On Optimal Control of a Free Surface Flow

Problem Statement, Optimization Approach, FreeFem++ Realization

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Fraunhofer Institut Techno- und Wirtschaftsmathematik

Technische Universität Kaiserslautern

Motivation

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- Polymer melt is extruded through a flat die.
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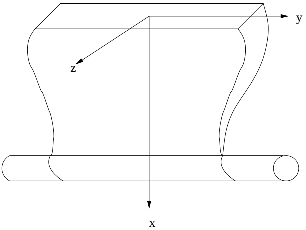

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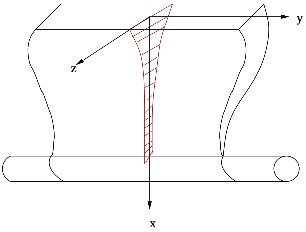

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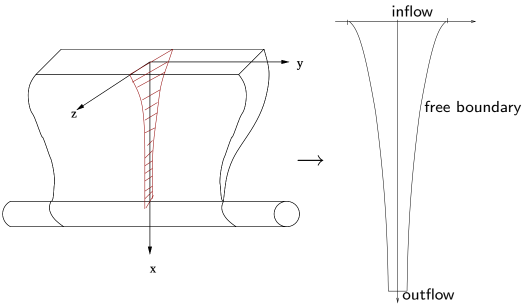

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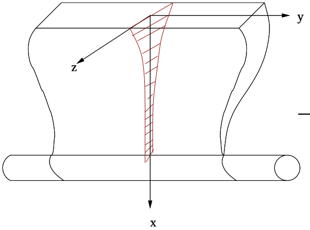

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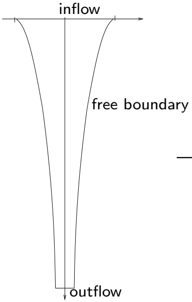

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Optimal control of the free boundary

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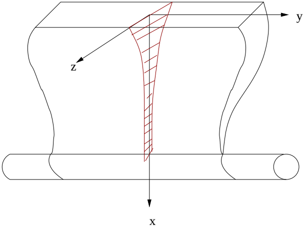
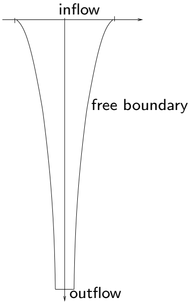
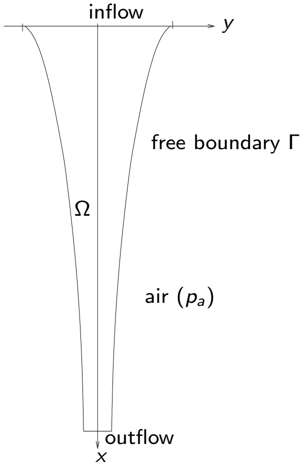




Optimal control of the free boundary

Mathematical Model



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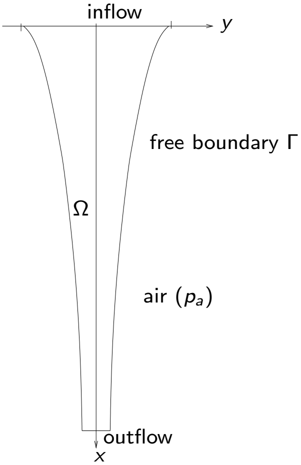

Stokes equations:

$$\nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,$$

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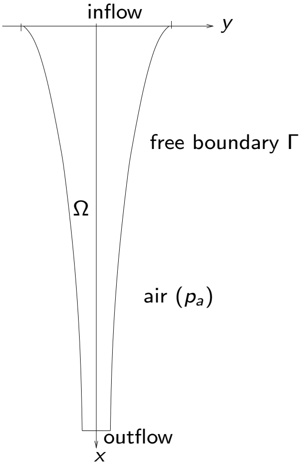

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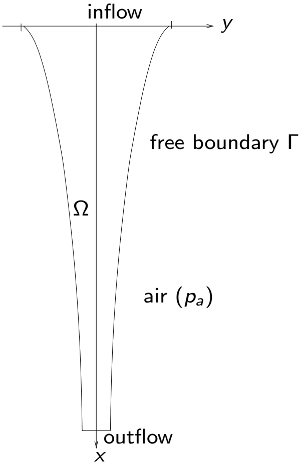

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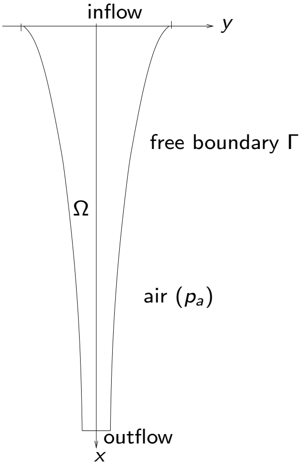

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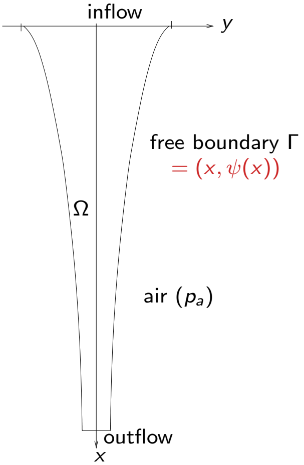

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Optimization Approach

Graph approach[1]: Model the free boundary Γ as

$$\Gamma = \{(x, \pm\psi(x)) \,|\, x \in (0,1)\}.$$

The graph of the desired boundary is denoted by $\pm\bar{\psi}$.

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Optimal Control Problem:

$\min J(\psi, p_a)$ subject to (Ma), (Mo), (In), (Out), (Dyn), (Kin).

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Lagrange Formalism

Lagrange function: $L\left(\psi, p_a, p, \mathbf{v}, \zeta_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3, \boldsymbol{\zeta}_4, \zeta_5, \boldsymbol{\zeta}_6\right) :=$

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Gradient of reduced cost function:

$$J'(p_a) = \beta p_a - \mathbf{n} \cdot \hat{\mathbf{v}}|_{(x,\psi(x))} \sqrt{1 + (\psi'(x))^2}, \quad x \in [0,1]$$

Optimization Algorithm

Result: optimal p_a

Initialization of $p_a^{(0)}$;

$i = 0$ (Iteration counter);

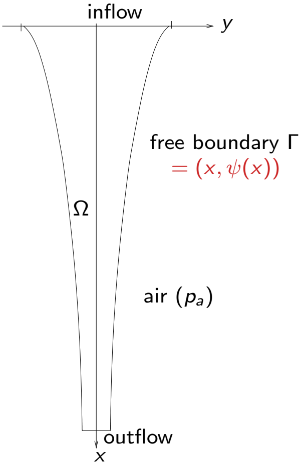
FWD: Forward step 0: computation of $\mathbf{v}^{(0)}, p^{(0)}, \psi^{(0)}$ using $p_a^{(0)}$;

repeat

- $i = i + 1$;
- **BWD:** Solve the adjoint system;
- **GRD:** Compute the gradient $J'(p_a^{(i-1)}) =: -d^{(k)}$;
- **SL:** Compute the step length λ^k using e.g. Armijo rule;
- **UP:** Update the control: $p_a^{(i)} = p_a^{(i-1)} + \lambda^{(k)} d^{(k)}$;
- **FWD:** Forward step i: compute $\mathbf{v}^{(i)}, p^{(i)}, \psi^{(i)}$ using $p_a^{(i)}$;

until $\left\| J'(p_a^{(i-1)}) \right\| < tol$;

Reminder: Mathematical Model



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Algorithm for Forward Problem (FWD)

Ansatz: split off the kinematic boundary condition $\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0$

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What is $\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0$ in terms of ψ?

- $\Gamma = \{(x, \pm\psi(x)) \,|x \in (0,1)\,\}$
- tangential vector: $(1 \quad \psi'(x))^\mathsf{T}, \quad (1 \quad -\psi'(x))^\mathsf{T}$
- outward unit normal vector:

$$\mathbf{n}|_\Gamma = \begin{pmatrix} -\psi'(x) \\ \pm 1 \end{pmatrix} \frac{1}{\sqrt{1 + \psi'(x)^2}}$$

$\mathbf{v}\cdot\mathbf{n} = 0 \Rightarrow$ ODE for ψ

$$0 \stackrel{!}{=} \mathbf{v}\cdot\mathbf{n}\big|_\Gamma = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big|_\Gamma \cdot \begin{pmatrix} -\psi'(x) \\ \pm 1 \end{pmatrix} \frac{1}{\sqrt{1+\psi'(x)^2}}$$

$$= \frac{1}{\sqrt{1+\psi'(x)^2}}\Big(- v_1\psi'(x) \pm v_2\Big)$$

$\Rightarrow$ ODE : $\quad \psi'(x) = \pm \frac{v_2}{v_1}\Big|_{(x,\pm\psi(x))}, \quad \psi(0) = R$

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Simplification:

We use the quantities of the last iteration step to compute the new boundary, i.e.

$$(\psi'(x))^{(k)} = \pm \frac{v_2^{(k-1)}}{v_1^{(k-1)}}\Bigg|_{(x, \pm\psi^{(k-1)}(x))}, \quad \psi^{(k)}(0) = R$$

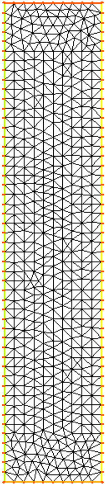
$\Rightarrow$ Simplified ODE can be solved by explicit integration.

Algorithm to Solve FWD step i

Input: $p_a^{(i)}$ as a function of x
$k = 0$ (iteration counter);
Initial boundary $\psi^{(0)}$ (e.g. $\psi^{(0)} \equiv R$) this yield $\Omega^{(0)}$;
FEM-Solve: use $\Omega^{(0)}$ to compute $\mathbf{v}^{(0)}, p^{(0)}$;
repeat
  $k = k + 1$;
  ODE Solve: use $\mathbf{v}^{(k-1)}$ on $\psi^{(k-1)}$ to obtain $\psi^{(k)}$;
  move $m = \psi^{(k-1)} - \psi^{(k)}$;
  $\Omega^{(k)} =$movemesh$(\Omega^{(k-1)}, [x, y - m])$;
  FEM-Solve: use $\Omega^{(k)}$ to compute $\mathbf{v}^{(k)}, p^{(k)}$;
until $max(\mathbf{v} \cdot \mathbf{n}) < tol$;
$\mathbf{v}^{(i)} := \mathbf{v}^{(k)}$, $p^{(i)} := p^{(k)}$, $\psi^{(i)} := \psi^{(k)}$;
Output: $\mathbf{v}^{(i)}, p^{(i)}, \psi^{(i)}$

Sample Step of Forward Algorithm

initial domain:



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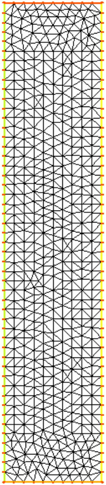

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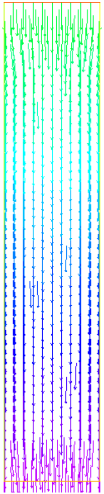

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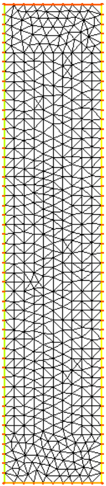

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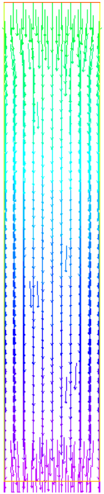
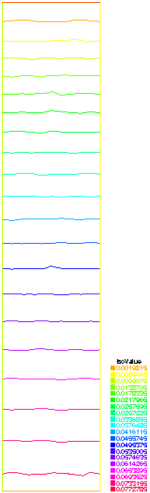
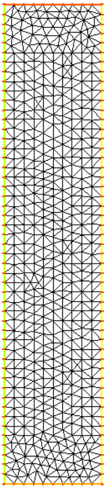


deformation:

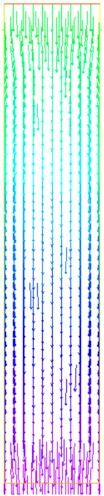


Sample Step of Forward Algorithm

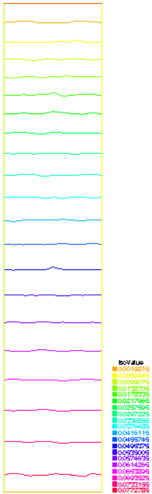
initial domain:



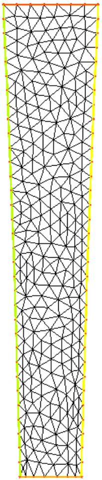
velocity **v**:



deformation:



new domain:

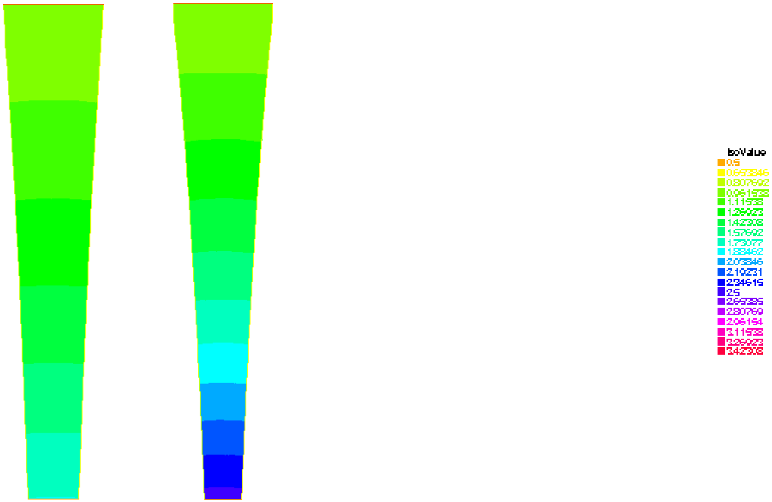


Numerical Results of Forward Problem

$p_a = 5$:

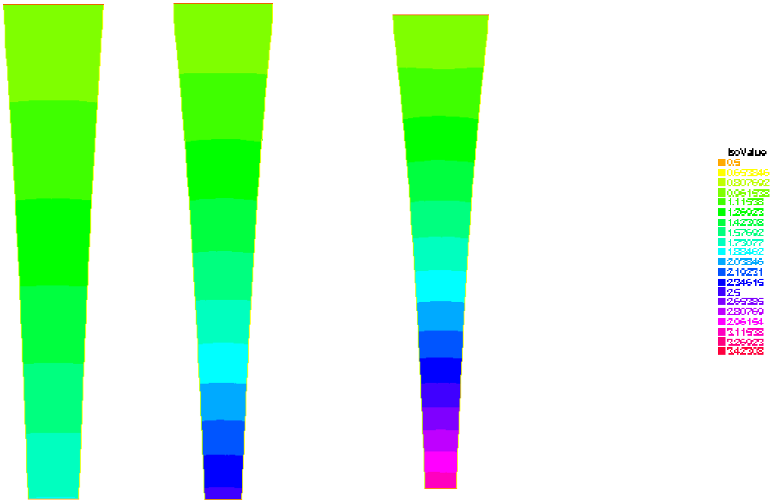
Numerical Results of Forward Problem

$p_a = 5$: $p_a = 10$:



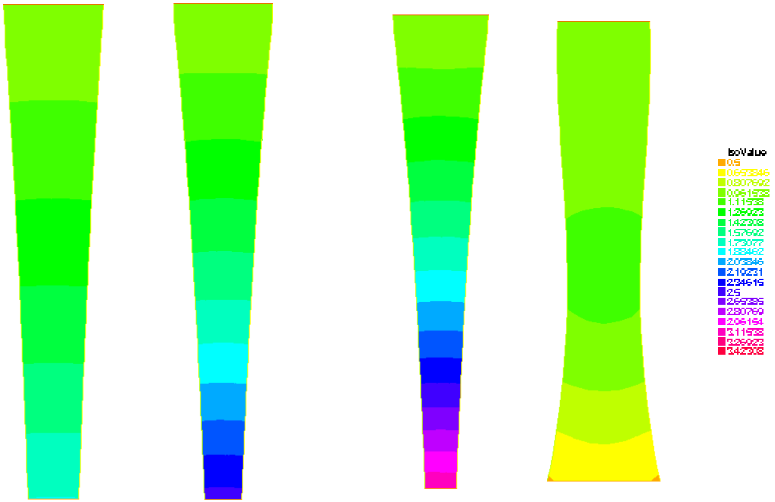
Numerical Results of Forward Problem

$p_a = 5$: $p_a = 10$: $p_a = 15$:



Numerical Results of Forward Problem

$p_a = 5$: $p_a = 10$: $p_a = 15$: $p_a = 5sin(5(1-x))$:



Conclusion

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- Numerics with FreeFem++ work very well for the forward problem (due to movemesh and adaptmesh functions).
- Adjoint equation system is comparably easy.

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Outlook:

- Implementation of adjoint system and gradient method
  $\longrightarrow$ update of control might bring some difficulties.
  $\longrightarrow$ need method to convert discrete data from the boundary into a function of x.

Outlook

Reminder: Adjoint System and Gradient of Reduced Cost Function

Adjoint system:

Stokes equations:

$$\nabla \cdot \hat{\mathbf{v}} = 0 \quad \text{in } \Omega,$$
$$\nabla \cdot \hat{\mathbf{T}}^{\mathsf{T}} = \mathbf{0} \quad \text{in } \Omega,$$
$$\left(\hat{\mathbf{T}} = -\hat{p}\mathbf{I} + \left(\nabla\hat{\mathbf{v}} + \nabla\hat{\mathbf{v}}^{\mathsf{T}}\right)\right)$$

Boundary conditions:

$$\hat{\mathbf{v}} = \mathbf{0} \quad \text{on inflow},$$
$$\hat{\mathbf{T}} \cdot \mathbf{n} = \mathbf{0} \quad \text{on outflow},$$
$$\hat{\mathbf{T}} \cdot \mathbf{n} - \hat{\psi} \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma,$$
$$\alpha\left(\psi - \bar{\psi}\right) + (p_a)_x \hat{v}_1 + \left(\hat{\mathbf{v}}_x \cdot (\mathbf{T} + p_a\mathbf{I})\right)_1 - \hat{\psi}_x \hat{v}_1 = 0 \quad \text{on } \Gamma,$$
$$\left(p_a \hat{v}_1 - \hat{\psi} v_1\right)\Big|_{(x,\psi(x))} = 0 \quad \text{for } x = 1.$$

Gradient of reduced cost function:

$$J'(p_a) = \beta p_a - \mathbf{n} \cdot \hat{\mathbf{v}}\big|_{(x,\psi(x))} \sqrt{1 + (\psi'(x))^2}, \quad x \in [0,1]$$