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INTERNAL REPORT No 492
2002

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A Robin-Robin preconditioner for advection-diffusion equations with discontinuous coefficients

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Abstract

We consider an advection-diffusion problem with discontinuous viscosity coefficients. We apply a substructuring technique and we extend to the resulting Schur complement the Robin-Robin preconditioner used for problems with constant viscosity. A quasi optimal convergence analysis is performed in the case of uniform convection by means of Fourier techniques. The variational formulation in order to generalize the preconditioner to an arbitrary number of subdomains is also addressed, as well as some numerical tests in 3D.

1 Introduction

The main goal of domain decomposition techniques is the efficient solution on parallel machines of problems issued from Computational Mechanics set on complex geometries and discretized on very fine grids. Most of these methods are based on iterative substructuring, which consists in splitting the original domain into small disjoint subdomains without overlap and reducing the original problem to an interface one to be solved by an iterative method. The parallel efficiency of the methods depends mainly on the choice of the preconditioner for the interface problem, which should have good parallel properties, should be able to handle arbitrary elliptic operators and discretization grids, and whose performance should not be affected neither by the discretization parameter h nor the number of subdomains. Many such preconditioners have been proposed during the years following the early work of Bramble, Pasciak and Schatz ([7]), dealing with both symmetric ([1, 6, 10, 11, 13, 14]) and non symmetric operators ([2, 3, 8]).

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We present herein an extension of both the Robin-Robin preconditioner (introduced in [3] and [2]) and the generalized Neumann-Neumann one (see [6] and [15]) to an advection-diffusion equation with discontinuous viscosity coefficients. The problem, important in itself in both engineering and environmental sciences, arises from the modeling of the diffusion and transport through heterogeneous media, where different materials with different physical properties are present in the computational domain. The basic decomposition is therefore given by the physics of the problem and we are mainly interested here on what's going on along the discontinuity interface. We extend the generalized Neumann-Neumann preconditioner for the Schur complement, which deals with heterogeneity in the coefficients, by replacing the local Neumann condition with suitable Robin conditions which take into account the non-symmetry of the problem. When the viscosity is continuous we recover the Robin-Robin preconditioner, as well as the generalized Neumann-Neumann one as long as the operator is symmetric.

The paper is organized as follows. In Section 2 we state the problem, while in Section 3 the method is defined and analyzed at the continuous level by means of a Fourier analysis, showing that the preconditioned Schur complement system has a condition number independent of the coefficients of the problem. In Section 4 we introduce the variational formulation for the problem to make it possible the generalization to an arbitrary number of subdomains. Some numerical results illustrating the performance of the proposed method conclude the paper.

2 Statement of the model problem

Let Ω be bounded domain in \mathbf{R}^2 . We consider the following general advection-diffusion problem

$$\begin{aligned} -\operatorname{div}(\nu(x)\nabla u) + \vec{b} \cdot \nabla(u) + au &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{aligned} \tag{1}$$

where \vec{b} is the convective field $\vec{b} = (b_x, b_y)$ while the constant a may arise from an Euler implicit time discretization for the time dependent problem, and represent the inverse of the time step, i.e. $a = 1/\Delta t$.

We assume the function $\nu(x)$ to be piecewise constant

$$\nu(x) = \begin{cases} \nu_1 & \text{if } x \in \Omega_1 \\ \nu_2 & \text{if } x \in \Omega_2 \end{cases}$$

with $\nu_1 < \nu_2$, where Ω_1 and Ω_2 are two subset of Ω such that

$$\Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega}_1 \cup \overline{\Omega}_2 = \Omega.$$

We denote with Γ the interface between the two subdomains, i.e.

$$\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2,$$

and, finally, we denote with \mathcal{L}_j ($j = 1, 2$) the operators

$$\mathcal{L}_j(w) := -\nu_j \Delta w + \vec{b} \cdot \nabla w + aw$$

3 Analysis of the Preconditioner

3.1 The Continuous Algorithm

We introduce, at the continuous level, the global interface operator

$$\begin{aligned} \Sigma : H_{00}^{1/2}(\Gamma) \times L^2(\Omega) &\longrightarrow H^{-1/2}(\Gamma) \\ (u_\Gamma, f) &\longmapsto \left(\nu_1 \frac{\partial u_1}{\partial n_1} + \nu_2 \frac{\partial u_2}{\partial n_2} \right)_\Gamma \end{aligned} \quad (2)$$

where u_j ($j = 1, 2$) is the solution to problem

$$\begin{aligned} \mathcal{L}_j(u_j) &= f && \text{in } \Omega_j \\ u_j &= 0 && \text{on } \partial\Omega \cap \partial\Omega_j \\ u_j &= u_\Gamma && \text{on } \Gamma \end{aligned} \quad (3)$$

Since the operator Σ is linear with respect to both variables, we can easily reduce (1) to the Steklov-Poincaré formulation of a coupled problem on the interface

$$\mathcal{S}(u_\Gamma) = \chi \quad (4)$$

where we have set $\mathcal{S}(\cdot) := \Sigma(\cdot, 0)$ as well as $\chi := -\Sigma(0, f)$. In order to solve equation (4) with an iterative procedure, we split the operator \mathcal{S} into

$$\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 \quad (5)$$

where

$$\mathcal{S}_j : u_\Gamma \mapsto \left(\nu_j \frac{\partial u_j}{\partial n_j} - \frac{\vec{b} \cdot \vec{n}_j}{2} u_j \right)_\Gamma$$

(for $j = 1, 2$). Since $\vec{n}_1 = -\vec{n}_2$, the terms $\frac{1}{2} \vec{b} \cdot \vec{n}_j u_j$ vanish in the sum and we recover the operator \mathcal{S} .

Following [2], [3] and [15], we propose as a preconditioner for the Steklov-Poincaré equation at the continuous level an approximate inverse of \mathcal{S} , which is the weighted sum of the inverses of the operators \mathcal{S}_1 and \mathcal{S}_2 , namely

$$\mathcal{T} = D_1 \mathcal{S}_1^{-1} D_1 + D_2 \mathcal{S}_2^{-1} D_2 \quad (6)$$

where

$$D_1 = \frac{\nu_1}{\nu_1 + \nu_2}, \quad D_2 = \frac{\nu_2}{\nu_1 + \nu_2}$$

are constant operators on the interface satisfying $D_1 + D_2 = Id$. The approximate inverse \mathcal{T} is therefore defined as follows

$$\begin{aligned} \mathcal{T} : H^{-1/2}(\Gamma) &\longrightarrow H_{00}^{1/2}(\Gamma) \\ g &\longmapsto \frac{\nu_1}{\nu_1 + \nu_2} v_{1|\Gamma} + \frac{\nu_2}{\nu_1 + \nu_2} v_{2|\Gamma} \end{aligned} \quad (7)$$

where v_j ($j = 1, 2$) is the solution to

$$\begin{aligned} \mathcal{L}_j(v_j) &= 0 && \text{in } \Omega_j \\ v_j &= 0 && \text{on } \partial\Omega \cap \partial\Omega_j \end{aligned} \quad (8)$$

$$\left(\nu_j \frac{\partial v_j}{\partial n_j} - \frac{\vec{b} \cdot \vec{n}_j}{2} v_j \right)_{\Gamma} = \frac{\nu_j}{\nu_1 + \nu_2} g \quad \text{on } \Gamma$$

3.2 The vertical strip case - Uniform velocity

In this section we consider the case where $\Omega = \mathbf{R}^2$ is decomposed into the left ($\Omega_1 =]-\infty, 0[\times \mathbf{R}$) and right ($\Omega_2 =]0, +\infty[\times \mathbf{R}$) half-planes, we assume the convective field to be uniform $\vec{b} = (b_x, b_y)$, with the additional requirement on the solutions u_j to be bounded as $|x| \rightarrow +\infty$.

We can express the action of the operator \mathcal{S} in terms of its Fourier transform in the y direction as

$$\mathcal{S}u_{\Gamma} = \mathcal{F}^{-1} \left(\hat{\mathcal{S}}(\xi) \hat{u}_{\Gamma}(\xi) \right), \quad u_{\Gamma} \in H_{00}^{1/2}(\Gamma)$$

where we have denoted with ξ the Fourier variable and with \mathcal{F}^{-1} the inverse Fourier transform. We consider, for $j = 1, 2$, the problem

$$\begin{aligned} \mathcal{L}_j(u_j) &= 0 && \text{in } \Omega_j \\ u_j &= u_{\Gamma} && \text{on } \Gamma, \end{aligned} \quad (9)$$

and we have to compute the Fourier transform of $(\nu_1(\partial u_1/\partial n_1) + \nu_2(\partial u_2/\partial n_2))_{\Gamma}$. Performing a Fourier transform in the y direction on the operators \mathcal{L}_j , we get

$$(a + b_x \partial_x - \nu_j \partial_{xx} + ib_y \xi + \nu_j \xi^2) \hat{u}_j(x, \xi) = 0, \quad (10)$$

for $j = 1, 2$, where $i^2 = -1$. For a given ξ , equation (10) is an ordinary differential equation in x whose solutions have the form $\alpha_j(\xi) \exp\{\lambda_j^-(\xi)x\} + \beta_j(\xi) \exp\{\lambda_j^+(\xi)x\}$, where

$$\lambda_j^\pm(\xi) = \frac{b_x \pm \sqrt{b_x^2 + 4a\nu_j + 4\nu_j^2\xi^2 + 4ib_y\nu_j\xi}}{2\nu_j}, \quad (11)$$

with $\text{Re}(\lambda_j^\pm) \geq 0$, as $\text{Re}(z)$ indicates the real part of a complex number z . The boundedness assumption on the solutions u_j ($j = 1, 2$) for $x \rightarrow \pm\infty$, implies $\alpha_1(\xi) = \beta_2(\xi) = 0$, while the Dirichlet condition on the interface provides $\beta_1(\xi) = \alpha_2(\xi) = \hat{u}_\Gamma$. Hence,

$$\begin{aligned} \nu_1 \left(\frac{\partial \hat{u}_1}{\partial n_1} \right)_\Gamma &= \nu_1 \left(\frac{\partial \hat{u}_1}{\partial x} \right)_{|x=0} \\ &= \frac{1}{2} \hat{u}_\Gamma \left(b_x + \sqrt{b_x^2 + 4a\nu_1 + 4\nu_1^2\xi^2 + 4ib_y\nu_1\xi} \right) \end{aligned}$$

as well as

$$\begin{aligned} \nu_2 \left(\frac{\partial \hat{u}_2}{\partial n_2} \right)_\Gamma &= \nu_2 \left(-\frac{\partial \hat{u}_2}{\partial x} \right)_{|x=0} \\ &= -\frac{1}{2} \hat{u}_\Gamma \left(b_x - \sqrt{b_x^2 + 4a\nu_2 + 4\nu_2^2\xi^2 + 4ib_y\nu_2\xi} \right) \end{aligned}$$

and we have the following expression for $\hat{\mathcal{S}}$:

$$\hat{\mathcal{S}}\hat{u}_\Gamma = \frac{1}{2} \left(\sqrt{b_x^2 + 4a\nu_1 + 4\nu_1^2\xi^2 + 4ib_y\nu_1\xi} + \sqrt{b_x^2 + 4a\nu_2 + 4\nu_2^2\xi^2 + 4ib_y\nu_2\xi} \right) \hat{u}_\Gamma \quad (12)$$

Let \mathcal{T} be the operator introduced in (6). In order to evaluate its performance as a preconditioner we define

$$N_1 = \left[\frac{\nu_1}{\nu_1 + \nu_2} \right]^2, \quad N_2 = \left[\frac{\nu_2}{\nu_1 + \nu_2} \right]^2, \quad (13)$$

and the symbol of the preconditioned operator is easily determined as

$$\begin{aligned} \Phi(\xi) &= N_1 \left(1 + \frac{\sqrt{b_x^2 + 4a\nu_2 + 4\nu_2^2\xi^2 + 4ib_y\nu_2\xi}}{\sqrt{b_x^2 + 4a\nu_1 + 4\nu_1^2\xi^2 + 4ib_y\nu_1\xi}} \right) \\ &\quad + N_2 \left(1 + \frac{\sqrt{b_x^2 + 4a\nu_1 + 4\nu_1^2\xi^2 + 4ib_y\nu_1\xi}}{\sqrt{b_x^2 + 4a\nu_2 + 4\nu_2^2\xi^2 + 4ib_y\nu_2\xi}} \right). \end{aligned} \quad (14)$$

Remark 3.1 Notice that, for $a = b_x = b_y = 0$, we have $\Phi(\xi) = 1$, implying exact preconditioning in this simple case.

Due to the presence of the first order term, the resulting linear system is non-symmetric, and we use at the discrete level an iterative method of Krylov type, such as GMRES. In that order, we recall that the reduction factor in a GMRES iteration, for a positive real matrix A with symmetric part M , is bounded from above (see [16]) by

$$\rho_{\text{GMRES}} \leq 1 - \frac{(\lambda_{\min}(M))^2}{\lambda_{\max}(A^T A)},$$

The reduction factor for the associated GMRES algorithm preconditioned by \mathcal{T} can therefore be estimated, in the Fourier space, by

$$\rho_{\text{GMRES}} \leq 1 - \frac{(\min_{\xi} \operatorname{Re} \Phi(\xi))^2}{\max_{\xi} |\Phi(\xi)|^2}.$$

where the function $\Phi(\xi)$ is the symbol of the preconditioned operator, defined in (14). We can prove the following lemma, ensuring that the above reduction factor can be bounded from above by a constant independent of the parameters of the problem.

Lemma 3.1 *Let $\Phi(\xi)$ be the function defined in (14). Then*

$$\frac{\max_{\xi} |\Phi(\xi)|^2}{(\min_{\xi} \operatorname{Re} \Phi(\xi))^2} \in O(1) \quad (15)$$

independently of a , b_x , b_y , ν_1 and ν_2 .

Proof. Assume $b_y \neq 0$, and let

$$z(\xi) := \sqrt{\frac{b_x^2 + 4a\nu_2 + 4\nu_2^2 \xi^2 + 4ib_y \nu_2 \xi}{b_x^2 + 4a\nu_1 + 4\nu_1^2 \xi^2 + 4ib_y \nu_1 \xi}}.$$

We have

$$\begin{aligned} |z(\xi)| &= \left[\frac{(b_x^2 + 4a\nu_2 + 4\nu_2^2 \xi^2)^2 + (4b_y \nu_2 \xi)^2}{(b_x^2 + 4a\nu_1 + 4\nu_1^2 \xi^2)^2 + (4b_y \nu_1 \xi)^2} \right]^{1/4} \\ &= \left(\frac{[b_x^2 + 4a\nu_2]^2 + 8[b_x^2 + 2b_y^2 + 4a\nu_2] \nu_2^2 \xi^2 + 16 \nu_2^4 \xi^4}{[b_x^2 + 4a\nu_1]^2 + 8[b_x^2 + 2b_y^2 + 4a\nu_1] \nu_1^2 \xi^2 + 16 \nu_1^4 \xi^4} \right)^{1/4}, \end{aligned} \quad (16)$$

which is bounded, as

$$1 < |z(\xi)| \leq \frac{\nu_2}{\nu_1}, \quad (17)$$

and it is not difficult (althought rather tedious) to see that its first derivative is given by

$$\frac{d|z(\xi)|}{d\xi} = \frac{1}{2} |z(\xi)|^{-3/4} \frac{F + 2G\xi^2 + H\xi^4}{[Q(\xi)]^2} (\nu_2 - \nu_1) \xi,$$

where

$$F = [b_x^2(b_x^2 + 2b_y^2) + 32a^2\nu_1\nu_2] [b_x^2(\nu_1 + \nu_2) + 2a\nu_1\nu_2] + 2ab_x^2 [b_x^2(2\nu_2^2 + 5\nu_1\nu_2 + 2\nu_1^2) + 6b_y^2\nu_1\nu_2]$$

$$G = [b_x^4(\nu_2^2 + \nu_1^2) + 16a^2\nu_1^2\nu_2^2] (\nu_1 + \nu_2) + 8ab_x^2\nu_1\nu_2 (\nu_2^2 + \nu_1\nu_2 + \nu_1^2)$$

$$H = [b_x^2 + 2b_y^2] \nu_1^2\nu_2^2 (\nu_2 + \nu_1) + 4a\nu_1^3\nu_2^3$$

$$Q(\xi) = [b_x^2 + 4a\nu_1]^2 + 8 [b_x^2 + 2b_y^2 + 4a\nu_1] \nu_1^2\xi^2 + 16 \nu_1^4\xi^4.$$

As the coefficients F , G and H are positive, the function $|z(\xi)|$ is decreasing in $(-\infty, 0)$, increasing in $(0, +\infty)$, and we have

$$\min_{\xi} |z(\xi)| = |z(0)| = \sqrt{\frac{b_x^2 + 4a\nu_2}{b_x^2 + 4a\nu_1}}, \quad \sup_{\xi} |z(\xi)| = \lim_{\xi \rightarrow \pm\infty} |z(\xi)| = \frac{\nu_2}{\nu_1}.$$

Since $z^{-1} = \bar{z}/|z|^2$, the complex valued function $\Phi(\xi)$ can be written as

$$\Phi(\xi) = N_1 [1 + z(\xi)] + N_2 \left[1 + \frac{\bar{z}(\xi)}{|z(\xi)|^2} \right], \quad (18)$$

hence

$$\operatorname{Re} \Phi(\xi) = N_1 + N_2 + \left[N_1 + \frac{N_2}{|z(\xi)|^2} \right] \operatorname{Re} z(\xi), \quad (19)$$

as well as

$$\operatorname{Im} \Phi(\xi) = \left[N_1 - \frac{N_2}{|z(\xi)|^2} \right] \operatorname{Im} z(\xi). \quad (20)$$

Since $\operatorname{Re} z(\xi) \geq 0$, we have from (19)

$$\operatorname{Re} \Phi(\xi) \geq N_1 + N_2 = \frac{\nu_1^2}{(\nu_1 + \nu_2)^2} + \frac{\nu_2^2}{(\nu_1 + \nu_2)^2} > \frac{\nu_2^2}{(\nu_1 + \nu_2)^2}. \quad (21)$$

So far, let us focus on $|\Phi(\xi)|^2$, which we must prove to be bounded from above. We have from (19) and (20)

$$|\Phi(\xi)|^2 = [N_1 + N_2 + \psi_1(\xi) \cos \vartheta]^2 + [\psi_2(\xi) \sin \vartheta]^2, \quad (22)$$

where $\vartheta = \vartheta(\xi)$ is the argument of $z(\xi)$, and $\psi_1(\xi)$ and $\psi_2(\xi)$ are defined as

$$\psi_1(\xi) = N_1 |z(\xi)| + \frac{N_2}{|z(\xi)|} \quad (23)$$

and

$$\psi_2(\xi) = N_1 |z(\xi)| - \frac{N_2}{|z(\xi)|}. \quad (24)$$

Hence we have, for all ξ

$$|\Phi(\xi)|^2 \leq [N_1 + N_2 + \psi_1(\xi)]^2 + [\psi_2(\xi)]^2 = \Psi(\xi). \quad (25)$$

The left inequality in (17) entails $\psi_1(\xi) > 0$, as well as the right one entails $\psi_2(\xi) < 0$, for all $\xi \in \mathbf{R}$. More, since

$$\psi_1'(\xi) = \left[\frac{N_1 |z(\xi)|^2 - N_2}{|z(\xi)|^2} \right] \frac{d|z(\xi)|}{d\xi}$$

and

$$\psi_2'(\xi) = \left[\frac{N_1 |z(\xi)|^2 + N_2}{|z(\xi)|^2} \right] \frac{d|z(\xi)|}{d\xi},$$

the same argument shows that $\psi_1(\xi)$ is increasing in $(-\infty, 0)$ and decreasing in $(0, +\infty)$, while $\psi_2(\xi)$ behaves in the opposite way.

The function $\Psi(\xi)$ is therefore increasing in $(-\infty, 0)$ and decreasing in $(0, +\infty)$, as

$$\Psi'(\xi) = 2 [N_1 + N_2 + \psi_1(\xi)] \psi_1'(\xi) + 2 [\psi_2(\xi)] \psi_2'(\xi),$$

where the two terms on the right hand side have the same sign. This entails

$$\max_{\xi} |\Phi(\xi)|^2 \leq \Psi(0),$$

and we have to focus on the calculation of $\Psi(0)$, considering two different cases.

i) If $b_x \neq 0$, let us define $\eta := 4a/b_x^2$. We have

$$\Psi(0) = \left[N_1 \left(1 + \sqrt{\frac{1 + \eta\nu_2}{1 + \eta\nu_1}} \right) + N_2 \left(1 + \sqrt{\frac{1 + \eta\nu_1}{1 + \eta\nu_2}} \right) \right]^2 + \left[N_1 \sqrt{\frac{1 + \eta\nu_2}{1 + \eta\nu_1}} - N_2 \sqrt{\frac{1 + \eta\nu_1}{1 + \eta\nu_2}} \right]^2 \quad (26)$$

It can be easily verified that the right hand term is decreasing as a function of η : since η is positive, it attains its maximum when $\eta = 0$. This provides

$$\begin{aligned} \max_{\xi} |\Phi(\xi)|^2 &\leq (2N_1 + 2N_2)^2 + (N_1 - N_2)^2 \\ &= 5N_1^2 + 6N_1N_2 + 5N_2^2 \\ &= 5 \frac{\nu_1^4}{(\nu_1 + \nu_2)^4} + 6 \frac{\nu_1^2\nu_2^2}{(\nu_1 + \nu_2)^4} + 5 \frac{\nu_2^4}{(\nu_1 + \nu_2)^4}. \end{aligned} \quad (27)$$

So far, gathering together (21) and (27), we can conclude

$$\begin{aligned} \frac{\max_{\xi} |\Phi(\xi)|^2}{(\min_{\xi} \operatorname{Re} \Phi(\xi))^2} &\leq \frac{(\nu_1 + \nu_2)^4}{\nu_2^4} \left[5 \frac{\nu_1^4}{(\nu_1 + \nu_2)^4} + 6 \frac{\nu_1^2 \nu_2^2}{(\nu_1 + \nu_2)^4} + 5 \frac{\nu_2^4}{(\nu_1 + \nu_2)^4} \right] \\ &= 5 + 6 \left(\frac{\nu_1}{\nu_2} \right)^2 + 5 \left(\frac{\nu_1}{\nu_2} \right)^4 < 16, \end{aligned} \quad (28)$$

where the last inequality follows from the assumption $\nu_1 < \nu_2$.

ii) If $b_x = 0$, namely the flux term is parallel to the interface, $|z(0)| = \sqrt{\nu_2/\nu_1}$, and we have

$$\begin{aligned} \max_{\xi} |\Phi(\xi)|^2 &\leq \left[N_1 \left(1 + \sqrt{\frac{\nu_2}{\nu_1}} \right) + N_2 \left(1 + \sqrt{\frac{\nu_1}{\nu_2}} \right) \right]^2 + \left[N_1 \sqrt{\frac{\nu_2}{\nu_1}} - N_2 \sqrt{\frac{\nu_1}{\nu_2}} \right]^2 \\ &= \frac{1}{(\nu_1 + \nu_2)^4} [\nu_1^4 + \nu_2^4 + 2\nu_1^3\nu_2 + 2\nu_1\nu_2^3 + 2\nu_1^{7/2}\nu_2^{1/2} + 2\nu_1^2\nu_2^2 \\ &\quad + 2\nu_1^{3/2}\nu_2^{5/2} + 2\nu_1^{3/2}\nu_2^{5/2} + 2\nu_1^{1/2}\nu_2^{7/2}]. \end{aligned} \quad (29)$$

Gathering together (21) and (29), we get

$$\frac{\max_{\xi} |\Phi(\xi)|^2}{(\min_{\xi} \operatorname{Re} \Phi(\xi))^2} \leq 1 + 2 \sum_{n=1}^7 \left(\frac{\nu_1}{\nu_2} \right)^{n/2} + \left(\frac{\nu_1}{\nu_2} \right)^4 < 16. \quad (30)$$

A better estimate can be obtained when $b_y = 0$. The complex valued function $\Phi(\xi)$ reduces here to a real one

$$\Phi(\xi) = N_1 \left(1 + \frac{\sqrt{b_x^2 + 4a\nu_2 + 4\nu_2^2\xi^2}}{\sqrt{b_x^2 + 4a\nu_1 + 4\nu_1^2\xi^2}} \right) + N_2 \left(1 + \frac{\sqrt{b_x^2 + 4a\nu_1 + 4\nu_1^2\xi^2}}{\sqrt{b_x^2 + 4a\nu_2 + 4\nu_2^2\xi^2}} \right).$$

The function $\Phi(\xi)$ is symmetric in ξ , it can be easily proved that is decreasing in $[0, +\infty)$ and satisfies $\Phi(\xi) \geq 1$ for all ξ . Hence

$$\frac{\max_{\xi} |\Phi(\xi)|^2}{(\min_{\xi} \operatorname{Re} \Phi(\xi))^2} = \left[\frac{\max_{\xi} \Phi(\xi)}{\min_{\xi} \Phi(\xi)} \right]^2 \leq \left[\max_{\xi} \Phi(\xi) \right]^2 = [\Phi(0)]^2.$$

If $b_x \neq 0$, we define $\eta := 4a/b_x^2$, and we have

$$\Phi(0) = N_1 (1 + \varphi(\eta)) + N_2 \left(1 + \frac{1}{\varphi(\eta)} \right) \quad (31)$$

where

$$\varphi(\eta) := \sqrt{\frac{1 + \eta\nu_2}{1 + \eta\nu_1}}. \quad (32)$$

The right hand side in (31) is decreasing as a function of η , since

$$\varphi'(\eta) = \frac{1}{2} \sqrt{\frac{1 + \eta\nu_1}{1 + \eta\nu_2}} \frac{\nu_2 - \nu_1}{(1 + \eta\nu_1)^2} > 0$$

and it is not difficult to see that

$$\nu_1^2 [\varphi(\eta)]^2 - \nu_2^2 < 0.$$

Hence

$$\Phi(0) < N_1 (1 + \varphi(0)) + N_2 \left(1 + \frac{1}{\varphi(0)}\right). \quad (33)$$

We therefore have from (32) and (33)

$$\begin{aligned} \frac{\max_{\xi} \Phi(\xi)}{\min_{\xi} \Phi(\xi)} &< 2 \left[\frac{\nu_1}{\nu_1 + \nu_2} \right]^2 + 2 \left[\frac{\nu_2}{\nu_1 + \nu_2} \right]^2 \\ &= 2 \frac{\nu_1^2 + \nu_2^2}{(\nu_1 + \nu_2)^2} \\ &< 2. \end{aligned} \quad (34)$$

On the other hand, if $b_x = 0$ (*i.e.* if there is no convective term) we simply have

$$\begin{aligned} \Phi(0) &= \left[\frac{\nu_1}{\nu_1 + \nu_2} \right]^2 \left(1 + \sqrt{\frac{\nu_2}{\nu_1}}\right) + \left[\frac{\nu_2}{\nu_1 + \nu_2} \right]^2 \left(1 + \sqrt{\frac{\nu_1}{\nu_2}}\right) \\ &< \frac{1}{(\nu_1 + \nu_2)^2} [\nu_1^2 + \nu_2^2 + (\nu_1 + \nu_2) \sqrt{\nu_1 \nu_2}] \\ &\leq \frac{\nu_1^2 + \nu_2^2}{(\nu_1 + \nu_2)^2} + \frac{1}{2} \\ &< 1 + \frac{1}{2} < 2. \end{aligned} \quad (35)$$

□

We have therefore proved the following results.

Theorem 3.1 *Let \mathcal{T} be the operator defined in (6). In the case where the plane \mathbf{R}^2 is decomposed into the left and right half-planes and the convective field is uniform, the reduction factor for the associated GMRES preconditioned by \mathcal{T} can be bounded from above by a constant independent of the time step Δt , the convective field \vec{b} and the viscosity coefficients ν_1 and ν_2 .*

Proof. Straightforward from Lemma 3.1. \square

Corollary 3.1 *When the convective field is normal to the interface, we have*

$$\text{cond}(\mathcal{T} \circ \mathcal{S}) < 2. \quad (36)$$

Proof. When the convective field is perpendicular to the interface we have $b_y = 0$ and the symbol of the preconditioned operator $\Phi(\xi)$ is real. As a consequence, the condition number of $\mathcal{T} \circ \mathcal{S}$ can be evaluated as

$$\text{cond}(\mathcal{T} \circ \mathcal{S}) \sim \frac{\max_{\xi} \Phi(\xi)}{\min_{\xi} \Phi(\xi)},$$

and we conclude by (34)-(35). \square

Remark 3.2 The argument above is based only on the assumption $\nu_1 < \nu_2$, and it can be easily seen that a symmetric argument would give the same result as long as $\nu_2 < \nu_1$. Even more interesting, it appears that the reduction factor of the GMRES algorithm for the preconditioned system improves with the growth of the ratio ν_2/ν_1 . This allows the treatment of large discontinuities.

4 Variational Generalization

4.1 The continuous problem

Let us consider in \mathbf{R}^d (with $d = 2, 3$) the domain partition

$$\Omega = \bigcup_{k=1}^N \Omega_k,$$

with $\Omega_j \cap \Omega_k = \emptyset$ for $j \neq k$, on which we are solving the general advection-diffusion problem

$$\begin{aligned} -\text{div}(\nu(x)\nabla u) + \vec{b}(x) \cdot \nabla(u) + a(x)u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega_D \\ \nu(x)\frac{\partial u}{\partial n} &= \varphi && \text{on } \partial\Omega_N \end{aligned} \quad (37)$$

with piecewise constant viscosity

$$\nu(x) := \sum_{k=1}^N \nu_k \mathbf{1}_{\Omega_k}(x)$$

where $\mathbf{1}_{\Omega_k}$ is the characteristic function of the domain Ω_k .

In order to restrict ourselves to well-posed problems, we assume that the velocity field $\vec{b} \in W^{1,\infty}(\Omega)$ is of bounded divergence, that

$$a - \frac{1}{2}\operatorname{div}(\vec{b}) \geq \mu > 0,$$

for some $\mu \in \mathbf{R}$, and that the Neumann boundary conditions are given only on a subset $\partial\Omega_N$ of the domain boundary where we have outflow conditions,

$$\vec{b} \cdot \vec{n} \geq 0 \quad \forall x \in \partial\Omega_N,$$

where as usual \vec{n} denotes the unit vector normal to $\partial\Omega$ pointing outwards. The variational formulation of (37) reads

$$\text{Find } u \in \mathbb{H}(\Omega) : \quad a(u, v) = L(v) \quad \forall v \in \mathbb{H}(\Omega), \quad (38)$$

where

$$\mathbb{H}(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega_D} = 0\},$$

and

$$a(u, v) = \int_{\Omega} \nu \nabla u \nabla v + (\vec{b} \cdot \nabla u)v + auv,$$

$$L(v) = \int_{\Omega} fv + \int_{\partial\Omega_N} \varphi v.$$

In order to extend the sub-structuring technique discussed in the previous section to this general partitioning, we define the interfaces

$$\Gamma_k := \partial\Omega_k \setminus \partial\Omega, \quad \Gamma = \cup_k \Gamma_k,$$

and we have to describe the action of the advection-diffusion operator on each subdomain Ω_k . The simple restriction of the bilinear form $a(u, v)$ to Ω_k

$$\hat{a}_k(u, v) = \int_{\Omega_k} \nu_k \nabla u \nabla v + (\vec{b} \cdot \nabla u)v + auv$$

is not satisfactory because of its lack of positiveness. To overcome this problem, an integration by parts of the advective term $1/2(\vec{b}(x) \cdot \nabla u)v$ leads to the local symmetrized form

$$\begin{aligned} a_k(u, v) &:= \int_{\Omega_k} \nu_k \nabla u \nabla v + \frac{1}{2} \left[(\vec{b} \cdot \nabla u)v - (\vec{b} \cdot \nabla v)u \right] + \left(a - \frac{1}{2}\operatorname{div} \vec{b} \right) uv + \frac{1}{2} \int_{\partial\Omega_N \cap \partial\Omega_k} \vec{b} \cdot \vec{n}_k uv \\ &= \hat{a}_k(u, v) - \int_{\Gamma_k} \frac{1}{2} \vec{b} \cdot \vec{n}_k uv. \end{aligned}$$

Summing up on k , and letting

$$L_k(v) := \int_{\Omega_k} f v + \int_{\partial\Omega_N \cap \partial\Omega_k} \varphi v,$$

the variational problem (38) is equivalent to

$$\text{Find } u \in \mathbb{H}(\Omega) : \quad \sum_{k=1}^n \{a_k(u, v) - L_k(v)\} = 0 \quad \forall v \in \mathbb{H}(\Omega), \quad (39)$$

since the interface terms $-\int_{\Gamma_k} 1/2 \vec{b} \cdot \vec{n}_k u v$ added locally to each form \hat{a}_k cancel each other by summation. However, since we have by construction

$$a_k(u, u) = \int_{\Omega_k} \left\{ \nu_k |\nabla u|^2 + \left(a - \frac{1}{2} \operatorname{div} \vec{b} \right) u^2 \right\} + \int_{\partial\Omega_N \cap \partial\Omega_k} \frac{1}{2} \vec{b} \cdot \vec{n}_k u^2 \quad \forall u \in \mathbb{H}(\Omega_k)$$

where we have denoted with $\mathbb{H}(\Omega_k) = \{v_k = v|_{\Omega_k}, v \in \mathbb{H}(\Omega)\}$ the space of restrictions, their presence is very important since it guarantees that the local bilinear form $a_k(u, v)$ is positive on $\mathbb{H}(\Omega_k)$.

4.2 Finite Element Approximation

In order to get a numerical solution, the variational problem (39) above must be approximated by finite elements methods, which amounts to replace the space $\mathbb{H}(\Omega)$ with a suitable finite element space $\mathbb{H}_h(\Omega)$. We will use herein second order isoparametric finite elements defined on regular triangulations of Ω , as they are a good compromise between accuracy and cost-efficiency. Other choices are of course possible, but in any case the triangulations will respect the geometry of subdomain decomposition: the interfaces Γ_k will coincide with interelement boundaries, which means that each subdomain can be obtained as the union of a given subset of elements in the original triangulation.

When problem (37) is advection-dominated, these finite elements techniques must be stabilized. In the following we will use *Galerkin Least-Squares* techniques (*GALS*), but different choices (such as *Streamline Diffusion*) can be made. The *GALS* technique consists in adding to the original variational formulation the element residuals

$$\int_{T_i} \delta_i(h) \left(-\operatorname{div}(\nu(x)\nabla u) + \vec{b}(x) \cdot \nabla u + a(x)u - f \right) \left(-\operatorname{div}(\nu(x)\nabla v) + \vec{b}(x) \cdot \nabla v + a(x)v \right)$$

where T_i is an element of the triangulation, with a suitable choice of the local positive stabilization parameter $\delta_i(h)$. The stabilized finite elements formulation then reads

$$\text{Find } u_h \in \mathbb{H}_h(\Omega) : \quad \sum_{k=1}^n \{a_{kh}(u_h, v_h) - L_{kh}(v_h)\} = 0 \quad \forall v_h \in \mathbb{H}_h(\Omega), \quad (40)$$

where

$$\begin{aligned}
a_{kh}(u, v) &= a_k(u, v) \\
&+ \sum_{T_i \subset \Omega_k} \int_{T_i} \delta_i(h) \left(-\operatorname{div}(\nu_k \nabla u) + \vec{b} \cdot \nabla u + au \right) \left(-\operatorname{div}(\nu_k \nabla v) + \vec{b} \cdot \nabla v + av \right), \\
L_{kh}(v) &= L_k(v) + \sum_{T_i \subset \Omega_k} \int_{T_i} \delta_i(h) f \left(-\operatorname{div}(\nu_k \nabla v) + \vec{b} \cdot \nabla v + av \right).
\end{aligned}$$

Notice that the variational structure of the original problem and of its finite elements discretization are very similar. From now on, since this will be true also for the numerical domain decomposition we introduce in the following section, we will use the same notation for both the continuous and the discrete problem and omit all the subscripts h in all finite elements formulations. The reader should just remember that, when dealing with finite elements, the bilinear and linear forms $a_k(\cdot, \cdot)$ and $L_k(\cdot)$ should be replaced by their discrete counterparts $a_{kh}(\cdot, \cdot)$ and $L_{kh}(\cdot)$ as defined in the present section.

4.3 Substructuring

The variational structure of problems (39) and (40) allows to reduce them to an interface problem by means of standard substructuring techniques. For that purpose, following [2], we consider the local space of restrictions $\mathbb{H}(\Omega_k)$ defined in the previous section and we introduce the space

$$\mathbb{H}^0(\Omega_k) = \left\{ v_k \in \mathbb{H}(\Omega), v_k = 0 \text{ in } \overline{\Omega \setminus \Omega_k} \right\}$$

consisting of functions of $\mathbb{H}(\Omega_k)$ with zero continuous extension in $\overline{\Omega \setminus \Omega_k}$, the global trace space $\mathbb{V} = \operatorname{Tr} \mathbb{H}(\Omega)|_{\Gamma}$, the local trace spaces

$$\mathbb{V}_k = \left\{ \bar{v}_k = \operatorname{Tr} v_k|_{\Gamma_k}, v_k \in \mathbb{H}(\Omega_k) \right\} = \left\{ \bar{v}_k = \operatorname{Tr} v|_{\Gamma_k}, v \in \mathbb{H}(\Omega) \right\},$$

the restriction operators

$$R_k : \mathbb{H}(\Omega) \rightarrow \mathbb{H}(\Omega_k), \quad \bar{R}_k : \mathbb{V} \rightarrow \mathbb{V}_k,$$

the a_k -harmonic extension $\operatorname{Tr}_k^{-1} : \mathbb{V}_k \rightarrow \mathbb{H}(\Omega_k)$, defined as

$$a_k(\operatorname{Tr}_k^{-1} \bar{u}_k, v_k) = 0 \quad \forall v_k \in \mathbb{H}^0(\Omega_k), \quad \operatorname{Tr}(\operatorname{Tr}_k^{-1} \bar{u}_k)|_{\Gamma_k} = \bar{u}_k, \quad \operatorname{Tr}_k^{-1} \bar{u}_k \in \mathbb{H}(\Omega_k) \quad (41)$$

as well as its adjoint Tr_k^{-*} , defined by

$$a_k(v_k, \operatorname{Tr}_k^{-*} \bar{u}_k) = 0 \quad \forall v_k \in \mathbb{H}^0(\Omega_k), \quad \operatorname{Tr}(\operatorname{Tr}_k^{-*} \bar{u}_k)|_{\Gamma_k} = \bar{u}_k, \quad \operatorname{Tr}_k^{-*} \bar{u}_k \in \mathbb{H}(\Omega_k). \quad (42)$$

Since the bilinear form a_k is elliptic on $\mathbb{H}^0(\Omega_k)$ by construction, problems (41) and (42) are well-posed, and we can define the local Schur complement operator $S_k : \mathbb{V}_k \rightarrow \mathbb{V}'_k$ as

$$\langle S_k \bar{u}_k, \bar{v}_k \rangle = a_k(\text{Tr}_k^{-1} \bar{u}_k, \text{Tr}_k^{-*} \bar{v}_k) \quad \forall \bar{u}_k, \bar{v}_k \in \mathbb{V}_k.$$

If we decompose the local degrees of freedom U_k of $u_k = R_k u$ into internal (U_k^0) and interface (\bar{U}_k) degrees of freedom, the matrix A_k associated to the bilinear form a_k can be decomposed into

$$A_k = \begin{bmatrix} A_k^0 & B_k \\ \tilde{B}_k^T & \bar{A}_k \end{bmatrix},$$

and we have

$$\text{Tr}_k^{-1} = \begin{pmatrix} -(A_k^0)^{-1} B_k \\ Id \end{pmatrix},$$

as well as

$$S_k \bar{U}_k = \left(\bar{A}_k - \tilde{B}_k^T (A_k^0)^{-1} B_k \right) \bar{U}_k.$$

We can therefore decompose each restriction of the solution u and test function v into $R_k u = u_k^0 + \text{Tr}_k^{-1}(\bar{R}_k u)$ and $R_k v = v_k^0 + \text{Tr}_k^{-*}(\bar{R}_k v)$, and eliminate the local internal component u_k^0 since it is solution of the local well-posed problem

$$a_k(u_k^0, v_k) = L_k(v_k) \quad \forall v_k \in \mathbb{H}^0(\Omega_k), \quad u_k^0 \in \mathbb{H}^0(\Omega_k).$$

Thus, we can introduce the global Schur complement operator

$$S = \sum_{k=1}^N \bar{R}_k^T S_k \bar{R}_k$$

and we reduce problems (39) and (40) to the interface problem

$$S \bar{u} = F \quad \text{in } \mathbb{V}, \tag{43}$$

where the right-hand side is defined as

$$\begin{aligned} \langle F, \bar{v} \rangle &= \sum_k L_k(\text{Tr}_k^{-*}(\bar{R}_k \bar{v})) \\ &= \sum_k [L_k(v_k) - L_k(v_k - \text{Tr}_k^{-*}(\bar{R}_k \bar{v}))] \\ &= \sum_k [L_k(v_k) - a_k(u_k^0, v_k - \text{Tr}_k^{-*}(\bar{R}_k \bar{v}))] \quad (\text{construction of } u_k^0) \\ &= \sum_k [L_k(v_k) - a_k(u_k^0, v_k)] \quad (\text{definition of } \text{Tr}_k^{-*}), \end{aligned}$$

where v_k is any function in $\mathbb{H}(\Omega_k)$ such that $v_k = \bar{v}$ on Γ_k .

4.4 Definition of the preconditioner

The preconditioner we propose here for the solution of (43) is an extension of the ones proposed in [2] and [15] and a generalization of the one discussed in the previous section to an arbitrary number of subdomains. We precondition the interface operator $S = \sum_{k=1}^N \bar{R}_k^T S_k \bar{R}_k$ with a weighted sum of inverses:

$$T = \sum_{k=1}^N D_k^T (S_k)^{-1} D_k, \quad (44)$$

with

$$\sum_{k=1}^N D_k \bar{R}_k = Id_{\Gamma}. \quad (45)$$

Notice that, as well known in the domain decomposition literature, for any $F_k \in \mathbb{V}'_k$ the action of the operator $(S_k)^{-1} F_k$ is simply equal to the trace on Γ_k of the solution w_k of the local variational problem

$$a_k(w_k, v_k) = \langle F_k, Tr_k v_k \rangle \quad \forall v_k \in \mathbb{H}(\Omega_k), w_k \in \mathbb{H}(\Omega_k),$$

which, by construction of the bilinear form a_k , is associated to the operator

$$-\operatorname{div}(\nu_k \nabla w) + \vec{b} \cdot \nabla w + aw$$

with Robin boundary condition on the interface

$$\nu_k \frac{\partial w}{\partial n_k} - \frac{1}{2} \vec{b} \cdot \vec{n}_k w = F_k \quad \text{on } \Gamma_k.$$

In order to achieve good parallelization properties for the preconditioned algorithm, as the bilinear form changes with the subdomains, the weights D_k should be chosen as local as possible. The following section is dedicated to their construction.

4.4.1 Construction of the weights D_k

Since we have to take into account what happens in the neighborhood of each interface point, the map D_k is defined on each degree of freedom of the interface Γ_k . For $P \in \Gamma_k$ we define the set

$$J_P := \{j \in \{1, \dots, N\} \mid P \in \Gamma_j\},$$

consisting of all indices corresponding to the subdomains Ω_j whose interface boundary contains P . We define the weight D_k on the degree of freedom $\bar{u}(P)$ by

ν_1, ν_2	ν_1/ν_2	$\vec{b} = (\pm 1, 0, 0)$		$\vec{b} = (0, 1, 1)$	$\vec{b} = (\pm 1, 3, 5)$	
$10^{-1}, 10^{-5}$	10^4	10	11	17	15	17
$10^{-2}, 10^{-6}$		12	16	13	7	8
$10^{-1}, 10^{-6}$	10^5	10	11	17	15	17
$10^{-6}, 10^{-11}$		5	5	2	7	7
$10^{-1}, 10^{-7}$	10^6	10	11	17	15	17
$10^3, 10^{-3}$		3	3	3	3	3
$1, 10^{-7}$	10^7	6	7	9	11	11

Table 1: Number of iterations for the two-domain 3D model problem: $\text{res} < 10^{-10}$

$$D_k \bar{u}(P) = C_P \frac{\nu_k}{\sum_{j \in J_P} \nu_j} \bar{u}(P),$$

where the constant C_P is chosen in a suitable way in order to satisfy (45), and it depends only on the number of subdomains to which the point P belongs. As an example, consider a domain $\Omega \in \mathbf{R}^3$ decomposed into N parallelepipedal subdomains: if the point P is a vertex that belongs to 8 subdomains, the set J_P will consist of 8 indices and we choose $C_P = 1/3$, if P lies on a side which separates 4 subdomains, J_P consists of 4 indices and we choose $C_P = 1/2$, and finally if P belongs to a face and separates only two subdomains we choose $C_P = 1$.

5 Numerical results in three-dimensions

The advection-diffusion problem (37) is discretized by means of the stabilized Galerkin Least-Squares technique described in Section 4.2 using second order elements on an hexahedral decomposition. The interface problem (43) is solved by a GMRES algorithm preconditioned by the operator \mathcal{T} . The algorithm stops when the ℓ^2 norm on the interface of the initial residual is reduced by a factor of 10^{-10} .

5.1 A two-domains model problem

The first experiment deals with a partition of the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ into two subdomains $\Omega_1 = [0, 0.5] \times [0, 1] \times [0, 1]$ and $\Omega_2 = [0.5, 1] \times [0, 1] \times [0, 1]$. We choose different convective fields

- i) $\vec{b} = \vec{e}_1$: the velocity is perpendicular to the interface,
- ii) $\vec{b} = \vec{e}_2 + \vec{e}_3$: the velocity is parallel to the interface,
- iii) $\vec{b} = \vec{e}_1 + 3\vec{e}_2 + 5\vec{e}_3$: we refer to this velocity as “*oblique*”,

as well as $a = 1$. We consider large jumps between the viscosity coefficients, we choose $f \equiv 0$ in the whole Ω and we impose $u = 1$ on the bottom face of the cube as well as

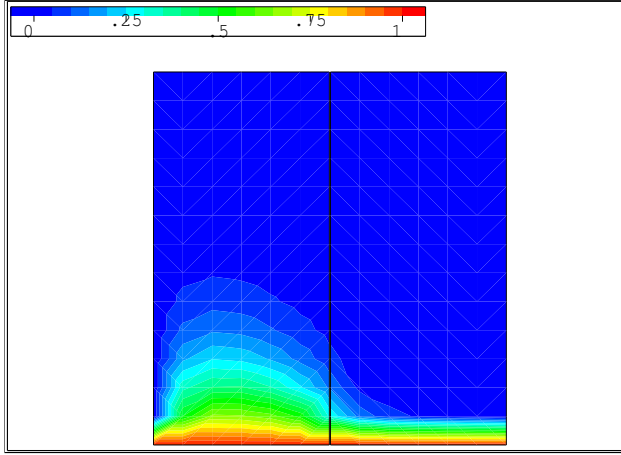


Figure 1: $\vec{b} = (-1, 0, 0)$, $\nu_1 = 10^{-1}$, $\nu_2 = 10^{-6}$. Section: $y = 0.5$.

homogeneous Dirichlet conditions on the rest of the boundary $\partial\Omega$.

The total number of finite elements is 1728, the total number of degrees of freedom is 14023 and the number of degrees of freedom on the interface is 625. The number of iterations is reported in Table 1: when two results are present, the first one refers to a convective field directed from the more viscous region to the less viscous one, while the second refers to the opposite case. The results show that the preconditioner is almost insensitive to the choice of the convective fields, although it performs slightly better when the flux is directed towards the less viscous region. Nevertheless, a strong improvement in the number of iterations is observed when one of the two subproblems is not advection-dominated as well as when both subdomains have very little viscosity. However, the number of iterations is reasonable in all cases and it appears to be, as we expected from the theory, fairly insensitive to the viscosity jumps. Finally, we have represented in Figure 1 and Figure 2 two cross-sections (which take into account the direction of the convective field) of the results for $\vec{b} = (-1, 0, 0)$, $\nu_1 = 10^{-1}$, $\nu_2 = 10^{-6}$ and for $\vec{b} = (1, 3, 5)$, $\nu_1 = 1$, $\nu_2 = 10^{-7}$ respectively; in both cases, Ω_1 is on the left side of the figure.

5.2 Influence of the number of subdomains

We investigate here the robustness of the preconditioner with respect to the number of subdomains and to their mutual position. We consider the cube $\Omega = [-0.5, 0.5] \times [-0.5, 0.5] \times [0, 1]$ partitioned into 8 subdomains, numbered in a clockwise helicoidal way from $\Omega_1 = [-0.5, 0] \times [-0.5, 0] \times [0, 0.5]$ to $\Omega_8 = [0, 0.5] \times [-0.5, 0] \times [0.5, 1]$. We consider the velocity field

$$\vec{b} = -2\pi y \vec{e}_1 + 2\pi x \vec{e}_2 + \sin(2\pi x) \vec{e}_3.$$

and we consider the cube as constituted of two different materials disposed in the following ways:

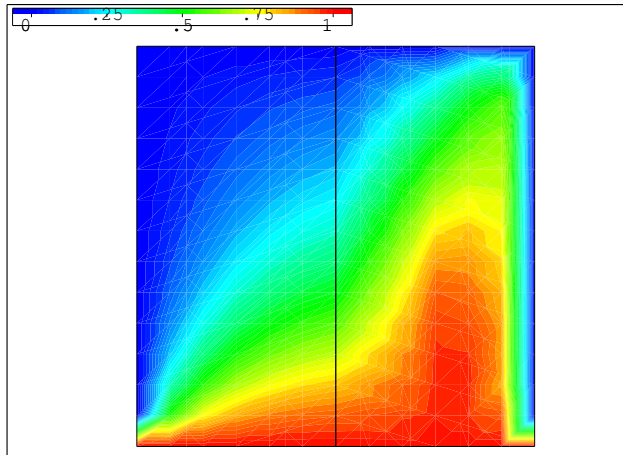


Figure 2: $\vec{b} = (1, 3, 5)$, $\nu_1 = 1, \nu_2 = 10^{-7}$. Section: $3x - y = 0.5$.

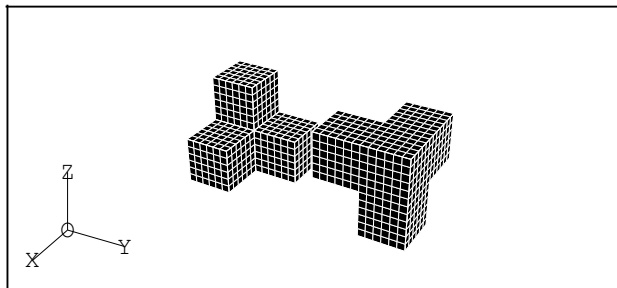


Figure 3: The subdomains Ω_1 (left) and Ω_2 (right) in Test 2.

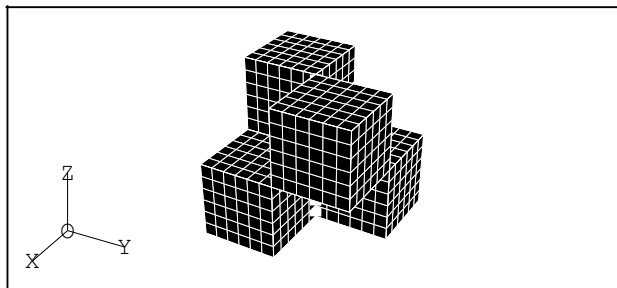


Figure 4: The domain Ω_2 in Test 3.

ν_1, ν_2	ν_1/ν_2	Test 1	Test 2	Test 3
$10^{-1}, 10^{-5}$	10^4	33	33	34
$10^{-1}, 10^{-6}$	10^5	32	33	34
$10^{-1}, 10^{-7}$	10^6	32	33	34
$10^3, 10^{-3}$	10^6	29	28	21
$1, 10^{-7}$	10^7	29	31	29

Table 2: Number of iterations for the multidomain model problem: $\text{res} < 10^{-10}$

ν_1	$\nu_2, \nu_4, \nu_5, \nu_7$	ν_3	ν_6	ν_8	ITER
10^{-1}	10^{-6}	10^{-2}	10^{-3}	10^{-4}	34

Table 3: Number of iterations in Test 4. Residual $< 10^{-10}$

Test 1: $\nu_1 = \nu_4 = \nu_5 = \nu_8$, and $\nu_2 = \nu_3 = \nu_6 = \nu_7$: this is the configuration considered in the previous section, but each physical domain here is decomposed into four smaller subdomains.

Test 2: $\nu_1 = \nu_5 = \nu_6 = \nu_8$, and $\nu_2 = \nu_3 = \nu_4 = \nu_7$: the homogeneous subdomains Ω_1 and Ω_2 are shown in Figure 3.

Test 3: $\nu_1 = \nu_3 = \nu_6 = \nu_8$, and $\nu_2 = \nu_4 = \nu_5 = \nu_7$: this case is a black and white decomposition where each subdomain of one kind is surrounded by subdomains of the other one. Figure 4 shows Ω_2 .

Test 4: We choose $\nu_1 = 10^{-1}$, $\nu_3 = 10^{-2}$, $\nu_6 = 10^{-3}$, $\nu_8 = 10^{-4}$ and $\nu_2 = \nu_4 = \nu_5 = \nu_7 = 10^{-6}$.

We choose again $f \equiv 0$ in the whole Ω , and Dirichlet conditions $u = 1$ on the bottom face and $u = 0$ on the rest of $\partial\Omega$. The total number of finite elements is still 1728, the total number of degrees of freedom 14023, but the number of interface degrees of freedom has raised to 1801. We report in Table 2 the number of iterations, and we observe that the preconditioner is sensitive to the number of subdomains (33 against 20), but it appears once again insensitive to the jumps in the viscosity coefficients. Finally, the preconditioned system is not affected by the larger number of different viscosity coefficients (see the results of Test 4 in Table 3).

5.3 A three layers model problem

The third experiment deals with a parallelepipedal domain $\Omega = [0, 1.5] \times [0, 1] \times [0, 1]$ which is partitioned into three layers $\Omega_1 = [0, 0.5] \times [0, 1] \times [0, 1]$, $\Omega_2 = [0.5, 1] \times [0, 1] \times [0, 1]$, $\Omega_3 = [1, 1.5] \times [0, 1] \times [0, 1]$. It is a very simplified model of transport and diffusion of a species through different layers of materials. For instance, a three layer model arises from the far field modeling of a nuclear waste disposal, where the repositories are stocked into a central layer of clay, surrounded below by dogger and above by limestone and marl,

Partition	NE	NDF	NIDF	ITER
$3 \times 1 \times 1$	2808	14275	838	13

Table 4: A three layers model problem

whose viscosity coefficients are very different from each other (in the case of Iodine¹²⁹, for instance, these are $9.48 \cdot 10^{-7}$ m²/year in the clay and $5 \cdot 10^{-4}$ m²/year in the other materials). We choose $\nu_1 = \nu_3 = 0.1$, $\nu_2 = 10^{-4}$, a discontinuous convective field given by

$$\begin{cases} \vec{b} = 3\vec{e}_2 - 2\vec{e}_3 & \text{in } \Omega_1 \text{ and } \Omega_3 \\ \vec{b} = -\vec{e}_1 & \text{in } \Omega_2, \end{cases}$$

and a discontinuous reaction term given by

$$\begin{cases} a = .001 & \text{in } \Omega_1 \text{ and } \Omega_3 \\ a = .1 & \text{in } \Omega_2. \end{cases}$$

We choose $f \equiv 1$ and we impose the following boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial n} = 0 & \text{on } [0, 0.5] \times [0, 1] \times \{0\}, [1, 1.5] \times [0, 1] \times \{0\} \text{ and } \{0\} \times [0, 1] \times [0, 1] \\ u = 0 & \text{elsewhere} \end{cases}$$

We report in Table 4 the total number of finite elements (NE), the total number of degrees of freedom (NDF), the number of interface degrees of freedom (NIDF) and the number of iterations (ITER). Once again the result is quite satisfactory: discontinuity in all coefficients appears not to affect the performance of the preconditioner.

6 Conclusions

The proposed preconditioner is a generalization of the Robin-Robin preconditioner to advection-diffusion problems with discontinuous coefficients. We have shown its robustness, assessed first theoretically by a Fourier analysis in the special case of the two half-planes, then by some numerical tests in 3D, where the preconditioner has shown fair insensitivity to the jumps in the viscosity coefficients as well as to the convective field. It remains sensitive to the number of subdomains, but this seems unavoidable in the case of advection-dominated problems without coarse grid correction, although far less spectacular than for the case of pure diffusion. The preconditioner transforms naturally into the Robin-Robin one when the viscosity is continuous, and, probably being its most interesting feature, it has the same algebraic structure as this latter one. Therefore it can be easily

implemented into a software which contains the Robin-Robin or the Neumann-Neumann preconditioner.

However, if the extension to systems of advection-diffusion equations appears to be quite straightforward, further work is required in the following directions:

- a convergence analysis in a more general setting is not yet available (and it appears to be quite difficult)
- the extension to the case of discontinuous convective fields should be addressed
- the introduction of a coarse space to reduce the sensitivity to the number of subdomains should be analyzed
- the algorithm should be tested on more complex situations

Acknowledgements

This work was made during the first author's visit at CMAP of the École Polytechnique (Palaiseau, France). The authors and especially the first one would like to thank M. Vidrascu at the INRIA in Rocquencourt for many valuable suggestions and comments: without her precious help the numerical tests up above would have barely seen the light.

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