A New Cement to Glue Nonconforming Grids with Robin Interface Conditions: The Finite Element Case

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Summary. We present and analyze a new nonconforming domain decomposition method based on a Schwarz method with Robin transmission conditions. We prove that the method is well posed and convergent. Our error analysis is valid in two dimensions for piecewise polynomials of low and high order and also in three dimensions for P_1 elements. We further present an efficient algorithm in two dimensions to perform the required projections between arbitrary grids. We finally illustrate the new method with numerical results.

1 Introduction

We propose a domain decomposition method based on the Schwarz algorithm that permits the use of optimized interface conditions on nonconforming grids. Such interface conditions have been shown to be a key ingredient for efficient domain decomposition methods in the case of conforming approximations (see Després [1991], Nataf et al. [1995], Japhet [1998], Chevalier and Nataf [1998]). Our goal is to use these interface conditions on nonconforming grids, because this simplifies greatly the parallel generation and adaptation of meshes per subdomain. The mortar method, first introduced in Bernardi et al. [1994], also permits the use of nonconforming grids, and it is well suited to the use of "Dirichlet-Neumann" (Gastaldi et al. [1996]) or "Neumann-Neumann" methods applied to the Schur complement matrix. But the mortar method can not be used easily with optimized transmission conditions in the framework of Schwarz methods. In Achdou et al. [2002], the case of finite volume discretizations has been introduced and analyzed. This paper is a first step in the finite element case; we consider only interface conditions of order 0 here.

2 Definition of the method and the iterative solver

We consider the model problem

$$\begin{aligned} (Id - \Delta)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where f is given in $L^2(\Omega)$ and Ω is a $\mathcal{C}^{1,1}$ (or convex) domain in $\mathbb{R}^d$, $d = 2$ or 3. We assume that it is decomposed into K non-overlapping subdomains $\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}^k$, where Ω_k, $1 \leq k \leq K$ are $\mathcal{C}^{1,1}$ or convex polygons in two or polyhedrons in three dimensions. We also assume that this domain decomposition is conforming. Let $\mathbf{n}_k$ be the unit outward normal for Ω^k and $\Gamma^{k,\ell} = \partial\Omega^k \cap \partial\Omega^\ell$.

The variational statement of problem (1) consists of writing the problem as follows: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (\nabla u \nabla v + uv)\, dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega). \tag{2}$$

We introduce now the space $H_*^1(\Omega^k) = \{\varphi \in H^1(\Omega^k),\ \varphi = 0 \text{ over } \partial\Omega \cap \partial\Omega^k\}$, and the constrained space

$$\mathcal{V} = \{(\mathbf{v}, \mathbf{q}) \in (\prod_{k=1}^{K} H_*^1(\Omega^k)) \times (\prod_{k=1}^{K} H^{-1/2}(\partial\Omega^k)),\ v_k = v_\ell \text{ and } q_k = -q_\ell \text{ on } \Gamma^{k,\ell}\}.$$

Problem (2) is then equivalent to the following: Find $(\mathbf{u}, \mathbf{p}) \in \mathcal{V}$ such that

$$\begin{aligned} \sum_{k=1}^{K} \int_{\Omega^k} (\nabla u_k \nabla v_k + u_k v_k)\, dx - \sum_{k=1}^{K} {}_{H^{-1/2}(\partial\Omega^k)} < p_k, v_k >_{H^{1/2}(\partial\Omega^k)} \\ = \sum_{k=1}^{K} \int_{\Omega^k} f_k v_k dx, \quad \forall \mathbf{v} \in \prod_{k=1}^{K} H_*^1(\Omega^k). \end{aligned}$$

Being equivalent with the original problem, where $p_k = \frac{\partial u}{\partial \mathbf{n}_k}$ over $\partial\Omega^k$, this problem is naturally well posed. We now describe the iterative procedure in the continuous case, and then its discrete, non-conforming analog.

2.1 The continuous case

We introduce for $\alpha \in \mathbb{R}$, $\alpha > 0$, the zeroth order transmission condition

$$p_k + \alpha u_k = -p_\ell + \alpha u_\ell \quad \text{over } \Gamma^{k,\ell}$$

and the following algorithm: let $(u_k^n, p_k^n) \in H_*^1(\Omega^k) \times H^{-1/2}(\partial\Omega^k)$ be an approximation of (u, p) in Ω^k at step n. Then, (u_k^{n+1}, p_k^{n+1}) is the solution in $H_*^1(\Omega^k) \times H^{-1/2}(\partial\Omega^k)$ of

$$\begin{aligned}\int_{\Omega^k}\left(\nabla u_k^{n+1}\nabla v_k + u_k^{n+1}v_k\right)dx - &< p_k^{n+1}, v_k >_{H^{-1/2}(\partial\Omega^k)} \\ &= \int_{\Omega^k} f_k v_k dx, \quad \forall v_k \in H^1_*(\Omega^k), \\ < p_k^{n+1} + \alpha u_k^{n+1}, v_k >_{\Gamma^{k,\ell}} = &< -p_\ell^n + \alpha u_\ell^n, v_k >_{\Gamma^{k,\ell}}, \quad \forall v_k \in H^{1/2}_{00}(\Gamma^{k,\ell}).\end{aligned} \tag{3}$$

Convergence of this algorithm is shown in Després [1991] using energy estimates and summarized in the following

Theorem 1. *Assume that f is in $L^2(\Omega)$ and $(p_k^0)_{1\le k\le K} \in \prod_\ell H^{1/2}(\Gamma^{k,\ell})$. Then, algorithm (3) converges in the sense that*

$$\lim_{n\longrightarrow\infty}\left(\|u_k^n - u_k\|_{H^1(\Omega^k)} + \|p_k^n - p_k\|_{H^{-1/2}(\partial\Omega^k)}\right) = 0, \quad \textit{for } 1\le k\le K,$$

where u solves (1), $u_k = u_{|\Omega^k}$, $p_k = \frac{\partial u_k}{\partial \mathbf{n}_k}$ on $\partial\Omega^k$ for $1\le k\le K$.

2.2 The discrete case

We introduce now the discrete spaces: each Ω_k is provided with its own mesh $\mathcal{T}_h^k$, $1\le k\le K$, such that $\overline{\Omega}_k = \cup_{T\in\mathcal{T}_h^k} T$. For $T\in\mathcal{T}_h^k$, let h_T be the diameter of T and h the discretization parameter, $h = \max_{1\le k\le K}(\max_{T\in\mathcal{T}_h^k} h_T)$. Let ρ_T be the diameter of the circle in two dimensions or sphere in three dimensions inscribed in T. We suppose that $\mathcal{T}_h^k$ is uniformly regular: there exists σ and τ independent of h such that $\forall T\in\mathcal{T}_h^k$, $\sigma_T \le \sigma$ and $\tau h \le h_T$. We consider that the sets belonging to the meshes are of simplicial type (triangles or tetrahedra), but the following analysis can be applied as well for quadrangular or hexahedral meshes. Let $\mathcal{P}_M(T)$ denote the space of all polynomials defined over T of total degree less than or equal to M for our Lagrangian finite elements. Then, we define over each subdomain two conforming spaces Y_h^k and X_h^k by

$$\begin{aligned}Y_h^k &= \{v_{h,k}\in\mathcal{C}^0(\overline{\Omega}_k),\ v_{h,k|T}\in\mathcal{P}_M(T),\ \forall T\in\mathcal{T}_h^k\},\\ X_h^k &= \{v_{h,k}\in Y_h^k,\ v_{h,k|\partial\Omega_k\cap\partial\Omega} = 0\}.\end{aligned} \tag{4}$$

The space of traces over each $\Gamma^{k,\ell}$ of elements of Y_h^k is denoted by $\mathcal{Y}_h^{k,\ell}$. In the sequel we assume for the sake of simplicity that referring to a pair (k,ℓ) implies that $\Gamma^{k,\ell}$ is not empty. With each such interface we associate a subspace $\tilde{W}_h^{k,\ell}$ of $\mathcal{Y}_h^{k,\ell}$ like in the mortar element method; for two dimensions, see Bernardi et al. [1994], and for three dimensions see Belgacem and Maday [1997] and Braess and Dahmen [1998]. To be more specific, we recall the situation in two dimensions: if the space X_h^k consists of continuous piecewise polynomials of degree $\le M$, then it is readily noticed that the restriction of X_h^k to $\Gamma^{k,\ell}$ consists of finite element functions adapted to the (possibly curved) side $\Gamma^{k,\ell}$ of piecewise polynomials of degree $\le M$. This side has two end points which we denote by $x_0^{k,\ell}$ and $x_n^{k,\ell}$ and which belong to the set of vertices of the

corresponding triangulation of $\Gamma^{k,\ell}$: $x_0^{k,\ell}, x_1^{k,\ell}, \ldots, x_{n-1}^{k,\ell}, x_n^{k,\ell}$. The space $\tilde{W}_h^{k,\ell}$ is then the subspace of those elements of $\mathcal{Y}_h^{k,\ell}$ that are polynomials of degree $\leq M-1$ over both $[x_0^{k,\ell}, x_1^{k,\ell}]$ and $[x_{n-1}^{k,\ell}, x_n^{k,\ell}]$. As before, the space $\tilde{W}_h^k$ is the product space of the $\tilde{W}_h^{k,\ell}$ over each ℓ such that $\Gamma^{k,\ell} \neq \emptyset$.

The discrete constrained space is then defined by

$$\mathcal{V}_h = \{(\mathbf{u}_h, \mathbf{p}_h) \in (\prod_{k=1}^{K} X_h^k) \times (\prod_{k=1}^{K} \tilde{W}_h^k),$$

$$\int_{\Gamma^{k,\ell}} ((p_{h,k} + \alpha u_{h,k}) - (-p_{h,\ell} + \alpha u_{h,\ell}))\psi_{h,k,\ell} = 0, \ \forall \psi_{h,k,\ell} \in \tilde{W}_h^{k,\ell}\},$$

and the discrete problem is the following: Find $(\mathbf{u}_h, \mathbf{p}_h) \in \mathcal{V}_h$ such that $\forall \mathbf{v}_h = (v_{h,1}, \ldots v_{h,K}) \in \prod_{k=1}^{K} X_h^k$,

$$\sum_{k=1}^{K} \int_{\Omega^k} (\nabla u_{h,k} \nabla v_{h,k} + u_{h,k} v_{h,k})\, dx - \sum_{k=1}^{K} \int_{\partial\Omega^k} p_{h,k} v_{h,k} ds = \sum_{k=1}^{K} \int_{\Omega^k} f_k v_{h,k} dx. \quad (5)$$

The discrete algorithm is then as follows: let $(u_{h,k}^n, p_{h,k}^n) \in X_h^k \times \tilde{W}_h^k$ be a discrete approximation of $(\mathbf{u}, \mathbf{p})$ in Ω^k at step n. Then, $(u_{h,k}^{n+1}, p_{h,k}^{n+1})$ is the solution in $X_h^k \times \tilde{W}_h^k$ of

$$\int_{\Omega^k} \Big(\nabla u_{h,k}^{n+1} \nabla v_{h,k} + u_{h,k}^{n+1} v_{h,k}\Big) dx - \int_{\partial\Omega^k} p_{h,k}^{n+1} v_{h,k} ds = \int_{\Omega^k} f_k v_{h,k} dx, \ \forall v_{h,k} \in X_h^k, \quad (6)$$

$$\int_{\Gamma^{k,\ell}} (p_{h,k}^{n+1} + \alpha u_{h,k}^{n+1}) \psi_{h,k,\ell} = \int_{\Gamma^{k,\ell}} (-p_{h,\ell}^{n} + \alpha u_{h,\ell}^{n}) \psi_{h,k,\ell}, \ \forall \psi_{h,k,\ell} \in \tilde{W}_h^{k,\ell}. \quad (7)$$

Remark 1. Let $\pi_{k,\ell}$ denote the orthogonal projection operator from $L^2(\Gamma^{k,\ell})$ onto $\tilde{W}_h^{k,\ell}$. Then (7) corresponds to

$$p_{h,k}^{n+1} + \alpha \pi_{k,\ell}(u_{h,k}^{n+1}) = \pi_{k,\ell}(-p_{h,\ell}^{n} + \alpha u_{h,\ell}^{n}) \quad \text{over } \Gamma^{k,\ell}. \quad (8)$$

Remark 2. A fundamental difference between this method and the original mortar method in Bernardi et al. [1994] is that the interface conditions are chosen in a symmetric way: there is no master and no slave, see also Gander et al. [2001]. Equation (8) is the transmission condition on $\Gamma^{k,\ell}$ for Ω^k, and the transmission condition on $\Gamma^{k,\ell}$ for Ω^ℓ is

$$p_{h,\ell}^{n+1} + \alpha \pi_{\ell,k}(u_{h,\ell}^{n+1}) = \pi_{\ell,k}(-p_{h,k}^{n} + \alpha u_{h,k}^{n}) \quad \text{over } \Gamma^{k,\ell}. \quad (9)$$

In order to analyze the convergence of this iterative scheme, we define for any $\mathbf{p}$ in $\prod_{k=1}^{K} L^2(\partial\Omega_k)$ the norm

$$\|\mathbf{p}\|_{-\frac{1}{2},*} = (\sum_{k=1}^{K} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{K} \|p_k\|^2_{H_*^{-\frac{1}{2}}(\Gamma^{k,\ell})})^{\frac{1}{2}},$$

where $\|.\|_{H_*^{-\frac{1}{2}}(\Gamma^{k,\ell})}$ stands for the dual norm of $H_{00}^{\frac{1}{2}}(\Gamma^{k,\ell})$. Convergence of the algorithm (6)-(7) can be shown again using an energy estimate, see Japhet et al. [2003].

Theorem 2. *Assume that $\alpha h \leq c$ for some constant c small enough. Then, the discrete problem (5) has a unique solution $(\mathbf{u}_h, \mathbf{p}_h) \in \mathcal{V}_h$. The algorithm (6)-(7) is well posed and converges in the sense that*

$$\lim_{n \longrightarrow \infty} (\|u_{h,k}^n - u_{h,k}\|_{H^1(\Omega^k)} + \sum_{\ell \neq k} \|p_{h,k}^n - p_{h,k}\|_{H_*^{-\frac{1}{2}}(\Gamma^{k,\ell})}) = 0, \text{ for } 1 \leq k \leq K.$$

3 Best approximation properties

In this part we give best approximation results of $(\mathbf{u}, \mathbf{p})$ by elements in $\mathcal{V}_h$. The proofs can be found in Japhet et al. [2003] for the two dimensional case with the degree of the finite element approximations $M \leq 13$ and in three dimensions for first order approximations.

Theorem 3. *Assume that the solution $\mathbf{u}$ of (1) is in $H^2(\Omega) \cap H_0^1(\Omega)$ and $u_k = \mathbf{u}_{|\Omega^k} \in H^{2+m}(\Omega^k)$ with $M - 1 \geq m \geq 0$, and let $p_{k,\ell} = \frac{\partial u}{\partial \mathbf{n}_k}$ over each $\Gamma^{k,\ell}$. Then, there exists a constant c independent of h and α such that*

$$\|\mathbf{u}_h - \mathbf{u}\|_* + \|\mathbf{p}_h - \mathbf{p}\|_{-\frac{1}{2},*} \leq c(\alpha h^{2+m} + h^{1+m}) \sum_{k=1}^{K} \|\mathbf{u}\|_{H^{2+m}(\Omega_k)} + c(\frac{h^m}{\alpha} + h^{1+m}) \sum_{k=1}^{K} \sum_{\ell} \|p_{k,\ell}\|_{H^{\frac{1}{2}+m}(\Gamma^{k,\ell})}.$$

Assuming more regularity on the normal derivatives on the interfaces, we have

Theorem 4. *Assume that the solution $\mathbf{u}$ of (1) is in $H^2(\Omega) \cap H_0^1(\Omega)$ and $u_k = \mathbf{u}_{|\Omega^k} \in H^{2+m}(\Omega^k)$ with $M - 1 \geq m \geq 0$, and $p_{k,\ell} = \frac{\partial u}{\partial \mathbf{n}_k}$ is in $H^{\frac{3}{2}+m}(\Gamma_{k,\ell})$. Then there exists a constant c independent of h and α such that*

$$\|\mathbf{u}_h - \mathbf{u}\|_* + \|\mathbf{p}_h - \mathbf{p}\|_{-\frac{1}{2},*} \leq c(\alpha h^{2+m} + h^{1+m}) \sum_{k=1}^{K} \|\mathbf{u}\|_{H^{2+m}(\Omega_k)} + c(\frac{h^{1+m}}{\alpha} + h^{2+m})(\log h)^{\beta(m)} \sum_{k=1}^{K} \sum_{\ell} \|p_{k,\ell}\|_{H^{\frac{3}{2}+m}(\Gamma^{k,\ell})}.$$

4 Efficient projection algorithm

The projection (8) between non conforming grids is not an easy task in an algorithm, already for two dimensional problems, since one needs to find the

intersections of corresponding arbitrary grid cells. A short and efficient algorithm has been proposed in Gander et al. [2001] in the finite volume case with projections on piecewise constant functions. In our case, we denote by n the dimension of $W_h^{k,\ell}$, and we introduce the shape functions $\{\psi_i^{k,\ell}\}_{1\le i\le n}$ of $W_h^{k,\ell}$. Then, to compute the right hand side in (7), we need to compute the interface matrix

$$M = (\int_{\Gamma^{k,\ell}} \psi_i^{k,\ell}\psi_j^{\ell,k})_{1\le i,j\le n}.$$

In the same spirit as in Gander et al. [2001], the following short algorithm in Matlab computes the interface matrix M for non-matching grids in one pass.

```
function M=InterfaceMatrix(ta,tb);
n=length(tb);
m=length(ta);
ta(m)=tb(n);                          % must be numerically equal
j=1;
M=zeros(n,length(ta));
for i=1:n-1,
  tm=tb(i);
  while ta(j+1)<tb(i+1),
    M(i:i+1,j:j+1)=M(i:i+1,j:j+1)+intMortar(ta(j),ta(j+1),...
      tb(i),tb(i+1),tm,ta(j+1),j==1|j==m-1,i==1|i==n-1);
    j=j+1;
    tm=ta(j);
  end;
  M(i:i+1,j:j+1)=M(i:i+1,j:j+1)+intMortar(ta(j),ta(j+1),...
    tb(i),tb(i+1),tm,tb(i+1),j==1|j==m-1,i==1|i==n-1);
end;
```

It takes two vectors `ta` and `tb` with ordered entries, which represent two non-matching grids at the interface, with `ta(1)=tb(1)`, `ta(end)=tb(end)`, and computes the matrix `M(i,j)`$=\int_{\Gamma^{k,\ell}} b^i a^j$, where b^i is the hat function for the node `tb(i)` and a^j is the hat function for the node `ta(j)`. The mortar condition of constant shape functions at the corners is taken into account, and from the resulting matrix M the first and last row and column needs to be removed. This algorithm has linear complexity; it does a single pass without any special cases or any additional grid. It advances automatically on whatever side the next cell boundary is coming and handles any possible cases of non-matching grids at a one dimensional interface.

5 Numerical results

On the unit square $\Omega = (0,1)\times(0,1)$ we consider the problem

$$(Id - \Delta)u(x,y) = x^3(y^2-2) - 6xy^2 + (1+x^2+y^2)sin(xy), \quad (x,y)\in\Omega,$$
$$u = x^3y^2 + sin(xy), \quad (x,y)\in\partial\Omega,$$

whose exact solution is $u(x,y) = x^3y^2 + sin(xy)$. We decompose the unit square into four non-overlapping subdomains with meshes generated in an independent manner, as shown in Figure 1 on the left. The computed solution

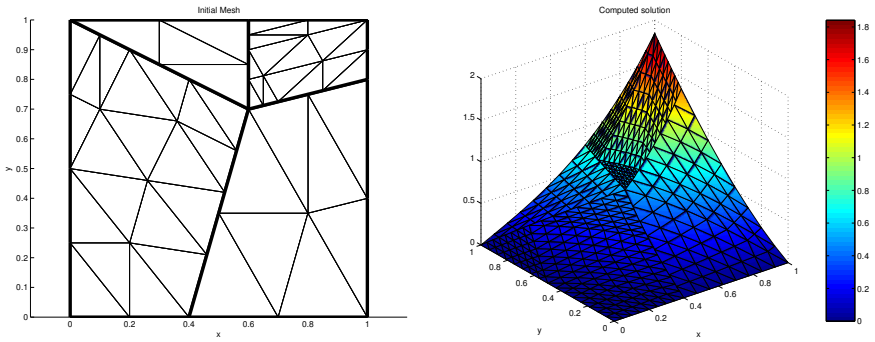

Fig. 1. Initial mesh and computed solution after two refinements.

is the solution at convergence of the discrete algorithm (6)-(7), with stopping criterion $\max_{k,\ell/\Gamma^{k,\ell}\neq\emptyset}\left(\int_{\Gamma^{k,\ell}}((p_{h,k}+\alpha u_{h,k})-(-p_{h,\ell}+\alpha u_{h,\ell}))\psi_{k,\ell}\right) < 10^{-8}$, and $\alpha = 10$. On Figure 1 on the right, we show the computed solution.

Figure 2 on the left corresponds to the best approximation error of Theorem 4. On the right, we compare in the case of two subdomains the optimal

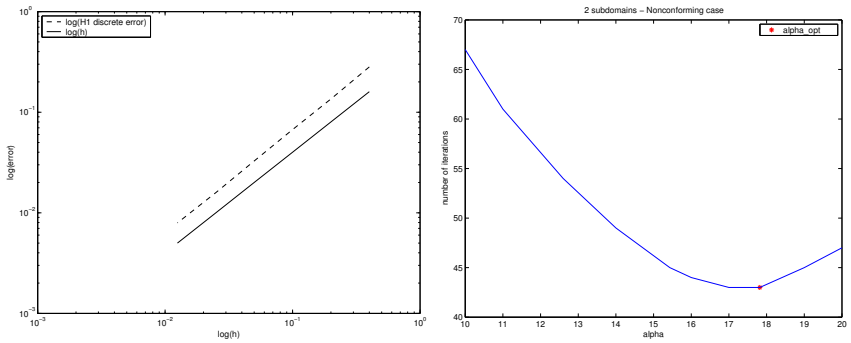

Fig. 2. H^1 error versus h on the left and number of iterations versus α on the right.

numerical α to the theoretical value, which minimizes the convergence rate at the continuous level: $\alpha_{opt} = [(\pi^2+1)((\frac{\pi}{h_{min}})^2+1)]^{\frac{1}{4}}$. The nonconforming meshes have 289 and 561 nodes respectively, and the discretization parameters are $h_1 = 0.065$ and $h_2 = 0.032$. We observe that the optimal numerical α is very close to α_{opt}.

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