x)^j D_\eta^\beta \varrho\left(\frac{x}{h} \langle h\eta \rangle\right) \right] \hat{v}(\eta).$$

The derivatives of $(h\partial_x)^j D_\eta^\beta \varrho(\frac{x}{h}\langle h\eta\rangle)$ can be obtained recursively in the form

$$(h\partial_x)^j D_\eta^\beta \varrho\left(\frac{x}{h}\langle h\eta\rangle\right) = h^{|\beta|} \langle h\eta \rangle^{j-|\beta|} \sum_{k \leq 2^{|\beta|}} f_{k,j,\beta}(h\eta) g_{k,j,\beta}\left(\frac{x}{h}\langle h\eta\rangle\right)$$

where the $f_{k,j,\beta}(z)$ all belong to $S(1, \frac{dz^2}{\langle z\rangle^2})$ and the $g_{k,j,\beta}$ all satisfy $g_{k,j,\beta} \in \mathcal{S}(\mathbb{R})$. Let G_β denote the $L^2(\mathbb{R}_+)$-function $G_\beta = \sup_{\substack{k \leq 2^{|\beta|} \\ j=0,1,2}} |g_{k,j,\beta}|(z)$. We end as usual by

$$\begin{aligned}
\|(h\partial_x)^j \, \mathrm{ad}_{y,\varepsilon}^\beta \, E_\varrho^h v\|^2_{L^2(\mathbb{R}_+, H^{2+|\beta|-j,h})} &\leq C_\beta h^{2|\beta|} \int_{\mathbb{R}_+^{d+1}} |\hat{v}(\eta)|^2 \langle h\eta\rangle^4 |G_\beta(\frac{x}{h}\langle h\eta\rangle)|^2 \, dx \, đ\eta \\
&\leq C_\beta \|G_\beta\|^2_{L^2} h^{2|\beta|+1} \int_{\mathbb{R}_+^{d+1}} |\hat{v}(\eta)|^2 \langle h\eta\rangle^3 \, đ\eta.
\end{aligned}$$

We have proved

$$(1.13) \qquad \|\, \mathrm{ad}_{y,\varepsilon}^\beta \, \mathrm{ad}_{hD_y,\varepsilon}^\alpha \, E_\varrho^h v\|_{H^{(2,|\beta|),h}} \leq C_{\alpha,\beta} h^{|\alpha|+|\beta|+1/2} \|v\|_{H^{3/2,h}}.$$

b)$\underline{\mathrm{ad}_{y,\varepsilon}^\beta \, \mathrm{ad}_{hD_y,\varepsilon}^\alpha \, \mathcal{L}_{x_0}^h}$: The differential operator $\mathcal{L}_{x_0}^h$ writes $L(x + x_0, hD_x, y, hD_y; h)$ with $L \in \mathbb{P}^2 S^{h_0}(1, g_\eta)$ and the operators $\chi(\varepsilon y) y_i$ and $\chi(\varepsilon h D_y) h D_{y_{i'}}$, $1 \leq i, i' \leq d$, commute with hD_x. Moreover the symbols $\chi(\varepsilon\eta)\eta_{i'}$ are uniformly bounded in $S(\langle\eta\rangle, g_\eta)$ so that the multi-commutator $\mathrm{ad}_{hD_y,\varepsilon}^\alpha \mathcal{L}_{x_0}^h$ is treated by referring to semi-classical calculus in the metric g_η. It remains to study the commutator $\mathrm{ad}_{\chi(\varepsilon y) y_i} \varphi(y, hD_y; h)$, $1 \leq i \leq d$, when the h-symbol $\varphi(y, \eta; h)$ belongs to $S^{h_0}(\langle\eta\rangle^n, g_\eta)$. Its Schwarz-kernel writes

$$\int_{\mathbb{R}^d} e^{\frac{i}{h}(y-y')\eta} \varphi(y,\eta;h) \left[\chi(\varepsilon y) y_i - \chi(\varepsilon y') y'_i\right] \frac{đ\eta}{h^d}.$$

We have

$$\chi(\varepsilon y)y_i - \chi(\varepsilon y')y'_i = (y-y').\int_0^1 e_i\chi(\varepsilon y_t) + F_i(\varepsilon y_t)dt$$

by setting $y_t = ty + (1-t)y'$ and $F_i(y) = y_i\partial_y\chi(y)$. Integration by parts transforms the above kernel into

$$h\int_{\mathbb{R}^d} e^{\frac{i}{h}(y-y')\eta} D_\eta\varphi(y,\eta;h).\left[\int_0^1 e_i\chi(\varepsilon y_t) + F_i(\varepsilon y_t)dt\right]\frac{d\eta}{h^d}$$

where the kernel-symbol $D_\eta\varphi(y,\eta;h).\left[\int_0^1 e_i\chi(\varepsilon y_t) + F_i(\varepsilon y_t)dt\right]$ is uniformly bounded in $S(\langle\eta\rangle^{n-1}, dy^2 + dy'^2 + \frac{d\eta^2}{\langle\eta\rangle^2})$. The semi-classical version of [18]-Proposition 2.1 then gives

$$\mathrm{ad}_{\chi(\varepsilon y)y_i}\,\varphi(y,hD_y;h) = h\psi(y,hD_y;h,\varepsilon),$$

where $\psi(\varepsilon)$ is bounded in $S^{h_0}(\langle\eta\rangle^{n-1}, g_\eta)$ uniformly with respect to $\varepsilon\in(0,1)$. As a conclusion, we obtain

$$\mathrm{ad}_{y,\varepsilon}^\beta\,\mathrm{ad}_{hD_y,\varepsilon}^\alpha\,\mathcal{L}_{x_0}^h = h^{|\alpha|+|\beta|}L_{\alpha,\beta}(x+x_0,hD_x,y,hD_y;h,\varepsilon)$$

with $L_{\alpha,\beta}(\varepsilon)$ uniformly bounded in $\mathbb{P}^2S^{h_0}(\langle\eta\rangle^{-|\beta|}, g_\eta)$. Proposition A.7 about continuity of h-pseudo-differential operators yields

$$\|\,\mathrm{ad}_{y,\varepsilon}^\beta\,\mathrm{ad}_{hD_y,\varepsilon}^\alpha\,\mathcal{L}_{x_0}^h v\|_{H^{(0,s+|\beta|),h}} \le C_{\alpha,\beta,s}h^{|\alpha|+|\beta|}\|v\|_{H^{(2,s),h}},\quad s\in\mathbb{R}. \tag{1.14}$$

c) $\underline{\mathrm{ad}_{y,\varepsilon}^\beta\,\mathrm{ad}_{hD_y,\varepsilon}^\alpha\,\mathcal{L}_{D,x_0}^{h\;-1}}$: Leibnitz formula applied to the identity

$$\mathcal{L}_{D,x_0}^{h\;-1}\mathcal{L}_{D,x_0}^h = Id \quad \text{on } H_0^1(\mathbb{R}_+^{d+1})\cap H^{(2,s),h}(\mathbb{R}_+^{d+1}),\ s\ge 0$$

leads to

$$\begin{aligned}&\mathrm{ad}_{y,\varepsilon}^\beta\,\mathrm{ad}_{hD_y,\varepsilon}^\alpha\,\mathcal{L}_{D,x_0}^{h\;-1}\\ &= -\sum_{(\alpha_1,\beta_1)+\cdots+(\alpha_l,\beta_l)=(\alpha,\beta)} C_{\alpha_i,\beta_i}\left(\mathcal{L}_{D,x_0}^{h\;-1}\right)\left(\mathrm{ad}_{y,\varepsilon}^{\beta_1}\,\mathrm{ad}_{hD_y,\varepsilon}^{\alpha_1}\,\mathcal{L}_{D,x_0}^h\right)\left(\mathcal{L}_{D,x_0}^{h\;-1}\right)\ldots\\ &\qquad\ldots\left(\mathrm{ad}_{y,\varepsilon}^{\beta_l}\,\mathrm{ad}_{hD_y,\varepsilon}^{\alpha_l}\,\mathcal{L}_{D,x_0}^h\right)\left(\mathcal{L}_{D,x_0}^{h\;-1}\right).\end{aligned}$$

The operator $\mathcal{L}_{D,x_0}^h$ is nothing but the restriction to $H_0^1(\mathbb{R}_+^{d+1})\cap H^{(2,s),h}(\mathbb{R}_+^{d+1})$ of $\mathcal{L}_{x_0}^h$. Hence by referring to (1.14), we obtain for every $s\ge 0$ the estimates

$$\|\,\mathrm{ad}_{y,\varepsilon}^\beta\,\mathrm{ad}_{hD_y,\varepsilon}^\alpha\,\mathcal{L}_{D,x_0}^{h\;-1}v\|_{H^{(2,s+|\beta|),h}} \le C_{\alpha,\beta,s}h^{|\alpha|+|\beta|}\|v\|_{H^{(0,s),h}}. \tag{1.15}$$

Estimates (1.14)(1.15) hold uniformly with respect to $\varepsilon\in(0,1)$ and combining them with (1.11)(1.12) provides the expected estimate of $\mathrm{ad}_y^{\beta'}\,\mathrm{ad}_{hD_y}^{\alpha'}\,F^{-,h}(x_0)$. We conclude by using (1.9) and (1.13). $\square$

1.2 Approximate Dirichlet-to-Neumann Operator and Applications

The symbol $L \in \mathbb{P}^2 S^{h_0}(1, g_\eta)$ such that $\mathcal{L}^h = L(x, hD_x, y, hD_y; h)$ is explicitely determined by

$$\begin{aligned}
\mathcal{L}^h &= a + b_x(h\partial_x) + {}^tB_y(h\partial_y) - (h\partial_x)^2 - (h\partial_x)C_{xy}(h\partial_y) \\
&\quad - {}^t(h\partial_y)C_{yx}(h\partial_x) - {}^t(h\partial_y)C_{yy}(h\partial_y) \\
&= -(h\partial_x)^2 + \left[-2C_{xy}(h\partial_y) + b_x - h({}^t\partial_y C_{yx})\right](h\partial_x) \\
&\quad -C_{yy} : \left[(h\partial_y)^t(h\partial_y)\right] + {}^tB_y(h\partial_y) + a - h\left[\partial_x C_{xy} + {}^t\partial_y C_{yy}\right](h\partial_y),
\end{aligned}$$

A representant of its principal symbol $(1,0)$-symbol is given by

$$\sigma_{2,1}(L) = \xi^2 + i\left[-2iC_{xy}\eta + b_x\right]\xi + {}^t\eta C_{yy}\eta + i^tB_y\eta + a.$$

Its discriminant equals

$$\begin{aligned}
\Delta &= -\left[-2iC_{xy}\eta + b_x\right]^2 - 4^t\eta C_{yy}\eta - 4i^tB_y\eta - 4a. \\
&= -\left[4^t\eta C_{yy}\eta - 4(C_{xy}\eta)^2 + (b_x)^2 + 4a\right] - 4i\left[-b_xC_{xy} + {}^tB_y\right]\eta \\
&= -\left[4^t\eta\tilde{C}\eta + (b_x)^2 + 4a\right] - 4i\left[-b_xC_{xy} + {}^tB_y\right]\eta
\end{aligned}$$

where the matrix

$$\tilde{C} = C_{yy} - C_{yx}C_{xy} = {}^t\begin{pmatrix} -C_{xy} \\ Id_y \end{pmatrix} C \begin{pmatrix} -C_{xy} \\ Id_y \end{pmatrix} \tag{1.16}$$

still satisfies ${}^t\tilde{C} = \tilde{C} \geq \gamma^2 Id_y$. On $\{z \in \mathbb{C}, \mathrm{Re}(z) \geq 0\}$ we choose $\sqrt{z} = \sqrt{(\varrho e^{i\theta})} = \varrho^{1/2}e^{i\frac{\theta}{2}}$, $|\theta| \leq \frac{\pi}{2}$, and the roots of the principal symbol of L write

$$\xi^-(x, y, \eta; h) = -C_{x,y}\eta + i\frac{-b_x + \sqrt{-\Delta}}{2} \tag{1.17}$$

$$\xi^+(x, y, \eta; h) = -C_{x,y}\eta + i\frac{-b_x - \sqrt{-\Delta}}{2}. \tag{1.18}$$

Our smoothness assumptions on the coefficients a, B, C, yield $\xi^\pm \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle, g_\eta)$. Meanwhile we have

$$\mathrm{Im}\ \xi^\mp = \frac{-b_x \pm \mathrm{Re}\sqrt{-\Delta}}{2} \gtrless \frac{-b_x \pm \sqrt{\mathrm{Re}(-\Delta)}}{2} \gtrless \pm\sqrt{\alpha^2 + \gamma^2|\eta|^2}, \tag{1.19}$$

which implies $|\xi^+ - \xi^-| = \sqrt{|\Delta|} \geq 2\inf\{\alpha, \gamma\}\langle\eta\rangle$. By Theorem B.1 and Theorem B.4 one can construct four sequences of symbols $(A_k^{\pm,\pm})_{k\in\mathbb{N}}$, with $A_k^{\pm,\pm} \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{1-k}, g_\eta)$ so that

$$\lambda_N^\pm = -i\sum_{k<N} h^k A_k^{\pm,+} \text{ and } \kappa_N^\pm = -i\sum_{k<N} h^k A_k^{\pm,-}$$

satisfy the two properties:
a) $\sigma_{0,\langle\eta\rangle}(\lambda_N^{\pm}) = \sigma_{0,\langle\eta\rangle}(\kappa_N^{\pm}) = i\xi^{\pm}$.
b) $L + \left(i\xi - \kappa_N^{\mp}\right) \#^h_{(1,0)} \left(i\xi - \lambda_N^{\pm}\right) = h^N R_N^{\pm}$ with $R_N^{\pm} \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{2-N}, g_\eta)$.
We set $\Lambda_N^{\pm,h}(x) = \lambda_N^{\pm}(x,y,hD_y;h)$ and $K_N^{\pm,h}(x) = \kappa_N^{\pm}(x,y,hD_y;h)$ and the former equality gives the approximate factorization

$$\mathcal{L}^h = -(h\partial_x - \kappa_N^{\mp,h}(x))(h\partial_x - \Lambda_N^{\pm,h}(x)) + h^N R_N^{\pm}(x,y,hD_y;h). \tag{1.20}$$

Remark 1.6 *The approximate factorization of $\mathcal{L}^h_{x_0} = \tau_{-x_0}\mathcal{L}^h\tau_{x_0}$ is deduced from (1.20) by simply changing the argument x into $x+x_0$. Meanwhile the approximate factorization of $e^{-c\frac{x}{h}}\mathcal{L}^h e^{c\frac{x}{h}}$ is obtained by replacing $\lambda_N^{\pm}$ and $\kappa_N^{\pm}$ by $\lambda_N^{\pm} - c$ and $\kappa_N^{\pm} - c$.*

Proposition C.3 applies, with suitable signs, to the symbols $\lambda_N^{\pm}$ involved in the approximate factorization (1.20).

Proposition 1.7 *There exists $h_0 > 0$ so that, for any $(N,s,h) \in \mathbb{N}\times\mathbb{R}\times(0,h_0)$, the initial value problem*

$$\begin{cases} h\partial_x u - \Lambda_N^{\mp,h}(x)u = 0, \ x > x' \ (resp.\ x < x') \\ \qquad\qquad u\Big|_{x=x'} = v \end{cases}$$

defines an evolution system $S_N^{-,h}(x'',x')$, $x' \le x''$, (resp. $S_N^{+,h}(x'',x')$, $x' \ge x''$) on $H^{s,h}(\mathbb{R}^d)$ with all the regularity properties of Proposition C.3. Moreover for any $\varepsilon > 0$ one can find $h_{N,s,\varepsilon}$ so that the estimates

$$\|S_N^{-,h}(x'',x')\|_{\mathcal{L}(H^{s,h})} \le e^{-\frac{(\alpha-\varepsilon)(x''-x')}{h}}, \quad x'' \ge x' \tag{1.21}$$

$$\text{and} \quad \|S_N^{+,h}(x'',x')\|_{\mathcal{L}(H^{s,h})} \le e^{\frac{(\alpha-\varepsilon)(x''-x')}{h}}, \quad x'' \le x', \tag{1.22}$$

hold for every $h \in (0, h_{N,s,\varepsilon})$.

Proof : We just have to prove (1.21)(1.22). The $+$ and $-$ cases are symmetric and we focus on $S_N^{-,h}$. By (1.19), we can find, for any $\varepsilon > 0$, $c_\varepsilon > 0$ so that

$$-\operatorname{Re}\left[i\xi^-\right] - (\alpha - \varepsilon) \ge c_\varepsilon\langle\eta\rangle.$$

By Lemma C.2-b) this provides for any $(N,s) \in \mathbb{N}^*\times\mathbb{R}$ the existence of $h_{N,s,\varepsilon}$ such that

$$\operatorname{Re}(u, \Lambda_N^{-,h}(x)u)_{H^{s,h}} \le -(\alpha-\varepsilon)\|u\|^2_{H^{s,h}}, \quad \forall u \in H^{s+1,h}(\mathbb{R}^d).$$

Differentiating $\|S_N^{-,h}(x'',x')v\|^2_{H^{s,h}}$ with respect to $x'' > x'$ and referring to the continuity at $x'' = x'$ yield (1.21). $\square$

Proposition 1.8 *a) There exists $h_N > 0$ so that $v_N^{\mp}(x) = S_N^{\mp,h}(x+x_0, x_0)u_0$, $\pm x > 0$, is the variational solution of*

$$\left\{\begin{aligned} \mathcal{L}_{x_0}^h v_N^{\mp} &= h^N R_N^{\mp}(x+x_0, y, hD_y; h)v_N^{\mp}, \quad x \gtrless 0 \\ v_N^{\mp}\Big|_{x=0} &= u_0 \end{aligned}\right. \tag{1.23}$$

as soon as $u_0 \in H^{1/2,h}(\mathbb{R}^d)$ and $h \in (0, h_N)$.
b) Let $p_{k,l}$, $k \in \mathbb{N}$, $l \in \mathbb{Z}$, denote the seminorms on $S(\langle\eta\rangle^l, g_\eta)$ defined by (A.2). For any $(k, N) \in \mathbb{N} \times \mathbb{N}^*$, there exists $h_{k,N} \in (0, h_0)$ so that the estimate

$$p_{k,1-N}(\lambda^{\pm}(x,h) - \lambda_N^{\pm}(x,h)) \leq C_{k,N}h^N \tag{1.24}$$

holds for $h \in (0, h_{k,N})$ with $C_{k,N} > 0$ independent of $(x, h) \in \mathbb{R} \times (0, h_{k,N})$.

Proof : a) We take $h_N > 0$ small enough so that Proposition 1.7 estimate (1.21) hold for $s = 0$ and $\varepsilon = \alpha/2$. For $u_0 \in H^{1/2,h}(\mathbb{R}^d)$, $v_N^-(x) = S_N^{-,h}(x+x_0, x_0)u_0$ is a classical solution to

$$h\partial_x v_N^- - \Lambda_N^{-,h}(x+x_0)v_N^- = 0, \quad \text{for } x > 0. \tag{1.25}$$

Thus the approximate factorization (1.20) of $\mathcal{L}_{x_0}^h$ leads to

$$\mathcal{L}_{x_0}^h v_N^- = h^N R_N^-(x+x_0, y, hD_y; h)v_N^- \quad \text{in } \mathcal{D}'(\mathbb{R}_+^{d+1}),$$

while the continuity of the evolution system $S_N^{-,h}$ gives $v_N^-\Big|_{x=0} = u_0$. It remains to check that $v_N^- \in H^{1,h}(\mathbb{R}_+^{d+1})$ and that the right-hand side belongs to $L^2(\mathbb{R}_+^{d+1})$. By making use of equation (1.25), we get for $k + |k'| \leq 1$

$$\|(h\partial_x)^k(h\partial_y)^{k'}v_N^-\|_{L^2(\mathbb{R}_+^{1+d})}^2 \leq \int_0^\infty C_N\|v_N^-(x)\|_{H^{1,h}}^2 dx,$$

while $R_N^- \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{2-N}, g_\eta)$, $N \geq 1$, yields

$$\|R_N^-(x+x_0, y, hD_y; h)v_N^-\|_{L^2(\mathbb{R}_+^{1+d})} \leq \int_0^\infty C_N\|v_N^-(x)\|_{H^{1,h}dx}^2.$$

By referring to Lemma C.4, we have

$$\int_0^\infty \|v_N^-(x)\|_{H^{1,h}}^2 \leq C_N h\left[\|u_0\|_{H^{1/2,h}}^2 + h\int_0^\infty \|v_N^-(x)\|_{L^2}^2 dx\right]$$

while estimate (1.21) implies

$$\int_0^\infty \|v_N^-(x)\|_{L^2}^2 dx \leq \|u_0\|_{L^2}^2 \int_0^\infty e^{-\frac{\alpha x}{h}dx}.$$

b) Since $h^{-N}(\lambda_N^-(x,h) - \lambda_M^-(x,h))$ belongs to $\mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{1-N}, g_\eta)$ for $M \geq N$, we just have to find $M \geq N$ large enough so that $p_{k,1-N}(\lambda^-(x,h) - \lambda_M^-(x,h))$ is an $O(h^N)$. This number M will be fixed further as a function of (k, N). By Beals criterion as stated in

Proposition A.11, there exists $\nu = \nu(k,N) \in \mathbb{N}$ so that $p_{k,1-N}(\lambda^-(x,h) - \lambda^-_M(x,h))$ is estimated by

$$\sup_{|\alpha|+|\beta|\leq\nu} h^{-|\alpha|-|\beta|} \| \operatorname{ad}_y^\beta \operatorname{ad}_{hD_y}^\alpha (\Lambda^{-,h}(x) - \Lambda^{-,h}_M(x))\|_{\mathcal{L}(H^{3/2,h},H^{1/2+|\beta|+N,h})}.$$

By taking the difference between (1.2) and (1.23), $e^-_M = u - v^-_M$, with $h \in (0,h_M)$, appears as the variational solution to

$$\begin{cases} \mathcal{L}^h_{x_0} e_M = h^M R^-_M(x+x_0,y,hD_y;h) v^-_M \\ \qquad\qquad e_{M,x_0}\Big|_{x=0} = 0 \end{cases}$$

and we have $(\Lambda^{-,h}(x_0) - \Lambda^{-,h}_M(x_0))u_0 = \gamma_1^h e_M$. Hence we get for $h \in (0,h_M)$

$$(\Lambda^{-,h}(x_0) - \Lambda^{-,h}_M(x_0))u_0 = \gamma_1^h \left(\mathcal{L}^h_{D,x_0}\right)^{-1} \left[h^M R^{-,h}_M(x+x_0,y,hD_y;h) S^{-,h}_M(x+x_0,x_0)u_0\right].$$

We next develop the same techniques as in Theorem 1.3. We introduce the mollified commutators (1.10) so that every factor of

$$(1.26)\ \operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon}(\Lambda^{-,h}(x) - \Lambda^{-,h}_M(x)) = \gamma_1^h \sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha \\ \beta_1+\beta_2+\beta_3=\beta}} \left[C_{\alpha_i\beta_i} \operatorname{ad}^{\beta_1}_{y,\varepsilon} \operatorname{ad}^{\alpha_1}_{hD_y,\varepsilon} \left(\mathcal{L}^h_{D,x_0}\right)^{-1}\right.$$
$$\left.\operatorname{ad}^{\beta_2}_{y,\varepsilon} \operatorname{ad}^{\alpha_2}_{hD_y,\varepsilon} \left(h^M R^{-,h}_M(x+x_0,y,hD_y;h)\right) \operatorname{ad}^{\beta_3}_{y,\varepsilon} \operatorname{ad}^{\alpha_3}_{hD_y,\varepsilon} \left(S^{-,h}_M(x+x_0,x_0)\right)\right].$$

make sense for $\varepsilon > 0$ and $|\alpha| + |\beta| \leq \nu$. We need estimates for each of the three factors which are uniform with respect to $\varepsilon \in (0,1)$. The first one is estimated by (1.15) while the second one only involves semi-classical operators. Let us have a look at $\operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon} \left(S^{-,h}_M(x+x_0,x_0)\right)$ for $|\alpha|+|\beta| \leq \nu$. For $x > 0$, we have

$$\begin{aligned} &h\partial_x \left[\operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon} S^{-,h}_M(x+x_0,x_0)\right] = \operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon} \left[\Lambda^{-,h}_M(x+x_0) S^{-,h}_M(x+x_0,x_0)\right] \\ &= \Lambda^{-,h}_M(x+x_0) \left[\operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon} S^{-,h}_M(x+x_0,x_0)\right] + \\ &\sum_{(\alpha',\beta')<(\alpha,\beta)} C_{\alpha',\beta'} \operatorname{ad}^{\beta-\beta'}_{y,\varepsilon} \operatorname{ad}^{\alpha-\alpha'}_{hD_y,\varepsilon} \Lambda^{-,h}_M(x+x_0) \operatorname{ad}^{\beta'}_{y,\varepsilon} \operatorname{ad}^{\alpha'}_{hD_y,\varepsilon} S^{-,h}_M(x+x_0,x_0), \end{aligned}$$

while $\operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon} S^{-,h}_M(x_0,x_0) = 0$ as soon as $(\alpha,\beta) \neq 0$. Hence we get

$$\operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon} S^{-,h}_M(x''+x_0,x_0) = \frac{1}{h}\int_{x_0}^{x''+x_0} S^{-,h}_M(x''+x_0,x) \sum_{(\alpha',\beta')<(\alpha,\beta)} [\ldots] dx.$$

Note that the factor $\operatorname{ad}^{\beta-\beta'}_{y,\varepsilon} \operatorname{ad}^{\alpha-\alpha'}_{hD_y,\varepsilon} \Lambda^{-,h}_M(x+x_0)$ may produce a loss of regularity, especially for $\beta = \beta'$. Let $F_\mu(x)$, $\mu \in \{0,1\ldots\nu\}$, denote the quantity

$$\sup_{|\alpha|+|\beta|=\mu} \| \operatorname{ad}^\beta_{y,\varepsilon} \operatorname{ad}^\alpha_{hD_y,\varepsilon} S^{-,h}_M(x+x_0,x_0)\|_{\mathcal{L}(H^{3/2,h},H^{3/2-\mu,h})}.$$

The previous identity implies

$$F_\mu(x'') \le C_{M,\mu} \int_{x_0}^{x''+x_0} \sup_{0\le\mu'\le\mu} \|S_M^{-,h}(x''+x_0,x)\|_{\mathcal{L}(H^{3/2-\mu',h})} \sup_{0\le\mu'<\mu} F_{\mu-1}(x)dx.$$

We now take $h_{M,\nu}$ so that (1.21) holds with $s=3/2-\mu$, $\mu\in\{0,1\ldots\nu\}$, and $\varepsilon=\alpha/2$. With such a choice, we get at once by induction the uniform boundedness of the $F_\mu(x)$, that is

$$\|\operatorname{ad}_{y,\varepsilon}^\beta \operatorname{ad}_{hD_y,\varepsilon}^\alpha S_M^{-,h}(x+x_0,x_0)\|_{\mathcal{L}(H^{3/2,h},H^{3/2-|\alpha|-|\beta|,h})} \le C_{\alpha,\beta}$$

for all (α,β), $|\alpha|+|\beta|\le\nu$. We conclude by taking $M=M(N,\nu(k,N))$ large enough in order to balance the above loss of regularity by the second factor of (1.26). $\square$

Theorem 1.9 *a) There exists $h_0>0$ so that, for any $(s,h)\in\mathbb{R}\times(0,h_0)$, the initial value problem*

$$\begin{cases} h\partial_x u - \Lambda^{\mp,h}(x)u = 0, \ x>x' \text{ (resp. } x'<x) \\ u\big|_{x=x'} = v \end{cases}$$

defines an evolution system $S^{-,h}(x'',x')$, $x''\ge x'$ (resp. $S^{+,h}(x'',x')$, $x''\le x'$), on $H^{s,h}(\mathbb{R}^d)$ which satisfies all the regularity properties of Proposition C.3.
b) For any $\varepsilon>0$ one can find $h_{s,\varepsilon}$ so that the estimates

$$\|S^{-,h}(x'',x')\|_{\mathcal{L}(H^{s,h})} \le e^{-\frac{(\alpha-\varepsilon)(x''-x')}{h}}, \quad x''\ge x', \tag{1.27}$$

$$\text{and} \quad \|S^{+,h}(x'',x')\|_{\mathcal{L}(H^{s,h})} \le e^{\frac{(\alpha-\varepsilon)(x''-x')}{h}}, \quad x''\le x', \tag{1.28}$$

hold for every $h\in(0,h_{s,\varepsilon})$.
c) There exists $h_0>0$ so that $u(x)=S^{\mp,h}(x+x_0,x_0)u_0$ coincides with the variational solution of (1.7) as soon as $u_0\in H^{1/2,h}(\mathbb{R}^d)$ and $h\in(0,h_0)$.

Proof : Applying Proposition 1.8-b) with $k=0$ and $N=1$ gives

$$\|\lambda^\pm - i\xi^\pm\|_{L^\infty} \le C_0 h.$$

According to (1.19), we can take h_0 small enough so that $\mathbb{R}\mathrm{e}\,\lambda^\pm \gtrless \pm\frac{\alpha\wedge\gamma}{2}\langle\eta\rangle$. Assertions a) and b) come from the same arguments as the one developed for $S_N^{-,h}$. For part c), we first notice that if u is the variational solution of (1.7) with $u_0\in\mathcal{S}(\mathbb{R}^d)$, then $u(x)$ is the classical solution to

$$\begin{cases} h\partial_x u - \Lambda^{\mp,h}(x+x_0)u = 0, \ x>x' \text{ (resp. } x'<x) \\ u\big|_{x=0} = u_0 \end{cases}$$

and coincides with $S^{\mp,h}(x+x_0,x_0)u_0$. The half-space $H^{1,h}$ estimate for $S^{\mp,h}(x+x_0,x_0)u_0$ is derived from Lemma C.4 like in Propostion 1.8-a) and the equality carries over for any $u_0\in H^{1/2,h}(\mathbb{R}^d)$. $\square$

2 Domain Decomposition Algorithms

In this section, we give the precise definitions of the domain decomposition methods with which we are concerned and exhibit their basic properties.

2.1 Description

We want to solve the whole space problem:

$$\mathcal{L}^h(u) = f \text{ in } \mathbb{R}^{d+1} \tag{2.1}$$

where f is given in $L^2(\mathbb{R}^{d+1})$. The space $\mathbb{R}^{d+1}$ is decomposed into N vertical strips: Let $\Omega_i = (l_i, L_i) \times \mathbb{R}^d$, $1 \le i \le N$ with $-\infty = l_1 < l_2 \le L_1 < \ldots < L_{i-2} < l_i \le L_{i-1} < \ldots < L_N = +\infty$. We have $\mathbb{R}^{d+1} = \cup_{i=1}^N \bar{\Omega}_i$. As interface conditions, we take for the left (resp. right) boundary of a subdomain $h\partial_x - \Pi^{+,h}$ (resp. $h\partial_x - \Pi^{-,h}$) where the operators $\Pi^{\pm,h}$ $(0 < h < h_0)$ are h-pseudodifferential operators whose symbols $\pi^\pm$ satisfy (We set $j \vee 1 = \max\{j, 1\}$) :

H1 *There exists* $j \ge 0$ *such that* $\pi^\pm \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^j, g_\eta)$ *with* $|\sigma_{\langle\eta\rangle^{j\vee 1}}(\pi^\pm - i\xi^\mp)| \ge c_\pi \langle\eta\rangle^{j\vee 1}$, *for some positive constant* c_π.
H2 *There exists* k, $j \vee 1 - 1 \le k \le j$ *so that* $\mathrm{Re}(\pm\pi^\pm \mp \frac{b_x}{2}) \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^k, g_\eta)$, $\mathrm{Re}(\pm\pi^\pm \mp \frac{b_x}{2}) \ge c'_\pi \langle\eta\rangle^k$, *for some positive constant* c'_π.

In the sequel, we may have to restrict the range of the small parameter h according to estimates of the symbols π^+ and π^-. In such a case, we write $h \in (0, h_\pi)$.

Three domain decomposition methods are considered. They are defined recursively by starting from the initial estimates $u_i^0 \in H^2(\Omega_i)$ of the solution u of (2.1) in the domain Ω_i, $i = 1, \ldots, N$.
The first one is the additive Schwarz method (abbreviated to ASM) and writes:

$$\begin{cases} u_i^{n+1} \in H^2(\Omega_i),\ 1 \le i \le N \\ \mathcal{L}^h(u_i^{n+1}) = f \text{ in } \Omega_i \\ (h\partial_x - \Pi^{+,h})(u_i^{n+1}) = (h\partial_x - \Pi^{+,h})(u_{i-1}^n) \text{ at } x = l_i,\ 2 \le i \le N \\ (h\partial_x - \Pi^{-,h})(u_i^{n+1}) = (h\partial_x - \Pi^{-,h})(u_{i+1}^n) \text{ at } x = L_i,\ 1 \le i \le N-1 \end{cases} \tag{2.2}$$

In the ASM, u_i^n is updated at same time in every subdomain. In the second algorithm, the value of u_i^n is updated in one subdomain at a time. We proceed by double sweeps over the subdomains:
<u>left to right sweep</u>

$$\begin{cases} u_i^{n+1/2} \in H^2(\Omega_i),\ 1 \le i \le N \\ \mathcal{L}^h(u_i^{n+1/2}) = f \text{ in } \Omega_i \\ (h\partial_x - \Pi^{+,h})(u_i^{n+1/2}) = (h\partial_x - \Pi^{+,h})(u_{i-1}^{n+1/2}) \text{ at } x = l_i,\ 2 \le i \le N \\ (h\partial_x - \Pi^{-,h})(u_i^{n+1/2}) = (h\partial_x - \Pi^{-,h})(u_{i+1}^n) \text{ at } x = L_i,\ 1 \le i \le N-1 \end{cases} \tag{2.3}$$

right to left sweep

$$(2.4)\quad \begin{cases} u_i^{n+1} \in H^2(\Omega_i),\ 1 \le i \le N \\ \mathcal{L}^h(u_i^{n+1}) = f \text{ in } \Omega_i \\ (h\partial_x - \Pi^{+,h})(u_i^{n+1}) = (h\partial_x - \Pi^{+,h})(u_{i-1}^{n+1/2}) \text{ at } x = l_i,\ 2 \le i \le N \\ (h\partial_x - \Pi^{-,h})(u_i^{n+1}) = (h\partial_x - \Pi^{-,h})(u_{i+1}^{n+1}) \text{ at } x = L_i,\ 1 \le i \le N-1. \end{cases}$$

Let us remark that $u_N^{n+1/2} = u_N^{n+1}$. This algorithm is called DSA (for Double Sweep Algorithm).
The third algorithm denoted shortly by FDA is a slight modification of the second one. It consists only in flow directed sweeps:
left to right sweep

$$(2.5)\quad \begin{cases} u_i^{n+1} \in H^2(\Omega_i),\ 1 \le i \le N \\ \mathcal{L}^h(u_i^{n+1}) = f \text{ in } \Omega_i \\ (h\partial_x - \Pi^{+,h})(u_i^{n+1}) = (h\partial_x - \Pi^{+,h})(u_{i-1}^{n+1}) \text{ at } x = l_i,\ 2 \le i \le N \\ (h\partial_x - \Pi^{-,h})(u_i^{n+1}) = (h\partial_x - \Pi^{-,h})(u_{i+1}^{n}) \text{ at } x = L_i,\ 1 \le i \le N-1 \end{cases}$$

2.2 Well-posedness of the algorithms

The well-posedness relies on the study of the boundary value problem

$$(2.6)\qquad \begin{cases} \mathcal{L}^h(v) = f \text{ in } \Omega_i \\ (h\partial_x - \Pi^{+,h})(v) = g_l \text{ at } x = l_i \\ (h\partial_x - \Pi^{-,h})(v) = g_r \text{ at } x = L_i. \end{cases}$$

Proposition 2.1 *For any $m \ge 0$. there exists $h_{\pi,m} \in (0, h_0)$ so that the boundary value problem (2.6) admits a unique solution $v \in H^{2+m,h}(\Omega_i)$ as soon as $f \in H^{m,h}(\Omega_i)$, g_l and $g_r \in H^{3/2+m-j\vee 1,h}(\mathbb{R}^d)$ and $h \in (0, h_{\pi,m})$. Moreover the mapping: $(g_l, g_r) \in \left(H^{3/2+m-j\vee 1,h}(\mathbb{R}^d)\right)^2 \to v \in H^{2+m,h}(\Omega_i)$, where one forgets g_l for $i = 1$ and g_r for $i = N$, is uniformly continuous.*

Proof : According to Lemma C.2 we take $h_\pi > 0$ small enough so that there exists $c^\pm \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{-j\vee 1}, g_\eta)$ with

$$(\pi^\pm - i\xi^\mp)\#^h c^\pm = 1, \quad \forall h \in (0, h_\pi).$$

Existence: Let u denote the solution of the whole space problem (2.1). We seek a solution to (2.6) of the form $v = u\big|_{\Omega_i} + w$ with

$$w = S^{-,h}(x, l_i)\alpha_l + S^{+,h}(x, L_i)\alpha_r$$

and $\alpha_l, \alpha_r \in H^{3/2+m,h}(\mathbb{R}^d)$. By Theorem 1.9, w is then the variational solution of $\mathcal{L}^h w = 0$ with Dirichlet-boundary conditions

$$\begin{aligned} w\big|_{x=l_i} &= \alpha_l + S^{+,h}(l_i, L_i)\alpha_r \\ w\big|_{x=L_i} &= S^{-,h}(L_i, l_i)\alpha_l + \alpha_r. \end{aligned}$$

In such a case, we get like in Lemma 1.5 that w and v belong to $H^{2+m,h}(\Omega_i)$. We next construct α_l and α_r. The boundary conditions of (2.6) are equivalent to

$$\begin{aligned} & \left[\Lambda^{-,h}(l_i) - \Pi^{+,h}\right]\alpha_l + \left[\Lambda^{+,h}(l_i) - \Pi^{+,h}\right] S^{+,h}(l_i, L_i)\alpha_r \\ & \qquad = g_l - (h\partial_x - \Pi^{+,h})u = \widetilde{g_l} \\ \text{and} \quad & \left[\Lambda^{-,h}(l_i) - \Pi^{-,h}\right] S^{-,h}(L_i, l_i)\alpha_l + \left[\Lambda^{+,h}(l_i) - \Pi^{-,h}\right]\alpha_r \\ & \qquad = g_r - (h\partial_x - \Pi^{-,h})u = \widetilde{g_l}. \end{aligned}$$

this system also writes

$$\left[Id + \begin{pmatrix} 0 & \left[\Lambda^{-,h}(l_i) - \Pi^{+,h}\right]^{-1}\left[\Lambda^{+,h}(l_i) - \Pi^{+,h}\right] S^{+,h}(l_i, L_i) \\ \left[\Lambda^{+,h}(l_i) - \Pi^{-,h}\right]^{-1}\left[\Lambda^{-,h}(l_i) - \Pi^{-,h}\right] S^{-,h}(L_i, l_i) & 0 \end{pmatrix}\right]\begin{pmatrix} \alpha_l \\ \alpha_r \end{pmatrix} = \begin{pmatrix} \left[\Lambda^{-,h}(l_i) - \Pi^{+,h}\right]^{-1}\widetilde{g_l} \\ \left[\Lambda^{+,h}(l_i) - \Pi^{-,h}\right]^{-1}\widetilde{g_r} \end{pmatrix} \tag{2.7}$$

where the right-hand-side belongs to $\left(H^{3/2+m,h}(\mathbb{R}^d)\right)^2$. According to Theorem 1.9, the above perturbation of identity is bounded on $\left(H^{3/2+m,h}(\mathbb{R}^d)\right)^2$ with a norm uniformly estimated by $C_\pi e^{-\frac{\alpha(L_i - l_i)}{2h}}$, as soon as $h \in (0, h_m)$. We take $h_{\pi,m} \leq h_m \wedge h_\pi$ small enough and we obtain for $h \in (0, h_{\pi,m})$

$$\begin{pmatrix} \alpha_l \\ \alpha_r \end{pmatrix} = \left[Id + O(e^{-\frac{\alpha(L_i - l_i)}{2h}})\right]\begin{pmatrix} \left[\Lambda^{-,h}(l_i) - \Pi^{+,h}\right]^{-1}\widetilde{g_l} \\ \left[\Lambda^{+,h}(l_i) - \Pi^{-,h}\right]^{-1}\widetilde{g_r} \end{pmatrix}. \tag{2.8}$$

Uniqueness: By linearity, it is done when a solution $v \in H^{2,h}(\Omega_i)$ of (2.6) with $f = 0$, $g_l = 0$ and $g_r = 0$ necessarily equals 0. By multiplying $\mathcal{L}^h(v)$ by $\overline{v}$, integrating by parts and taking the real part, we obtain

$$0 \geq C_0\|v\|^2_{H^{1,h}(\Omega_i)} + \left[h\,\mathbb{R}\mathrm{e}\left(v, \frac{b_x}{2}v - h\partial_x v - C_{xy}h\partial_y v\right)_{L^2}\right]_{l_i}^{L_i},$$

with $C_0 > 0$ and $h \in (0, h_0)$, h_0 small enough. From the boundary conditions, we get

$$(\Pi^{\pm,h} - \Lambda^{\mp,h})v = (h\partial_x - \Lambda^{\mp,h})v \quad \text{at } x = l_i \text{ or } L_i$$

so that $v\big|_{l_i \text{ or } L_i} \in H^{1/2+j\vee 1,h}(\mathbb{R}^d)$ for any $h \in (0, h_\pi)$. Thus, the estimate

$$0 \geq C_0\|v\|^2_{H^{1,h}(\Omega_i)} + \left[h\,\mathbb{R}\mathrm{e}\left(v, \frac{b_x}{2}v - \Pi^{\pm}v - C_{xy}h\partial_y v\right)_{L^2}\right]_{l_i}^{L_i}$$

makes sense. We now combine hypothesis H2 with Lemma C.2 in order to get positive boundary terms for $h \in (0, h_\pi)$, h_π small enough. This yields $v = 0$. $\square$

2.3 Substructuring

For the sake of simplicity, we next work with $e_i^n = u_i^n - u\big|_{\Omega_i}$, which amounts to take $f = 0$. We set

$$\begin{aligned} g_{i,l}^n &= (h\partial_x - \Pi^{+,h})e_i^n\Big|_{x=l_i} \\ \text{and} \qquad g_{i,r}^n &= (h\partial_x - \Pi^{-,h})e_i^n\Big|_{x=L_i}. \end{aligned}$$

Let Q_i^h, $\alpha_{i,l}^h$ and $\alpha_{i,r}^h$, $1 \leq i \leq N$ respectively denote the operator defined in Proposition 2.1 and by (2.7)(2.8) with $m = 0$ and $f = 0$. Then we have $e_i^n = Q_i^h(g_{i,l}^n, g_{i,r}^n)$ and more precisely

$$e_i^n = S^{-,h}(x, l_i)\alpha_{i,l}^h(g_{i,l}^n, g_{i,r}^n) + S^{+,h}(x, L_i)\alpha_{i,r}^h(g_{i,l}^n, g_{i,r}^n). \tag{2.9}$$

Now, the ASM described in (2.2) writes

$$\begin{array}{rcl} g_{2,l}^{n+1} &=& (h\partial_x - \Pi^{+,h})Q_1^h(g_{1,r}^n)\Big|_{x=l_2} \\ g_{3,l}^{n+1} &=& (h\partial_x - \Pi^{+,h})(Q_2^h(g_{2,l}^n, 0) + Q_2^h(0, g_{2,r}^n))\Big|_{x=l_3} \\ &\vdots& \\ g_{N,l}^{n+1} &=& (h\partial_x - \Pi^{+,h})(Q_{N-1}^h(g_{N-1,l}^n, 0) + Q_{N-1}^h(0, g_{N-1,r}^n))\Big|_{x=l_N} \\ g_{1,r}^{n+1} &=& (h\partial_x - \Pi^{-,h})(Q_2^h(g_{2,l}^n) + Q_2^h(0, g_{2,r}^n))\Big|_{x=L_1} \\ &\vdots& \\ g_{N-2,r}^{n+1} &=& (h\partial_x - \Pi^{-,h})(Q_{N-1}^h(g_{N-1,l}^n) + Q_{N-1}^h(0, g_{N-1,r}^n))\Big|_{x=L_{N-2}} \\ g_{N-1,r}^{n+1} &=& (h\partial_x - \Pi^{-,h})Q_N^h(g_{N,l}^n)\Big|_{x=L_{N-1}}. \end{array} \tag{2.10}$$

We consider $2(N-1)$-uplets $G = (g_{2,l}, \ldots, g_{N-1,r}) \in \left(H^{3/2-j\vee 1,h}(\mathbb{R}^d)\right)^{2(N-1)}$ and define the operator $\mathcal{T}$ by

$$\mathcal{T}G = \begin{bmatrix} (h\partial_x - \Pi^{+,h})Q_1(g_{1,r})\Big|_{x=l_2} \\ (h\partial_x - \Pi^{+,h})(Q_2(g_{2,l}, 0) + Q_2(0, g_{2,r}))\Big|_{x=l_3} \\ \vdots \\ (h\partial_x - \Pi^{+,h})(Q_{N-1}(g_{N-1,l}, 0) + Q_2(0, g_{N-1,r}))\Big|_{x=l_N} \\ (h\partial_x - \Pi^{-,h})(Q_2(0, g_{2,r}) + Q_2(g_{2,l}, 0))\Big|_{x=L_1} \\ \vdots \\ (h\partial_x - \Pi^{-,h})(Q_{N-1}(0, g_{N-1,r}) + Q_{N-1}(g_{N-1,l}, 0))\Big|_{x=L_{N-2}} \\ (h\partial_x - \Pi^{-,h})Q_N(g_{N,l}, 0)\Big|_{x=L_{N-1}} \end{bmatrix}.$$

The operator $\mathcal{T}$ is bounded on $\left(H^{3/2-j\vee 1,h}(\mathbb{R}^d)\right)^{2(N-1)}$ according to (2.8)(2.9) and Theorem 1.9. It can be written as an operator valued matrix

$$\mathcal{T}(G) = \begin{bmatrix} 0 & & & 0 & \times & & & 0 \\ \times & \ddots & & & & \ddots & & \\ & \ddots & \ddots & & & & \ddots & \\ 0 & & \times & 0 & 0 & & & \times \\ \times & & & 0 & 0 & \times & & 0 \\ & \ddots & & & & \ddots & \ddots & \\ & & \ddots & & & & \ddots & \times \\ 0 & & & \times & 0 & & & 0 \end{bmatrix} \begin{bmatrix} g_{2,l} \\ \vdots \\ \vdots \\ g_{N,l} \\ g_{1,r} \\ \vdots \\ \vdots \\ g_{N-1,r} \end{bmatrix}$$

where the crosses correspond to non zero operators.

From (2.10), we see that the additive Schwarz method corresponds to a Jacobi algorithm:

$$G^{n+1} = \mathcal{T}(G^n) \tag{2.11}$$

Consider now the DSA (2.3)-(2.4) and the FDA (2.5). In order to write them in a compact form, we introduce four $(2N-2) \times (2N-2)$ operator valued matrices:

$$\mathcal{M}l = (\mathcal{M}l)_{1 \le m,\, n \le 2N-2} \text{ with } \mathcal{M}l_{mn} = \begin{cases} \mathcal{T}_{mn} & 1 \le m,\, n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{A}l = (\mathcal{A}l)_{1 \le m,\, n \le 2N-2} \text{ with } \mathcal{A}l_{mn} = \begin{cases} \mathcal{T}_{mn} & 1 \le m \le N-1,\ N \le n \le 2N-2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{A}r = (\mathcal{A}r)_{1 \le m,\, n \le 2N-2} \text{ with } \mathcal{A}r_{mn} = \begin{cases} \mathcal{T}_{mn} & N \le m \le 2N-2,\ 1 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{M}r = (\mathcal{M}r)_{1 \le m,\, n \le 2N-2} \text{ with } \mathcal{M}r_{mn} = \begin{cases} \mathcal{T}_{mn} & N \le m,\, n \le 2N-2 \\ 0 & \text{otherwise} \end{cases}$$

so that we have $\mathcal{T} = \mathcal{M}l + \mathcal{A}l + \mathcal{M}r + \mathcal{A}r$. From the structure of $\mathcal{T}$, we have the following important properties:

$$\begin{array}{lll} \mathcal{M}r^{N-1} = \mathcal{M}l^{N-1} = 0; & \mathcal{M}l\,\mathcal{M}r = \mathcal{M}r\,\mathcal{M}l = 0; & \mathcal{A}l^2 = \mathcal{A}r^2 = 0 \\ \mathcal{A}l\,\mathcal{M}l = \mathcal{A}r\,\mathcal{M}r = 0; & \mathcal{M}l\,\mathcal{A}r = \mathcal{M}r\,\mathcal{A}l = 0 & \end{array} \tag{2.12}$$

It is worth noticing that these relations come from the structure of the matrices and do not depend on the value of the components.
With these notations, we see the DSA (2.3)-(2.4) corresponds to the algorithm:

$$\begin{aligned} G^{n+1/2} &= (\mathcal{M}l + \mathcal{A}l)(G^{n+1/2}) + (\mathcal{A}r + \mathcal{M}r)(G^n) \\ G^{n+1} &= (\mathcal{M}l + \mathcal{A}l)(G^{n+1/2}) + (\mathcal{A}r + \mathcal{M}r)(G^{n+1}), \end{aligned}$$

that is

$$(2.13)\quad \begin{aligned} G^{n+1} &= (Id - \mathcal{A}r - \mathcal{M}r)^{-1}(Id - \mathcal{M}l - \mathcal{A}l)^{-1}(\mathcal{M}l + \mathcal{A}l)(\mathcal{A}r + \mathcal{M}r)\, G^n \\ &=: \mathcal{T}_{ds}\, G^n, \end{aligned}$$

while the FDA (2.5) corresponds to a Gauss-Seidel algorithm (see for instance [23], [5]):

$$G^{n+1} = (\mathcal{M}l + \mathcal{A}l)(G^{n+1}) + (\mathcal{A}r + \mathcal{M}r)(G^n),$$

that is

$$(2.14)\qquad G^{n+1} = (Id - \mathcal{M}l - \mathcal{A}l)^{-1}(\mathcal{A}r + \mathcal{M}r)\, G^n =: \mathcal{T}_{fl}\, G^n$$

The operators $(Id - \mathcal{M}l - \mathcal{A}l)$ and $(Id - \mathcal{M}r - \mathcal{A}r)$ are invertible since $\mathcal{M}l + \mathcal{A}l$ and $\mathcal{M}r + \mathcal{A}r$ are nilpotent.

3 Convergence Analysis

We begin the convergence analysis for the three algorithms ASM,FDA and DSA with a remark. If we take $\Pi^{\pm,h} = \Lambda^{\pm,h}$, then the operator $\begin{pmatrix} \alpha^h_{i,l} \\ \alpha^h_{i,r} \end{pmatrix}$ is diagonal and $\mathcal{A}_l = \mathcal{A}_r = 0$. Hence, the three operators $\mathcal{T}$, $\mathcal{T}_{fl}$ and $\mathcal{T}_{ds}$ are nilpotent. Actually we have the

Proposition 3.1 *If $\Pi^{\pm,h} = \Lambda^{\pm,h}$, there exists $h_0 > 0$ so that the algorithms ASM,FDA and DSA are well-posed for $h \in (0, h_0)$ and converge after finitely many iterations according to $\mathcal{T}^{N-1} = 0$, $\mathcal{T}_{fl}^{N-1} = 0$ and $\mathcal{T}_{ds} = 0$.*

The convergence for general $\Pi^{\pm,h}$ in the overlapping case is derived from a perturbative analysis of the above result. It relies on the nilpotency relationships (2.12) and on the exponential decay estimates (1.27)(1.28)(2.8). When the subdomains do not overlap, our results are weaker. They only hold for the ASM and FDA algortihms and require stronger assumptions. In this latter case, the convergence is proved via energy estimates like in [6][17].

Remark 3.2 *a) We are considering the non-constant coefficients case. Hence, the next convergence results do not hold only for domain decomposition into rectilinear strips but also for any situation which can be reduced to this case after a change of variables.*
b) While looking for uniform estimates in the previous sections, we did not fix the small parameter h. In the next convergence statements, h is supposed to be fixed so that the norm $\|.\|_{H^{s,h}}$ is equivalent to the usual H^s-norm.

3.1 The overlapping case

In this paragraph, we establish the geometric convergence for the three algorithms in the overlapping case.

Theorem 3.3 *There exist constants h_π, ξ_c, $\xi_c' > 0$ so that the following convergence result holds as soon as $0 < h < h_\pi$, $\inf_{1\leq i\leq N} \alpha(L_i - l_{i+1})/h > \xi_c'$, $\inf_{1\leq i\leq N} \alpha(L_i - l_i)/h > \xi_c$ and $L_i - l_i > 2(L_j - l_{j+1}) > 0$, $\forall i, j \in \{1, \ldots, N-1\}$.*
For initial data $u_i^0 \in H^{2,h}(\Omega_i), 1 \leq i \leq N$ and for $f \in L^2(\mathbb{R}^{d+1})$, the three algorithms achieve geometric convergence with the estimates:
$||u_i^n - u||_{H^{2,h}(\Omega_i)} \leq C_h \rho^{[\frac{n}{2(N-1)}]} \sup_j ||u_j^0 - u||_{H^{2,h}(\Omega_j)}$, *$n \geq 2N+1$, for the ASM,*
$||u_i^n - u||_{H^{2,h}(\Omega_i)} \leq C_h \rho^n \sup_j ||u_j^0 - u||_{H^{2,h}(\Omega_j)}$ *$n \geq 3$ for the DSA,*
$||u_i^n - u||_{H^{2,h}(\Omega_i)} \leq C_h \rho^{[\frac{n}{N-1}]} \sup_j ||u_j^0 - u||_{H^{2,h}(\Omega_j)}$ *$n \geq 2N-1$ for the FDA,*
(writing $[.]$ for the integer part) where $C_h > 0$, $\rho \in (0,1)$do not depend on (N, u_i^0, f).

On $\left(H^{3/2-j\vee 1,h}(\mathbb{R}^d)\right)^{2(N-1)}$ we use the $l^\infty(H^{3/2-j\vee 1,h}(\mathbb{R}^d))$-norm. Then, the Banach algebra $\mathcal{L}\left(\left(H^{3/2-j\vee 1,h}(\mathbb{R}^d)\right)^{2(N-1)}\right)$ is naturally endowed with the norm

$$\|K\| = \sup_{1\leq m\leq 2N-2} \sum_{1\leq n\leq 2N-2} \|K_{mn}\|_{\mathcal{L}(H^{1/2-j\vee 1,h}(\mathbb{R}^d))},$$

where K is considered as an operator-valued matrix $K = (K_{mn})_{1\leq m,n\leq 2N-2}$. Theorem 5.1 and Remark 5.10 of [16] which rely on a combinatorial analysis give as a consequence of the nilpotency relationships the

Lemma 3.4 *Suppose*

$$\rho = ||\mathcal{A}r||\,||\mathcal{A}l|| \left(\sum_{n=0}^{N-2} ||\mathcal{M}l||^n\right) \left(\sum_{n=0}^{N-2} ||\mathcal{M}r||^n\right) < 1.$$

Then the estimates

$$\begin{aligned} ||\mathcal{T}^n|| &\leq C\,\tfrac{1}{1-\rho}\,\rho^{\left[\frac{n}{2N-2}-\frac{3}{2}\right]}, \quad &\forall n \geq 2N \\ ||{\mathcal{T}_{ds}}^n|| &\leq C\,(1+\rho)\rho^{n-1}, \quad &\forall n \geq 2 \\ ||{\mathcal{T}_{fl}}^n|| &\leq C\,\tfrac{1}{1-\rho}\,\rho^{\left[\frac{n}{N-1}\right]-2}, \quad &\forall n \geq 2(N-1) \end{aligned}$$

hold with $C = \left(1 + \frac{\rho}{||\mathcal{A}l||}\right)\left(1 + \frac{\rho}{||\mathcal{A}r||} + \frac{\rho}{||\mathcal{A}r||||\mathcal{A}l||}\right)$.

Proof of Theorem 3.3: From (2.9)(2.10) we get at once

$$\begin{aligned} g_{i+1,l}^{n+1} = &\left[\Lambda^{-,h}(l_{i+1}) - \Pi^{+,h}(l_{i+1})\right] S^{-,h}(l_{i+1}, l_i)\alpha_{i,l}^h(g_{i,l}^n, g_{i,r}^n) \\ &+ \left[\Lambda^{+,h}(l_{i+1}) - \Pi^{+,h}(l_{i+1})\right] S^{+,h}(l_{i+1}, L_i)\alpha_{i,r}^h(g_{i,l}^n, g_{i,r}^n) \\ g_{i-1,r}^{n+1} = &\left[\Lambda^{-,h}(L_{i-1}) - \Pi^{-,h}(L_{i-1})\right] S^{-,h}(L_{i-1}, l_i)\alpha_{i,l}^h(g_{i,l}^n, g_{i,r}^n) \\ &+ \left[\Lambda^{+,h}(L_{i-1}) - \Pi^{-,h}(L_{i-1})\right] S^{+,h}(L_{i-1}, L_i)\alpha_{i,r}^h(g_{i,l}^n, g_{i,r}^n). \end{aligned}$$

Let $\delta = \inf_{1\leq i\leq N-1}(L_i - l_{i+1})$ and $L = \inf_{1\leq i\leq N}(L_i - l_i)$ satisfy $0 < \delta < \frac{L}{2}$. Putting together (1.27)(1.28) and (2.8) leads to

$$\|\mathcal{A}l_{n,n+N-1}\|_{\mathcal{L}(H^{1/2-j\vee 1,h}(\mathbb{R}^d))},\ \|\mathcal{A}_{n+N-1,n}\|_{\mathcal{L}(H^{1/2-j\vee 1,h}(\mathbb{R}^d))} \leq Ce^{-\frac{\alpha\delta}{2h}},$$
$$\|\mathcal{M}l_{n+1,n}\|_{\mathcal{L}(H^{1/2-j\vee 1,h}(\mathbb{R}^d))},\ \|\mathcal{M}r_{n+N-1,n+N}\|_{\mathcal{L}(H^{1/2-j\vee 1,h}(\mathbb{R}^d))} \leq Ce^{-\frac{\alpha(L-\delta)}{2h}} \leq Ce^{-\frac{\alpha L}{4h}}$$

for any $h \in (0, h_\pi)$, h_π small enough. With the notations of Lemma 3.4, this yields

$$\varrho \leq C^2 e^{-\frac{\alpha\delta}{h}} \left(\frac{1}{1 - Ce^{-\frac{\alpha L}{4h}}}\right)^2, \quad \forall h \in (0, h_\pi).$$

$\square$

Remark 3.5 *The convergence could be proved here with a simpler criterion than the one provided by Lemma 3.4. Indeed with the overlapping $\delta > 0$, the perturbations of the nilpotent matrices are always exponentially small with respect to h. However, the criterion of Lemma 3.4 has the advantage that it also gives convergence for $\delta = 0$ in the constant coefficients case (see [16]).*

3.2 The nonoverlapping case

In this paragraph, we still assume hypotheses **H1** and **H2** while enforcing $j \in [0, 2]$ and $2(j \vee 1 - 1) \leq k \leq j$. We split the difference $\pi^+ - \pi^-$ into its real and imaginary parts, $\pi^+ - \pi^- = q + ip$, and we set $r = \frac{p}{q}$ so that $\frac{\pi^+ - \pi^-}{q} = 1 + ir$. According to **H1** and **H2**, we have $q \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^j, g_\eta)$ with $q \geq C_\pi \langle\eta\rangle^k$, $p \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^j, g_\eta)$ and $r \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{j-k}, g_\eta)$, while $0 \leq 2(j \vee 1 - 1) \leq k \leq j \leq 2$ implies $j - k \leq 1$. The semi-classical quantization of p, q and r will be respectively denoted by P^h, Q^h and R^h. The next analysis requires additional assumptions. Although these assumptions will be verified in practical situations via semi-classical calculus, their general presentation is more convenient in terms of operators.

H3: The estimate

$$\mathbb{R}\mathrm{e}\left((1 + iR^h)u, (-b_x + 2C_{xy}h\partial_y + \Pi^{+,h} + \Pi^{-,h})u\right)_{L^2} \leq C_\pi h \|u\|^2_{H^{1/2,h}},\ \forall u \in H^{2,h}(\mathbb{R}^d),$$

holds for any $h \in (0, h_\pi)$.

H4: There exists $\mu \in (0, 1)$ so that

$$\text{(3.1)} \qquad \mathbb{R}\mathrm{e}\left(u, \left[R^h(-b_x C_{xy} + {}^t B_y)hD_y - R^h \frac{b_x^2}{4\mu^2} R^h\right] u\right)_{L^2} \geq -C_\pi h \|u\|^2_{H^{1,h}},$$
$$\forall u \in H^{2,h}(\mathbb{R}^d), h \in (0, h_\pi).$$

Theorem 3.6 *Let $\Pi^{\pm,h}$ satisfy the assumptions* **H1, H2, H3** *and* **H4**. *There exists $h_\pi > 0$, so that ASM and FDA algorithms achieve convergence by* $\lim_{n\to\infty} \|u_i^n - u\|_{H^{1,h}(\Omega_i)} = 0$ *as soon as $u_i^0 \in H^{2,h}(\Omega_i)$, $1 \leq i \leq N$, $f \in L^2(\mathbb{R}^{d+1})$ and $h \in (0, h_\pi)$.*

Remark 3.7 *This result does not hold for DSA. However, in the constant coefficients case, the convergence was proved in [16] for the three algorithms by methods based on Lemma 3.4 and Fourier analysis.*

The above result relies on the following energy estimate.

Proposition 3.8 *There exists $h_\pi \in (0, h_0)$ and $C_\pi > 0$ so that*

$$
\begin{aligned}
(3.2) \quad C_\pi \|u\|^2_{H^{1,h}(\Omega_i)} \quad & +h\,\mathbb{R}\mathrm{e}\left((h\partial_x - \Pi^{-,h})u, (Q^h)^{-1}(h\partial_x - \Pi^{-,h})u\right)_{L^2}\Big|_{x=l_i} \\
& +h\,\mathbb{R}\mathrm{e}\left((h\partial_x - \Pi^{+,h})u, (Q^h)^{-1}(h\partial_x - \Pi^{+,h})u\right)_{L^2}\Big|_{x=L_i} \\
\leq \; & h\,\mathbb{R}\mathrm{e}\left((h\partial_x - \Pi^{-,h})u, (Q^h)^{-1}(h\partial_x - \Pi^{-,h})u\right)_{L^2}\Big|_{x=L_i} \\
& +h\,\mathbb{R}\mathrm{e}\left((h\partial_x - \Pi^{+,h})u, (Q^h)^{-1}(h\partial_x - \Pi^{+,h})u\right)_{L^2}\Big|_{x=l_i}
\end{aligned}
$$

holds as soon as $u \in H^{2,h}(\Omega_i)$, $\mathcal{L}^h u = 0$ in Ω_i and $h \in (0, h_\pi)$.

Proof : Let us first remind that $(Q^h)^{-1}$ is a well-defined semi-classical operator with symbol in $\mathbb{P}^0 S^{h_\pi}(\langle\eta\rangle^{-k}, g_\eta)$ by taking $h_\pi \in (0, h_0)$ small enough. Hence the $h\,\mathbb{R}\mathrm{e}$-terms of (3.2) make sense as bounded forms on $H^{2,h}(\Omega_i)$ because $2(j \vee 1) - k \leq 2(j \vee 1) - 2(j \vee 1 - 1) = 2$. By mollifying $u\big|_{x=l_i}$ and $u\big|_{x=L_i}$, considered as Dirichlet boundary conditions, we can assume $u \in H^{m,h}(\Omega_i)$ with m large enough so that the next calculations make sense. The estimate for general $u \in H^{2,h}(\Omega_i)$ follows. The energy estimate is obtained by multiplying $\mathcal{L}^h(u)$ by $\overline{(Id + iR^h)u}$, integrating by parts and taking the real part:

$$
\begin{aligned}
(3.3) \qquad 0 \; = \; & \mathbb{R}\mathrm{e} \int\!\!\int_{\Omega_i} a|u|^2 - \frac{h}{2}{}^t\partial B|u|^2 + {}^t\overline{\partial u} C(h\partial u) + \overline{iR^h u} a u \\
& \qquad + \tfrac{1}{2}\overline{(iR^h u)} b_x h\partial_x u - \tfrac{1}{2} h\partial_x \left[b_x \overline{(iR^h u)}\right] u \\
& \quad + \overline{(iR^h u)}{}^t B_y(h\partial_y u) + \left[{}^t h\partial\overline{(iR^h u)}\right] C(h\partial u)\, dxdy \\
+ \; & \frac{h}{2}\mathbb{R}\mathrm{e}\int_{\mathbb{R}^d} \left[\overline{(u + iR^h u)}(b_x u - 2C_{xy} h\partial_y u - 2h\partial_x u)\right]_{l_i}^{L_i} dy.
\end{aligned}
$$

We first look at the boundary terms. Let us consider for $x \in \{l_i, L_i\}$ the expression

$$
\begin{aligned}
E(u) \; = \; & \left((h\partial_x - \Pi^{+,h})u, (Q^h)^{-1}(h\partial_x - \Pi^{+,h})u\right)_{L^2} \\
& \quad - \left((h\partial_x - \Pi^{-,h})u, (Q^h)^{-1}(h\partial_x - \Pi^{-,h})u\right)_{L^2} \\
= \; & \left((-\Pi^{+,h} + \Pi^{-,h})u, (Q^h)^{-1}(h\partial_x - \Pi^{+,h})u\right)_{L^2} \\
& \quad - \left((h\partial_x - \Pi^{-,h})u, (Q^h)^{-1}(\Pi^{+,h} - \Pi^{-,h})u\right)_{L^2}.
\end{aligned}
$$

Since the principal symbol, $\frac{1}{q}$, of $(Q^h)^{-1}$ is real-valued, we have $\left[(Q^h)^{-1}\right]^* - (Q^h)^{-1} = hc(x, y, hD_y; h)$ with $c \in \mathbb{P}^0 S^{h_\pi}(\langle\eta\rangle^{-k-1}, g_\eta)$. Hence, semi-classical calculus yields

$$
\begin{aligned}
\frac{h}{2}\mathbb{R}\mathrm{e}[E(u)] \; = \; & \tfrac{h}{2}\mathbb{R}\mathrm{e}\left((Q^h)^{-1}(\Pi^{+,h} - \Pi^{-,h})u, (-2h\partial_x + \Pi^{+,h} + \Pi^{-,h})u\right)_{L^2} \\
& \qquad + O(h^2)\|u\big|_x\|^2_{H^{1/2,h}}
\end{aligned}
$$

$$= \frac{h}{2}\,\mathbb{R}\mathrm{e}\left((1+iR^h)u, (b_x - 2C_{xy}h\partial_y u - 2h\partial_x)u\right)_{L^2} + O(h)\|u\|^2_{H^{1,h}(\Omega_i)},$$

where the last line is a consequence of **H3** and of the estimate (1.3). As a conclusion the boundary terms of (3.2) and (3.3) differ by an $O(h)\|u\|^2_{H^{1,h}(\Omega_i)}$ term. It remains to check that the volume integral of (3.3), now denoted by $V(u)$, is bounded from below by $C\|u\|^2_{H^{1,h}(\Omega_i)}$. It satisfies

$$\begin{aligned}V(u) \geq \mathbb{R}\mathrm{e}\int\int_{\Omega_i} \mu^2\overline{({}^th\partial u)}C(h\partial u) + \overline{(iR^h u)}au + \frac{1}{2}\overline{(iR^h u)}b_x h\partial_x u - \frac{1}{2}b_x\overline{(iR^h h\partial_x u)}u \\ +\overline{(iR^h u)}^t B_y(h\partial_y u) + \overline{[iR^h({}^th\partial u)]}C(h\partial u)\,dxdy \\ + \left[(1-\mu^2)\frac{\alpha\wedge\gamma}{2} - O(h)\right]\|u\|^2_{H^{1,h}(\Omega_i)}.\end{aligned}$$

Commutator terms were included in the $O(h)$ remainder by using $r \in \mathbb{P}^0S^{h_0}(\langle\eta\rangle^{j-k}, g_\eta)$, $j-k \leq 1$. By introducing $h\partial_C u = h\partial_x u + C_{xy}h\partial_y u$, we have

$$\begin{aligned}\int\int_{\Omega_i}\overline{({}^th\partial u)}C(h\partial u)\,dxdy &\geq \int\int_{\Omega_i}|h\partial_C u|^2 + \overline{({}^th\partial_y u)}\widetilde{C}(h\partial_y u)\,dxdy \\ &\geq \int\int_{\Omega_i}|h\partial_C u|^2\,dxdy,\end{aligned}$$

which leads to the lower bound of $V(u)$

$$\begin{aligned}\mathbb{R}\mathrm{e}\int_{l_i}^{L_i}\mu^2\|h\partial_C u\|^2_{L^2} + \left(iR^h u, au\right)_{L^2} + \frac{1}{2}\left(iR^h u, b_x h\partial_C u\right)_{L^2} - \frac{1}{2}\left(b_x(iR^h h\partial_C u), u\right)_{L^2} \\ + \left(iR^h u, {}^tB_y(h\partial_y u)\right)_{L^2} - \frac{1}{2}\left(iR^h u, b_x C_{xy}(h\partial_y u)\right)_{L^2} \\ +\frac{1}{2}\left(b_x(iR^h)C_{xy}(h\partial_y u), u\right)_{L^2} + \left(iR^h(h\partial u), C(h\partial u)\right)_{(L^2)^{d+1}}\,dx \\ + \left[(1-\mu^2)\frac{\alpha\wedge\gamma}{2} - O(h)\right]\|u\|^2_{H^{1,h}(\Omega_i)}.\end{aligned}$$

The symbol r of R^h is real-valued so that $R^h - (R^h)^* = hc(x,y,hD_y;h)$ with $c \in \mathbb{P}^0S^{h_\pi}(\langle\eta\rangle^{j-k-1}, g_\eta)$. The previous estimates and semi-classical calculus then imply that $V(u)$ is bounded from below by

$$\begin{aligned}&\textstyle\int_{l_i}^{L_i}\mu^2\|h\partial_C u\|^2_{L^2} + \mathbb{R}\mathrm{e}\left(h\partial_C u, ib_x R^h u\right)_{L^2} + \mathbb{R}\mathrm{e}(u, R^h(-b_x C_{xy} + {}^tB_y)hD_y u)_{L^2}\,dx \\ &\qquad + \left[(1-\mu^2)\tfrac{\alpha\wedge\gamma}{2} - O(h)\right]\|u\|^2_{H^{1,h}(\Omega_i)} \\ &\geq \textstyle\int_{l_i}^{L_i}\mathbb{R}\mathrm{e}\left(u, \left[R^h(-b_x C_{xy} + {}^tB_y)hD_y - R^h\frac{b_x^2}{4\mu^2}R^h\right]u\right)_{L^2}\,dx \\ &\qquad + \left[(1-\mu^2)\tfrac{\alpha\wedge\gamma}{2} - O(h)\right]\|u\|^2_{H^{1,h}(\Omega_i)}.\end{aligned}$$

We conclude by referring to **H4** and by taking h_π small enough. □

Proof of Theorem 3.6: We proceed as in [6] or [17]. Let

$$E_i^n = C_\pi\|u_i^n\|^2_{H^{1,h}(\Omega_i)},\ A_i^n = h\,\mathbb{R}\mathrm{e}\left((h\partial_x - \Pi^{-,h})u_i^n, (Q^h)^{-1}(h\partial_x - \Pi^{-,h})u_i^n\right)_{L^2}\Big|_{x=L_i}$$

$$B_i^n = h\,\mathbb{R}\mathrm{e}\left((h\partial_x - \Pi^{+,h})u_i^n, (Q^h)^{-1}(h\partial_x - \Pi^{+,h})u_i^n\right)_{L^2}\Big|_{x=l_i},$$

$$\mathcal{E}^n = \sum_i E_i^n, \ \mathcal{A}^n = \sum_i A_i^n \ \text{ and } \ \mathcal{B}^n = \sum_i B_i^n.$$

According to Proposition2.1, these quantities are well-defined for any $n \in \mathbb{N}$ as soon as $u_i^0 \in H^{2,h}(\Omega_i)$, $i = 1 \dots N$. Further since the principal symbol $\frac{1}{q}$ of $(Q^h)^{-1}$ is real-valued with $\frac{1}{q} \geq C_\pi \langle \eta \rangle^{-k}$, the quadratic term A_i^n, B_i^n, $\mathcal{A}^n$ and $\mathcal{B}^n$ are non-negative for any $h \in (0, h_\pi)$, h_π small enough (see Lemma C.1-b)). One easily checks by using $L_i = l_{i+1}$ that we have for ASM

$$\mathcal{E}^{n+1} + \mathcal{A}^{n+1} + \mathcal{B}^{n+1} \leq \mathcal{A}^n + \mathcal{B}^n$$

and for FDA

$$\mathcal{E}^{n+1} + \mathcal{A}^{n+1} \leq \mathcal{A}^n.$$

The convergence follows by summation over n. □

4 Local Approximations of Dirichlet-to-Neumann Operators

We saw in Proposition (3.1) that the three algortihms converge after a finite number of steps for $\Pi^{\pm,h} = \Lambda^{\pm,h}$. Nevertheless, local boundary conditions are preferred in numerical applications. Indeed numerical discretization truncates the high frequencies while h is usually small. It is thus sensible to approximate $\lambda^\pm$ by Taylor expansions at $\eta = 0$. Three Taylor expansions are considered up to the second order (see [17]). Higher order approximations are not used in practical algorithms. We first check that the operator $\Pi^{\pm,h}$ so constructed satisfy hypotheses **H1 H2 H3** and **H4**. Then we show that the differences $\Lambda^{\pm,h} - \Pi^{\pm,h}$ are small in some sense and that this choice of boundary conditions leads to a better convergence than arbitrary $\Pi^{\pm,h}$.

4.1 Adequacy with the general convergence analysis

We take

$$\pi_0^\pm = i\xi^\pm(x, y, 0; h), \tag{4.1}$$

$$\pi_1^\pm = i\xi^\pm(x, y, 0; h) + i\eta \frac{\partial \xi^\pm}{\partial \eta}(x, y, 0; h) + ihA_1^\pm(x, y, 0; h) \tag{4.2}$$

and

$$\begin{aligned} \pi_2^\pm = {} & i\xi^\pm(x, y, 0; h) + i\eta \frac{\partial \xi^\pm}{\partial \eta}(x, y, 0; h) + i\frac{\eta^2}{2}\frac{\partial^2 \xi^\pm}{\partial \eta^2}(x, y, 0; h) \\ & + ihA_1^\pm(x, y, 0; h) + ih\eta \frac{\partial A_1^\pm}{\partial \eta}(x, y, 0; h) + ih^2 A_2^\pm(x, y, 0; h). \end{aligned} \tag{4.3}$$

and we recall

$$\begin{aligned} i\xi^{\pm}(x,y,\eta;h) &= \frac{b_x}{2} - iC_{x,y}\eta \pm \frac{\sqrt{-\Delta}}{2}, \\ -\Delta &= \left[4^t\eta\widetilde{C}\eta + (b_x)^2 + 4a\right] + 4i\left[-b_xC_{xy} + {}^tB_y\right]\eta. \end{aligned}$$

In [7], the same approximations are made for the wave equation in the framework of classical pseudo-differential calculus. Here the h-pseudo-differential calculus also enables norm estimates for the difference between $\Lambda^{\pm,h}$ and the proposed local approximations. The complete expression of $\pi_l^{\pm}$, $l = 0,1,2$, are derived from Theorem B.1, TheoremB.4 and

$$\begin{aligned} & i\xi^{\pm}(x,y,0;h) = \frac{b_x \pm \sqrt{{b_x}^2 + 4a}}{2}, \\ & i\eta\frac{\partial\xi^{\pm}}{\eta}(x,y,0;h) = -iC_{xy}\eta \pm i\frac{(-b_xC_{xy} + {}^tB_y)\eta}{\sqrt{{b_x}^2 + 4a}} \\ \text{and} \qquad & i\frac{\eta^2}{2}\frac{\partial^2\xi^{\pm}}{\eta^2}(x,y,0;h) = \pm\frac{2^t\eta\widetilde{C}\eta(b_x^2 + 4a) + [(-b_xC_{xy} + {}^tB_y)\eta]^2}{({b_x}^2 + 4a)^{3/2}}. \end{aligned}$$

Proposition 4.1 *The symbol $\pi_l^{\pm}$ (or the operators $\Pi_l^{\pm,h}$), $l = 0,1,2$, all satisfy hypotheses* **H1**, **H2**, **H3** *and* **H4**.

Proof : By their construction the symbols $\pi_l^{\pm}$ belong to $\mathbb{P}^0S^{h_0}(\langle\eta\rangle^l, g_\eta)$ for $l = 0,1,2$. Assumption **H1** is true for $j = l$ while **H2** is satisfied by taking $k = 0$ for $j = l \in \{0,1\}$ and $k = 2$ for $j = l = 2$. In the case $l = 0$, we have $r = 0$ while $\sigma_{\langle\eta\rangle^0}(\pi^+ + \pi^-) = b_x$. Hence **H3** and **H4** are trivially satisfied. For $l = 1,2$, we have $\sigma_{\langle\eta\rangle^l}(\pi^+ + \pi^-) = b_x - 2iC_{xy}\eta$ so that **H3** is true. We now verify **H4** for $l = 1,2$. In these cases we can write

$$\left[-b_xC_{xy} + {}^tB_y\right]\eta = rr'$$

with $r' = \frac{1}{2}(b_x^2 + 4a)$ for $l = 1$ and $r' \geq \frac{1}{2}(b_x^2 + 4a) + {}^t\eta\widetilde{C}\eta$ for $l = 2$. Since $r' \in \mathbb{P}^0S^{h_\pi}(\langle\eta\rangle^{-j+k+1}, g_\eta)$ and $-j + k + 1 = 0$ for $l = 1$ and $-j + k + 1 = 2$ for $l = 2$, expression (3.1) equals

$$\mathbb{R}\mathrm{e}\left(R^hu, \left[r'(x,y,hD_y;h) - \frac{b_x^2}{4\mu^2}\right]R^hu\right)_{L^2} + O(h\|u\|^2_{H^{1/2,h}}).$$

By taking $\mu \in (\sqrt{2},1)$, **H4** comes at once from Lemma C.1-b). □

4.2 Quality of local approximations

In order to describe the low frequency behaviour of a semi-classical operator we introduce the

Definition 4.2 *The operator $a(x,y,hD_y;h)$, $a \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^j, g_\eta)$, is said to be an $O_{LF}(h^k)$ if the estimate*

$$\|a(x,y,hD_y;h)\chi(D_y)\|_{\mathcal{L}(L^2)} \leq C_\chi h^k, \quad \forall (x,h) \in \mathbb{R} \times (0,h_0),$$

holds for any $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.

Lemma 4.3 *For given $a \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^j, g_\eta)$, $j \in \mathbb{R}$, and $m \in \mathbb{N}$, we set*

$$b = a - \sum_{|\alpha| \leq m} \partial_\eta^\alpha a(x,y,0;h)\frac{\eta^\alpha}{\alpha!}.$$

Then the operators $(\partial_y^\beta b)(x,y,hD_y;h)$, $\beta \in \mathbb{Z}^d$, are $O_{LF}(h^{m+1})$. More precisely, the estimate

$$(4.4) \|\partial_y^\beta b(x,y,hD_y;h)u\|_{H^{s,h}} \leq C_{a,m,s} h^{m+1} \|\langle D_y\rangle^{m+1} u\|_{H^{s+j\vee m-(m+1),h}}, \ \forall u \in S(\mathbb{R}^d),$$

holds for any $(s,h) \in \mathbb{R} \times (0,h_0)$.

Proof : The whole lemma is contained in (4.4) because

$$\|\langle D_y\rangle^{m+1}\chi(D_y)u\|_{H^{j\vee m-(m+1),h}} \leq \|\langle hD_y\rangle^{j\vee m-(m+1)}\langle D_y\rangle^{m+1}\chi(D_y)u\|_{L^2} \leq C_\chi \|u\|_{L^2}.$$

We can reduce the analysis to $\beta = 0$. In order to treat separately low and high frequencies we consider the partition of unity $\chi_0 + \chi_\infty \equiv 1$ with $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\chi_0 \equiv 1$ in a neighbourhood of $\eta = 0$. Let $\tilde{\chi}_\infty \in \mathcal{C}^\infty(\mathbb{R}^d)$ be such that $\tilde{\chi}_\infty \equiv 1$ on supp χ_∞ and $\tilde{\chi}_\infty \equiv 0$ in a neighbourhood of $\eta = 0$. We write

$$b = b\chi_0 + \frac{b\tilde{\chi}_\infty(\eta)}{|\eta|^{m+1}}\chi_\infty(\eta)|\eta|^{m+1}.$$

With the $(1,0)$-quantization, we have

$$\mathrm{Op}_{(1,0)}^h\left[b\frac{\tilde{\chi}_\infty(\eta)}{|\eta|^{m+1}}\chi_\infty(\eta)|\eta|^{m+1}\right] = \mathrm{Op}_{(1,0)}^h\left[b\frac{\tilde{\chi}_\infty(\eta)}{|\eta|^{m+1}}\right] \circ \chi_\infty(hD_y)|hD_y|^{m+1}.$$

But $b\frac{\tilde{\chi}_\infty(\eta)}{|\eta|^{m+1}} \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{j\vee m-(m+1), g_\eta})$ implies

$$\|\mathrm{Op}_{(1,0)}^h\left[b\frac{\tilde{\chi}_\infty(\eta)}{|\eta|^{m+1}}\right] u\|_{H^{s,h}} \leq C_{a,m,s}\|u\|_{H^{s+j\vee m-(m+1),h}},$$

while we have

$$\|\chi_\infty(hD_y)|hD_y|^{m+1}u\|_{H^{s+j\vee m-(m+1),h}} \leq h^{m+1}\|\langle D_y\rangle^{m+1}u\|_{H^{s+j\vee m-(m+1),h}}.$$

For the low frequencies, Taylor formula gives

$$b\chi_0(\eta) = (m+1)\sum_{|\alpha|=m+1}\int_0^1 (1-t)^m \partial_\eta^\alpha a(x,y,t\eta;h)\,dt\ \chi_0(\eta)\frac{\eta^\alpha}{\alpha!}$$

where $\left[\int_0^1 (1-t)^m \partial_\eta^\alpha a(x,y,t\eta;h)\,dt\right] \chi_0(\eta)$ belongs to $\mathbb{P}^0 S^{h_0}(\langle\eta\rangle^{-\infty}, g_\eta)$. Hence we get

$$\|(b\chi_0)(x,y,hD_y;h)u\|_{H^{s,h}} \le C_{a,m,s} h^{m+1} \|\langle D_y\rangle^{m+1} u\|_{H^{s+j\vee m-(m+1),h}}.$$

$\square$

Proposition 4.4 *For any $\alpha \in \mathbb{Z}^d$, there exists $h_\alpha \in (0,h_0)$ so that $\mathrm{ad}_{D_y}^\alpha(\Lambda^{\pm,h} - \Pi_l^{\pm,h})$ is an $O_{LF}(h^{l+1})$ for $h \in (0,h_\alpha)$, $l = 0,1,2$. More precisely, for $(s,\alpha) \in \mathbb{R}\times\mathbb{Z}^d$ there exists $h_{s,\alpha} \in (0,h_0)$ so that the estimate*

$$\|\,\mathrm{ad}_{D_y}^\alpha(\Lambda^{\pm,h} - \Pi_l^{\pm,h})u\|_{H^{s,h}} \le C_{s,l} h^{l+1} \|\langle D_y\rangle^{l+1} u\|_{H^{s+l\vee 1-(l+1),h}}, \quad \forall u \in S(\mathbb{R}^d)$$

holds for any $h \in (0,h_{s,\alpha})$.

Proof : We write

$$\mathrm{ad}_{D_y}^\alpha(\Lambda^{\pm,h} - \Pi_l^{\pm,h}) = \frac{1}{h^{|\alpha|}}\,\mathrm{ad}_{hD_y}^\alpha(\Lambda^{\pm,h} - \Lambda_{l+1}^{\pm,h}) + \frac{1}{h^{|\alpha|}}\,\mathrm{ad}_{hD_y}^\alpha(\Lambda_{l+1}^{\pm,h} - \Pi_l^{\pm,h}).$$

The first term is estimated in $\mathcal{L}(H^{s+l\vee 1-(l+1),h}, H^{s,h})$, by some seminorm $p_k(\lambda^\pm - \lambda_{l+1}^\pm)$ with $k = k(s,\alpha)$. We refer to Proposition 1.8-b). For the second one, we use the previous lemma (Remind $\mathrm{ad}_{hD_y}^\alpha a(y,hD_y) = h^{|\alpha|}(\partial_y^\alpha a)(y,hD_y)$). $\square$

We next consider a simple domain decomposition problem with two nonoverlapping subdomains. The interface lays at $x = 0$. Though the general convergence result of Theorem 3.6 does not provide any order of convergence, the local approximations (4.1)(4.2)(4.3) lead to a convergence all the faster as h is small. It can be checked that $e_i^n = u_i^n - u$ ($i = 1,\,2$) satisfies

$$e_i^{n+2}\Big|_{x=0} = \left[\Lambda^{-,h}(0) - \Pi_l^{+,h}(0)\right]^{-1} \left[\Lambda^{+,h}(0) - \Pi_l^{+,h}(0)\right] \left[\Lambda^{+,h}(0) - \Pi_l^{-,h}(0)\right]^{-1}$$
$$\left[\Lambda^{-,h}(0) - \Pi_l^{-,h}(0)\right] e_i^n\Big|_{x=0} \equiv \mathcal{A}_l^h e_i^n\Big|_{x=0}.$$

Proposition 4.5 *There exists $h_0 > 0$ so that the operator $\mathcal{A}_l^h$ is an $O_{LF}(h^{2(l+1)})$ for $h \in (0,h_0)$ and $l = 0,1,2$. Indeed there exists for any $s \in \mathbb{R}$ $h_s \in (0,h_0)$ so that the estimate*

$$\|\mathcal{A}_l^h e\|_{H^{s,h}} \le C_s h^{2(l+1)} \|e\|_{H^{s-2(l+1),h}}, \quad \forall e \in S(\mathbb{R}^d),$$

holds for any $h \in (0,h_s)$.

Proof : For $e \in S(\mathbb{R}^d)$, we infer from **H1**, Lemma C.2-a) and Proposition 4.4

$$\begin{aligned}
\|\mathcal{A}_l^h e\|_{H^{s,h}} &\le C_s \|\left[\Lambda^{+,h}(0) - \Pi_l^{+,h}(0)\right]\left[\Lambda^{+,h}(0) - \Pi_l^{-,h}(0)\right]^{-1} \\
&\qquad \left[\Lambda^{-,h}(0) - \Pi_l^{-,h}(0)\right] e\|_{H^{s-l\vee 1,h}} \\
&\le C_s h^{l+1} \sum_{|\alpha|\le l+1} \|\partial_y^\alpha \left[\Lambda^{+,h}(0) - \Pi_l^{-,h}(0)\right]^{-1} \left[\Lambda^{-,h}(0) - \Pi_l^{-,h}(0)\right] e\|_{H^{s-(l+1),h}} \\
&\le C_s h^{l+1} \sum_{|\alpha|\le l+1} \|\partial_y^\alpha \left[\Lambda^{-,h}(0) - \Pi_l^{-,h}(0)\right] e\|_{H^{s-(l+1)-l\vee 1,h}}
\end{aligned}$$

We have

$$D_y c(y,hD_y) = \mathrm{ad}_{D_y} c(y,hD_y) + c(y,hD_y)D_y$$

while ad_{D_y} and $.D_y$ commute. Hence Leibnitz formula gives

$$D_y^\alpha = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \mathrm{ad}_{D_y}^{\alpha-\beta} c(y,hD_y) D_y^\beta.$$

We finally obtain by referring again to Proposition 4.4

$$\|\mathcal{A}_l^h e\|_{H^{s,h}} \leq C_s h^{2(l+1)} \sum_{|\alpha|\leq 2(l+1)} \|\partial_y^\alpha e\|_{H^{s-2(l+1),h}}.$$

□

A Semi-Classical Weyl Calculus and Beals Criterion

This appendix presents a semi-classical version of the work of Bony and Chemin in [2] where Beals criterion introduced in [1] was completely proved. It was done on the basis of confinement theory developed by Bony and Lerner in [3] to which we also refer. We introduce the notion of h-confinement for symbols depending on a small parameter $h \in (0,h_0)$ and write the semi-classical form of Beals criterion. We only give a summary of the arguments of [2][3] with indications in order to translate their proofs in the semi-classical framework.

Contrary to the rest of the paper, we consider here the Weyl quantization

$$\left(a^W(x,hD)u\right)(x) = \left(\mathrm{Op}_W^h[a]u\right)(x) = \int_{\mathbb{R}^{2d}} e^{\frac{i(x-y).\xi}{h}} a(\frac{x+y}{2},\xi)u(y)d\xi dy$$

The corresponding $\#_W^h$ operation, $h \in (0,h_0)$, defined by $\mathrm{Op}_W^h[a\#_W^h b] = \mathrm{Op}_W^h[a] \circ \mathrm{Op}_W^h[b]$ for $a,b \in S(T^*\mathbb{R}^d)$, writes according to [2][3][11]

$$\text{(A.1)} \qquad \begin{aligned} a\#_W^h b(X;h) &= \frac{1}{\pi^{2d}h^{2d}} \int\int e^{-\frac{2i}{h}[\![X-Y_1,X-Y_2]\!]} a(Y_1;h)b(Y_2;h)dY_1dY_2 \\ &= e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)} a(x,\xi;h)b(y,\eta;h)\Big|_{(y,\eta)=(x,\xi)}. \end{aligned}$$

where $\sigma(.,.)$ or $[\![.,.]\!]$ denote the canonical symplectic two-form on $T^*\mathbb{R}^d$.

The metric g and the weight $m \geq 0$ on $T^*\mathbb{R}^d$ are supposed independent of $h \in (0,h_0)$ and to satisfy the same assumptions as in [2]:
Slowness of g: There exists a constant $\overline{C}_0$ so that

$$g_X(X-Y) \leq \overline{C}_0^{-1} \Rightarrow (g_Y(.)/g_X(.))^{\pm 1} \leq \overline{C}_0.$$

Uncertainty Principle: $g_X(.) \leq g_X^\sigma(.)$ where $g_X^\sigma(.)$ is the σ-dual metric of g,

$$g_X^\sigma(T) = \sup_{W\neq 0} \frac{[\![T,W]\!]^2}{g_X(T)}.$$

If we set as usual $\lambda(X)^2 = \inf_{T\neq 0} \dfrac{g_X^\sigma(T)}{g_X(T)}$ it writes $\lambda \geq 1$.
We write the temperance condition in its symmetric form which is more natural according to [3] by introducing the metric $g_{XY}^\sigma = \left(\dfrac{g_X + g_Y}{2}\right)^\sigma$
Symmetric Temperance of g: There exist $\overline{C} > 0$ and $\overline{N} \in \mathbb{N}$ so that

$$(g_Y(.)/g_X(.))^{\pm 1} \leq \overline{C}\left(1 + g_{XY}^\sigma(X-Y)\right)^{\overline{N}}.$$

Slowness of m: There exists a constant $\tilde{C}_0 > 0$ so that

$$g_X(X-Y) \leq \overline{C}_0^{-1} \Rightarrow (m(Y)/m(X))^{\pm 1} \leq \tilde{C}_0.$$

Temperance of m: There exist $\tilde{C} > 0$ and $\tilde{N} \in \mathbb{N}$ so that

$$(m(Y)/m(X))^{\pm 1} \leq \tilde{C}\left(1 + g_{XY}^\sigma(X-Y)\right)^{\tilde{N}}.$$

With such a metric g and a weight m, one associates according to [2][3][11] the symbol class $S(m,g)$ of $\mathcal{C}^\infty$ functions $a(x,\xi)$ on $T^*\mathbb{R}^d$ such that the quantities

$$\|a\|_{k,S(m,g)} = \sup_{\substack{\nu\leq k,\\ x,\xi\in T^*\mathbb{R}^d\\ g_{x,\xi}(T_i)\leq 1, i=1\ldots\nu}} \frac{\left|a^{(\nu)}(x,\xi).T_1\ldots T_\nu\right|}{m(x,\xi)} \tag{A.2}$$

are bounded ($a^{(\nu)}$ denotes the ν^{th} derivative of a). It is a Fréchet space when endowed with these seminorms.

In the sequel we will use the following convention: constants depending on the slowness constants $\overline{C}_0$ and $\tilde{C}_0$ will be written with a $^{(0)}$ superscript; the one which also depend on the temperance constants $\overline{C}$, $\overline{N}$, $\tilde{C}$ and $\tilde{N}$ will have a $^{(1)}$ superscript; if moreover such a constant depends on h_0 we use a $^{(2)}$ superscript. Additional dependance will be indicated by the subscript.

A.1 h-confinement, partitions of unity and remarks

We set $U_{Y,r} = \{X \in T^*\mathbb{R}^d,\ g_Y(X-Y) \leq r^2\}$ where $r = r^{(0)}$ is a positive constant so that $r^2 \leq \overline{C}_0^{-1}$.

Definition A.1 *a) For a fixed $h \in (0,h_0)$, the space $\mathrm{Conf}^h(g,Y,r)$ of symbols h-confined in the g-ball $U_{Y,r}$ is $S(T^*\mathbb{R}^d)$ endowed with the topology given by the seminorms*

$$\|a\|_{k,\mathrm{Conf}^h(g,Y,r)} = \inf\{C>0,\ |\partial_{T_1}\ldots\partial_{T_l}a(X)| \leq C\left(1+h^{-2}g_Y^\sigma(X-U_{Y,r})\right)^{-k/2}\}$$

where $l \leq k$ and $g_Y(T_j) \leq 1$, $j = 1 \dots l$.
b) An h-symbol $(a(h))_{h\in(0,h_0)}$, that is a family of symbols indexed by $h \in (0, h_0)$, will be said h-confined in $U_{Y,r}$ if $a(h) \in S(T^*\mathbb{R}^d)$ and if the estimates

$$\|a\|_{k,\mathrm{Conf}^h(g,Y,r)} \leq C_k$$

hold uniformly with respect to $h \in (0, h_0)$.

Notice that the h-confinement corresponds semi-classically to the notion of support in the sense that the constant h-symbol $(a(h) = a)_{h\in(0,h_0)}$ is h-confined in $U_{Y,r}$ if and only if supp $a \subset U_{Y,r}$.

The construction of the partition of unity $(\varphi_Y)_{Y\in T^*\mathbb{R}^d}$ given in [3]-Theorem 3.1.3 relies only on the slowness of the metric g and does not depend on h. The functions φ_Y are uniformly bounded in $S(1, g)$ and satisfy

$$\int_{T^*\mathbb{R}^d} \varphi_Y |g_Y|^{1/2} dY = 1 \quad \text{and } \operatorname{supp} \varphi_Y \subset U_{Y,r}.$$

Proposition A.2 *a) If the h-symbol $a(h)$ belongs to $S(m, g)$ for any $h \in (0, h_0)$ then it writes*

$$a(X; h) = \int_{T^*\mathbb{R}^d} m(Y) a_Y(X; h) |g_Y|^{1/2} dY \tag{A.3}$$

where the symbols $a_Y(h)$ satisfy the uniform estimates

$$\|a_Y(h)\|_{k,\mathrm{Conf}^h(g,Y,r)} \leq C_k^{(0)} \|a(h)\|_{k,S(m,g)}.$$

b) Reciprocally, if the h-symbols $a_Y(h)$ are h-confined in $U_{Y,r}$ uniformly with respect to $Y \in T^*\mathbb{R}^d$ then the the integral (A.3) is uniformly bounded in $S(m, g)$ with

$$\forall k,\ \exists l, C_{k,l}^{(2)}, \quad \|a(h)\|_{k,S(m,g)} \leq C_{k,l}^{(2)} \sup_{\substack{Y\in T^*\mathbb{R}^d \\ h\in(0,h_0)}} \|a(h)\|_{l,\mathrm{Conf}^h(g,Y,r)}.$$

Proof : a) Take $a_Y(h) = m(Y)^{-1} a(h) \varphi_Y$. Then the estimates come from the slowness conditions by

$$|\partial_{T_1} \dots \partial_{T_l} a_Y(X; h)|$$
$$\begin{cases} \leq m(Y)^{-1} |\partial_{T_1} \dots \partial_{T_l} a\varphi_Y(X; h)| \leq C_k \frac{m(X)}{m(Y)} \|a(h)\|_{k,S(m,g)} \leq C_k \tilde{C}_0 \|a(h)\|_{k,S(m,g)} \\ = 0 \quad \text{if } X \notin U_{Y,r}. \end{cases}$$

b) We have

$$\|a(h)\|_{k,\mathrm{Conf}(g,Y,r)} \leq \sup\{1, h^k\} \|a(h)\|_{k,\mathrm{Conf}^h(g,Y,r)} \leq \sup\{1, h_0^k\} \|a(h)\|_{k,\mathrm{Conf}^h(g,Y,r)}$$

for all $h \in (0, h_0)$ and our assertions comes at once from the results [3][2] for $h = 1$. $\square$

We will not give the complete proofs of the next results which are rather long. We recall their general principle: 1) Establish the estimates for a constant metric γ, $\gamma \leq \gamma^\sigma$, with constants independent of γ; 2) For a general metric, reduce the problem to part 1) owing to the slowness and temperance conditions. Here are several remarks which will allow the reader to translate the proofs of [3]-Theorem 3.2.1 and [2]-Theorem 3.1 and Theorem 5.5 into the semi-classical framework.

Remark A.3 **a)** Like in Proposition A.2-b), some assertions can be deduced from the results for $h = 1$ by using

$$(1 + h^{-2}u)^{-k/2} \leq \sup\{1, h_0^k\}(1 + u)^{-k/2}, \quad \forall u \geq 0.$$

b) When the metric γ is constant, integration by part in the integral (A.1) is applied like in [3][2] with the operator $1 + \frac{1}{2}[\![T, D_{Y_2}]\!]$ where $T = T_{Y_1,X}$ is chosen so that $\gamma(T) = 1$ and $[\![T, Y_1 - X]\!] = \gamma^\sigma(Y_1 - X)^{1/2}$. Note that $h^{-2}\gamma^\sigma$ replaces γ^σ in the equality

$$\left(1 + \frac{1}{2}[\![T, D_{Y_2}]\!]\right)^k e^{-\frac{2i}{h}[\![Y_1 - X, Y_2 - X]\!]} = \left(1 + h^{-1}\gamma^\sigma(Y_1 - X)^{1/2}\right)^k e^{-\frac{2i}{h}[\![Y_1 - X, Y_2 - X]\!]}.$$

c) We need an h-dependent analogue of the quantity $\delta_r(X, Y) = 1 + g^\sigma_{XY}(U_{X,r} - U_{Y,r})$ which appears in the symmetric temperance condition

$$\delta^h_r(X, Y) = 1 + h^{-2}g^\sigma_{XY}(U_{X,r} - U_{Y,r}).$$

Remark a) applies to δ_r and δ^h_r so that Theorem 3.2.2 of [3] yields

$$\sup_{\substack{X, h\in(0,h_0)\\ r^2\leq C_0^{-1}}} \int_{T^*\mathbb{R}^d} \delta^h_r(X, Y)^{-N^{(1)}} |g_Y|^{1/2} dY \leq C^{(2)}. \tag{A.4}$$

We do not use the quantities

$$\begin{aligned} &\Delta^h_r(X, Y) = 1 + h^{-2}\sup\{g^\sigma_X(U_{X,r} - U_{Y,r}), g^\sigma_X(U_{X,r} - U_{Y,r})\}\\ \text{and}\quad &\Delta'^h_r(X, Y) = 1 + h^{-2}\inf\{g^\sigma_X(U_{X,r} - U_{Y,r}), g^\sigma_X(U_{X,r} - U_{Y,r})\}\end{aligned}$$

because it does not seem possible to make the logarithmic equivalence with δ^h_r, stated in [2] for $h = 1$, uniform with respect to h.

d) At some point, one is lead to estimate for a constant metric γ, $\gamma \leq \gamma^\sigma$, the quantity

$$\frac{1}{\pi^{2d}h^{2d}} \int\int \left(1 + h^{-2}\gamma^\sigma(Y_1 - X)\right)^{-(d+1)} \left(1 + h^{-2}\gamma^\sigma(Y_2 - U_2)\right)^{-(d+1)} dY_1 dY_2 \tag{A.5}$$

where $U_2 = U_{X_2,r'}$ for some $r' > 0$. We first use remark a) and write

$$\left(1 + h^{-2}\gamma^\sigma(Y_2 - U_2)\right)^{-(d+1)} \leq \sup\{1, h_0^{2d+2}\} \left(1 + \gamma^\sigma(Y_2 - U_2)\right)^{-(d+1)}.$$

Then the same argument as in the proof of Theorem 2.2.1 [3] gives

$$1+\gamma(Y_2-X_2)\leq C_{r'}\left(1+\gamma^\sigma(Y_2-U_2)\right).$$

Hence the integral (A.5) is less than

$$C_{d,r',h_0}\int\int\left(1+\gamma^\sigma(\tfrac{Y_1-X}{h})\right)^{-(d+1)}\left(1+\gamma(Y_2-X_2)\right)^{-(d+1)}d\left(\tfrac{Y_1-X}{h}\right)d\left(Y_2-X_2\right)$$

which is bounded by a constant C_{d,r',h_0} independent of X, X_2 and γ because $|\gamma||\gamma^\sigma|=1$.

e) The uniform L^2-continuity of $a^W(x,hD;h)$ when the h-symbol $a(h)$ is h-confined in the g-ball $U_{Y,r}$ is a consequence of Weyl's formula

$$a^W(x,hD;h)=\int_{\mathbb{R}^{2d}}\hat{a}(u,\theta;h)e^{i(u.x+h\theta.D)}đu đ\theta$$

which holds in $\mathcal{L}(L^2(\mathbb{R}^d))$ as soon as $\hat{a}(h)\in L^1(\mathbb{R}^{2d})$.

A.2 Results

Proposition A.4 *(h-biconfinement theorem) For any $\nu,k,N\in\mathbb{N}$, there exist $l=l^{(1)}_{\nu,k,N}\in\mathbb{N}$ and $C=C^{(2)}_{\nu,k,N}>0$ so that*

$$\text{(A.6)}\qquad \begin{aligned}&\|a\#^h_W b-\sum_{0\leq j<\nu}\frac{1}{j!}\left(\frac{ih[\![D_{X_1},D_{X_2}]\!]}{2}\right)^j a\otimes b\Big|_{\mathrm{Diagonal}}\|_{k,\mathrm{Conf}^h(g,Y,r)}\\&\leq Ch^\nu\lambda(Y)^{-\nu}\|a\|_{l,\mathrm{Conf}^h(g,Y,r)}\|b\|_{l,\mathrm{Conf}^h(g,Z,r)}\delta^h_r(Y,Z)^{-N}\end{aligned}$$

holds for all $a,b\in S(T^*\mathbb{R}^d)$.

The case $\nu=0$ expresses the semi-classical almost orthogonality: if the h-symbols $a_Y(h)$ and $b_Y(h)$ are h-confined in $U_{Y,r}$ uniformly with respect to $Y\in T^*\mathbb{R}^d$ then $a_Y(h)\#^h_W b_Z(h)$ is h-confined in $U_{Y,r}$ and $U_{Z,r}$ with h-confinement semi-norms estimated by $C^{(2)}_{k,N}\delta^h_r(Y,Z)^{-N}$. By taking N large enough and referring to Proposition A.2 and (A.4) one easily obtain

$$\forall k,\exists l=l^{(1)}_k,\exists C=C^{(2)}_k,\quad \|a\#^h_W b\|_{k,\mathrm{Conf}^h(g,Y,r)}\leq Cm(Y)\|a\|_{l,S(m,g)}\|b\|_{l,\mathrm{Conf}^h(g,Y,r)},$$
$$\forall k,\exists l=l^{(1)}_k,\exists C=C^{(2)}_k,\quad \|a\#^h_W b\|_{k,S(mm',g)}\leq C\|a\|_{l,S(m,g)}\|b\|_{l,S(m',g)}.$$

One also obtains that the remainder of the ν^{th}-order expansion writes

$$a\#^h_W b-\sum_{0\leq j<\nu}\frac{1}{j!}\left(\frac{ih[\![D_{X_1},D_{X_2}]\!]}{2}\right)^j a\otimes b\Big|_{Diagonal}=h^\nu R_\nu(a,b;h)$$

where the h-symbol $R_\nu(a,b;h)$ satisfies the estimates

$$\forall k,\exists l=l^{(1)}_{k,\nu},\exists C=C^{(2)}_{k,\nu},\quad \|R_\nu(a,b;h)\|_{k,S(mm'\lambda^{-\nu},g)}\leq C\|a\|_{l,S(m,g)}\|b\|_{l,S(m',g)}.$$

Proposition A.5 *(Decomposition of h-confined h-symbols) For any $K > 1$, $r' > 0$ small enough and any family $a_Y(h)$ of h-symbols uniformly h-confined in $U_{Y,r'}$, there exists two families $b_{Y,\nu}(h) \in S(T^*\mathbb{R}^d)$ and $c_{Y,\nu}(h) \in S(T^*\mathbb{R}^d)$ for $(Y,\nu) \in T^*\mathbb{R}^d \times \mathbb{N}$ so that*

$$a_Y(h) = \sum_{\nu\in\mathbb{N}} b_{Y,\nu}(h) \#^h_W c_{Y,\nu}(h)$$

and so that for all $N \in \mathbb{N}$, the h-symbols $(1+\nu)^N b_{Y,\nu}(h)$ and $(1+\nu)^N c_{Y,\nu}(h)$ are h-confined in $U_{Y,Kr'}$ uniformly with respect to (Y,ν).

Here is a remark about the proof which can be adapted from [2]. As we said, the small parameter h appears in the quantization and in the integrations by parts. The last argument of the proof of [2] relies on a decomposition in Fourier series. It is a technical trick required to inverse some tensor product operation, while the metric γ is constant. The small parameter must not appear in the phase of these Fourier series if one wants to get uniform estimates.

Next the constant $K > 1$ is fixed and one takes a g-partition of unity φ_Y, $Y \in T^*\mathbb{R}^d$, so that supp $\varphi_Y \subset U_{Y,r/K}$. The above Proposition provides two families of h-symbols $\psi_{Y,\nu}(h)$ and $\theta_{Y,\nu}(h)$ uniformly h-confined in $U_{Y,r}$ so that $\varphi_Y = \sum_{\nu\in\mathbb{N}} \psi_{Y,\nu}(h)\#^h_W\theta_{Y,\nu}(h)$.

Definition A.6 *For a weight m satisfying the slowness and temperance conditions and for any $h \in (0,h_0)$, we define the Sobolev space $H^h(m,g)$ as the space of $u \in S'(R^d)$ such that*

$$\sum_{\nu\in\mathbb{N}} \int_{T^*\mathbb{R}^d} m(Y)^2 \|\theta^W_{Y,\nu}(x,hD_x;h)u\|^2_{L^2} |g_Y|^{1/2} dY < \infty.$$

It is a Hilbert space when endowed with the scalar product

$$(u,v)_{H^h(m,g)} = \sum_{\nu\in\mathbb{N}} \int_{T^*\mathbb{R}^d} m(Y)^2 \left(\theta^W_{Y,\nu}(x,hD_x;h)u, \theta^W_{Y,\nu}(x,hD_x;h)v\right)_{L^2} |g_y|^{1/2} dY.$$

The h-biconfinement estimates given in Proposition A.4 with $\nu = 0$ are indeed sharper than the one obtained for $h = 1$ according to Remark A.3 a). Meanwhile the uniform L^2-continuity of h-confined h-symbols is given by Remark A.3 e). Hence the method developed in [2] which proves the continuity of pseudo-differential operators via Schur's lemma and almost orthogonality applies as it is in the semi-classical framework.

Proposition A.7 *a) If the weights m and m' satisfy the slowness and temperance conditions, then the quantized operator $a^W(x,hD_x)$ of a symbol $a \in S(m,g)$ sends continuously $H^h(m',g)$ into $H^h(m'/m,g)$. Moreover there exist $k = k^{(1)}_{m,m'} \in \mathbb{N}$ and $C = C^{(2)}_{m,m'} > 0$ so that*

$$\|a^W(x,hD_x)\|_{\mathcal{L}(H^h(m',g),H^h(m'/m,g))} \leq C\|a\|_{k,S(m,g)}.$$

b) The space $H^h(1,g)$ is nothing but $L^2(\mathbb{R}^d)$.

Next we write Beals criterion in the form proposed in [2]-Remark 5.6 which is more convenient for our purpose. With a vector $T_0 \in T^*\mathbb{R}^d$, $T_0 \neq 0$, we associate the operator $L^h_{T_0} = \mathrm{Op}^h_W([\![T_0, X]\!])$ where $[\![T_0, X]\!]$ denotes the linear form $X \to [\![T_0, X]\!]$. A finite sequence of such vectors will be written $T = \{T_j \neq 0,\ j = 1 \ldots |T|\}$.

Theorem A.8 *a) An operator $A : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ writes $a^W(x, hD)$ with $a \in S(m, g)$, $h \in (0, h_0)$ being fixed, iff the quantities*

$$h^{-|T|} m(Y)^{-1} \|\theta^W_{Y,\nu}(x, hD; h) \left(\prod_{j \leq |T|} \mathrm{ad}_{L^h_{T_j}}\ A\right) \|_{\mathcal{L}(L^2)}$$

are bounded by constants $C_{|T|}$ independent of $(Y, \nu) \in T^*\mathbb{R}^d$ and of the choice of T_j, $g_Y(T_j)$, for $j = 1 \ldots |T|$.
b) If we set

$$M^h_k(A) = \sup_{\substack{|T| \leq k,\ Y,\nu \\ g_Y(T_j) \leq 1}} h^{-|T|} \|\theta^W_{Y,\nu}(x, hD; h) \left(\prod_{j \leq |T|} \mathrm{ad}_{L^h_{T_j}}\ A\right) \|_{\mathcal{L}(L^2)}$$

the above condition is equivalent to the finiteness of these quantities for all $k \in \mathbb{N}$ and we have

$$\forall k,\ \exists l = l^{(1)}_k,\ \exists C = C^{(2)}_k,\ \|a\|_{k, S(m,g)} \leq C M^h_l(a^W(x, hD)). \tag{A.7}$$

Remark A.9 *a) The normalization factor $h^{-|T|}$ is here in order to compensate the gain of h induced by each commutator $\mathrm{ad}_{L^h_{T_j}}$.*
b) Since the estimates reverse to (A.7) are a consequence of semi-classical calculus, the second assertion means that the two topologies on $S(m, g)$ defined by the seminorms (A.2) and $M^h_k(a^W(x, hD))$ are uniformly equivalent with respect to $h \in (0, h_0)$.

A.3 Applications

We first develop the final remark of [2]-Section 5 concerned with diagonal metrics and finally apply our results to the metric $dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}$. Like in [2], we make the more general assumption that there exists a fixed basis $\mathcal{E} = \{e_i,\ i = 1 \ldots 2d\}$ of $T^*\mathbb{R}^d$ so that the convex hull of $\{\dot{e}_i = g_Y(e_i)^{-1/2} e_i,\ i = 1 \ldots 2d\}$ contains the g_Y-ball of radius ϱ for some $\varrho \in (0, 1)$ independent of Y. As an example if the metric g_Y is diagonal in the basis $\mathcal{E}$ it is true with $\varrho = (2d)^{-1/2}$. We consider $\mathcal{E}$-valued finite sequences $E = \{E_j \in \mathcal{E},\ j = 1 \ldots |E|\}$. With such a sequence we associate the weight $m_E(Y) = \Pi_{j \leq |E|}\, g_Y(E_j)^{1/2}$ which satisfies the slowness and temperance conditions and the normalized sequence given by $\dot{E}_j = g_Y(E_j)^{-1/2} E_j$, $j = 1 \ldots |E|$.

Proposition A.10 *Under the above assumptions, an operator $A : S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ writes $a^W(x, hD)$ with $a \in S(m, g)$, $h \in (0, h_0)$ fixed, iff the quantities*

$$N^h_k(A) = \sup_{|E| \leq k} h^{-|E|} \| \prod_{j \leq |E|} \mathrm{ad}_{L^h_{E_j}}\ A \|_{\mathcal{L}(L^2, H^h((m m_E)^{-1}, g))}$$

are all finite. Moreover the equivalence estimates

$$\forall k,\ \exists l = l_k^{(1)},\ \exists C = C_k^{(2)},\ \|a\|_{k,S(m,g)} \leq C N_l^h(a^W(x,hD)) \tag{A.8}$$

are uniform with respect to $h \in (0, h_0)$.

Proof : The h-confinement in $U_{Y,r}$ of $\theta_{Y,\nu}(h)$ uniform with respect to (Y,ν) combined with the temperance of mm_E which can be made uniform with respect to E, $|E| \leq k$ imply that $(m(Y)m_E(Y))^{-1}\theta_{Y,\nu}(h)$ is uniformly bounded is $S((mm_E)^{-1}, g)$. More precisely the k-th semi-norm is estimated by $C_{k,|E|}^{(2)}$. Hence Proposition A.7 yields

$$h^{-|E|}(m(Y)m_E(Y))^{-1}\|\theta_{Y,\nu}^W(x,hD;h)\left(\prod_{j\leq|E|} \mathrm{ad}_{L^h_{E_j}}\ A\right)\|_{\mathcal{L}(L^2)} \leq C_{|E|}^{(2)} N_{|E|}^h(A)$$

which also writes

$$h^{-|E|}m(Y)^{-1}\|\theta_{Y,\nu}^W(x,hD;h)\left(\prod_{j\leq|E|} \mathrm{ad}_{L^h_{\dot{E}_j}}\ A\right)\|_{\mathcal{L}(L^2)} \leq C_{|E|}^{(2)} N_{|E|}^h(A).$$

Our assumption ensures that any vector T_0 satisfying $g_Y(T_0) = 1$ writes $T_0 = \sum_{i\leq 2d} \lambda_i \dot{e}_i$ with $\lambda_i \geq 0$, $\sum_{i\leq 2d} \lambda_i \leq \varrho^{-1}$. We conclude with

$$M_k^h(A) \leq \varrho^{-k} C_k^{(2)} N_k^h(A), \quad \forall k \in \mathbb{N}.$$

□

For general metrics and weights, the above Proposition is not simpler than Theorem A.8 because Definition A.6 of Sobolev spaces $H^h(m,g)$ involves the family of h-symbols $\theta_{Y,\nu}(h)$. However in the case in which we are interested, that is with the metric $g_\xi = dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2}$ and the weights $\langle\xi\rangle^s$, $s \in \mathbb{R}$, the Sobolev spaces $H^h(\langle\xi\rangle^s, g_\xi)$ are nothing but the standard Sobolev spaces $H^s(R^d)$ with the h-dependent norms introduced in Section 2. This is a straightforward consequence of Proposition A.7: the operator $\langle hD\rangle^{-s} = \mathrm{Op}_W^h(\langle\xi\rangle^{-s})$ defines an isomorphism between $L^2(\mathbb{R}^d)$ and $H^h(\langle\xi\rangle^s, g_\xi)$ on one side uniform estimates with respect to $h \in (0,h_0)$ and an isometry between $L^2(\mathbb{R}^d)$ and $H^{s,h}(R^d)$ on the other side. In $T^*\mathbb{R}^d$, we take the basis $\mathcal{E} = \{e_1, \ldots, e_d, e_1^*, \ldots, e_d^*\}$, where $\{e_1, \ldots, e_d\}$ is the canonical basis and $\{e_1^*, \ldots, e_d^*\}$ its dual basis. Then we have $L^h(e_i) = hD_i$ and $L^h(e_i^*) = -x_i$ for $i = 1 \ldots d$ while Jacobi's identity gives

$$\mathrm{ad}_{x_i}\mathrm{ad}_{hD_{i'}} - \mathrm{ad}_{hD_{i'}}\mathrm{ad}_{x_i} = -\,\mathrm{ad}_{h\delta_{i,i'}} = 0.$$

Hence if E denotes an $\mathcal{E}$-valued finite sequence we have

$$\prod_{j\leq|E|} \mathrm{ad}_{L(E_j)} = \mathrm{ad}_{hD}^{\alpha}\,\mathrm{ad}_{-x}^{\beta} = \prod_{1\leq i\leq d} \mathrm{ad}_{hD}^{\alpha_i} \prod_{1\leq i'\leq d} \mathrm{ad}_{-x}^{\beta_{i'}}$$

where α_i and $\beta_{i'}$ are the occurence number of e_i and $e_{i'}^*$ in E. Finally we notice that the weight m_E defined above equals $m_E(x,\xi) = \langle\xi\rangle^{-|\beta|}$ and Proposition A.10 now has the form

Proposition A.11 *An operator* $A: S(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ *writes* $a^W(x, hD)$ *with* $a \in S(\langle\xi\rangle^s, g_\xi)$ *iff the quantities*

$$N_k^h(A) = \sup_{|\alpha|+|\beta|\leq k} h^{-|\alpha|-|\beta|} \|\operatorname{ad}_{hD}^\alpha \operatorname{ad}_x^\beta A\|_{\mathcal{L}(L^2, H^{|\beta|-s,h})}$$

are all finite. Moreover estimate (A.8) still holds uniformly with respect to $h \in (0, h_0)$.

Remark A.12 *a) Since* $\langle hD\rangle^m$, $m \in \mathbb{R}$, *satisfies the criterion and* ad_B *is a derivation, the norms in* $\mathcal{L}(L^2, H^{|\beta|-s,h})$ *can be replaced by norms in* $\mathcal{L}(H^{m,h}, H^{m+|\beta|-s,h})$.
b) In Theorem A.8, Proposition A.10 and Proposition A.11, the result holds with the standard $(1,0)$*-calculus and more generally with any* $(t, 1-t)$*-quantization as soon as the metric* g *is splitted,* $g_{x,\xi}(t_x, -t_\xi) = g_{x,\xi}(t_x, t_\xi)$, *owing to the equivalence of quantizations in this case.*

B Factorizing Semi-Classical Elliptic Operators

As it is well known, factorizing elliptic operators with an arbitrarily smooth or small remainder is related to the construction of Calderon projector. Instead of using complex integral and residue formula like in [4][11][21], we follow the method of Treves in [22] which is convenient in our case with flat boundaries. From a technical point of view, it gives at once uniform estimates in our framework. Secondly, it is completely constructive and provides a suitable algorithm for applications.

In order to use the $(1,0)$-calculus corresponding to the quantization

$$A(x, x'; h) = \int_{\mathbb{R}^d} e^{i\frac{(x-x').\xi}{h}} a(x, \xi; h) \frac{d\xi}{h^d}. \tag{B.1}$$

with which differential operators are easier to handle, we assume the metric g to be splitted $g_{x,\xi}(t_x, -t_\xi) = g_{x,\xi}(t_x, t_\xi)$. For $h \in (0, h_0)$, the Schwartz-kernel (B.1)defines an operator denoted by $a(x, hD; h)$ or $\operatorname{Op}^h[a(h)]$ continuous from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ when $a(h) \in \mathcal{S}'(T^*\mathbb{R}^d)$ and continuous from $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^d)$) into $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^d)$) when $a(h)$ belongs in a symbol class $S(m, g)$. We consider the space $S^{h_0}(m, g)$ of h-dependent symbols $a(h)$ uniformly bounded in $S(m, g)$ with respect to $h \in (0, h_0)$. Endowed with the semi-norms $p_\nu^h(a) = \sup_{h\in(0,h_0)} \|a(h)\|_{\nu, S(m,g)}$, where the $\|.\|_{\nu,S(m,g)}$ are defined by (A.2), $S^{h_0}(m, g)$ is a Fréchet space. The bilinear mapping $\#^h$ defined by

$$\operatorname{Op}^h\left[(a\#^h b)(h)\right] = \operatorname{Op}^h[a(h)] \circ \operatorname{Op}^h[b(h)],$$

satisfies

$$\begin{aligned} (a\#^h b)(x, \xi; h) &= e^{ihD_\xi.D_y} a(x, \xi; h) b(y, \eta; h)\Big|_{(y,\eta)=(x,\xi)} \\ &= \sum_{|\alpha|<N} \frac{(ih)^{|\alpha|}}{\alpha!} D_\xi^\alpha a D_x^\alpha b(x, \xi; h) + h^N R_N(a, b)(x, \xi; h) \end{aligned} \tag{B.2}$$

where, for any $N \in \mathbb{N}$, the mapping $(a,b) \to R_N(a,b)$ is continuous from $S^{h_0}(m,g) \times S^{h_0}(m',g)$ into $S^{h_0}(\frac{mm'}{\lambda^N}, g)$. Remind that when a and b are polynomials, that is when $a(x,hD)$ and $b(x,hD)$ are h-differential operators, formula (B.2) is nothing but the Leibnitz rule and the remainder R_N is zero for N large enough.

The principal symbol $\sigma_{m,g}(a)$ (or simply $\sigma_m(a)$ when the context is clear) of $a \in S^{h_0}(m,g)$ is defined as its class with respect to the equivalence relation

$$(a \equiv b) \Leftrightarrow (a - b = hc \text{ with } c \in S^{h_0}(\frac{m}{\lambda}, g)).$$

This equivalence relation is compatible with addition and multiplication of symbols and we can define $\sigma_{m,g}(a) + \sigma_{m,g}(b) = \sigma_{m,g}(a+b)$ and $\sigma_{m,g}(a)\sigma_{m',g}(b) = \sigma_{mm',g}(ab)$. The expansion (B.2) implies

$$\sigma_{mm',g}(a \#^h_{(0,1)} b) = \sigma_{m,g}(a)\sigma_{m',g}(b).$$

As a consequence of the equivalence with Weyl calculus [11] for a splitted g, the formal adjoint of $a(x,hD;h)$ writes $b(x,hD;h)$ where $b \in S^{h_0}(m,g)$ is a continuous function of $a \in S^{h_0}(m,g)$ and

$$\sigma_{m,g}(b) = \sigma_{m,g}(\overline{a}).$$

Note that in the sequel, we do not distinguish by notations the principal symbol $\sigma_{m,g}(a)$ as a class and an arbitrary representant.

B.1 A class of polynomial symbols

We consider semi-classical pseudo-differential operators on $\mathbb{R}^{d+1} = \mathbb{R}_x \times \mathbb{R}^d_y$ which are h-differential operators with respect to x. Their $(1,0)$-symbol in $T^*\mathbb{R}^{d+1}$ have the form

$$a(x,\xi,y,\eta;h) = \sum_{0\leq k\leq n} a_k(x,y,\eta;h)\xi^{n-k}$$

where for any $(j,k) \in \mathbb{N} \times \{0,\ldots,n\}$ $\partial_x^j a_k(x)$ is uniformly bounded in $S^{h_0}(m\lambda^k, g)$ with respect to $x \in \mathbb{R}$. We call $\mathbb{P}^n S^{h_0}(m,g)$ the space of such symbols endowed with seminorms

$$p^h_{\nu,j,k}(a) = \sup_{x\in\mathbb{R}} p^h_{\nu,k}(\partial_x^j a_k(x))$$

where $(p^h_{\nu,k})$ is the complete family of seminorms on $S^{h_0}(m\lambda^k, g)$ defined in the previous paragraph. The Leibnitz rule for differential operators, allows to define a $\#^h$ operation for the above polynomial symbols so that

$$\mathrm{Op}^h[(a\#^h b)(h)] = \mathrm{Op}^h[a(h)] \circ \mathrm{Op}^h[b(h)].$$

For $a \in \mathbb{P}^n S^{h_0}(m,g)$ and $b \in \mathbb{P}^{n'} S^{h_0}(m',g)$, we get

$$(a\#^h b)(x,y,\eta,\xi;h) = \sum_{\substack{0\leq l\leq n-k\leq n \\ 0\leq k'\leq n'}} h^l \binom{n-k}{l} \left[a_k \#^h_{(0,1)} D_x^l b_{k'}\right](x,y,\eta;h)\xi^{n+n'-(k+k'+l)}.$$

The mapping $\#^h$ is continuous from $\mathbb{P}^n S^{h_0}(m,g) \times \mathbb{P}^{n'} S^{h_0}(m',g)$ to $\mathbb{P}^{n+n'} S^{h_0}(mm',g)$. The asymptotic expansion is derived by referring to (B.2) for each term. Like in the previous paragraph, we introduce the equivalence relation on $\mathbb{P}^n S^{h_0}(m,g)$,

$$(a \equiv b) \Leftrightarrow (a - b = hc \text{ with } c \in \mathbb{P}^n S^{h_0}(\frac{m}{\lambda}, g)),$$

compatible with addition and multiplication. The principal symbol $\sigma_{n,m,g}(a)$ (or simply $\sigma_{n,m}(a)$) of $a \in \mathbb{P}^n S^{h_0}(m,g)$ is defined as the class of a and we have

$$\sigma_{n+n',mm',g}(a\#^h b) = \sigma_{n,m,g}(a)\sigma_{n',m',g}(b).$$

B.2 Factorization

In this paragraph we consider a semi-classical operator

$$P(x,hD_x,y,hD_y;h) = (hD_x)^n + \sum_{1\leq j\leq n} P_k(x,y,hDy;h)(hD_x)^{n-k}$$

of which the symbol equals $P(x,\xi,y,\eta;h) = \xi^n + \sum_{1\leq j\leq n} P_k(x,\xi,y,\eta;h)\xi^{n-k}$ and belongs to $\mathbb{P}^n S^{h_0}(1,g)$. We assume this operator to be elliptic in the following sense.
Ellipticity assumptions: a) There exists a representant of the principal symbol $\sigma_{n,1}(P)$ which considered as a polynomial function of ξ admits n^+ (resp.n^-) roots $\xi_i^{\pm}(x,y,\eta;h)$, $i = 1\dots n^{\pm}$, with a negative (resp. positive) imaginary part. The integers n^+ and n^- do not depend on $(x,y,\eta;h)$ and satisfy $n^+ + n^- = n$.
b) The roots $\xi_i^{\pm}(x,y,\eta;h)$, $i = 1\dots n^{\pm}$, belong to $\mathbb{P}^0 S^{h_0}(\lambda,g)$ with

$$\inf_{\substack{i=1\dots n^+\\ j=1\dots n^-}} |\xi_i^+ - \xi_j^-| \geq C\lambda$$

where $C > 0$ does not depend on $(x,y,\eta;h)$.
Notation: We set $M^{\pm}(x,\xi,y,\eta;h) = \prod_{i=1\dots n^{\pm}} (\xi - \xi_i(x,y,\eta;h))$ and $n^* = \min(n^+,n^-)$.

Theorem B.1 *There exist two sequences of symbols $(A_k^{\pm})_{k\in\mathbb{N}}$, $A_k^{\pm} \in \mathbb{P}^{n^{\pm}-1} S^{h_0}(\lambda^{1-k},g)$ so that:*
a) $\sigma_{n^{\pm},1}(\xi^{n^{\pm}} + A_0^{\pm}) = \sigma_{n^{\pm},1}(M^{\pm})$.
b) The symbols $M_N^{\pm} = \xi^{n^{\pm}} + \sum_{k<N} h^k A_k^{\pm} \in \mathbb{P}^{n^{\pm}} S^{h_0}(1,g)$ satisfy $P - M_N^+ \#^h M_N^- = h^N R_N$ with $R_N \in \mathbb{P}^{n^*-1} S^{h_0}(\lambda^{n-n^*+1-N},g) \subset \mathbb{P}^{n-1} S^{h_0}(\lambda^{1-N},g) \subset \mathbb{P}^n S^{h_0}(\lambda^{-N},g)$.

In order to get the remainder R_N precisely in $\mathbb{P}^{n^*-1} S^{h_0}(\lambda^{n-n^*+1-N},g)$ we will use a kind of euclidean division.

Lemma B.2 *a) Assume M_N^- to be defined as above and $\tilde{R}_N \in \mathbb{P}^{n-1} S^{h_0}(\lambda^{1-N},g)$, then there exist $Q_N^+ \in \mathbb{P}^{n^+-1} S^{h_0}(\lambda^{1-N},g)$ and $R_N^- \in \mathbb{P}^{n^--1} S^{h_0}(\lambda^{n-n^-+1-N},g)$ so that $\tilde{R}_N = Q_N^+ \#^h M_N^- + R_N^-$.*
b) Assume M_N^+ to be defined as above and $\tilde{R}_N \in \mathbb{P}^{n-1} S^{h_0}(\lambda^{1-N},g)$, then there exist $Q_N^- \in \mathbb{P}^{n^--1} S^{h_0}(\lambda^{1-N},g)$ and $R_N^+ \in \mathbb{P}^{n^+-1} S^{h_0}(\lambda^{n-n^++1-N},g)$ so that $\tilde{R}_N = M_N^+ \#^h Q_N^- + R_N^+$.

Proof : a) Indeed we prove that for $r_k \in \mathbb{P}^{n-k}S^{h_0}(\lambda^{k-N}, g)$, $1 \leq k \leq n^+$ there exist $q_k \in \mathbb{P}^{n^+-k}S^{h_0}(\lambda^{k-N}, g)$ and $r_{k+1} \in \mathbb{P}^{n-k-1}S^{h_0}(\lambda^{k+1-N}, g)$ so that $r_k = q_k \#^h M_N^- + r_{k+1}$. When it is done, it suffices to start from $r_1 = \tilde{R}_N$, to construct recursively the r_k's up to $r_{n^++1} = R_N^-$ and to take $Q_N^+ = \sum_{1\leq k\leq n^+} q_k$ which finally belongs to $\mathbb{P}^{n^+-1}S^{h_0}(\lambda^{1-N}, g)$ owing to the inclusions

$$\mathbb{P}^0 S^{h_0}(\lambda^{n^+-N}, g) \subset \ldots \subset \mathbb{P}^{n^+-2}S^{h_0}(\lambda^{2-N}, g) \subset \mathbb{P}^{n^+-1}S^{h_0}(\lambda^{1-N}, g).$$

So let $r_k = \sum_{j=0\ldots n-k} r_{k,j}\xi^{n-k-j}$ with $r_{k,j} \in \mathbb{P}^0 S^{h_0}(\lambda^{j+k-N}, g)$. We have

$$\begin{aligned} r_k &= \left[\sum_{0\leq j\leq n^+-k} r_{k,j}\xi^{n^+-k-j}\right]\xi^{n^-} + \sum_{n^+-k<j\leq n-k} r_{k,j}\xi^{n-k-j} \\ &= \left[\sum_{0\leq j\leq n^+-k} r_{k,j}\xi^{n^+-k-j}\right]\#^h\left[M_N^- - \sum_{1\leq j\leq n^-} M_{N,j}^-\xi^{n^--j}\right] \\ &\qquad + \sum_{n^+-k<j\leq n-k} r_{k,j}\xi^{n-k-j}. \end{aligned}$$

by writing $M_N^- = \xi^{n^-} + \sum_{1\leq j\leq n^-} M_{N,j}^-\xi^{n^--j}$. The fact that $q_k = \sum_{0\leq j\leq n^+-k} r_{k,j}\xi^{n^+-k-j}$ belongs to $\mathbb{P}^{n^+-k}S^{h_0}(\lambda^{k-N}, g)$ is a consequence of definitions while semi-classical calculus ensures that

$$r_{k+1} = \sum_{n^+-k<j\leq n-k} r_{k,j}\xi^{n-k-j} - \left[\sum_{0\leq j\leq n^+-k} r_{k,j}\xi^{n^+-k-j}\right]\#^h\left[\sum_{1\leq j\leq n^-} M_{N,j}^-\xi^{n^--j}\right]$$

belongs to $\mathbb{P}^{n-k-1}S^{h_0}(\lambda^{k+1-N}, g)$.
b) Due to the commutation relationship

$$\xi\#^h_{(0,1)}A(x, y, \eta; h) - A(x, y, \eta; h)\#^h_{(0,1)}\xi = h\partial_x A(x, y, \eta; h),$$

we can write

$$\tilde{R}_N = \sum_{0\leq j\leq n-1} (\xi\#^h)^{n-1-j}\tilde{R'}_{N,j}.$$

The same arguments as above works by transposing the operations on the left. Namely we replace the left factor $(\xi\#^h)^{n^+}$ by $M_N^+ - \sum_{1\leq j\leq n^+} M_{N,j}^+\xi^{n^+-j}$. $\square$

Proof of Theorem B.1: We construct the $A_k^\pm$' s recursively.
<u>$N = 1$:</u> By semi-classical calculus, we know $\sigma^h_{n,1}(P) = \sigma^h_{n^+,1}(M^+)\sigma^h_{n^-,1}(M^-)$ which means

$$P - M^+\#^h M^- = h\tilde{R}_1, \quad \tilde{R}_1 \in \mathbb{P}^n S^{h_0}(\lambda^{-1}, g).$$

Since P and $M^+\#^h M^-$ have the same leading term ξ^n, $\tilde{R}_1 \in \mathbb{P}^{n-1}S^{h_0}(\lambda^0, g)$. We conclude by referring to Lemma B.2: If $n^+ \geq n^-$ we take $M_1^- = M^-$, $A_0^- = M^- - \xi^{n^-}$, $A_0^+ = M^+ - \xi^{n^+} + hQ_1^+$, $M_1^+ = M^+ + hQ_1^+$ and $R_1 = R_1^-$. Otherwise, if $n^+ < n^-$ we take $M_1^+ = M^+$, $A_0^+ = M^+ - \xi^{n^+}$, $A_0^- = M^- - \xi^{n^-} + hQ_1^-$, $M_1^- = M^- + hQ_1^+$ and $R_1 = R_1^+$. Note that the principal symbol of $A_0^\pm$ and $M_1^\pm$ are not changed by the last operation.

$\underline{N \Rightarrow N+1:}$ We now suppose that the $A_k^{\pm}$' s are known for $k < N$, $N \geq 1$, and satisfy

$$P = M_N^+ \#^h M_N^- + h^N R_N, \quad R_N \in \mathbb{P}^{n^*-1} S^{h_0}(\lambda^{n-n^*+1-N}, g) \subset \mathbb{P}^{n-1} S^{h_0}(\lambda^{1-N}, g).$$

We want to construct $A_N^{\pm} \in \mathbb{P}^{n_\pm - 1} S^{h_0}(\lambda^{1-N}, g)$ so that $M_{N+1}^{\pm} = M_N^{\pm} + h^N A_N^{\pm}$ satisfy

$$P = M_{N+1}^+ \#^h M_{N+1}^- + h^{N+1} R_{N+1}, \quad R_{N+1} \in \mathbb{P}^{n^*-1} S^{h_0}(\lambda^{n-n^*-N}, g).$$

By the same argument as above relying on Lemma B.2, the problem is reduced to the construction of $\tilde{A}_N^{\pm} \in \mathbb{P}^{n_\pm - 1} S^{h_0}(\lambda^{1-N}, g)$ so that

$$P = \tilde{M}_{N+1}^+ \#^h \tilde{M}_{N+1}^- + h^{N+1} \tilde{R}_{N+1}, \quad \tilde{R}_{N+1} \in \mathbb{P}^{n-1} S^{h_0}(\lambda^{-N}, g),$$

by setting $\tilde{M}_{N+1}^{\pm} = M_N^{\pm} + h^N \tilde{A}_N^{\pm}$. One easily checks that the $\tilde{A}_N^{\pm}$ and $\tilde{R}_{N+1}$ have to solve

(B.3) $$\tilde{A}_N^+ \#^h M_N^- + M_N^+ \#^h \tilde{A}_N^- - R_N^h = -h \tilde{R}_{N+1} - h^N \tilde{A}_N^+ \#^h \tilde{A}_N^-.$$

From $\tilde{A}_N^{\pm} \in \mathbb{P}^{n_\pm - 1} S^{h_0}(\lambda^{1-N}, g)$, $R_N \in \mathbb{P}^{n-1} S^{h_0}(\lambda^{1-N}, g)$ and $N \geq 1$ we infer

$$\begin{aligned} \tilde{A}_N^+ \#^h M_N^- + M_N^+ \#^h \tilde{A}_N^- - R_N^h &\in \mathbb{P}^{n-1} S^{h_0}(\lambda^{1-N}, g) \\ \text{and} \quad \tilde{A}_N^+ \#^h \tilde{A}_N^- &\in \mathbb{P}^{n-2} S^{h_0}(\lambda^{2-2N}, g) \subset \mathbb{P}^{n-1} S^{h_0}(\lambda^{-N}, g). \end{aligned}$$

As a consequence, it suffices to find $\tilde{A}_N^{\pm}$ solving

(B.4) $$\begin{aligned} &\sigma_{n-1, \lambda^{1-N}} \left[\tilde{A}_N^+ \#^h M_N^- + M_N^+ \#^h \tilde{A}_N^- - R_N^h \right] = 0 \\ \text{or} \quad &\tilde{A}_N^- M^+ + \tilde{A}_N^+ M^- = R_N. \end{aligned}$$

When $(x, y, \eta; h)$ is fixed, the polynomials $M^{\pm}(\xi) = \Pi_{i=1\ldots n^\pm}(\xi - \xi_i)$ have no common root. Thus by Bezout Theorem, the linear mapping: $(f, g) \in \mathbb{C}^{n^- - 1}[\xi] \times \mathbb{C}^{n^+ - 1}[\xi] \to fM^+ + gM^- \in \mathbb{C}^{n-1}[\xi]$ is an isomorphism and $(\tilde{A}_N^+, \tilde{A}_N^-)$ is nothing but the inverse image of R_N. It remains to specify the behaviour of $\tilde{A}_N^{\pm}$ with respect to $(x, y, \eta; h)$. We set

$$\tilde{A}_N^{\pm} = \sum_{j=0\ldots n^\pm - 1} \tilde{A}_{N,j}^{\pm} \xi^{n^\pm - 1 - j}$$

and we will verify $\tilde{A}_{N,j}^{\pm} \in \mathbb{P}^{n^\pm - 1} S^{h_0}(\lambda^{1-N+j}, g)$. By writing equation (B.4) for $\xi = \xi_i^{\pm}$, $i = 1 \ldots n^{\pm}$, we get two indepent linear systems which determine the coefficients

$$\tilde{A}_{N,j}^{\pm} = \frac{\begin{vmatrix} M^{\mp}(\xi_1^{\pm}) {\xi_1^{\pm}}^{n^\pm - 1} & \ldots & M^{\mp}(\xi_1^{\pm}) {\xi_1^{\pm}}^{n^\pm - j} & R_N(\xi_1^{\pm}) & \ldots & M^{\mp}(\xi_1^{\pm}) \\ \vdots & & \vdots & \vdots & & \vdots \\ M^{\mp}(\xi_n^{\pm}) {\xi_{n^\pm}^{\pm}}^{n^\pm - 1} & \ldots & M^{\mp}(\xi_{n^\pm}^{\pm}) {\xi_{n^\pm}^{\pm}}^{n^\pm - j} & R_N(\xi_{n^\pm}^{\pm}) & \ldots & M^{\mp}(\xi_{n^\pm}^{\pm}) \end{vmatrix}}{\begin{vmatrix} M^{\mp}(\xi_1^{\pm}) {\xi_1^{\pm}}^{n^\pm - 1} & M^{\mp}(\xi_1^{\pm}) {\xi_1^{\pm}}^{n^\pm - 2} & \ldots & M^{\mp}(\xi_1^{\pm}) \\ \vdots & \vdots & & \vdots \\ M^{\mp}(\xi_n^{\pm}) {\xi_{n^\pm}^{\pm}}^{n^\pm - 1} & M^{\mp}(\xi_n^{\pm}) {\xi_{n^\pm}^{\pm}}^{n^\pm - 2} & \ldots & M^{\mp}(\xi_{n^\pm}^{\pm}) \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} M^{\mp}(\xi_1^{\pm})\xi_1^{\pm\, n^{\pm}-1} & \dots & R_N(\xi_1^{\pm}) & \dots & M^{\mp}(\xi_1^{\pm}) \\ \vdots & & \vdots & & \vdots \\ M^{\mp}(\xi_n^{\pm})\xi_{n^{\pm}}^{\pm\ \, n^{\pm}-1} & \dots & R_N(\xi_{n^{\pm}}^{\pm}) & \dots & M^{\mp}(\xi_{n^{\pm}}^{\pm}) \end{vmatrix}}{\left[\prod_{i=1\dots n^{\pm}} M^{\mp}(\xi_i^{\pm})\right]\left[\prod_{1\le k<k'\le n^{\pm}}(\xi_k^{\pm}-\xi_{k'}^{\pm})\right]}$$

where the polynomial R_N appears in the $(j+1)^{th}$ column, $j = 0\dots n^{\pm}-1$. By the multilinearity of the determinant, we can separate the contribution of the $(x, y, \eta; h)$-dependent coefficients of $M^{\mp}$ and R_N from the contribution of the $\xi_i^{\pm} \in S^{h_0}(\lambda, g)$. We set $M^{\mp} = \sum\limits_{0\le i\le n^{\mp}} M_i^{\mp}\xi^{n^{\mp}-i}$ and $R_N = \sum\limits_{0\le i\le n-1} R_{N,i}\xi^{n-1-i}$ and the above quotient splits as a sum whose terms write

$$\frac{M_{i_1}^{\mp}\dots M_{i_j}^{\mp}R_{N,i}M_{i_{j+2}}^{\mp}\dots M_{i_{n^{\pm}}}^{\mp}}{\left[\prod_{i=1\dots n^{\pm}} M^{\mp}(\xi_i^{\pm})\right]}\frac{\begin{vmatrix} \xi_1^{\pm\, n-1-i_1} & \dots & \xi_1^{\pm\, n-j-i_j} & \xi_1^{\pm\, n-1-i} & \dots & \xi_1^{\pm\, n^{\mp}-i_{n^{\pm}}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \xi_{n^{\pm}}^{\pm\ \, n-1-i_1} & \dots & \xi_{n^{\pm}}^{\pm\ \, n-j-i_j} & \xi_{n^{\pm}}^{\pm\ \, n-1-i} & \dots & \xi_{n^{\pm}}^{\pm\ \, n^{\mp}-i_{n^{\pm}}} \end{vmatrix}}{\left[\prod_{1\le k<k'\le n^{\pm}}(\xi_k^{\pm}-\xi_{k'}^{\pm})\right]}.$$

The assumptions on $M^{\mp}$ and R_N imply that the numerator of the first quotient belongs to $\mathbb{P}^0 S^{h_0}(\lambda^{i_1+\dots+i_j+1-N+i+i_{j+2}\dots+i_{n^{\pm}}}, g)$ while the second quotient is a universal symmetric polynomial of $(\xi_1^{\pm}, \dots, \xi_{n^{\pm}}^{\pm})$ of which the total degree equals $n^+n^- + j - (i_1 + \dots + i_j + i + i_{j+2}\dots + i_{n^{\pm}})$. The ellipticity assumption ensures $\frac{1}{(\xi_i^+ - \xi_k^-)} \in S^{h_0}(\lambda^{-1}, g)$, for all $i = 1\dots n^+$, $k = 1\dots n^-$, which yields $\frac{1}{M^{\mp}(\xi_i^{\pm})} \in S^{h_0}(\lambda^{-n^{\mp}}, g)$, for $i = 1\dots n^{\pm}$. We finally obtain $\tilde{A}_{N,j}^{\pm} \in S^{h_0}(\lambda^{j+1-N}, g)$ and $\tilde{A}_N^{\pm} \in \mathbb{P}^{n^{\pm}-1}S^{h_0}(\lambda^{1-N}, g)$. □

Remark B.3 *a) In the case $n = 2$, $n^{\pm} = 1$, the euclidean division described in Lemma B.2 has to be performed only once for $A_0^{\pm}$. Indeed if R_N is a polynomial of degree 0 with respect to ξ, the symbols $\tilde{A}_N^{\pm} \in \mathbb{P}^0 S^{h_0}(\lambda^{1-N}, g)$ given by equation (B.4)verify $\tilde{A}_N^+ + \tilde{A}_N^- = 0$ and R_{N+1} derived from identity (B.3) is again a polynomial of degree 0.*
b)In the above proofs, the $+$ and $-$ signs can be interchanged and we state the

Theorem B.4 *There exist two sequences of symbols $(B_k^{\pm})_{k\in\mathbb{N}}$, $B_k^{\pm} \in \mathbb{P}^{n^{\pm}-1}S^{h_0}(\lambda^{1-k}, g)$ so that:*
a) $\sigma_{n^{\pm},1}(\xi^{n^{\pm}} + B_0^{\pm}) = \sigma_{n^{\pm},1}(M^{\pm})$.
b) The symbols $M'^{\pm}_N = \xi^{n^{\pm}} + \sum_{k<N} h^k B_k^{\pm} \in \mathbb{P}^{n^{\pm}S^{h_0}(1,g)}$ satisfy $P - M'^{-}_N \#^h M'^{+}_N = h^N R'_N$ with $R'_N \in \mathbb{P}^{n^*-1}S^{h_0}(\lambda^{n-n^*+1-N}, g) \subset \mathbb{P}^n S^{h_0}(\lambda^{-N}, g)$.

C Functional Analysis of Semi-Classical Operators

This appendix is restricted to the metric $g_\eta = dy^2 + \frac{d\eta^2}{\langle\eta\rangle^2}$. Some results are standard (see a.e. [10]). However it provides the precise statements to which we often refer along our analysis. Let us first consider properties which are uniform with respect to $h \in (0, h_0)$

Lemma C.1 *a) Let the symbol $a \in S^{h_0}(\langle\eta\rangle^m, g_\eta)$, $m \geq 0$, satisfy the ellipticity condition $|K + a| \geq C_0\langle\eta\rangle^m$ for some constants $K \in \mathbb{C}$ and $C_0 > 0$. Then for every $h \in (0, h_0)$ and every $s \in \mathbb{R}$, the operator $A^h = a(y, hD_y; h)$ defined on $H^{s,h}(\mathbb{R}^d)$ with the domain $D(A^h) = H^{s+m,h}(\mathbb{R}^d)$ is closed and the domain of its adjoint $A^{*_s,h}$ with respect to the $H^{s,h}$-scalar product is also $D(A^{*_s,h}) = H^{s+m,h}(\mathbb{R}^d)$.*
b) If the principal symbol $\sigma_{\langle\eta\rangle^m}(a)$ (indeed one of its representative) of $a \in S^{h_0}(\langle\eta\rangle^m, g_\eta)$, $m \in \mathbb{R}$, is real-valued, then we have

$$\text{(C.1)} \qquad \mathrm{Im}(u, a(y, hD_y; h)u)_{H^{\frac{s}{2},h}} \leq C_{a,s} h \|u\|^2_{H^{\frac{s+m-1}{2},h}}, \quad \forall u \in H^{\frac{s}{2}+m,h}(\mathbb{R}^d).$$

If further this principal symbol is bounded from below by $C_0\langle\eta\rangle^m$, then we have

$$\text{(C.2)} \quad \mathbb{R}\mathrm{e}(u, a(y, hDy\ ; h)u)_{H^{\frac{s}{2},h}} \geq \|\sqrt{\sigma_{\langle\eta\rangle^m}(a)}(y, hD_y; h)u\|^2_{H^{\frac{s}{2},h}} - {C'}_{a,s} h \|u\|^2_{H^{\frac{s+m-1}{2},h}}, \quad \forall u \in H^{\frac{s}{2}+m,h}(\mathbb{R}^d).$$

The constants $C_{a,s}$ and ${C'}_{a,s}$ only depend on s and finitely many semi-norms of a

Proof : The problem can be reduced to the case $s = 0$, while replacing $a(y, hD_y; h)$ by $a_s(y, hD_y; h) = \langle hD_y\rangle^s a(y, hD_y; h)\langle hD_y\rangle^{-s}$. Since $\sigma_{\langle\eta\rangle^m}(a_s) = \sigma_{\langle\eta\rangle^m}(a)$, the assumptions of a) and b) are also satisfied by a_s, by possibly changing the constants.
a) For the sake of simplicity, we assume $K = 0$ and fix $h = 1$, which does not affect the validity of the next arguments. We have $|a| \geq C_0\langle\eta\rangle^m$ so that $a^{-1} \in S(\langle\eta\rangle^{-m}, g_\eta)$. Pseudo-differential calculus gives $a^{-1}\#^h a = 1 + b_1$, $b_1 \in S(\langle\eta\rangle^{-1}, g_\eta)$ and we can construct by induction $a_1 = a^{-1}$, b_1, $a_2 = a^{-1} - a^{-1}b_1$, b_2 $\ldots$, $a_k \in S(\langle\eta\rangle^{-m}, g_\eta)$, $b_k \in S(\langle\eta\rangle^{-k}, g_\eta)$ so that $a_k\#^h a = 1 + b_k$. Let $u^n \in L^2(\mathbb{R}^d)$ converge to $u \in L^2(\mathbb{R}^d)$ while $a(y, D_y)u^n \to v$ as $n \to \infty$. We have

$$u + b_m(y, D_y)u = \lim_{n\to\infty} a_m(y, D_y)a(y, D_y)u^n = a_m(y, D_y)v \quad \text{in } L^2(\mathbb{R}^d)$$

so that $u = a_m(y, D_y)v - b_m(y, D_y)u \in H^m(\mathbb{R}^d)$. Since the pseudo-differential operator $a(y, D_y)$ is continuous: $S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$, we also conclude $v = a(y, D_y)u$. The operator $A = a(y, D_y)$ defined on $L^2(\mathbb{R}^d)$ with domain $D(A) = H^m(\mathbb{R}^d)$ is closed. By the density of $S(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ and the imbedding $L^2(\mathbb{R}^d) \in S'(\mathbb{R}^d)$, the adjoint A^* of A is nothing but the formal adjoint $a(y, D_y^*)$ with domain $D(A^*) = \{u \in L^2(\mathbb{R}^d),\ a(y, D_y)^*u \in L^2(\mathbb{R}^d)\}$. The equivalence between the Weyl- and $(1,0)$- calculus implies $a(y, D_y)^* =$

$(\bar{a}+c)(y,D_y)$ with $c \in S(\langle\eta\rangle^{m-1}, g_\eta)$. Hence we can find a constant $K' \in \mathbb{C}$ so that $a' = K' + \bar{a} + c$ satisfies $|a'| \geq C'_0\langle\eta\rangle^m$ while $D(A^*) = \{u \in L^2(\mathbb{R}^d),\ a'(y,D_y)^* u \in L^2(\mathbb{R}^d)\}$. By taking $a'_m \in S(\langle\eta\rangle^{-m}, g_\eta)$ and $b_m \in S(\langle\eta\rangle^{-m}, g_\eta)$ so that $a'_m \# a = 1 + b'_m$, we get

$$u \in D(A^*) \ \Rightarrow\ u = a'_m(y,D_y)a(y,D_y)u - b_m(y,D_y)u \ \in\ H^m(\mathbb{R}^d)$$

and $D(A^*) = H^m(\mathbb{R}^d)$.
b) The difference between the symbol $a \in S^{h_0}(\langle\eta\rangle^m, g_\eta)$ and a representative of its principal part writes hb with $b \in S^{h_0}(\langle\eta\rangle^{m-1}, g_\eta)$. Hence we can suppose the symbol itself to be real-valued. The equivalence with the Weyl semi-classical calculus gives $a(y,hD_y;h)^* = (a+hc)(y,D_y;h)$ with $c \in S^{h_0}(\langle\eta\rangle^{m-1}, g_\eta)$. Hence for any $u \in S(\mathbb{R}^d)$, we have the estimates

$$\mathrm{Im}(u, a(y,hD_y;h)u)_{L^2} = \frac{ih}{2}(u, c(y,hD_y;h)u)_{L^2} \leq C_\alpha h\|u\|^2_{H^{\frac{m-1}{2},h}},$$

which extends to any $u \in H^{m,h}(\mathbb{R}^d)$ by density.
If $a \geq C_0\langle\eta\rangle^m$, then its square root $\sqrt{a}$ belongs to $S^{h_0}(\langle\eta\rangle^{\frac{m}{2}}, g_\eta)$. Semi-classical calculus yields

$$\frac{1}{2}[a(y,hD_y;h) + a(y,hD_y;h)^*] = (\sqrt{a})(y,hD_y;h)(\sqrt{a})(y,hD_y;h)^* + hc(y,hD_y;h)$$

with $c \in S^{h_0}(\langle\eta\rangle^{m-1}, g_\eta)$. We obtain for any $u \in S(\mathbb{R}^d)$

$$\begin{aligned} Re(u, a(y,hD_y;h)u)_{L^2} &= \|\sqrt{a}(y,hD_y;h)u\|^2_{L^2} + h(u, c(y,hD_y;h)u)_{L^2} \\ &\geq \|\sqrt{a}(y,hD_y;h)u\|^2_{L^2} - C_a h\|u\|^2_{H^{\frac{m-1}{2},h}} \end{aligned}$$

and we conclude by a density argument like above. □

We end with some refinements of the previous lemma which hold for h "small enough".

Lemma C.2 *a) Let $a \in S^{h_0}(\langle\eta\rangle^m, g_\eta)$, $m \in \mathbb{R}$, satisfy $|a| \geq C_0\langle\eta\rangle^m$ for some constant $C_0 > 0$. Then there exist h_a, $0 < h_a \leq h_0$, depending on finitely many seminorms of a, and $a' \in S^{h_a}(\langle\eta\rangle^{-r}, g_\eta)$ so that*

$$a\#^h a' = a'\#^h a = 1, \quad \forall h \in (0, h_a).$$

b) Let the symbol $a \in S^{h_0}(\langle\eta\rangle^m, g_\eta)$, $m \in \mathbb{R}$, satisfy $\mathbb{R}\mathrm{e}(a) \in S^{h_0}(\langle\eta\rangle^k, g_\eta)$ with $\mathbb{R}\mathrm{e}(a) \geq C_0\langle\eta\rangle^k$, $m-1 \leq k \leq m$ and $C_0 > 0$. Then there exist $h_{a,s} \in (0,h_0)$ and $C_{a,s} > 0$, depending on a finite number of seminorms of a and on s, so that

$$\mathbb{R}\mathrm{e}(u, a(y,hD_y;h)u)_{H^{\frac{s}{2},h}} \geq C_{a,s}\|u\|^2_{H^{\frac{s+k}{2},h}}, \quad \forall u \in H^{\frac{s}{2}+k,h}(\mathbb{R}^d), h \in (0, h_{a,s}).$$

c) Let the symbol $a \in S^{h_0}(\langle\eta\rangle, g_\eta)$, satisfy the uniform estimate $K + \mathbb{R}\mathrm{e}(a) \geq C_0\langle\eta\rangle$ for some constants $K \in \mathbb{R}$ and $C_0 \geq 0$. Let $A_s = a(y,hD_y;h)$ be the operator defined *on $H^{\frac{s}{2},h}(\mathbb{R}^d)$, $s \in \mathbb{R}$ with the domain $D(A_s) = H^{\frac{s+1}{2},h}(\mathbb{R}^d)$. Then there exists $h_{\sigma(a)} \in (0, h_0)$, depending only on a fixed number of seminorms of $\sigma_{\langle\eta\rangle}(\mathbb{R}\mathrm{e}(a))$, so that the resolvent of A_s is estimated by*

$$\|(z - \omega_{a,s} - A_s)^{-1}\|_{\mathcal{L}(H^{\frac{s}{2},h})} \leq \frac{C_{a,s}}{1+|z|}, \quad \forall z \in \mathbb{C}, \mathbb{R}\mathrm{e}(z) < 0, \tag{C.3}$$

with constants $\omega_{a,s} \in \mathbb{R}$ and $C_{a,s}$ possibly depending on a and s, for all $h \in (0, h_{\sigma(a)})$.

Proof : a) By considering $\langle hD_y\rangle^{-m} a(y, hD_y; h)$, the problem is reduced to the case $m = 0$. We write $(a^{-1})\#^h a = 1 + hb$ and $a\#^h(a^{-1}) = 1 + hc$ with $b, c \in S^{h_0}(\langle\eta\rangle^{-1}, g_\eta)$. For some h_a, $0 < h_a \leq h_0$, the operators $(1 + hb)(y, hD_y; h)$ and $(1 + hc)(y, hD_y; h)$ are invertible as soon as $h \leq h_a$. Hence $a(y, hD_y; h)$ admits a left and right inverse A'^h. We next check $A'^h = a'(y, hD_y; h)$ with $a' \in S^{h_a}(1, g_\eta)$ by referring to the Beals criterion of Proposition A.11. By applying multi-commutators to $a(y, hD_y; h)A'^h = 1$, we have

$$\begin{aligned}\mathrm{ad}_y^\alpha\, \mathrm{ad}_{hD_y}^\beta\, A'^h = -\sum_{\substack{(\alpha_1,\beta_1)+\cdots+(\alpha_l,\beta_l)=(\alpha,\beta)\\ (\alpha_1,\beta_1)\neq(0,0)}} & C_{\alpha_i,\beta_i} A'^h\, \mathrm{ad}_y^{\alpha_1}\, \mathrm{ad}_{hD_y}^{\beta_1}\, a(y, hD_y; h)\ldots \\ & A'^h\, \mathrm{ad}_y^{\alpha_l}\, \mathrm{ad}_{hD_y}^{\beta_l}\, a(y, hD_y; h) A'^h.\end{aligned}$$

Then the proof is done if A'^h is uniformly bounded on $H^{k,h}(\mathbb{R}^d)$ for $k \in \mathbb{N}$. It comes at once by induction from the identity

$$\begin{aligned}\langle hD_y\rangle A'^h &= \left[\langle hD_y\rangle, A'^h\right] + A'^h\langle hD_y\rangle \\ &= -A'^h\left[\langle hD_y\rangle, a(y, hD_y; h)\right] A'^h + A'^h\langle hD_y\rangle.\end{aligned}$$

b) We first write

$$\begin{aligned}\mathbb{R}\mathrm{e}\,(u, a(y, hD_y; h)u)_{H^{s,h}} = \mathbb{R}\mathrm{e}\,&(u, \mathbb{R}\mathrm{e}(a)(y, hD_y; h)u)_{H^{s,h}} \\ &- \mathrm{Im}\,(u, \mathrm{Im}(a)(y, hD_y; h)u)_{H^{s,h}}\,.\end{aligned}$$

By referring to estimates (C.1)(C.2), we deduce

$$\begin{aligned}\mathbb{R}\mathrm{e}\,(u, a(y, hD_y; h)u)_{H^{s,h}} &\geq \|\sqrt{\mathbb{R}\mathrm{e}(a)}\|^2_{H^{s,h}} - C_{a,s}h\left[\|u\|^2_{H^{\frac{s+m-1}{2},h}} + \|u\|^2_{H^{\frac{s+k-1}{2},h}}\right] \\ &\geq \|\sqrt{\mathbb{R}\mathrm{e}(a)}\|^2_{H^{s,h}} - C_{a,s}h\|u\|^2_{H^{\frac{s+k}{2},h}}.\end{aligned}$$

According to part a) we can choose h_a small enough so that $\sqrt{\mathbb{R}\mathrm{e}(a)}(y, hD_y; h)$ is an isomorphism from $H^{\frac{s+k}{2},h}(\mathbb{R}^d)$ onto $H^{\frac{s}{2},h}(\mathbb{R}^d)$, for all $0 < h \leq h_{\sigma(a)}$, with uniform estimates. Then $h_{a,s} \leq h_a$ is taken so that the negative term does not exceed half of the first one.

c) By possibly replacing the symbol a by $K + a$, we can assume $\mathbb{R}\mathrm{e}(a) \geq C_0\langle\eta\rangle$. For the sake of simplicity we consider $\mathbb{R}\mathrm{e}(a)$ as a representant of its prinsipal symbol. We take $h_{\sigma(a)} \in (0, h_0)$ so that $\sqrt{\mathbb{R}\mathrm{e}(a)}(y, hD_y; h)$ is an isomorphism from $H^{\frac{s}{2},h}(\mathbb{R}^d)$ onto $H^{\frac{s+1}{2},h}(\mathbb{R}^d)$. Due to the boundedness of $\langle hD_y\rangle^{-1/2} a(y, hD_y; h)\langle hD_y\rangle^{-1/2}$, $H^{\frac{s+1}{2},h}(\mathbb{R}^d)$ is a form domain for $q(u) = (u, a(y, hD_y; h)u)_{H^{\frac{s}{2},h}}$. We first check that q is strictly m-sectorial in the sense of [20]-Vol.I, that is

$$q(u) \in S_{a,s} = \{z \in \mathbb{C},\ -\theta_{a,s} \leq arg(z - \omega_{a,s}) \leq \theta_{a,s}\}, \quad \forall u \in H^{\frac{s+1}{2},h}(\mathbb{R}^d)$$

with $\theta_{a,s} < \frac{\pi}{2}$. Estimates (C.1) and (C.2) extended to $u \in H^{\frac{s+1}{2},h}(\mathbb{R}^d)$ give

$$\mathbb{R}\mathrm{e}\, q(u) \geq \|\sqrt{\mathbb{R}\mathrm{e}\, a}(y, hD_y; h)u\|^2_{H^{\frac{s}{2},h}} - C_{a,s} h \|u\|^2_{H^{\frac{s}{2},h}}$$

while we have $\mathrm{Im}\, q(u) \leq C'_{a,s}\|u\|^2_{H^{\frac{s+1}{2},h}}$. Hence we can find $h_{\sigma(a),s} > 0$, $\varepsilon_{a,s} > 0$ and $\omega_{a,s} \in \mathbb{C}$ so that $e^{i\varepsilon}(q + \omega_{a,s} - 1)$ is strictly m-accretive as soon as $-\varepsilon_{a,s} \leq \varepsilon \leq \varepsilon_{a,s}$ and $h \in (0, h_{\sigma(a),s})$. Then we take $\theta_{a,s} = \frac{\pi}{2} - \varepsilon_{a,s}$. By Lemma C.1 a), the operator $A = a(y, hD_y; h)$ with $D(A) = H^{\frac{s}{2}+1,h}(\mathbb{R}^d)$ is closed and $D(A^*) = H^{\frac{s}{2}+1,h}(\mathbb{R}^d)$. Meanwhile we have

$$q(u) = (u, Au)_{H^{\frac{s}{2},h}} = (A^*u, u)_{H^{\frac{s}{2},h}}, \quad \forall u \in D(A) = D(A^*) = H^{\frac{s}{2}+1,h}(\mathbb{R}^d).$$

By Proposition of [20], A is the unique stricly m-sectorial operator associated with the form q and therefore satisfies for $\mathbb{R}\mathrm{e}\, z < 0$

$$\|(z - \omega_{a,s} - A)^{-1}\|_{\mathcal{L}(H^{\frac{s}{2},h})} \leq dist(z - \omega_{a,s}, S_{a,s})^{-1} \leq \frac{C_{a,s}}{1 + |z|}.$$

□

We next use the above properties in order to study some parabolic evolution systems. Let $a \in \mathbb{P}^0 S^{h_0}(\langle\eta\rangle, g_\eta)$ be such that $\sigma_{\langle\eta\rangle}(\mathbb{R}\mathrm{e}\, a) \geq C_0\langle\eta\rangle$ where $C_0 > 0$ does not depend on $h \in (0, h_0)$ and $x \in \mathbb{R}$. We consider the operator $A^h(x) = a(x, y, hD_y; h)$ defined on $H^{s,h}(\mathbb{R}^d)$ with domain $D(A^h(x)) = H^{s+1,h}(\mathbb{R}^d)$. According to the above lemmas we know that this operator is closed and satisfies estimate (C.3) uniformly with respect to $(x, h) \in \mathbb{R} \times (0, h_{\sigma(a),s})$. According to Lemma C.2-a), the resolvent $(\omega_{a,s} + A^h(x))^{-1}$ is a uniformly bounded operator from $H^{s,h}(\mathbb{R}^d)$ into $H^{s+1,h}(\mathbb{R}^d)$ and we obtain

$$\text{(C.4)} \qquad \begin{aligned} &\|(A^h(x'') - A^h(x'))(\omega_{a,s} + A^h(x))^{-1}\|_{\mathcal{L}(H^{s,h})} \\ &\leq \|A^h(x'') - A^h(x')\|_{\mathcal{L}(H^{s+1,h}, H^{s,h})} \|(\omega_s + A^h(x))^{-1}\|_{\mathcal{L}(H^{s,h}, H^{s+1,h})} \\ &\leq C'_s|x'' - x'|. \end{aligned}$$

Hence the x-dependent operator $\omega_{a,s} + A^h(x)$ satisfies the three assumptions of Tanabe-Sobolevskii theorem (see [19][24]), that is : constant domain, resolvent estimate (C.3) and uniform smoothness (C.4). After a conjugation with $e^{\frac{\omega_{a,s} x}{h}}$ this theorem yields the

Proposition C.3 *Under the above assumptions on $a \in \mathbb{P}^0 S^{h^0}(\langle\eta\rangle, g_\eta)$, there exists $h_{\sigma(a)} \in (0, h_0)$, so that the initial value problem*

$$\text{(C.5)} \qquad \begin{cases} h\partial_x u + a(x, y, hD_y; h)u = 0, \ x > x' \\ \qquad\qquad u\Big|_{x=x'} = v \end{cases}$$

defines an evolution system $S^h(x'', x')$, $x' \leq x''$ on $H^{s,h}(\mathbb{R}^d)$ for every $s \in \mathbb{R}$ and $h \in (0, h_{\sigma(a)})$. Moreover this evolution system has the properties:
a)As a bounded operator on $H^{s,h}(\mathbb{R}^d)$, $S^h(x'', x')$ is strongly continuous with respect to x', x'', $x' \leq x''$.
b) For $x' < x''$, $S^h(x'', x')$ is a bounded operator: $H^{s,h}(\mathbb{R}^d) \to H^{s+1,h}(\mathbb{R}^d)$. As an element of $\mathcal{L}(H^{s,h}(\mathbb{R}^d))$, it is strongly differentiable with respect to x''. Its derivative $h\partial_{x''}S^h(x'', x')$ belongs to $\mathcal{L}(H^{s,h}(\mathbb{R}^d))$ and is strongly continuous with respect to x', x'', $x' < x''$, with the identity

$$h\partial_{x''}S^h(x'', x') + A^h(x'')S^h(x'', x') = 0, \quad x' < x''.$$

c) If $v \in H^{s+1,h}(\mathbb{R}^d)$, Then $S^h(x'', x')v$ is differentiable with respect to x', x'', for $x' \leq x''$ and we have

$$\begin{aligned} &h\partial_{x''}S^h(x'', x')v = -A^h(x'')S^h(x'', x')v \\ \text{and} \qquad &h\partial_{x'}S^h(x'', x')v = S^h(x'', x')A^h(x')v \qquad \text{for } x'' \geq x'. \end{aligned}$$

d) If $v \in H^{s+k,h}(\mathbb{R}^d)$, $k \in \mathbb{N}$, then $S^h(x'', x')v \in \bigcap_{j=0}^{k} \mathcal{C}^{k-j}(\{x' \leq x''\}, H^{s+j,h}(\mathbb{R}^d))$.

Accurate exponential decay estimates for the evolution systems involved in our problem are provided in the text. Here are some other estimates which will be useful.

Lemma C.4 *Let $u(x) = S^h(x, x_0)u_0$ for $x \geq x_0$ with $u_0 \in H^{s+1/2,h}(\mathbb{R}^d)$. We have for every $h \in (0, h_{\sigma(a)})$*

$$\int_{x_0}^{\infty} \|u(x)\|^2_{H^{s+1,h}} dx \leq C_{a,s} h \left[\|u_0\|^2_{H^{s+1/2,h}} + h \int_{x_0}^{\infty} \|u(x)\|^2_{H^{s,h}} dx \right].$$

Proof : For $x \geq x' > x_0$, $u(x) = S^h(x, x')u(x')$ is a classical solution in $H^{s,h}(\mathbb{R}^d)$ and in $H^{s+1/2,h}(\mathbb{R}^d)$ of

$$h\partial_x u + A^h(x)u = 0.$$

By taking the $H^{s,h}$-scalar product with $u(x)$, we obtain

$$h\partial_x \|u\|^2_{H^{s,h}} = -2\,\mathrm{Re}(u, A^h(x)u)_{H^{s,h}}$$

which implies

$$\begin{aligned} \|u(x')\|^2_{H^{s,h}} &\geq \frac{1}{h} \int_{x'}^{x''} 2\,\mathrm{Re}(u, A^h u)_{H^{s,h}}(x) dx \\ &\geq \frac{C_{a,s}}{h} \left[\int_{x'}^{x''} \|u(x)\|^2_{H^{s+1/2,h}} - C'_{a,s} h \|u(x)\|^2_{H^{s,h}} dx \right]. \end{aligned}$$

We do the same with the $H^{s+1/2,h}$-scalar product and we obtain

$$\|u(x')\|^2_{H^{s+1/2,h}} \geq \frac{C_{a,s}}{h}\left[\int_{x'}^{x''}\|u(x)\|^2_{H^{s+1,h}} - C'_{a,s}h\|u(x)\|^2_{H^{s+1/2,h}}\right].$$

Putting these two estimates together yields

$$\int_{x'}^{x''}\|u(x)\|^2_{H^{s+1,h}}dx \leq C_{a,s}h\left[\|u(x')\|^2_{H^{s+1/2,h}} + \int_{x'}^{x''} h\|u(x)\|^2_{H^{s,h}}dx\right]$$

and we conclude by taking the limit as $x' \to x_0$ and $x'' \to \infty$. □

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