Méthode TRAC: retournement temporel avec condition aux limites absorbante. Recréer le passé et application aux problèmes inverses en ouverture partielle

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Contents

Outline

1. Context and motivation
   - Forward Problem
   - Classical Time Reversal
2. What is the *TRAC* method ?
   - Time Reversed Absorbing Conditions (*TRAC*)
   - Stability Estimate
   - Illustration and application of the *TRAC* method
   - TRAC in the frequency domain
3. Application to inverse problems: objects discrimination
   - Criterion of objet discrimination in full aperture
   - Criterion of objet discrimination in partial aperture
   - Discrimination between one and two inclusions in partial aperture
4. Conclusion & Prospects

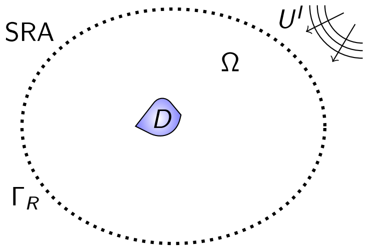
Outline

Forward problem

$\mathcal{L}$ is a hyperbolic equation: Maxwell, elasticity, wave equation, ...
An impinging wave U^I illuminates an unknwon inclusion D.

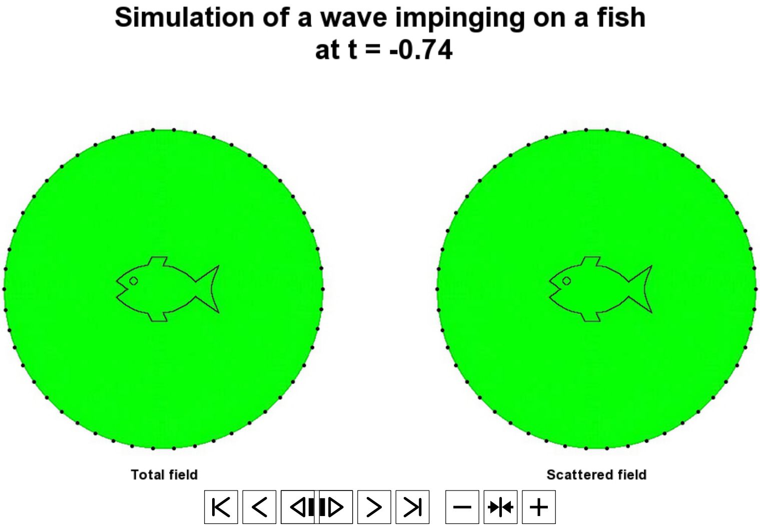
The total field U^T satisfies:

$$\begin{cases} \mathcal{L}(U^T) = 0 \text{ in } \mathbb{R}^d \\ \text{Radiation condition at infinity} \\ \text{Homogeneous Initial Conditions} \end{cases}$$



The scattered field $U^S := U^T - U^I$ is recorded from $t = 0$ to $t = T_f$ on a Source-Receiver Array (SRA) located on a surface Γ_R that encloses a domain Ω.

Goal: Reconstruct the scattered field from the recorded data on Γ_R.



Numerical simulations were made using `Freefem++` (F. Hecht). ▸ Next

Outline

Recreate the past from the SRA

Aim: From the recording on Γ_R, reconstruct the scattered field $U^S(t, x)$.

Problem: Inclusion D is unknown and thus $\mathcal{L}$ is known only in the surrounding medium $\Omega \setminus D$ where it is assumed to be homogeneous.

Three options:

- Find the properties of D by solving an inverse problem
- Time Reversal techniques
  - Classical Time Reversal (Larmat, Montagner, Fink, Capdeville, Tourin and Clévédé, 2006)
  - our idea: Time Reversed Absorbing Conditions (*TRAC*)

Recreate the past from the SRA

Based on Time reversibility of hyperbolic equations.
Example: the wave equation

$$\mathcal{L}(U) = 0 \longrightarrow \rho\, u_{tt} - \mathrm{div}\,(\mu \nabla u) = 0$$

If $u(t,x)$ is a solution, $u(-t,x)$ is a solution as well since:

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(-t,x)}{\partial t^2}$$

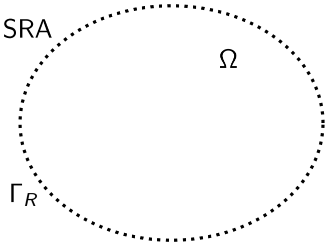
TR approach: Find a BVP whose solution is the time reversed scattered field $U_R^S(t,\cdot) := U^S(T_f - t, \cdot)$. Thus, U_R^S satisfies:

$$\begin{cases} \mathcal{L}_0(U_R^S) & = & 0 \text{ in } (0, T_f) \times \Omega \backslash D \\ U_R^S(t,\cdot) & = & U^S(T_f - t, \cdot) \text{ on } (0, T_f) \times \partial\Omega \end{cases}$$

Problem: No boundary condition on ∂D.
This boundary value problem is underdetermined.

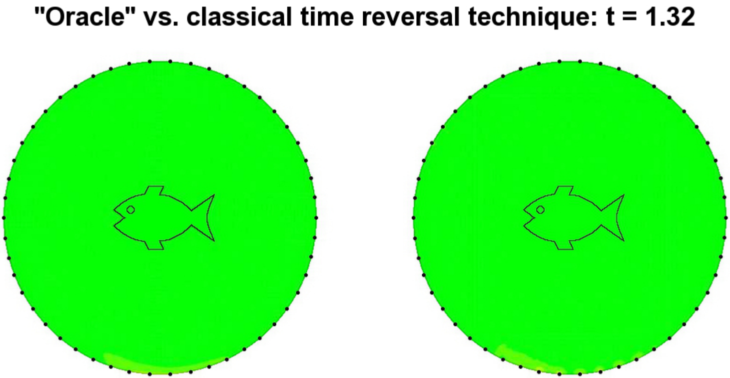
Classical Time Reversal

Ignore there is an inclusion and solve in whole domain Ω



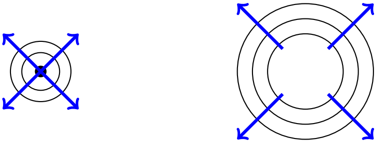
$$
\begin{aligned}
\mathcal{L}_0(W_R^S) &= 0 \text{ in } (0, T_f) \times \Omega \\
W_R^S(t, \cdot) &= U^S(T_f - t, \cdot) \text{ on } (0, T_f) \times \partial\Omega
\end{aligned}
$$

This classical time-reversed solution W_R^S differs from U_R^S.



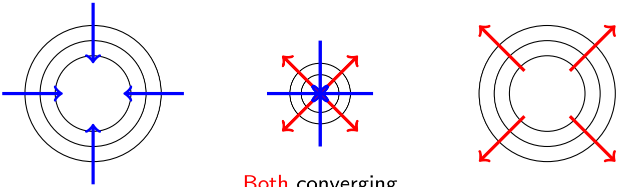
Left: “Oracle” time reversal, Right: Classical time reversal (20% noise)

“Oracle” means the desired time reverse scattered field. ▸ Next



Diverging only   Diverging waves

Figure : Forward problem



Converging only   Both converging and diverging waves interfere   Diverging only

Figure : Problem in the numerical TR: diffraction limit – focal spot of size $\lambda/2$

Outline

Outline

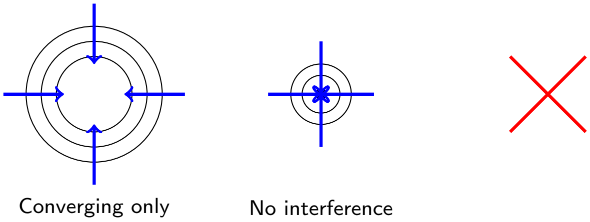
TRAC method

TRAC combines time reversal techniques (Fink, Montagner et al.) and absorbing boundary conditions (Engquist & Majda, Bayliss & Turkel)

TRAC works without knowing about the “scatterers”

- the shape or/and
- the location or/and
- the boundary conditions it satisfies.

Perfect Numeric Time Reversal



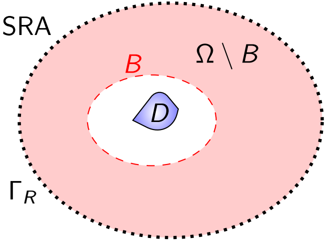
$\Rightarrow$ No diffraction limit

Recall, U_R^S satisfies :

$$\begin{aligned} \mathcal{L}_0(U_R^S) &= 0 \text{ in } (0, T_f) \times \Omega \backslash D \\ U_R^S(t,\cdot) &= U^S(T_f - t, \cdot) \text{ on } (0, T_f) \times \partial\Omega \end{aligned}$$

In order to remove the underdetermination, we introduce an artificial domain B enclosing D and solve the reversed problem in $\Omega \backslash B$.

Which boundary condition
on the artificial boundary ∂B ?



TRAC

Note that the forward scattered field U^S satisfies

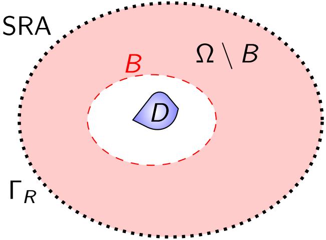
$$\begin{cases} \mathcal{L}_0(U^S) = 0 \text{ in } \mathbb{R}^d \setminus D \\ U^S \text{ satisfies a radiation condition at } \infty \end{cases}$$

Let ABC denote an absorbing boundary condition, we have :

$$\mathrm{ABC}(U^S) = 0 \text{ on } \partial B\,,$$

We time reverse it :

$$\mathrm{TRAC}(U_R^S) = 0 \text{ on } \partial B$$

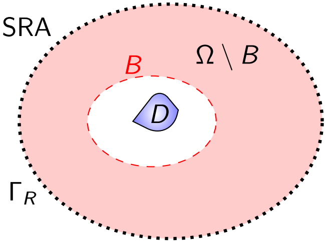


$\mathrm{TRAC}(U_R^S) = 0$ is the missing boundary condition on ∂B.

The time reversed solution U_R^S satisfies the following BVP in a restricted domain :

$$\begin{aligned} \mathcal{L}_0(U_R^S) &= 0 \text{ in } (0, T_f) \times \Omega \backslash B \\ U_R^S(t, \cdot) &= U^S(T_f - t, \cdot) \text{ on } \Gamma_R \\ \mathrm{TRAC}(U_R^S) &= 0 \text{ on } \partial B\,. \end{aligned} \tag{1}$$

By solving (1), we are able to recreate the past, namely reconstruct U^S in domain $\Omega \setminus B$.



Example: the wave equation

$$\mathcal{L}(U) \longrightarrow \rho\, u_{tt} - \mathrm{div}\,(\mu \nabla u)$$

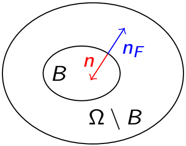
The *TRAC* problem reads

$$\begin{cases} \rho_0 \dfrac{\partial^2 u_R^S}{\partial t^2} - \mathrm{div}\,\left(\mu_0 \nabla u_R^S\right) = 0 & \text{in } \Omega \backslash B \\ u_R^S(t,\cdot) = u^S(T_f - t, \cdot) & \text{on } \Gamma_R \\ \mathrm{TRAC}(u_R^S) := \dfrac{\partial u_R^S}{\partial t} + c_0 \dfrac{\partial u_R^S}{\partial n} - c_0\, \kappa\, \dfrac{u_R^S}{2} = 0 & \text{on } \partial B \\ \text{zero Cauchy Data} & \end{cases} \quad (2)$$

where $c_0 = \sqrt{\mu_0/\rho_0}$ and κ is the curvature of ∂B.

By solving (2), we are able to recreate the past, namely reconstruct u^S in domain $\Omega \backslash B$.

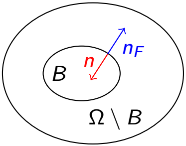
The outward normal to B is denoted by n_F whereas the outward normal to $\Omega \setminus B$ is denoted by n.



We assume that the forward scattered field, u^S satisfies BT1

$$\frac{\partial}{\partial t}(u^S(t,x)) + c_0 \frac{\partial}{\partial n_F}(u^S(t,x)) + \kappa \frac{c_0}{2}(u^S(t,x)) = 0$$

The outward normal to B is denoted by n_F whereas the outward normal to $\Omega \setminus B$ is denoted by n.



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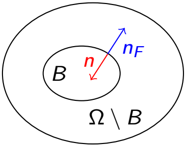
using $u_R^S(t,x) := u^S(T_f - t, x)$

$$-\frac{\partial}{\partial t}(u_R^S(t,x)) + c_0 \frac{\partial}{\partial n_F}(u_R^S(t,x)) + \kappa \frac{c_0}{2}(u_R^S(t,x)) = 0$$

The outward normal to B is denoted by n_F whereas the outward normal to $\Omega \setminus B$ is denoted by n.



We assume that the forward scattered field, u^S satisfies BT1

$$\frac{\partial}{\partial t}(u^S(t,x)) + c_0 \frac{\partial}{\partial n_F}(u^S(t,x)) + \kappa \frac{c_0}{2}(u^S(t,x)) = 0$$

using $u^S_R(t,x) := u^S(T_f - t, x)$

$$-\frac{\partial}{\partial t}(u^S_R(t,x)) + c_0 \frac{\partial}{\partial n_F}(u^S_R(t,x)) + \kappa \frac{c_0}{2}(u^S_R(t,x)) = 0$$

since $\partial/\partial n_F = -\partial/\partial n$

$$\frac{\partial}{\partial t}(u^S_R(t,x)) + c_0 \frac{\partial}{\partial n}(u^S_R(t,x)) - \kappa \frac{c_0}{2}(u^S_R(t,x)) = 0$$

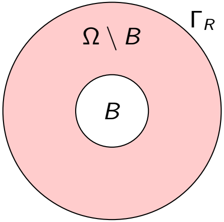
Outline

The *TRAC* boundary condition differs from BT1 by the minus sign and well posedness might be an issue.

Let $B \in \mathbb{R}^3$ be a ball of radius r_0 and let u satisfy the following equations:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 & \text{in } \Omega \setminus B \\ u = 0 & \text{on } \Gamma_R \\ \dfrac{\partial u}{\partial t} + c\,(\dfrac{\partial u}{\partial n} - \dfrac{u}{r}) = g & \text{on } \partial B \end{cases} \tag{3}$$

with $u = u_0$ and $\dfrac{\partial u}{\partial t} = u_1$ at $t = 0$.



Stability Estimate

Proposition

If Ω is a ball of radius R, we have the following stability estimate written for the BVP (3) *in spherical coordinates :*

$$\frac{1}{2}\frac{d}{dt}\left(\iiint r^2 \sin\phi\,(\frac{\partial u}{\partial t})^2 + c^2 \sin\phi(\frac{\partial(ru)}{\partial r})^2 + c^2 \sin\phi(\frac{\partial u}{\partial \phi})^2 + \frac{c^2}{\sin\phi}(\frac{\partial u}{\partial \theta})^2\right)$$

$$+\iint_{r=r_0} cr^2 \sin\phi(\frac{\partial u}{\partial t})^2 \;=\; \iint_{r=r_0} cr^2 \sin\phi\,\frac{\partial u}{\partial t}\,g$$

This stability estimate is not the standard energy estimate.

Proof: see *Time Reversed Absorbing Conditions*, Assous, Kray, Nataf, Turkel, Comptes Rendus Mathématiques, Serie I (2010)

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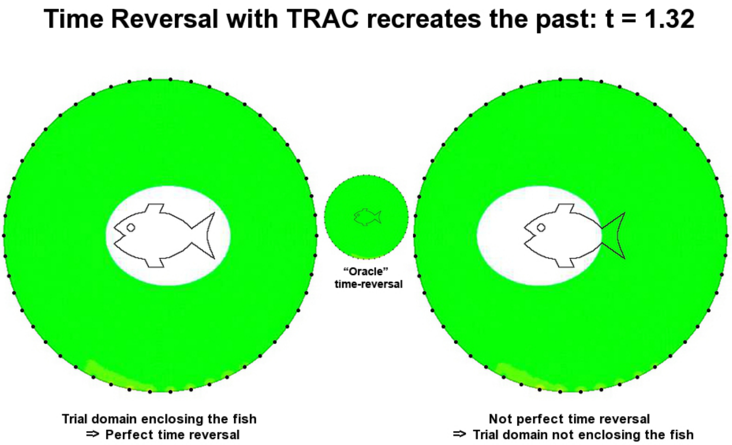
Time reversal with *TRAC* (20% noise)



Left: B encloses the inclusion D ; *TRAC* recreates the past

Right: B does not enclose D ; past is not correctly recreated ▸ Next

Inverse problem via *TRAC*

TRAC has at least two applications in inverse problems :

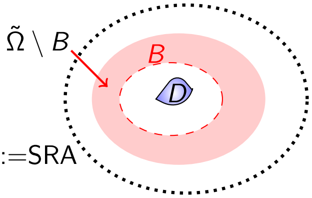
1. The first application is the reduction of the size of the computational domain by redefining the reference surface on which the receivers appear to be located. This is reminiscent of the redatuming method, see Berryhill, 1979.

2. The second application is to identify an unknown inclusion D from boundary measurements.This is achieved by using a trial and error procedure on the trial domain B.

u_R^T time-reversed total field (exact)
v_R^T *TRAC* reconstruction

relative L^2-error :

$$E(v_R) = \frac{\|u_R^T - v_R^T\|_{L^2(\tilde{\Omega}\setminus B)}}{\|u_R^T\|_{L^2(\tilde{\Omega}\setminus B)}}$$



Noise/Test	1	2	3	4	5	6	Mean value
0%	6.28%	3.14%	1.93%	3.92%	7.37%	5.18%	4.64%
5%	6.39%	3.37%	2.31%	4.15%	7.50%	5.31%	4.84%
10%	6.72%	4.03%	3.21%	4.68%	7.84%	5.78%	5.38%
20%	8.04%	5.98%	5.32%	6.28%	8.87%	7.21%	6.95%
Mean value	6.86%	4.13%	3.19%	4.76%	7.90%	5.87%	5.45%

Remark : penetrable inclusions such as $c_D = 3$ and $c_0 = 1$.

Outline

Example : Helmholtz equation

$$\mathcal{L}(U) \longrightarrow -\omega^2 \hat{u} - c^2 \Delta \hat{u}$$

Time Reversal amounts then to phase conjugation, see SAR.

$$\begin{aligned}\hat{v}_R(\omega, \vec{x}) &= \int v(-t, \vec{x}) e^{-i\omega t} dt = \int v(t, \vec{x}) e^{i\omega t} dt \\ &= \overline{\int v(t, \vec{x}) e^{-i\omega t} dt} = \overline{\hat{v}}(\omega, \vec{x}) \,.\end{aligned}$$

TRAC Helmholtz problem :

$$\begin{cases} -\omega^2 \hat{u}_R - c^2 \Delta \hat{u}_R = 0 & \text{in } \Omega \setminus B \\ i\omega \hat{u}_R + c \dfrac{\partial \hat{u}_R}{\partial n} - \kappa \dfrac{c \hat{u}_R}{2} = 0 & \text{on } \partial B \\ \hat{u}_R = \overline{\hat{u}} & \text{on } \Gamma_R \end{cases}$$

TRAC in the frequency domain

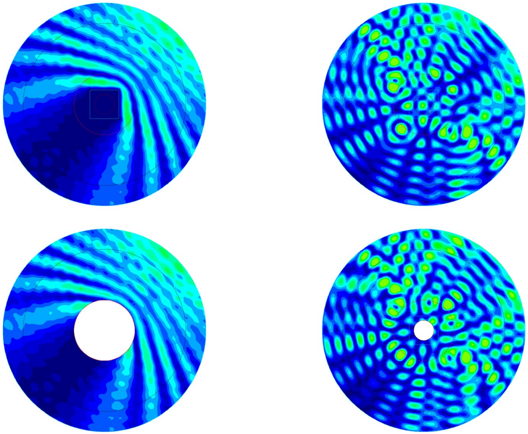


Figure : Soft square shaped inclusion D; of length $2\,\lambda$. From left to right, from top to bottom : oracle, classical phase conjugation, *TRAC* with B enclosing the inclusion, *TRAC* with B inside the inclusion.

Outline

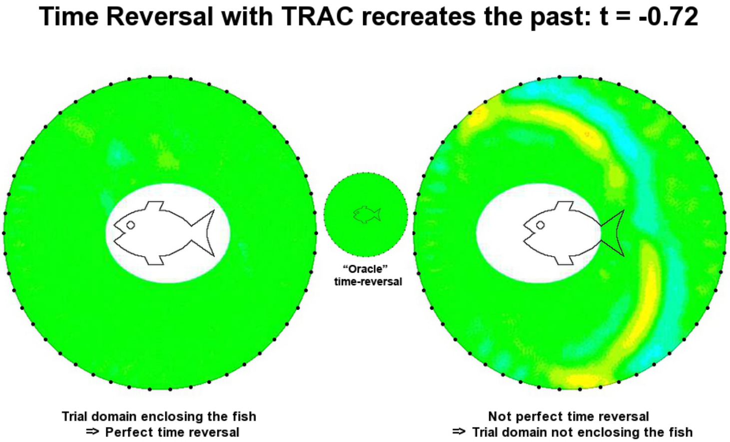
Inverse problem via *TRAC*

Principle to identify the inclusion:

- If B encloses the object D, the solution u_R^S to the time reversed BVP coincides with the time reversed of the "forward" solution u^S ("oracle").
- Conversely, if there is a difference between u_R^S and the reverse of the "forward" solution u^S, we know that domain B does not contain the inclusion.

Outline

Final time solutions

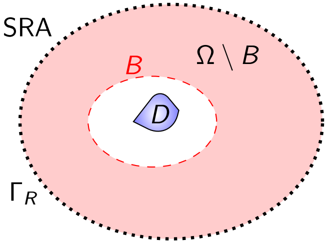


Criterion of objet discrimination

In full aperture : final time criterion

$$J_{FT}(B) := \frac{\|v_R^S(T_f, \cdot)\|_{L^\infty(\Omega\setminus B)}}{\sup_{t\in[0,T_f]} \|u^I(t,\cdot)\|_{L^\infty(\Omega)}}$$

v_R^S computed time-reverse scattered field
u^I known forward incident wave



Only Final time solutions are displayed

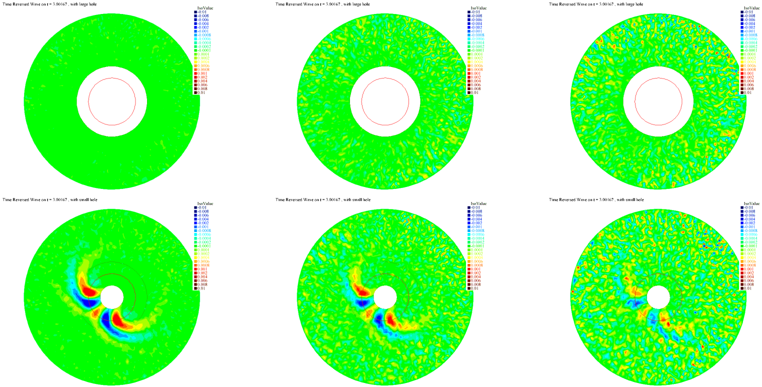
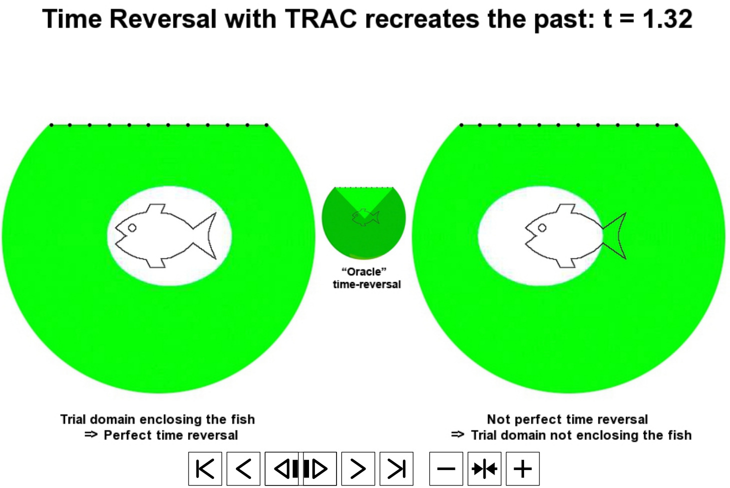


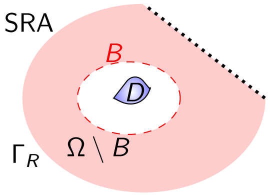
Figure : $Coeff = 10\%$, $Coeff = 30\%$, $Coeff = 50\%$

Outline



Left: B encloses the inclusion D ; *TRAC* recreates the past
Right: B does not enclose D ; past is not correctly recreated ▸ Next

In partial aperture :



cross-correlation image

$$f(\vec{x}) := \int_{t=0}^{t=T_f} v_R^S(T_f - t, \vec{x}) \times u^I(t, \vec{x}) \; dt \,,$$

cross-correlation criterion

$$J_{CC}(B) \; := \; \frac{\left\| \displaystyle\int_{t=0}^{t=T_f} v_R^S(T_f - t, .) \times u^I(t, .) \; dt \right\|_{L^\infty(\Omega \setminus B)}}{\left\| \displaystyle\int_{t=0}^{t=T_f} |u^I(t, .)|^2 \; dt \right\|_{L^\infty(\Omega)}} \,,$$

v_R^S computed time-reverse scattered field
u^I known forward incident wave

Resulting image with the cross-correlation function

Imaging by cross-correlation to detect the interfaces of the inclusion

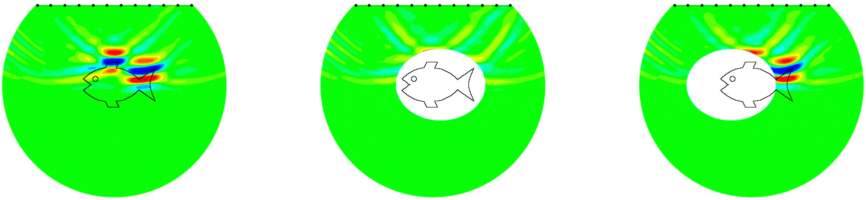


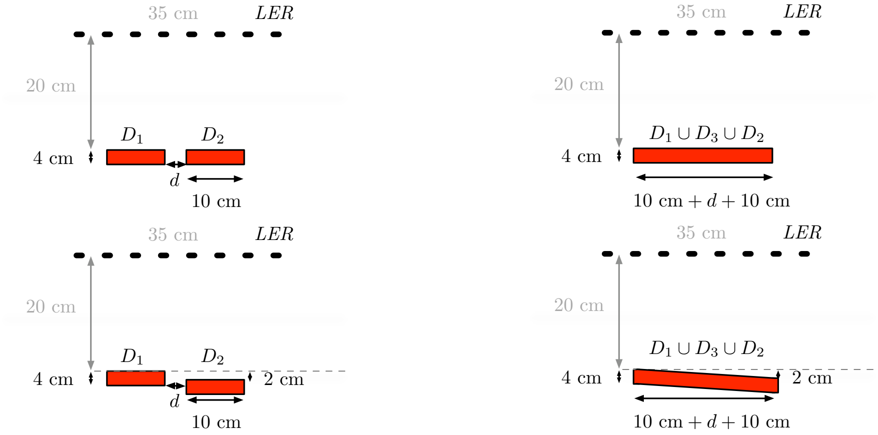
Figure : Classical TR, *TRAC* enclosing the inclusion, *TRAC* missing the tail

$J_{CC}(B) = 23.01\%$ $J_{CC}(B) = 7.95\%$ $J_{CC}(B) = 21.87\ \%$

Outline

Objects Discrimination with Partial Aperture SRA

Four configurations: $\lambda \simeq 10\ cm$, LER := SRA



Two distinct inclusions and a unique one, aligned and not aligned.
Data from Igel'08, (12th Int. Conf. on Ground Penetrating Radar)

Objects Discrimination with Partial Aperture SRA

simplified electromagnetic waves in first approximation:

- ε the electric permittivity, μ the magnetic permeability
- $c = \sqrt{1/\varepsilon\mu}$ the velocity.

$$\varepsilon \frac{\partial^2 u(t,r)}{\partial t^2} - \text{div}(\frac{1}{\mu}\nabla u(t,r)) = 0. \qquad \text{(EM)}$$

Properties of the mines (iron or plastic):

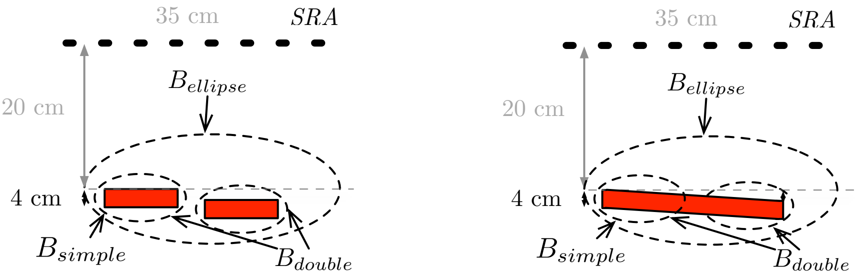
$$\varepsilon = \varepsilon_0 \varepsilon_r \ , \qquad \mu = \mu_0 \mu_r$$

	soil	iron mine	plastic mine
ε_r	5	1	3
μ_r	1	10,000	1
c_r	0.44	0.01	0.57

See Igel'08.

Objects Discrimination with Partial Aperture SRA

Trial domains B

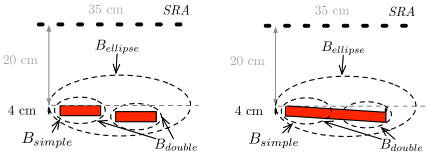


Criterion $J(B)$ designed such that:
$J(B) >> 1 \Rightarrow$ domain B encloses the inclusion(s)
$J(B) \simeq 1, 10 \Rightarrow$ domain B does not enclose the inclusion(s)

Criterion:

$J(B) >> 1 \Rightarrow D \subset B$

$J(B) \simeq 1,\ 10 \Rightarrow D \not\subset B$

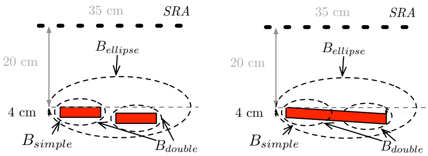


$d = \lambda/4$		aligned inclusions		non aligned inclusions	
Trial domain B	Noise	2 inclusions	1 inclusion	2 inclusions	1 inclusion
$B_{ellipse}$	0%	1513327,65	4964577,64	23816,93	90352,37
	20%	1442376,08	4668092,94	23962,34	99995,21
B_{simple}	0%	2,07	6,72	0,72	1,96
	20%	2,01	6,33	0,68	2,03
B_{double}	0%	163,33	13,68	324,33	17,87
	20%	166,14	14,02	334,11	17,97

Table : Criterion J for discriminating between one or two plastic mines distant from $\lambda/4 \Rightarrow$ Subwavelength resolution.

Discrimination between one iron mine and two

Criterion:
$J(B) >> 1 \Rightarrow D \subset B$
$J(B) \simeq 1,\ 10 \Rightarrow D \not\subset B$



$d = \lambda/8$		aligned inclusions		non aligned inclusions	
Trial domain B	Noise	2 inclusions	1 inclusion	2 inclusions	1 inclusion
$B_{ellipse}$	0%	112293,94	197912,87	6899,20	8613,34
	20%	101868,59	194473,61	6371,37	9341,23
B_{simple}	0%	1,90	3,97	0,99	1,48
	20%	1,73	3,83	1,00	1,46
B_{double}	0%	54,05	7,86	200,72	12,52
	20%	49,46	7,23	224,23	12,96

Table : Criterion J for discriminating between one or two iron mines distant from $\lambda/8 \Rightarrow$ Subwavelength resolution.

Outline

Conclusion & prospects

- stability and energy estimate
- time and frequency domain method
- redatuming without paraxial approximation
- for solid or penetrable object of any shape
- robust with respect to noise on the recorded data
- full and partial aperture of the SRA
- application to discrimination between one and two close inclusions

Actual work & prospects

- combination of the *TRAC* method and an inverse problem technique (Work by Marie Kray and Maya de Buhan, Paris 5)
- comparison with experimental data (real data)
- other equations: Maxwell, elasticity

Thanks !

Publications :

- F. Assous, M. Kray, F. Nataf and E. Turkel, *Time Reversed Absorbing Conditions*, CR. Acad. Sci., (2010)
- F. Assous, M. Kray, F. Nataf and E. Turkel, *Time Reversed Absorbing Condition : Application to Inverse Problems*, Inverse problems (2011)
- F. Assous, M. Kray and F. Nataf, *Time Reversed Absorbing Condition in the Partial Aperture Case*, Wave Motion (2012)

all papers can be found on HAL.