A Schwarz additive method with high order interface conditions and nonoverlapping subdomains

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We prove the convergence of a Schwarz additive method for a nonoverlapping decomposition into rectangles with interface conditions of order two in the tangential direction.

1. Introduction

The rate of convergence of Schwarz type algorithms is very sensitive to the choice of the interface conditions. The original Schwarz method is based on the use of Dirichlet boundary conditions. In order to increase the efficiency of the algorithm, it has been proposed to replace the Dirichlet boundary conditions by more general boundary conditions, see [8] (or in a different context [5]). Choosing artificial boundary conditions as interface conditions is a good choice. In [10], it is shown that using exact artificial boundary conditions leads in some situations to the convergence of the Schwarz method in a number of steps equals to the number of subdomains. The use of such interface conditions is then optimal. Unfortunately, the exact artificial boundary conditions are non local in space and they have to be approximated at various orders by partial differential operators using techniques developed for artificial boundaries, see e.g. [2]. When the interface conditions thus obtained do not involve any derivation in the direction tangential to the boundary (low order approximation), convergence has been proved in [1] for an arbitrary nonoverlapping decomposition of the domain. For higher order interface conditions convergence proofs were, to our knowledge, restricted to decompositions of the domain into strips (see [11]). Nevertheless, numerical tests of such interface conditions for decomposition into rectangles show their superiority compared to low order interface conditions (see [9]).

In this paper, we prove the convergence of the additive Schwarz method with high order interface conditions for a domain decomposed into rectangles. We consider the equation

$$\mathcal{L}(u) \equiv \frac{u}{\epsilon^2} - \Delta u = f \text{ in } \Omega_d, \; u = 0 \text{ on } \partial\Omega_d \tag{1}$$

where $\Omega_d =]0, L_X[\times]0, H_Y[$, $\epsilon > 0$. We want to solve (1) by a nonoverlapping additive Schwarz method with interface conditions of order 2 with respect to the tangential direction

$$\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}$$

which is the local approximation of order 2 with respect to ϵ of the exact artificial boundary condition.
The domain Ω_d is decomposed into rectangles: $\bar{\Omega}_d = \overline{\cup_{i,j}]l_i, L_i[\times]h_j, H_j[} = \overline{\cup_{i,j}\Omega_{i,j}}$.

The paper is organized as follows: in § 2, we introduce some notations used throughout the paper. In § 3, the algorithm is defined and is proved to be well-posed. In § 4, convergence is proved by an energy method.

2. Notations

In dealing with boundary value problems on rectangles with mixed boundary conditions, we shall make a constant use of some notations (see [4]).
Let Ω be the rectangle $]l, L[\times]h, H[$. We denote

$$\Gamma_1 =]l, L[\times\{h\}\,,\ \Gamma_2 = \{L\}\times]h, H[\,,\ \Gamma_3 =]l, L[\times\{H\}\,,\ \Gamma_4 = \{l\}\times]h, H[$$

and $\Gamma = \cup_i \Gamma_i$. The segments are thus numbered in such a way that Γ_{i+1} ($\Gamma_5 = \Gamma_1$) follows Γ_i according to the positive orientation.
We denote by S_i the vertex which is the endpoint of Γ_i:

$$S_1 = (L, h),\ S_2 = (L, H),\ S_3 = (l, H) \text{ and } S_4 = (l, h).$$

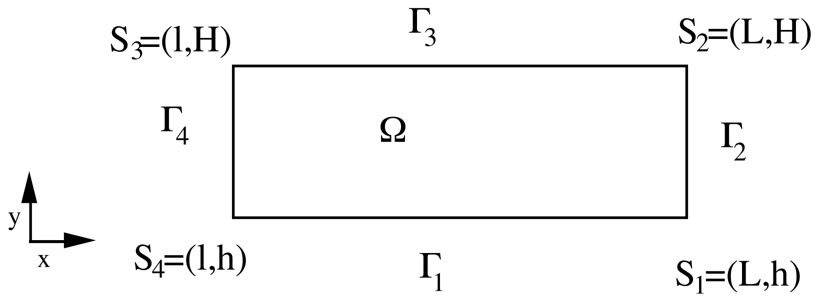

FIGURE 1 - Notation

Furthermore n_i (resp. τ_i) is the unit outward normal (resp. tangent) vector on Γ_i so that $(n_i,\ \tau_i)$ is positively oriented.
We denote by $(x_i(\sigma), y_i(\sigma))$ the point of Γ which, for small enough $|\sigma|$ is at distance σ (counted algebraically) of S_i along $\partial\Omega$. Consequently $(x_i(\sigma), y_i(\sigma)) \in \Gamma_i$ when $\sigma < 0$ and $(x_i(\sigma), y_i(\sigma)) \in \Gamma_{i+1}$ when $\sigma > 0$. We say that two functions ϕ_j and ϕ_{j+1} defined on Γ_i and Γ_{i+1} respectively are equivalent at S_i if

$$\int_0^{\delta_i} |\phi_i(x_i(-\sigma), y_i(-\sigma)) - \phi_{i+1}(x_i(\sigma), y_i(\sigma))|^2/\sigma \, d\sigma < \infty.$$

for some $\delta_i > 0$. We shall then write

$$\phi_i \equiv \phi_{i+1} \text{ at } S_i.$$

In considering mixed boundary conditions, it will be convenient to fix a partition of $\{1, 2, 3, 4\}$ in two subsets $\mathcal{D}$ and $\mathcal{A}$. The union of the Γ_i with $i \in \mathcal{D}$ (resp. $\mathcal{A}$) is going to be the boundary where we consider a Dirichlet (resp. artificial) boundary conditions. We have either $u = 0$ on Γ_i if $i \in \mathcal{D}$ or, if $i \in \mathcal{A}$

$$\frac{\partial u}{\partial n_i} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau_i^2} = g_i$$

for some $g_i \in L^2(\Gamma_i)$. Accordingly and concerning corners, we fix, $\mathcal{A}c$, a subset of $\{1,2,3,4\}$ so that corner conditions are written on S_i, $i \in \mathcal{A}c$. The set $\mathcal{A}c$ is such that $\cup_{i\in\mathcal{A}c}\{S_i\}$ is the set of vertices which do not touch an edge Γ_i with $i \in \mathcal{D}$. We define for $m = 1$ or 2

$$\mathcal{H}^m(\Omega) = \{u \in H^m(\Omega)/\ u_{|\Gamma_i} \in H^m(\Gamma_i) \text{ for } i \in \mathcal{A} \text{ and } u_{|\Gamma_i} = 0 \text{ for } i \in \mathcal{D}\}$$

which, endowed with its natural norm $\|u\|_{\mathcal{H}^m} = \sqrt{\|u\|^2_{H^m} + \sum_{i\in\mathcal{A}} \|u_{|\Gamma_i}\|^2_{H^m(\Gamma_i)}}$ and the associated scalar product, is a Hilbert space.

3. Definition of the algorithm

The Schwarz algorithm is defined by:

Definition 1. *Let $u^n_{i,j}$ be an approximation to u at step n in the interior subdomain $\Omega_{i,j}$, $u^{n+1}_{i,j}$ is defined by:*

$$\begin{aligned}
\mathcal{L}(u^{n+1}_{i,j}) &= f \ \text{in } \Omega_{i,j}\\
(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n+1}_{i,j}) &= (\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n}_{i+1,j}) \ \text{on } \Gamma_{i,j,2}\\
(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n+1}_{i,j}) &= (\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n}_{i-1,j}) \ \text{on } \Gamma_{i,j,4}\\
(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n+1}_{i,j}) &= (\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n}_{i,j+1}) \ \text{on } \Gamma_{i,j,3}\\
(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n+1}_{i,j}) &= (\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u^{n}_{i,j-1}) \ \text{on } \Gamma_{i,j,1}\\
(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})(u^{n+1}_{i,j}) &= (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})(u^{n}_{i+1,j}) \ \text{at } (x,y) = (L_i, h_j)\\
(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(u^{n+1}_{i,j}) &= (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(u^{n}_{i+1,j}) \ \text{at } (x,y) = (L_i, H_j)\\
(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(u^{n+1}_{i,j}) &= (-\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(u^{n}_{i-1,j}) \ \text{at } (x,y) = (l_i, H_j)\\
(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y})(u^{n+1}_{i,j}) &= (-\frac{\partial}{\partial x} - \frac{\partial}{\partial y})(u^{n}_{i-1,j}) \ \text{at } (x,y) = (l_i, h_j).
\end{aligned}$$

For the other subdomains, the definition is similar except on $\partial\Omega_d \cap \partial\Omega_{i,j}$ where $u^{n+1}_{i,j} = 0$.

The following theorem shows that the algorithm is well posed in $\prod_{i,j}\mathcal{H}^2(\Omega_{i,j})$

Theorem 2. *Let $l < L$, $h < H \in \mathbf{R}$, $\Omega =]l, L[\times]h, H[$, $f \in L^2(\Omega)$, $g_i \in L^2(\Gamma_i)$ for $i \in \mathcal{A}$ and $h_i \in \mathbf{R}$, $i \in \mathcal{A}c$. There exists a unique $u \in \mathcal{H}^2(\Omega)$ satisfying:*

$$\begin{aligned}
\mathcal{L}(u) &= f \ \text{in } \Omega\\
(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial\tau^2})(u) &= g_i \ \text{on } \Gamma_i,\ i \in \mathcal{A}\\
(\frac{\partial}{\tau_i} - \frac{\partial}{\tau_{i+1}})(u) &= h_i \ \text{at } S_i,\ i \in \mathcal{A}c.
\end{aligned}$$

Proof: We first consider the variational formulation in $\mathcal{H}^1(\Omega)$ of the above boundary value problem:
Find $u \in \mathcal{H}^1(\Omega)$ such that:

$$\int\int_\Omega \frac{uv}{\epsilon^2} + \nabla u.\nabla v + \sum_{i\in\mathcal{A}}\int_{\Gamma_i} \frac{uv}{\epsilon} + \frac{\epsilon}{2}\frac{\partial u}{\partial\tau_i}\frac{\partial v}{\partial\tau_i} = \int\int fv + \sum_{i\in\mathcal{A}}\int_{\Gamma_i} g_i v + \frac{\epsilon}{2}\sum_{i\in\mathcal{A}c} h_i v(S_i),\ \forall v \in \mathcal{H}^1(\Omega). \quad (2)$$

The term $v(S_i)$ makes sense. Indeed, v is a continuous function on each edge since $v_{|\Gamma_i} \in H^1(\Gamma_i),\ i=1,\ldots,4$. Moreover, since $v \in H^1(\Omega)$ we have near S_i $(i=1,\ldots,4)$ that

$$\int_0^\delta |v_i(x_i(-\sigma), y_i(-\sigma)) - v_{i+1}(x_i(\sigma), y_i(\sigma))|^2/\sigma \, d\sigma < \infty$$

for some $\delta > 0$ (see e.g. [4]). Thus, v as a function of the boundary of Ω is continuous at S_i and (2) is well defined.

Lemma 3. *Problem (2) is well posed.*

Proof: The result follows from an easy application of the Lax-Milgram theorem in the Hilbert space $\mathcal{H}^1(\Omega)$. □

It remains to prove the $\mathcal{H}^2(\Omega)$-regularity. Our proof follows that of [6] where the case $h_i = 0$ was considered. We use interpolation results of [7] and regularity results for elliptic problems on nonsmooth domains of [4]. We will proceed in three steps.

Step 1. *Let u denote the solution to problem (2). On each edge Γ_i, $i=1,\ldots,4$, $u_{|\Gamma_i} \in H^{3/2}(\Gamma_i)$.*

Proof: For $i \in \mathcal{D}$, the statement is obvious since $u_{|\Gamma_i} = 0$.
Otherwise, in the sense of distributions, we have

$$\frac{u}{\epsilon^2} - \Delta u = f \text{ in } \Omega.$$

Since $\Delta u \in L^2(\Omega)$ and $u \in H^1(\Omega)$, we have (see [4]) that $\frac{\partial u}{\partial n} \in \tilde{H}^{-1/2}(\Gamma_i)$, $i=1,\ldots,4$ where $\tilde{H}^{-1/2}(]s,t[)$ is the dual of

$$\tilde{H}^{1/2}(]s,t[) = \{u \in H_0^{1/2}(]s,t[) \text{ s.t. } u(y)/\sqrt{(t-y)(y-s)} \in L^2(]s,t[)\}.$$

Hence in the sense of distributions

$$\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau_i^2} = g_i \text{ on } \Gamma_i,\ \ i = 1 \in \mathcal{A}.$$

and $\partial^2 u/\partial \tau_i^2 \in \tilde{H}^{-1/2}(\Gamma_i)$, $i \in \mathcal{A}$. Let Pr^2 denote a right inverse to $\frac{\partial^2 u}{\partial \tau_i^2}$. The operator Pr^2 is continuous from $H^{-1}(\Gamma_i)$ into $H^1(\Gamma_i)$ and from $L^2(\Gamma_i)$ into $H^2(\Gamma_i)$. Thus, by interpolation, Pr^2 is continuous from $\tilde{H}^{-1/2}(\Gamma_i)$ into $H^{3/2}(\Gamma_i)$. Since Pr^2 is unique up to an affine function, $u_{|\Gamma_i} \in H^{3/2}(\Gamma_i)$, $i \in \mathcal{A}$.

Step 2. *Let u denote the solution to problem (2). Then, $u \in H^2(\Omega)$.*

Proof: It follows from the fact that $u \in H^{3/2}(\Gamma_i)$, u as a function of the boundary is continuous at the vertices S_i and regularity results for boundary value problems on polygon (see [4], p 58).

Step 3. *Let u denote the solution to problem (2). $u_{|\Gamma_i} \in H^2(\Gamma_i)$, $i=1,\ldots,4$.*

Proof: From $u \in H^2(\Omega)$, it follows that for $i=1,\ldots,4$, $\partial u/\partial n \in H^{1/2}(\Gamma_i)$. Thus, $\partial^2 u/\partial \tau_i^2 \in L^2(\Gamma_i)$. From standard regularity results, we have $u_{|\Gamma_i} \in H^2(\Gamma_i)$.
Then, it is easy to check that u is also the solution to the problem stated in Theorem 2. □

4. Convergence proof

The proof lies on the energy estimate of Lemma 6. In order to prove it, we shall need two results.

Theorem 4. *$H^m(]l,L[\times]h,H[) \cap \mathcal{H}^2(]l,L[\times]h,H[)$ is dense into $\mathcal{H}^2(]l,L[\times]h,H[)$ for $m \geq 4$.*

Proof: The proof is given in the Annex.

Lemma 5. *For all $v \in \mathcal{H}^2(]l,L[\times]h,H[)$, we have*

$$\int\int_{]l,L[\times]h,H[} \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} = \int\int_{]l,L[\times]h,H[} (\frac{\partial^2 v}{\partial x \partial y})^2 - \int_{\Gamma_1 \cup \Gamma_3} \frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial n} + \int_{\Gamma_2 \cup \Gamma_4} \frac{\partial v}{\partial n}\frac{\partial^2 v}{\partial y^2}$$

Proof: By Theorem 4, it suffices to prove the equality for $v \in H^4 \cap \mathcal{H}^2$. The equality is obtained by integrating by parts first in the x direction and then in the y direction. □

We can now prove

Lemma 6. *Let* $u \in \mathcal{H}^2(]l,L[\times]h,H[)$ *such that*

$$\frac{u}{\epsilon^2} - \Delta u = 0.$$

Then, we have the following energy estimate:

$$\begin{aligned}\int\int \frac{3u^2}{\epsilon^3} + 4\frac{|\nabla u|^2}{\epsilon} + \epsilon((\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 + 2(\frac{\partial^2 u}{\partial x \partial y})^2) - \int_\Gamma (\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2})^2 - (-\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2})^2 \\ +\frac{\epsilon}{2}\left((\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(L,h) - (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(L,h) + ((\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(L,H) - (\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(L,H)) \right. \\ \left. +((\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(l,H) - (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(l,H)) + ((\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(l,h) - (\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(l,h))\right) = 0.\end{aligned}$$

Proof: Equation (1) is multiplied by $\frac{3u}{\epsilon} - \epsilon\Delta u$ and is integrated over $]l,L[\times]h,H[$:

$$\int\int \frac{3u^2}{\epsilon^3} + 4\frac{|\nabla u|^2}{\epsilon} + \epsilon(\Delta u)^2 - \int \frac{4}{\epsilon} u \frac{\partial u}{\partial n} = 0$$

Lemma 5 applied to the integral of the term $\epsilon\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}$ yields:

$$\int\int \frac{3u^2}{\epsilon^3} + 4\frac{|\nabla u|^2}{\epsilon} + \epsilon((\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 + 2(\frac{\partial^2 u}{\partial x \partial y})^2) - 2\epsilon\int_{\Gamma_1 \cup \Gamma_3} \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial n} + \int_{\Gamma_2 \cup \Gamma_4} 2\epsilon\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial y^2} - \int_\Gamma \frac{4}{\epsilon} u \frac{\partial u}{\partial n} = 0$$

By integrating by parts over $\Gamma_1 \cup \Gamma_3$, we obtain

$$\begin{aligned}\int\int \frac{3u^2}{\epsilon^3} + 4\frac{|\nabla u|^2}{\epsilon} + \epsilon((\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 + 2(\frac{\partial^2 u}{\partial x \partial y})^2) + \int_\Gamma 2\epsilon\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial \tau^2} - \frac{4}{\epsilon} u\frac{\partial u}{\partial n} \\ +2\epsilon\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(L,h) - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(L,H) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(l,H) - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(l,h)\right) = 0\end{aligned}$$

The boundary and corner terms can be written as differences of squares:

$$\begin{aligned}\int\int \frac{3u^2}{\epsilon^3} + 4\frac{|\nabla u|^2}{\epsilon} + \epsilon((\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 + 2(\frac{\partial^2 u}{\partial x \partial y})^2) - \int_\Gamma (\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2})^2 - (-\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2})^2 \\ +\frac{\epsilon}{2}\left((\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(L,h) - (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(L,h) - ((\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(L,H) - (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(L,H)) \right. \\ \left. +((\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(l,H) - (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(l,H)) - ((\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y})^2(l,h) - (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})^2(l,h))\right) = 0\end{aligned}$$

□

We can now prove the :

Theorem 7. *Assume* $u^0_{i,j} \in \mathcal{H}^2(\Omega_{i,j})$.
Then, the additive Schwarz method (Definition 1) converges in $\mathcal{H}^2$.

Proof: We proceed as in [1]. Equation (1) and the additive Schwarz method are linear so that it suffices to take $f=0$ and to prove the convergence to zero of $u^n_{i,j}$ as n tends to infinity. Let

$$E^n = \sum_{i,j}\int\int \frac{3{u^n_{i,j}}^2}{\epsilon^3} + 4\frac{|\nabla u^n_{i,j}|^2}{\epsilon} + \epsilon((\frac{\partial^2 u^n_{i,j}}{\partial x^2})^2 + (\frac{\partial^2 u^n_{i,j}}{\partial y^2})^2 + 2(\frac{\partial^2 u^n_{i,j}}{\partial x \partial y})^2),$$

$$B^n = \sum_{i,j}\int_{\Gamma_{i,j}\backslash\partial\Omega_d}(-\frac{\partial u^n_{i,j}}{\partial n} + \frac{u^n_{i,j}}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u^n_{i,j}}{\partial \tau^2})^2$$

and

$$\begin{aligned}C^n = \frac{\epsilon}{2}\sum_{i,j,l_i\neq 0,\, L_i\neq L_X,\, h_i\neq 0,\, H_i\neq H_Y} & (\frac{\partial u^n_{i,j}}{\partial x} + \frac{\partial u^n_{i,j}}{\partial y})^2(L_i,h_j) + (\frac{\partial u^n_{i,j}}{\partial x} - \frac{\partial u^n_{i,j}}{\partial y})^2(L_i,H_j) \\ & + (\frac{\partial u^n_{i,j}}{\partial x} + \frac{\partial u^n_{i,j}}{\partial y})^2(l_i,H_j) + (\frac{\partial u^n_{i,j}}{\partial x} - \frac{\partial u^n_{i,j}}{\partial y})^2(l_i,h_j).\end{aligned}$$

The estimate of Lemma 6 and the definition of the algorithm show that we have

$$E^{n+1} + B^{n+1} + C^{n+1} = B^n + C^n.$$

Hence, after summation over n

$$\sum_n E^n \le B^0 + C^0,$$

and $\lim_{n\to\infty} E_n = 0$. $\square$

References

[1] B. Despres, *Domain Decomposition Method and the Helmholtz Problem*, Mathematical and Numerical aspects of wave propagation phenomena, SIAM (1991), 44-52.

[2] B. Engquist and A. Majda, *Absorbing Boundary Conditions for the Numerical Simulation of Waves*, Math. Comp. **31** (139), (1977), 629-651.

[4] P. Grisvard, Singularities in Boundary Value Problems, RMA 22, Masson & Springer Verlag (1992).

[5] T. Hagstrom, R.P. Tewarson and A. Jazcilevich, *Numerical Experiments on a Domain Decomposition Algorithm for Nonlinear Elliptic Boundary Value Problems*, Appl. Math. Lett., **1, No 3** (1988), 299-302.

[6]K. Lemrabet, *Problémes aux limites de Ventcel dans un domaine non régulier*, C.R. Acad. Sc. Paris, **t. 300**, Série I, n^0 15, (1985), 531-534.

[7] J.L. Lions and E. Magenes, Problèmes aux limites non homogènes et applications, vol **1**, Dunod, Paris, (1968).

[8] P.L. Lions, *On the Schwarz Alternating Method III: A Variant for Nonoverlapping Subdomains*, Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM (1989), 202-223.

[9] F. Nataf and F. Rogier, *Outflow Boundary Conditions and Domain Decomposition Method*, Cont. math. **180**, "Proceedings of the Seventh International Conference on Domain Decomposition, (1993), 289-293.

[10] F. Nataf, F. Rogier and E. de Sturler, *Optimal Interface Conditions for Domain Decomposition Methods*, Internal report CMAP Ecole Polytechnique, **301**, (1994).

[11] F. Nataf and F. Rogier, *Factorization of the Convection-Diffusion Operator and the Schwarz Algorithm*, M^{3}AS, **5, No 1** (1995), 67-93.

Annex

The goal of the annex is to prove

Theorem A1. *$H^m(]l,L[\times]h,H[) \cap \mathcal{H}^2(]l,L[\times]h,H[)$ is dense into $\mathcal{H}^2(]l,L[\times]h,H[)$ for $m \geq 4$.*

Proof: The proof is adapted from [4]. We first define

$$\gamma : \mathcal{H}^2 \longrightarrow \prod_{i=1}^{4} H^2(\Gamma_i) \times H^{1/2}(\Gamma_i)$$
$$u \longmapsto (\gamma_i(u), \gamma_i(\frac{\partial u}{\partial n_i}))_{1\leq i\leq 4}$$

where γ_i is the trace operator on Γ_i. We know that $Ker(\gamma) = H_0^2(]l,L[\times]h,H[)$. Let

$$Z^2(\Gamma) = \{(g_i,h_i)_{1\leq i\leq 4} \in \prod_{i=1}^{4} H^2(\Gamma_i)\times H^{1/2}(\Gamma_i)/\ g_i(S_i) = g_{i+1}(S_i),\ \frac{\partial g_i}{\partial \tau_i} \equiv h_{i+1} \text{ at } S_i,$$
$$-\frac{\partial g_{i+1}}{\partial \tau_{i+1}} \equiv h_i \text{ at } S_i,\ i = 1,\dots,4 \text{ and } g_i = 0 \text{ for } i \in \mathcal{D}\}$$

We know (see e.g. [4]) that $Im(\gamma) \subset Z^2(\Gamma)$. Conversely, let $(g,h) = (g_i,h_i)_{1\leq i\leq 4} \in Z^2(\Gamma)$, there exists $u \in \{u \in H^2(]l,L[\times]h,H[)/\gamma_i(u) = 0 \text{ on } \Gamma_i,\ i \in \mathcal{D}\}$ such that $\gamma(u) = (g,h)$. Since $\gamma_i(u) = g_i$, we have that $u \in \mathcal{H}^2$. Finally, $Im(\gamma) = Z^2(\Gamma)$. The vector space $Z^2(\Gamma)$ is endowed with the norm

$$||a||_\gamma = \inf_{u\in\mathcal{H}^2/\ \gamma(u)=a} ||u||_{\mathcal{H}^2}.$$

Since $Ker(\gamma) = H_0^2$ is a closed subspace of the Hilbert space $\mathcal{H}^2$, for each $a \in Z^2(\Gamma)$ there exists a unique $u \in \mathcal{H}^2$ such that $||a||_\gamma = ||u||_{\mathcal{H}^2}$. Let ρ be a right inverse to γ defined as follows

$$\rho : Z^2 \longrightarrow \mathcal{H}^2$$
$$a \longmapsto u \ \text{ s.t. } ||a||_\gamma = ||u||_{\mathcal{H}^2}$$

The operator ρ is by definition a linear continuous operator. It is easy to check that $(Z^2(\Gamma), ||\ ||_\gamma)$ is a Hilbert space.

The vector space $Z^2(\Gamma)$, endowed with the norm

$$||(g,h)||_{Z^2}^2 = \sum_{i=1}^{4} ||g_i||^2_{H^2(\Gamma_i)} + ||h_i||^2_{H^{1/2}(\Gamma_i)} + |g_i(S_i) - g_{i+1}(S_i)|^2$$
$$+ \int_0^{\delta_i} |\frac{\partial g_i}{\partial \tau_i}(x_i(-\sigma), y_i(-\sigma)) - h_{i+1}(x_i(\sigma), y_i(\sigma))|^2/\sigma\, d\sigma$$
$$+ \int_0^{\delta_i} |\frac{\partial g_{i+1}}{\partial \tau_{i+1}}(x_i(\sigma), y_i(\sigma)) - h_i(x_i(-\sigma), y_i(-\sigma))|^2/\sigma\, d\sigma,$$

is also a Hilbert space. We show now that the norms $||\ ||_\gamma$ and $||\ ||_{Z^2}$ are equivalent. We know (see e.g. [4]) that there exists $K > 0$ such that $\forall u \in \mathcal{H}^2$, $||\gamma(u)||_{Z^2} \leq K||u||_{\mathcal{H}^2}$. Hence, $\forall a \in Z^2$, $||a||_{Z^2} \leq K||a||_\gamma$. Since $Z^2(\Gamma)$ is a Hilbert space, there exists $c > 0$ such that

$$c||a||_\gamma \leq ||a||_{Z^2} \leq K||a||_\gamma.$$

Thus, $\mathcal{H}^2$ can be written as a direct sum

$$\mathcal{H}^2 = H_0^2 \oplus \rho(Z^2),$$

and any continuous linear form l on $\mathcal{H}^2$ can be represented as

$$< l, u >=< l_1, u - \rho(\gamma(u)) > + < l_2, \gamma(u) >$$

where $l_1 \in H^{-2}$ and $l_2 \in Z^{2'}$.

Let l be a linear form on $\mathcal{H}^2$ that vanishes on $H^m \cap \mathcal{H}^2$, $m \geq 4$. We show that l vanishes also on $\mathcal{H}^2$ and thus the dense inclusion of $H^m \cap \mathcal{H}^2$ in $\mathcal{H}^2$. The linear form l is decomposed as above into l_1 and l_2. The form l vanishes on $\mathcal{D}(]l, L[\times]h, H[) \subset H^m \cap \mathcal{H}^2$ and therefore we have $l_1 = 0$. In other words, $< l, u >$ depends only on $\gamma(u)$.
In order to prove that the linear form l vanishes everywhere, it suffices to prove that $\gamma(H^m \cap \mathcal{H}^2)$ is dense into Z^2.
We first study $\gamma(H^m \cap \mathcal{H}^2)$. We know that

$$\gamma(H^m \cap \mathcal{H}^2) = \{(g_i, h_i)_{1 \leq i \leq 4} \in \prod_{i=1}^{4} H^{m-1/2}(\Gamma_i) \times H^{m-3/2}(\Gamma_i) / \ g_i(S_i) = g_{i+1}(S_i),$$
$$\frac{\partial g_i}{\partial \tau_i} = h_{i+1} \text{ at } S_i, \ -\frac{\partial g_{i+1}}{\partial \tau_{i+1}} = h_i \text{ at } S_i, \ \frac{\partial h_i}{\partial \tau_i} + \frac{\partial h_{i+1}}{\partial \tau_{i+1}} = 0 \text{ for } i = 1, \ldots, 4 \text{ and } g_i = 0 \text{ for } i \in \mathcal{D}\}.$$

In order to prove the density, we only have to look at things locally near each corner S_i depending on the kind of the corner. Let $(g_i, h_i, g_{i+1}, h_{i+1}) \in Z^2$ near S_i.

If we assume i and $i+1$ belong to $\mathcal{A}$, the functions $\sigma \mapsto \frac{\partial g_i}{\partial \tau_i}(x_i(-\sigma), y_i(-\sigma)) - h_{i+1}(x_i(\sigma), y_i(\sigma))$ and $\sigma \mapsto \frac{\partial g_{i+1}}{\partial \tau_{i+1}}(x_i(\sigma), y_i(\sigma)) + h_i(x_i(-\sigma), y_i(-\sigma))$ belong to $\tilde{H}^{1/2}(\mathbf{R}_+)$ near zero. There exist two sequences $(\alpha_n)_{n \in \mathbf{N}}$ and $(\beta_n)_{n \in \mathbf{N}}$ in $\mathcal{D}(\mathbf{R}_+)$ which converge to $\frac{\partial g_i}{\partial \tau_i} - h_{i+1}$ and $\frac{\partial g_{i+1}}{\partial \tau_{i+1}} + h_i$ respectively.
The function $g_i + g_{i+1}$ belongs to $H^2(\mathbf{R}_+)$ near zero. Let $(\delta_n)_{n \in \mathbf{N}} \in \mathcal{D}(\bar{\mathbf{R}}_+)$ converge to $g_i + g_{i+1}$ in H^2.
The function $g_i - g_{i+1}$ belongs to $H^2 \cap H_0^1(\mathbf{R}_+)$ near zero. We use the

Lemma A2. *The space*

$$\mathcal{D}_c = \{\lambda \in \mathcal{D}(\bar{\mathbf{R}}_+) / \ \lambda(0) = 0 \text{ and } \lambda''(0) = 0\}$$

is dense in $H^2 \cap H_0^1(\mathbf{R}_+)$.

Proof: Let $\eta \in \mathcal{D}_c$ s.t. $\eta'(0) = 1$. Let $u \in H^2(\mathbf{R}_+)$. The function $u - u'(0)\eta \in H_0^2(\mathbf{R}_+)$. Let $(\phi_n)_{n \in \mathbf{N}} \in \mathcal{D}(\mathbf{R}_+)$ be a sequence that converges to $u - u'(0)\eta$ in $H_0^2(\mathbf{R}_+)$. The sequence $(u'(0)\eta + \phi_n)_{n \in \mathbf{N}} \in \mathcal{D}_c$ converges to u in $H^2(\mathbf{R}_+)$. □

Let $(\lambda_n)_{n \in \mathbf{N}} \in \mathcal{D}_c(\bar{\mathbf{R}}_+)$ converge to $g_i - g_{i+1}$ in $H^2 \cap H_0^1(\mathbf{R}_+)$.
We now define an approximating sequence $(g_i^n, h_i^n, g_{i+1}^n, h_{i+1}^n)$ of $(g_i, h_i, g_{i+1}, h_{i+1})$ as follows:

$$\begin{aligned} g_i^n &= (\lambda_n + \delta_n)/2 \\ g_{i+1}^n &= (-\lambda_n + \delta_n)/2 \\ h_i^n &= \beta^n - (-\lambda_n' + \delta_n')/2 \\ h_{i+1}^n &= -\alpha^n + (\lambda_n' + \delta_n')/2 \end{aligned}$$

Let us first check that the sequence belongs to $\gamma(H^m \cap \mathcal{H}^2)$ locally near S_i. The regularity of the functions is clear. Moreover, at the corner S_i we have:

$$\begin{aligned} g_i^n(S_i) - g_{i+1}^n(S_i) &= \lambda_n(0) = 0 \\ \frac{\partial g_i}{\partial \tau_i}(S_i) - h_{i+1}(S_i) &= (\lambda_n' + \delta_n')/2 + \alpha_n - (\lambda_n' + \delta_n')/2 = \alpha_n(0) = 0 \\ \frac{\partial g_{i+1}}{\partial \tau_{i+1}}(S_i) + h_i(S_i) &= -(-\lambda_n' + \delta_n')/2 + \beta_n + (-\lambda_n' + \delta_n')/2 = \beta_n(0) = 0 \\ \frac{\partial h_i}{\partial \tau_i}(S_i) + \frac{\partial h_{i+1}}{\partial \tau_{i+1}}(S_i) &= \beta_n' - (-\lambda_n'' + \delta_n'')/2 - \alpha_n' + (\lambda_n'' + \delta_n'')/2 = \lambda_n''(0) = 0. \end{aligned}$$

The convergence of $(g_i^n, h_i^n, g_{i+1}^n, h_{i+1}^n)$ to $(g_i, h_i, g_{i+1}, h_{i+1})$ can easily be checked.

If we assume $i \in \mathcal{D}$ and $i+1 \in \mathcal{A}$, the proof is very similar. It suffices to take $\delta = -\lambda$.

If we assume i and $i+1$ belong to $\mathcal{D}$, the proof can be found in [4]. $\square$