New constructions of perfectly matched layers for the linearized Euler equations

Nouvelles constructions de couches parfaitement adaptées pour le système d'Euler linéarisé

F. Nataf [a]

[a] *CMAP, Ecole Polytechnique, 91128 Palaiseau Cedex, France*

Abstract

Based on a PML for the advective wave equation, we propose two PML models for the linearized Euler equations. The derivation of the first model can be applied to other physical models. The second model was implemented. Numerical results are shown. *To cite this article: F. Nataf, C. R. Acad. Sci. Paris, Ser. I XXX (2005).*

Résumé

A partir d'une couche adaptée pour l'équation des ondes advectives, nous proposons deux modèles de telles couches pour les équations d'Euler linéarisées. La construction du premier modèle peut être appliqué à d'autres systèmes d'équations aux dérivées partielles. Le second modèle a été implémenté. Des résultats numériques illustrent l'intérêt de cette construction.*Pour citer cet article : F. Nataf, C. R. Acad. Sci. Paris, Ser. I XXX (2005).*

1. Introduction

Since the work by Berenger on perfectly matched layer for the Maxwell equations [2] in a computational box, many works have been devoted to this subject. We consider here the linearized Euler equations, see [4], [6] and references therein. The key difficulty is the possible instability of vorticity waves especially for oblique flows. We address this question and propose two ways to design PML for the Euler equations that are based on the use of a PML for the underlying advective wave equation. The derivation of the first model can be applied to other physical models. The second model was implemented and numerical results are shown.

Email address: `nataf@cmap.polytechnique.fr` (F. Nataf).

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2. Analysis of the Euler system via Smith factorization

We write the linearized Euler equations as:

$$\begin{pmatrix} \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & \bar{\rho}\bar{c}^2\partial_x & \bar{\rho}\bar{c}^2\partial_y \\ \frac{1}{\bar{\rho}}\partial_x & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & 0 \\ \frac{1}{\bar{\rho}}\partial_y & 0 & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix} = \begin{pmatrix} f_p \\ f_u \\ f_v \end{pmatrix} \tag{1}$$

We first recall the definition of the Smith factorization of a matrix with polynomial entries and apply it to systems of PDEs:

Theorem 2.1 *Let n be an integer and A an invertible $n \times n$ matrix with polynomial entries with respect to the variable λ: $A = (a_{ij}(\lambda))_{1 \le i,j \le n}$.*
Then, there exist three matrices with polynomial entries E, D and F with the following properties: det(E)=det(F)=1, D is a diagonal matrix and $A = EDF$.

This factorization is different from the diagonalization of a matrix which could involve, for instance, taking the square root of a polynomial, see [7] for more details. We first take formally the Fourier transform of the system in (1) with respect to y and t (dual variables are k and ω resp.). We keep the partial derivatives in x since in the sequel we shall design a PML for a truncation of the domain in the x direction. We note

$$\hat{\hat{A}}_{Euler} = \begin{pmatrix} i\omega + \bar{u}\partial_x + ik\bar{v} & \bar{\rho}\bar{c}^2\partial_x & i\bar{\rho}\bar{c}^2 k \\ \frac{1}{\bar{\rho}}\partial_x & i\omega + \bar{u}\partial_x + ik\bar{v} & 0 \\ \frac{ik}{\bar{\rho}} & 0 & i\omega + \bar{u}\partial_x + i\bar{v}k \end{pmatrix} \tag{2}$$

We can perform the Smith factorization of $\hat{\hat{A}}_{Euler}$ by considering it as a matrix with polynomials in ∂_x entries. We have

$$\hat{\hat{A}}_{Euler} = EDF \tag{3}$$

where $D_{11} = D_{22} = 1$ and $D_{33} = \hat{\hat{\mathcal{G}}}\hat{\hat{\mathcal{L}}}$, $\hat{\hat{\mathcal{G}}} = i\omega + \bar{u}\partial_x + ik\bar{v}$ and $\hat{\hat{\mathcal{L}}} = -\omega^2 + 2ik\bar{u}\bar{v}\partial_x + 2i\omega(\bar{u}\partial_x + ik\bar{v}) + (\bar{c}^2 - \bar{v}^2)k^2 - (\bar{c}^2 - \bar{u}^2)\partial_{xx}$. The operators showing up in the diagonal matrix have a physical meaning, $\mathcal{G}$ is a first order transport operator and $\mathcal{L}$ is the advective wave operator.

3. PMLs for the Euler System

Among the two operators $\mathcal{L}$ and $\mathcal{G}$, the only operator which generates waves propagating in both positive x and negative x directions is the operator $\mathcal{L}$. This suggests that designing a PML for the Euler equation can be reduced to the design of a PML for the advective wave operator $\mathcal{L}$. This question has been the subject of several works [1] [3] and references therein. Following these works, we use for operator $\mathcal{L}$ a PML defined by replacing the x derivatives by a "pml" x derivative. The definition is as follows:

$$\mathcal{L}_{pml} = \partial_{tt} + 2\bar{u}\bar{v}\partial_y(\partial_x^{pml}) + 2\partial_t(\bar{u}\partial_x^{pml} + \bar{v}\partial_y) - (\bar{c}^2 - \bar{v}^2)\partial_{yy} - (\bar{c}^2 - \bar{u}^2)(\partial_x^{pml})^2 \tag{4}$$

where

$$\partial_x^{pml} = \alpha(x)[\partial_x - \frac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(\partial_t + \bar{v}\partial_y)] + \frac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(\partial_t + \bar{v}\partial_y) \tag{5}$$

where the operator $\alpha(x)$ is a pseudo-differential operator in the t and y variables:

$$\alpha(x)(\phi) = \mathcal{F}^{-1}\left(\frac{\bar{c}(i\omega + ik\bar{v})}{\bar{c}(i\omega + ik\bar{v}) + (\bar{c}^2 - \bar{u}^2)\sigma(\omega, x, k)}\,\hat{\hat{\phi}}\right) \tag{6}$$

where $\sigma(\omega, x, k) \geq 0$ is the damping parameter of the PML.
Based on (3), a first possibility is to define a PML for the Euler system by substitution of $\mathcal{L}$ with $\mathcal{L}^{pml}$ in matrix D. In matrices E and F, the x derivatives are not modified. Modifying only the advective wave operator avoids instability problems with the vorticity wave. We thus define:

$$\hat{\hat{A}}^{pml1}_{Euler} = ED^{pml}F \tag{7}$$

where $D^{pml}_{11} = D^{pml}_{22} = 1$ and $D^{pml}_{33} = \hat{\hat{\mathcal{G}}}\hat{\hat{\mathcal{L}}}^{pml}$. A direct computation yields:

$$\hat{\hat{A}}^{pml1}_{Euler} = \hat{\hat{A}}_{Euler} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & C_2 & 0 \end{pmatrix} \tag{8}$$

where

$$C_1 = \frac{(\partial_x - \partial_x^{pml})\hat{\hat{\mathcal{G}}}[(\bar{u}^2 - \bar{c}^2)(\partial_x + \partial_x^{pml}) + 2\bar{u}(i\omega + i\bar{v}k)]}{i\bar{\rho}\bar{c}^2 k(i\omega + ik\bar{v})} \text{ and } C_2 = \frac{C_1}{\bar{\rho}\bar{u}}$$

The difference with the Euler system concerns only the last equation on the variable v, but it implies a division by $i\bar{\rho}\bar{c}^2 k(i\omega + ik\bar{v})$ which can be zero. Taking $\sigma(\omega, x, k) = \tilde{\sigma}(x)\left(\bar{\rho}\bar{c}^2 k(\omega + k\bar{v})\right)^2$ would prevent C_1 and C_2 from being singular. But it would be at the expense of the damping of the PML. Indeed, $\sigma(\omega, x, k)$ would be small for small values of k or of $i\omega + ik\bar{v}$. The present first model raises difficulties. Nevertheless, it should deserve interest since it corresponds to a systematic way to design a PML for systems of PDEs. Moreover, since matrices E and F are not unique, it is quite possible that a more suitable Smith factorization when used in formula (7) would lead to a practicable PML.
The rationale for the second model we introduce now is that the pressure p satisfies an advective wave equation which is the only equation that demands a PML. Indeed, let multiply (2) by the matrix

$$El = \begin{pmatrix} \hat{\hat{\mathcal{G}}} & -\bar{\rho}\bar{c}^2\partial_x & -i\bar{\rho}\bar{c}^2 k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

We get:

$$El\,\hat{\hat{A}}_{Euler} = \begin{pmatrix} \hat{\hat{\mathcal{L}}} & 0 & 0 \\ \frac{1}{\bar{\rho}}\partial_x & i\omega + \bar{u}\partial_x + ik\bar{v} & 0 \\ \frac{ik}{\bar{\rho}} & 0 & i\omega + \bar{u}\partial_x + i\bar{v}k \end{pmatrix} \tag{10}$$

We substitute $\hat{\hat{\mathcal{L}}}$ with $\hat{\hat{\mathcal{L}}}^{pml}$ and apply El^{-1} and we are thus led to define:

$$\hat{\hat{A}}^{pml2}_{Euler} = \begin{pmatrix} \hat{\hat{\mathcal{G}}}^{-1}(\hat{\hat{\mathcal{L}}}^{pml} + \bar{c}^2(\partial_{xx} - k^2) & \bar{\rho}\bar{c}^2\partial_x & i\bar{\rho}\bar{c}^2 k \\ \frac{1}{\bar{\rho}}\partial_x & \hat{\hat{\mathcal{G}}} & 0 \\ \frac{ik}{\bar{\rho}} & 0 & \hat{\hat{\mathcal{G}}} \end{pmatrix} = \hat{\hat{A}}_{Euler} + \begin{pmatrix} (\hat{\hat{\mathcal{L}}}^{pml} - \hat{\hat{\mathcal{L}}})\hat{\hat{\mathcal{G}}}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

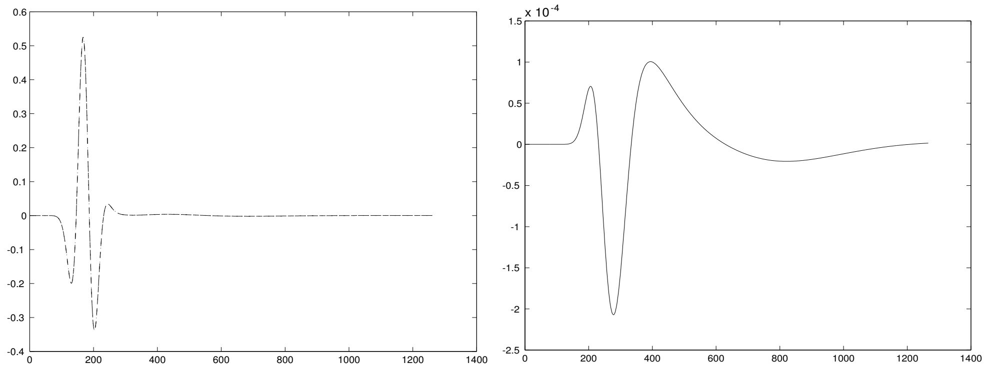

Figure 1. Pressure (left) and error on the pressure (right) near the corner for an oblique velocity $M = 0.9$ vs. time — Pression (gauche) et erreur sur la pression (droite) en fonction du temps près du coin pour une vitesse oblique $M = 0.9$

In order to get rid of the operator $\hat{\hat{\mathcal{G}}}^{-1}$, we introduce a new variable $\mathcal{P}$ such that $\mathcal{G}(\mathcal{P}) = p$ with the following interface conditions between the Euler media and the PML: $\mathcal{P} = 0$, p and u are continuous, $\partial_x(p_{Euler}) = \partial_x^{pml}(p_{pml})$. This procedure leads to a perfectly matched layer if the layer is infinite, see [5].

4. Numerical Results

The 2D linearized Euler equations are discretized on a uniform staggered grid using a Yee Scheme. The convective derivatives are discretized using an upwind scheme both in the Euler region and in the PMLs. The reference solution is obtained by computing the solution on a much larger domain. The initial solutions are zero. Let $f(t, x, y) = (1 - 2\pi^2(f_c t - 1)^2)e^{-\pi^2(f_c t-1)^2}\delta_M(x, y)$ for $t < T_s$ and zero for $t > T_s$ with $T_s = 0.05$, $f_c = 4/T_s$ and δ_M is the Dirac mass located in the middle of the computational domain. The right handside was $f(t, x, y)$ on all three equations of system (1). For an oblique velocity $u_0 = v_0 = 270$, pressure near the upperleft corner is shown on Figure 1. The stability of the PML was assessed by computing on time intervals much longer than those used for generating the figures. Both PML models have a straightforward three-dimensional extension and could be used with variable coefficients but have not been tested in these cases.

References

[1] E. Bécache, A.-S. Bonnet-Ben Dhia, and G. Legendre. Perfectly matched layers for the convected Helmholtz equation. *SIAM J. Numer. Anal.*, 42(1):409–433 (electronic), 2004.

[2] J.P. Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 114(2), 1994.

[3] Julien Diaz and Patrick Joly. Stabilized perfectly matched layer for advective acoustics. In *Mathematical and numerical aspects of wave propagation—WAVES 2003*, pages 115–119. Springer, Berlin, 2003.

[4] Th. Hagstrom. A new construction of perfectly matched layers for hyperbolic systems with applications to the linearized Euler equations. In *Math. and num. aspects of wave propagation—WAVES 2003*, pages 125–129. Springer, 2003.

[5] F. Nataf. A new construction of perfectly matched layers for the linearized Euler equations. http://www.cmap.polytechnique.fr/preprint/repository/566.pdf, http://hal.ccsd.cnrs.fr/ccsd-00004155, submitted.

[6] Adib N. Rahmouni. An algebraic method to develop well-posed PML models. Absorbing layers, perfectly matched layers, linearized Euler equations. *J. Comput. Phys.*, 197(1):99–115, 2004.

[7] J. T. Wloka, B. Rowley, and B. Lawruk. *Boundary value problems for elliptic systems.* CUP, Cambridge, 1995.