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A new construction of perfectly matched layers for the linearized Euler equations

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A new construction of perfectly matched layers for the linearized Euler equations

Frédéric Nataf*

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Abstract

Based on a PML for the advective wave equation, we propose two PML models for the linearized Euler equations. The derivation of the first model can be applied to other physical models. The second model was implemented. Numerical results are shown.

Contents

1 Introduction

Since the work by Berenger on perfectly matched layer for the Maxwell equations [Ber94] and [Ber96] in a computational box, many works have been devoted to a better understanding of their principle and behaviour see [MPV98], [ZC96], [CW94], [LS00] [MC98] [BFJ03][BJ02] [AGH02] to extensions to other geometries, see [ST04] [CM98], [Pet00] or equations see [HN02] [AGH99][DJ03] . We consider here the linearized Euler equations which has been the subject of many works, see [Rah04], [Hu01], [TAC98] [Hes98] [Hu96] [Hag03] (and references therein) but is still a challenging problem for oblique flows. One of the key difficulty is the possible instability of the vorticity waves. We address this question and propose two ways to design PML for the

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Euler equations that are based on the use of a PML for the underlying advective wave equation. The derivation of the first model can be applied to other physical models. The second model was implemented.

More precisely, in section 2 we introduce the Smith factorization of the Euler equation. This tool is used in section 3 to propose two ways to design PML for the Euler equation. In section 4, numerical results are shown for the second model.

2 Analysis of the Euler system via Smith factorization

We write the linearized Euler equations as:

$$\begin{pmatrix} \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & -\bar{\rho}\bar{c}^2\partial_x & -\bar{\rho}\bar{c}^2\partial_y \\ \frac{1}{\bar{\rho}}\partial_x & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & 0 \\ \frac{1}{\bar{\rho}}\partial_y & 0 & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix} = \begin{pmatrix} f_p \\ f_u \\ f_v \end{pmatrix} \tag{1}$$

2.1 Smith factorization

We first recall the definition of the Smith factorization of a matrix with polynomial entries and apply it to systems of PDEs, see [Gan66b, Gan66a, Gan98] or [WRL95] and references therein.

Theorem 2.1 *Let n be an integer and A an invertible $n \times n$ matrix with polynomial entries in the variable λ : $A = (a_{ij}(\lambda))_{1 \le i,j \le n}$.*
Then, there exist three matrices with polynomial entries E, D and F with the following properties:

- *det(E)=det(F)=1*
- *D is a diagonal matrix.*
- *$A = EDF$.*

Morevoer, D is uniquely defined up to a reordering and multiplication of each entry by a constant by a formula defined as follows. Let $1 \le k \le n$,

- *S_k is the set of all the submatrices of order $k \times k$ extracted from A.*
- *$Det_k = \{Det(B_k) \backslash B_k \in S_k\}$*
- *LD_k is the largest common divisor of the set of polynomials Det_k.*

Then,

$$D_{kk}(\lambda) = \frac{LD_k(\lambda)}{LD_{k-1}(\lambda)}, 1 \le k \le n \tag{2}$$

(by convention, $LD_0 = 1$).

Application to the Euler system We first take formally the Fourier transform of the system (1) with respect to y and t (dual variables are k and ω resp.). We keep the partial derivatives in x since in the sequel we shall design a PML for a truncation of the domain in the x direction. We note

$$\hat{\hat{A}}_{Euler} = \begin{pmatrix} i\omega + \bar{u}\partial_x + ik\bar{v} & -\bar{\rho}\bar{c}^2\partial_x & -i\bar{\rho}\bar{c}^2 k \\ \frac{1}{\bar{\rho}}\partial_x & i\omega + \bar{u}\partial_x + ik\bar{v} & 0 \\ \frac{ik}{\bar{\rho}} & 0 & i\omega + \bar{u}\partial_x + i\bar{v}k \end{pmatrix} \tag{3}$$

We can perform a Smith factorization of $\hat{\hat{A}}_{Euler}$ by considering it as a matrix with polynomials in ∂_x entries. We have

$$\hat{\hat{A}}_{Euler} = EDF \tag{4}$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \hat{\hat{\mathcal{G}}}\hat{\hat{\mathcal{L}}} \end{pmatrix} \tag{5}$$

and

$$E = \frac{1}{(\bar{u}(\bar{c}^2 - \bar{u}^2))^{1/3}} \begin{pmatrix} i\bar{\rho}\bar{c}^2 k & 0 & 0 \\ 0 & \bar{u} & 0 \\ i\omega + \bar{u}\partial_x + i\bar{v}k & E_2 & \dfrac{\bar{c}^2 - \bar{u}^2}{ik\bar{\rho}\bar{c}^2} \end{pmatrix}$$

and

$$F = -\begin{pmatrix} \dfrac{i\omega + \bar{u}\partial_x + ik\bar{v}}{ik\bar{\rho}\bar{c}^2} & \dfrac{\partial_x}{ik} & 1 \\ \dfrac{\partial_x}{\bar{\rho}\bar{u}} & \dfrac{i\omega + \bar{u}\partial_x + ik\bar{v}}{\bar{u}} & 0 \\ \dfrac{\bar{u}}{i\omega + ik\bar{v}} & \dfrac{\bar{\rho}\bar{u}^2}{i\omega + ik\bar{v}} & 0 \end{pmatrix}$$

where

$$E_2 = \bar{u}\frac{(-\bar{u}\bar{c}^2 + \bar{u}^3)\partial_{xx} + (2\bar{u}^2 - \bar{c}^2)(i\omega + ik\bar{v})\partial_x + \bar{u}((i\omega + ik\bar{v})^2 + k^2\bar{c}^2)}{\bar{c}^2(i\omega + ik\bar{v})},$$

$$\hat{\hat{\mathcal{G}}} = i\omega + \bar{u}\partial_x + ik\bar{v} \tag{6}$$

and

$$\hat{\hat{\mathcal{L}}} = -\omega^2 + 2ik\bar{u}\bar{v}\partial_x + 2i\omega(\bar{u}\partial_x + ik\bar{v}) - (\bar{c}^2 - \bar{v}^2)\partial_{yy} - (\bar{c}^2 - \bar{u}^2)\partial_{xx} \tag{7}$$

The operators showing up in the diagonal matrix have a physical meaning:

$$\mathcal{G} = \partial_t + \bar{u}\partial_x + \bar{v}\partial_y$$

is a first order transport operator and

$$\mathcal{L} = \partial_{tt} + 2\bar{u}\bar{v}\partial_{xy} + 2\partial_t(\bar{u}\partial_x + \bar{v}\partial_y) - (\bar{c}^2 - \bar{v}^2)\partial_{yy} - (\bar{c}^2 - \bar{u}^2)\partial_{xx}$$

is the advective wave operator. Let us remark that we have:

$$\mathcal{L} = \mathcal{G}^2 - \bar{c}^2(\partial_{xx} + \partial_{yy}) \tag{8}$$

Remark 1 *This decomposition is not unique. Indeed, by denoting*

$$E_1 = -1/\bar{\rho}\partial_y(i\omega + \bar{u}\partial_x + \bar{v}\partial_y),\ \tilde{E}_2 = \bar{\rho}\bar{c}^2\partial_x\partial_y,\ E_3 = (i\omega + \bar{u}\partial_x + \bar{v}\partial_y)^2 - \bar{c}^2\partial_{xx} \tag{9}$$

and then if we apply the operator E_1 to the first equation of (1), $\tilde{E}_2$ to the second equation of (1) and E_3 to the third equation of (1) we obtain $\mathcal{G}\mathcal{L}v = \tilde{f}$. We obtain thus a Smith decomposition where the third variable is a physical one.

2.2 Modes via Smith factorization

In the PML analysis of section 3, we shall use the expression of solutions to the homogeneous Euler equation. In order to illustrate the previous section, we make use of the Smith factorization to compute them. We take the Fourier transform in t and y of (1) and seek non zero solutions to

$$\hat{\hat{A}}_{Euler} \begin{pmatrix} \hat{\hat{p}}(\omega, x, k) \\ \hat{\hat{u}}(\omega, x, k) \\ \hat{\hat{v}}(\omega, x, k) \end{pmatrix} = 0 \qquad x \in \mathbf{R},\ \omega \in \mathbf{R},\ k \in \mathbf{R}$$

Using Smith factorization (4), we have

$$EDF \begin{pmatrix} \hat{\hat{p}}(\omega, x, k) \\ \hat{\hat{u}}(\omega, x, k) \\ \hat{\hat{v}}(\omega, x, k) \end{pmatrix} = 0 \qquad x \in \mathbf{R},\ \omega \in \mathbf{R},\ k \in \mathbf{R}$$

Since $\det(E) = 1$, E^{-1} is still a matrix with polynomials in ∂_x entries so that we can apply it to the above equation and get:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \hat{\hat{\mathcal{G}}}\hat{\hat{\mathcal{L}}} \end{pmatrix} F \begin{pmatrix} \hat{\hat{p}}(\omega, x, k) \\ \hat{\hat{u}}(\omega, x, k) \\ \hat{\hat{v}}(\omega, x, k) \end{pmatrix} = 0 \qquad x \in \mathbf{R},\ \omega \in \mathbf{R},\ k \in \mathbf{R}$$

This implies that

$$F \begin{pmatrix} \hat{\hat{p}}(\omega, x, k) \\ \hat{\hat{u}}(\omega, x, k) \\ \hat{\hat{v}}(\omega, x, k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sum_i \alpha_i(\omega, k) e^{\lambda_i(\omega,k)x} \end{pmatrix} \qquad x \in \mathbf{R},\ \omega \in \mathbf{R},\ k \in \mathbf{R} \tag{10}$$

where $\hat{\hat{\mathcal{G}}}\hat{\hat{\mathcal{L}}}(e^{\lambda_i(\omega,k)x}) = 0$. Since $\mathcal{GL}$ is of third order in the x direction, we have three possible values for λ_i:

$$\lambda_1 = -\frac{i\omega + ik\bar{v}}{\bar{u}} \tag{11}$$

$$\lambda_2 = \begin{cases} \dfrac{u(i\omega + ik\bar{v}) - \bar{c}(i\omega + ik\bar{v})\sqrt{1 - \frac{k^2(\bar{c}^2 - \bar{v}^2)}{(\omega + k\bar{v})^2}}}{\bar{c}^2 - \bar{u}^2} & \text{for } |k|\sqrt{\bar{c}^2 - \bar{v}^2} < |\omega + k\bar{v}| \\[2ex] \dfrac{u(i\omega + ik\bar{v}) - \bar{c}\sqrt{k^2(\bar{c}^2 - \bar{v}^2) - (\omega + k\bar{v})^2)}}{\bar{c}^2 - \bar{u}^2} & \text{for } |k|\sqrt{\bar{c}^2 - \bar{v}^2} > |\omega + k\bar{v}| \end{cases} \tag{12}$$

$$\lambda_3 = \begin{cases} \dfrac{u(i\omega + ik\bar{v}) + \bar{c}(i\omega + ik\bar{v})\sqrt{1 - \frac{k^2(\bar{c}^2 - \bar{v}^2)}{(\omega + k\bar{v})^2}}}{\bar{c}^2 - \bar{u}^2} & \text{for } |k|\sqrt{\bar{c}^2 - \bar{v}^2} < |\omega + k\bar{v}| \\[2ex] \dfrac{u(i\omega + ik\bar{v}) + \bar{c}\sqrt{k^2(\bar{c}^2 - \bar{v}^2) - (\omega + k\bar{v})^2)}}{\bar{c}^2 - \bar{u}^2} & \text{for } |k|\sqrt{\bar{c}^2 - \bar{v}^2} > |\omega + k\bar{v}| \end{cases} \tag{13}$$

Remark 2 *λ_1 comes from the transport operator $\mathcal{G}$ whereas $\lambda_{2,3}$ come from the advective wave operator $\mathcal{L}$.*

Since $\det(F) = 1$, F^{-1} is still a matrix with polynomials in ∂_x entries so that we can apply it to equation (10) and get:

$$\begin{pmatrix} \hat{\hat{p}}(\omega,x,k) \\ \hat{\hat{u}}(\omega,x,k) \\ \hat{\hat{v}}(\omega,x,k) \end{pmatrix} = \sum_{i=1}^{3} \alpha_i(\omega,k) F^{-1} \begin{pmatrix} 0 \\ 0 \\ e^{\lambda_i(\omega,k)x} \end{pmatrix} \tag{14}$$

We shall call, for $i = 1, 2, 3$

$$W_i(\omega,x,k) = F^{-1} \begin{pmatrix} 0 \\ 0 \\ e^{\lambda_i(\omega,k)x} \end{pmatrix} \tag{15}$$

the modes of the Euler system.

Remark 3 *In section 3.3, we shall use that*

$$(F^{-1})_{13} = \frac{\hat{\hat{\mathcal{G}}}}{\bar{u}} \tag{16}$$

3 PMLs for the Euler System

The Smith factorization of the Euler system (4) and the computations of the previous section show that the modes correspond either to operator $\mathcal{L}$ or to operator $\mathcal{G}$. Among these two operators, the only operator which generates waves propagating in both positive x and negative x directions is the operator $\mathcal{L}$. This suggests that designing a PML for the Euler equation can be reduced to the design of PML for the advective wave operator $\mathcal{L}$. This question has been the subject of several works, [HN02], [BBBDL04] [DJ03] [BBBDL03] and references therein. Following these works, we use for operator $\mathcal{L}$ a PML defined by replacing the x derivatives by a "pml" x derivative. The definition is as follows:

$$\mathcal{L}_{pml} = \partial_{tt} + 2\bar{u}\bar{v}\partial_y(\partial_x^{pml}) + 2\partial_t(\bar{u}\partial_x^{pml} + \bar{v}\partial_y) - (\bar{c}^2 - \bar{v}^2)\partial_{yy} - (\bar{c}^2 - \bar{u}^2)(\partial_x^{pml})^2 \tag{17}$$

where

$$\partial_x^{pml} = \alpha(x)[\partial_x - \frac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(\partial_t + \bar{v}\partial_y)] + \frac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(\partial_t + \bar{v}\partial_y) \tag{18}$$

where the operator $\alpha(x)$ is a pseudo-differential operator in the t and y variables:

$$\alpha(x)(\phi) = \mathcal{F}^{-1}\left(\frac{\bar{c}(i\omega + ik\bar{v})}{\bar{c}(i\omega + ik\bar{v}) + (\bar{c}^2 - \bar{u}^2)\sigma(\omega,x,k)}\ \hat{\hat{\phi}}\right) \tag{19}$$

where $\sigma(\omega,x,k) \geq 0$ is the damping parameter of the PML and $= \mathcal{F}^{-1}$ is the inverse Fourier transform in the variables ω and k. Let us notice that we have

$$\partial_x^{pml} - \partial_x = \gamma(x)[\partial_x - \frac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(\partial_t + \bar{v}\partial_y)] \tag{20}$$

where the operator $\gamma(x)$ is a pseudo-differential operator in the t and y variables:

$$\gamma(x)(\phi) = \mathcal{F}^{-1}\left(\frac{-(\bar{c}^2 - \bar{u}^2)\sigma(\omega,x,k)}{\bar{c}(i\omega + ik\bar{v}) + (\bar{c}^2 - \bar{u}^2)\sigma(\omega,x,k)}\ \hat{\hat{\phi}}\right) \tag{21}$$

A PML used for truncating the domain in the y direction would consist in replacing in the operator $\mathcal{L}$ the y derivatives by a "pml" y derivative defined as follows:

$$\partial_y^{pml} = \alpha(y)[\partial_y - \frac{\bar{v}}{\bar{c}^2 - \bar{v}^2}(\partial_t + \bar{u}\partial_x)] + \frac{\bar{v}}{\bar{c}^2 - \bar{v}^2}(\partial_t + \bar{u}\partial_x) \tag{22}$$

3.1 First PML model

Based on (4), a first possibility is to define a PML for the Euler system by substitution of $\mathcal{L}$ with $\mathcal{L}^{pml}$ in matrix D (see formula (5)). In matrices E and F and in the operator $\mathcal{G}$, the x derivatives are not modified. Modifying only the advective wave operator avoids instability problems with the vorticity wave. We thus define:

$$\hat{\hat{A}}^{pml1}_{Euler} = ED^{pml}F \tag{23}$$

where

$$D^{pml} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \hat{\hat{\mathcal{G}}}\hat{\hat{\mathcal{L}}}^{pml} \end{pmatrix} \tag{24}$$

A direct computation yields:

$$\hat{\hat{A}}^{pml1}_{Euler} = \hat{\hat{A}}_{Euler} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & C_2 & 0 \end{pmatrix} \tag{25}$$

where

$$C_1 = \frac{(\partial_x - \partial_x^{pml})\hat{\hat{\mathcal{G}}}[(\bar{u}^2 - \bar{c}^2)(\partial_x + \partial_x^{pml}) + 2\bar{u}(i\omega + i\bar{v}k)]}{i\bar{\rho}\bar{c}^2 k(i\omega + ik\bar{v})} \text{ and } C_2 = \frac{C_1}{\bar{\rho}\bar{u}}$$

The difference with the Euler system concerns only the last equation on the variable v, but :

1. The formula is complex and involves third order derivatives on both the pressure p and the normal velocity u.
2. The formula implies a division by $i\bar{\rho}\bar{c}^2 k(i\omega + ik\bar{v})$ which can be zero.

As for the first point, one might argue that it is just a matter of implementation. The second point seems more serious. A possible cure could be to take:

$$\sigma(\omega, x, k) = \tilde{\sigma}(x)\left(i\bar{\rho}\bar{c}^2 k(i\omega + ik\bar{v})\right)$$

where $\tilde{\sigma}(x) \geq 0$. From formulas for C_1 and C_2 and formula (20)-(21), we see that it would regularize C_1 and C_2. But it would be at the expense of the damping of the PML. Indeed, $\sigma(\omega, x, k)$ would be small for small values of k or of $i\omega + ik\bar{v}$. The present first model raises difficulties. Nevertheless, it should deserve interest since it corresponds to a systematic way to design a PML for systems of PDEs. Moreover, since matrices E and F are not unique, it is quite possible that a more suitable Smith factorization when used in formula (23) would lead to a practicable PML. In the next section, we design another PML for the Euler system whose numerical results will be given in section 4.

3.2 Second PML model

The rationale for this model is that the pressure p satisfies an advective wave equation which is the only equation that demands a PML. Indeed, let multiply (3) by the matrix

$$El = \begin{pmatrix} \hat{\hat{\mathcal{G}}} & -\bar{\rho}\bar{c}^2\partial_x & -i\bar{\rho}\bar{c}^2 k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{26}$$

We get:

$$El\,\hat{\hat{A}}_{Euler} = \begin{pmatrix} \hat{\hat{\mathcal{L}}} & 0 & 0 \\ \frac{1}{\bar\rho}\partial_x & i\omega + \bar u \partial_x + ik\bar v & 0 \\ \frac{ik}{\bar\rho} & 0 & i\omega + \bar u\partial_x + i\bar v k \end{pmatrix} \tag{27}$$

We substitute $\hat{\hat{\mathcal{L}}}$ with $\hat{\hat{\mathcal{L}}}^{pml}$ and apply

$$El^{-1} = \begin{pmatrix} \hat{\hat{\mathcal{G}}}^{-1} & -\bar\rho\bar c^2\partial_x\hat{\hat{\mathcal{G}}}^{-1} & -i\bar\rho\bar c^2 k\hat{\hat{\mathcal{G}}}^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we are thus led to define:

$$\hat{\hat{A}}^{pml2}_{Euler} = \begin{pmatrix} \hat{\hat{\mathcal{G}}}^{-1}(\hat{\hat{\mathcal{L}}}^{pml} + \bar c^2(\partial_{xx} - k^2)) & \bar\rho\bar c^2\partial_x & i\bar\rho\bar c^2 k \\ \frac{1}{\bar\rho}\partial_x & \hat{\hat{\mathcal{G}}} & 0 \\ \frac{ik}{\bar\rho} & 0 & \hat{\hat{\mathcal{G}}} \end{pmatrix} \tag{28}$$

A direct computation yields:

$$\hat{\hat{A}}^{pml2}_{Euler} = \hat{\hat{A}}_{Euler} + \begin{pmatrix} (\hat{\hat{\mathcal{L}}}^{pml} - \hat{\hat{\mathcal{L}}})\hat{\hat{\mathcal{G}}}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

In order to get rid of the operator $\hat{\hat{\mathcal{G}}}^{-1}$, we introduce a new variable $\mathcal{P}$ such that $\mathcal{G}(\mathcal{P}) = p$ so that in the physical space the enlarged PML system we consider reads:

$$\mathcal{A}^{pml2}_{Euler}\begin{pmatrix} \mathcal{P} \\ p \\ u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{G} & -1 & 0 & 0 \\ \mathcal{L}^{pml} - \mathcal{L} & \mathcal{G} & \bar\rho\bar c^2\partial_x & \bar\rho\bar c^2\partial_y \\ 0 & \frac{1}{\bar\rho}\partial_x & \mathcal{G} & 0 \\ 0 & \frac{1}{\bar\rho}\partial_y & 0 & \mathcal{G} \end{pmatrix}\begin{pmatrix} \mathcal{P}(t,x,y) \\ p(t,x,y) \\ u(t,x,y) \\ v(t,x,y) \end{pmatrix} = 0, \quad t > 0, x > 0, y \in \mathbf{R} \tag{30}$$

with the following interface conditions between the Euler media and the PML

$$\mathcal{P} = 0,\ p \text{ and } u \text{ are continuous, } \partial_x(p_{Euler}) = \partial_x^{pml}(p_{pml})$$

Study of the PML media We now proceed to an analysis of the PML system similar to that of § 2.2 for the Euler system. The Smith factorization of $\hat{\hat{\mathcal{A}}}^{pml2}_{Euler}$ reads

$$\hat{\hat{\mathcal{A}}}^{pml2}_{Euler} = \tilde E\tilde D\tilde F$$

where

$$\tilde D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \hat{\hat{\mathcal{G}}} & 0 \\ 0 & 0 & 0 & \hat{\hat{\mathcal{G}}}\hat{\hat{\mathcal{L}}}^{pml} \end{pmatrix} \tag{31}$$

and $\tilde E$ and $\tilde F$ are matrices with polynomial in ∂_x entries and their determinants are one. This will enable us to give the general form of the solutions to the homogeneous PML equations.

Indeed, let us denote $\hat{\hat{W}} = (\hat{\hat{\mathcal{P}}}, \hat{\hat{p}}, \hat{\hat{u}}, \hat{\hat{v}})^T$ such a solution. From the Smith factorization, there exist $(\beta_i(\omega, k))_{i=0,\ldots,3}$ such that

$$\tilde{F}(\hat{\hat{W}}) = \beta_0 \begin{pmatrix} 0 \\ 0 \\ e^{\lambda_0^{pml} x} \\ 0 \end{pmatrix} + \sum_{i=1}^{3} \beta_i \begin{pmatrix} 0 \\ 0 \\ 0 \\ e^{\lambda_i^{pml} x} \end{pmatrix}$$

where

$$\begin{cases} \lambda_0^{pml} = \lambda_1^{pml} = \lambda_1 \\ \lambda_{2,3}^{pml} = \dfrac{\bar{c}(i\omega + ik\bar{v})}{\bar{c}(i\omega + ik\bar{v}) + (\bar{c}^2 - \bar{u}^2)\sigma}(\lambda_{2,3} - \dfrac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(i\omega + i\bar{v}k)) + \dfrac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(i\omega + i\bar{v}k) \end{cases}$$

By applying $\tilde{F}^{-1}$ to the above equation, we see that there exist vectors $W_i^{pml}(\omega, x, k)$, $i = 0, \ldots, 3$ such that

$$\hat{\hat{W}} = \sum_{i=0}^{3} \beta_i(\omega, k) W_i^{pml}(\omega, x, k) e^{\lambda_i^{pml}(\omega,k)x} \tag{32}$$

If we consider the solution in the positive x half space, its boundedness as x tends to infinity implies that $\beta_3 = 0$.

3.3 PMLness of the second model

A key property of a PML is that there is no reflection at the interface between the Euler media and the PML media. We will prove that it is the case for a truncation of the space with an infinite PML starting at $x = 0$ with a constant damping parameter σ. We have to consider the following coupled problem:
Find $(W_l, W_r) = ((p_l, u_l, v_l), (\mathcal{P}_r, p_r, u_r, v_r))$ such that:

$$\begin{aligned} & A_{Euler} W_l = 0, \quad t > 0,\ x < 0,\ y \in \mathbf{R} \\ & \mathcal{A}_{Euler}^{pml2} W_r = 0, \quad t > 0,\ x > 0,\ y \in \mathbf{R} \\ & \text{at } x = 0, \quad \mathcal{P}_r = 0,\ p_l = p_r,\ \partial_x(p_l) = \partial_x^{pml}(p_r),\ u_l = u_r, \quad t > 0,\ y \in \mathbf{R} \end{aligned} \tag{33}$$

We take the Fourier transform in t and y of the above coupled system and get:

$$\begin{aligned} & \hat{\hat{A}}_{Euler} \hat{\hat{W}}_l = 0, \quad x < 0,\ \omega,\ k \in \mathbf{R} \\ & \hat{\hat{\mathcal{A}}}_{Euler}^{pml2} \hat{\hat{W}}_r = 0,\ x > 0,\ \omega,\ k \in \mathbf{R} \\ & \text{at } x = 0, \quad \hat{\hat{\mathcal{P}}}_r = 0,\ \hat{\hat{p}}_l = \hat{\hat{p}}_r,\ \partial_x(\hat{\hat{p}}_l) = \partial_x^{pml}(\hat{\hat{p}}_r),\ \hat{\hat{u}}_l = \hat{\hat{u}}_r,\ \omega,\ k \in \mathbf{R} \end{aligned} \tag{34}$$

From section 2.2, we know that the general solution to the Euler system is :

$$\hat{\hat{W}}_l = \sum_{i=1}^{3} \alpha_i(\omega, k) W_i(\omega, x, k) e^{\lambda_i(\omega,k)x}$$

where W_i is defined in (15). As for the solution in the PML media, we know from (32) and the boundedness of the solution as x tends to infinity that

$$\hat{\hat{W}}_r = \sum_{i=0}^{2} \beta_i(\omega, k) W_i^{pml}(\omega, x, k) e^{\lambda_i^{pml}(\omega,k)x}$$

We study the adequacy of the PML by considering α_1 and α_2 to be given. This corresponds to ingoing waves from the Euler media and moving towards the interface between the Euler media and the PML media. The four other quantities $(\alpha_3, (\beta_i)_{i=0,\dots,2})$ are determined by the interface conditions. The media is perfectly matched if we have no reflection in the Euler media, i.e. if $\alpha_3 = 0$. We now prove that this is indeed the case. We focus on the equation satisfied by the pressure. By applying matrix

$$\begin{pmatrix} \hat{\hat{\mathcal{G}}} & \hat{\hat{\mathcal{G}}} & -\bar{\rho}\bar{c}^2 \partial_x & -i\bar{\rho}\bar{c}^2 k \end{pmatrix}$$

to $\hat{\hat{\mathcal{A}}}^{pml2}_{Euler}$, we have that $\hat{\hat{p}}_r$ satisfies the equation of advective Helmholtz PML media:

$$\hat{\hat{\mathcal{L}}}^{pml}(\hat{\hat{p}}_r) = 0$$

From (27), we also have that

$$\hat{\hat{\mathcal{L}}}(\hat{\hat{p}}_l) = 0$$

The interface conditions on the pressure are $\hat{\hat{p}}_l = \hat{\hat{p}}_r$ and $\partial_x(\hat{\hat{p}}_l) = \partial_x^{pml}(\hat{\hat{p}}_r)$. We know from works on PML for the convective Helmholtz that there is no reflection at the interface for the pressure. Therefore there exists $\beta_p(\omega, k)$ such that

$$\hat{\hat{p}}_l = \beta_p(\omega, k) e^{\lambda_2(\omega,k)x} \tag{35}$$

We prove now that as a consequence, α_3 is zero. Indeed, taking the first component of (32) we have that

$$\hat{\hat{p}}_l = \sum_{i=1}^{3} \alpha_i \, (W_i)_1 \, e^{\lambda_i x} \tag{36}$$

From (15) and (16), we have

$$(W_i)_1 = \frac{1}{\bar{u}} \hat{\hat{\mathcal{G}}}(e^{\lambda_i x}) = \frac{1}{\bar{u}} (\lambda_i - \lambda_1) \, e^{\lambda_i x}$$

So that we have $(W_1)_1 = 0$ whereas $(W_i)_1 \neq 0$ for $i = 2, 3$. Thus we can infer from (35) and (36) that $\alpha_3 = 0$. This shows that there is no reflection at the interface between the Euler and an infinite PML media.

In practice, the PML has a finite width. As a result, the coefficient β_3 will not be zero in general. But, the corresponding mode $W_3^{pml} \, e^{\lambda_3^{pml} x}$ is exponentially decreasing as x is decreasing. Its contribution to the solution on the interface can be made as small as necessary simply by increasing the width of the PML.

4 Numerical Results

We have taken $c = 300$ and $\rho = 1$. The 2D linearized Euler equations are discretized on a uniform staggered grid using a Yee Scheme. The convective derivatives are discretized using

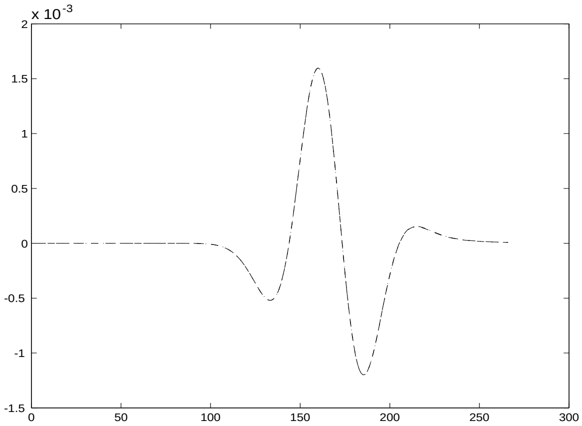

Figure 1: Pressure fields for the reference solution and the PML solution near the upperleft corner (left) and in the PML (right) for an oblique flow $M = 0.33$ vs. time steps

an upwind scheme both in the Euler region and in the PMLs. The computational domain is the square $[0, 1.2] \times [0, 1.2]$, PMLs have a width 0.9. The reference solution is obtained by computing the solution on a much larger domain. The initial solutions are zero. Let $f(t, x, y) = (1 - 2\pi^2(f_c t - 1)^2)e^{-\pi^2(f_c t-1)^2}\delta_M(x, y)$ for $t < T_s$ and zero for $t > T_s$ with $T_s = 0.05$, $f_c = 4/T_s$ and δ_M is the Dirac mass located in the middle of the computational domain. The PML solution is compared with a reference solution that is computed on a much larger domain. In figure 4, the pressure for both solutions are plotted as a function of time 4 points from the upperleft corner of the domain (left figure) and inside the PML (right figure). The velocity field is $u_0 = v_0 = 1/3$. In the Euler region, both curves are nearly identical. In the PML, we see the damping of the PML solution. Of course, for the reference, solution, this corresponds to an Euler region and there is no damping. For this computation, the right handside was $f(t, x, y)$ in the equation on the pressure p and 0 for the equations on the velocity. In the other computations, the right handside was $f(t, x, y)$ on all three equations of system (1). In Figure 2, we show the pressure at different times of the computation for an oblique velocity $u_0 = v_0 = 270$. For the same computation, pressure near the upperleft corner is shown on Figure 3. Figure 4 is a similar figure for a horizontal flow in a duct. The stability of the PML was assessed by computing on time intervals much longer than those used for generating the figures.

5 Conclusion

The first PML model proposed in § 3.1 is obtained by using the Smith factorization of the Euler equations and a PML for the advective wave equation. This method can be applied to many systems of partial differential equations. The second PML model we have proposed for the Euler linearized equations are based on the PML for the advective wave equation. Thus, the PML for Euler inherits the properties from the latter. This second model was implemented and numerical results illustrate the efficiency of the approach.

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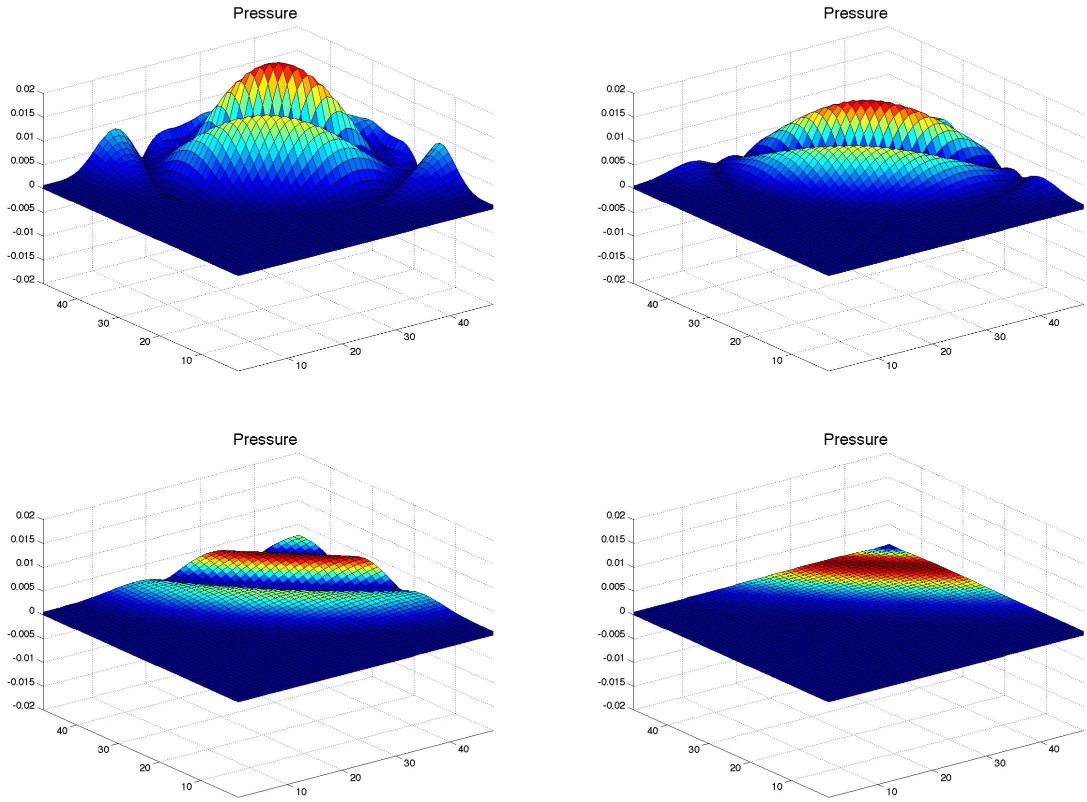

Figure 2: Pressure field for an oblique velocity $M = 0.9$ at successive time steps

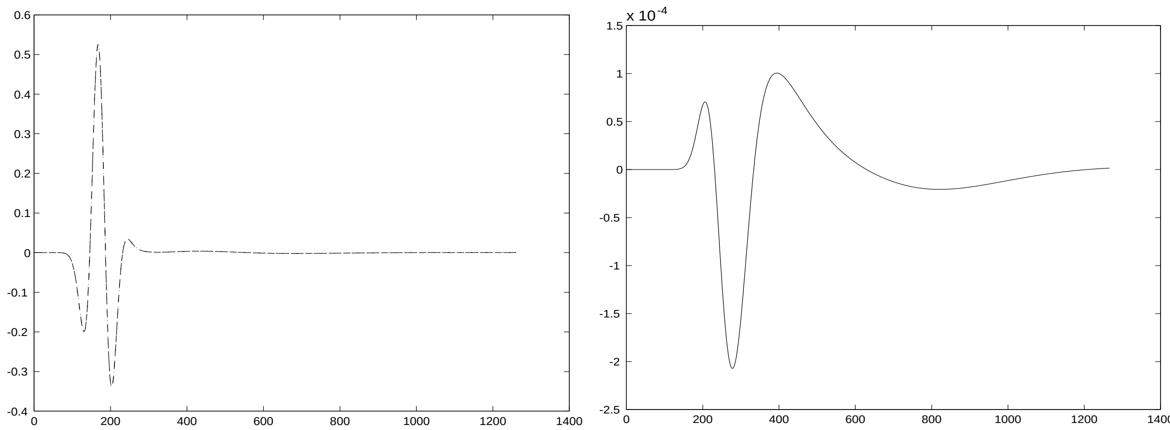

Figure 3: Pressure field (left) and error on the pressure (right) near the upperleft corner for an oblique velocity $M = 0.9$ vs. time steps

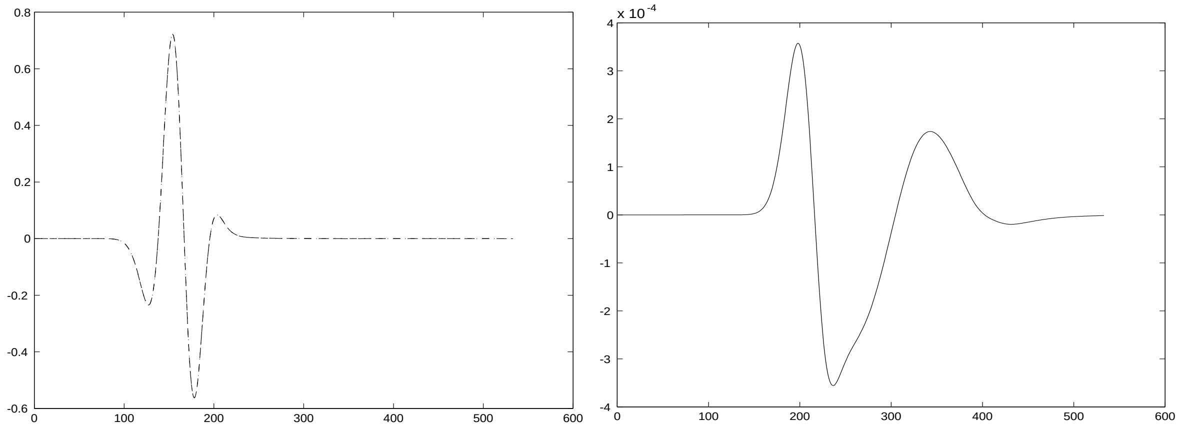

Figure 4: Pressure field (left) and error on the pressure (right) near the upperleft corner for a horizontal flow $M = 0.33$ vs.time steps

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