The Preconditioned Conjugate Gradients Algorithmes

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October 22, 2007

The Preconditioned Conjugate Gradients Algorithmes:

Problem to be solved: arg $min_{x\in\mathbb{R}^N} E(x)$ where E is quadratic in z (i.e: $\nabla E(x) = Ax - b$, where A is a symetric positive matrix and $b \in \mathbb{R}^n$).

Algorithme 1 *The Preconditioned Conjugate Gradients* Let be $x^0 \in \mathbb{R}^n$, $\varepsilon \in \mathbb{R}$ let be C given matrix symetric positve

$$G^0 = Ax^0 - b = \nabla_z E(x^0)$$

$$H^0 = -CG^0$$

- *pour* $i = 0$ *à* n

$$\rho^i = -\frac{(G^i, H^i)}{(H^i, AH^i)} \qquad \textit{we have also } AH^i = \nabla E(x^i) - \nabla E(0))$$
$$x^{i+1} = x^i + \rho^i H^i$$
$$G^{i+1} = G^i + \rho^i AH^i \quad = \nabla E(x^{i+1});$$
$$\gamma^{i+1} = \frac{(G^{i+1}, G^{i+1})_C}{(G^i, G^i)_C}$$
$$H^{i+1} = -CG^{i+1} + \gamma^{i+1} H^i$$
si $(G^{i+1}, G^{i+1})_C < \varepsilon$ *stop*

Theoreme 1 *Denote* $\mathcal{E}_C(x) = \sqrt{(Ax - \overline{x}, x - \overline{x})_C}$*, the erreur in the precondition norm of* A*, where* $\overline{x}$ *is the solution of the problem. Then the erreur at the iteration* k *of Preconditioned Conjugate Gradients is major by :*

$$\mathcal{E}_C(x^k) \le 2 \left(\frac{\sqrt{K_C(A) - 1}}{\sqrt{K_C(A) + 1}} \right)^k \mathcal{E}_C(x^0)$$

where $K_C(A)$ *is the condition number of matrix* CA*,i.e.* $K_C(A) = \frac{\lambda_1^C}{\lambda_n^C}$ *with* λ_1^C *(resp.* λ_n^C*) is the smaller (resp. larger) eigen value of matrix* CA *.*

Exercice 1 *Write a small program with computed science language (C++,C, scilab, mathlab, freefem++) a small program to solve the following PDE by finite element.*

find u *a function de* $\Omega =]0,1[^2$ *dans* $\mathbb{R}$ *such that*

$$-\Delta u = f, \qquad dans \quad \Omega$$

and such that $u = g$ *on the border* $\partial\Omega$*.*

For that we use the finite difference schema. We compute the approximation of u *at points* $x_{n,m}$*, call* $u_{n,m}$*, where the points* $x_{n,m}$ *are* $(h_1 n, h_2 m)$ *with* $h_1 = 1/N$ *and* $h_2 = 1/M$*, for* $n = 0, ..., N$*, and* $m = 0, ..., M$*.*

The finite difference schema to approche Δ *by* Δ_d *at internal point* $x_{n,m}$ *i.e;* n *is different of* 0 *or* N *and* m *different of* 0 *or* M*).*

The schema:

for $n = 1, ..., N-1$*, and* $m = 1, ..., M-1$ *:*

$$\Delta_d u_{n,m} = \frac{u_{n-1,m} + u_{n+1,m} - 2u_{n,m}}{h_1^2} + \frac{u_{n,m-1} + u_{n,m-1} - 2u_{n,m}}{h_2^2} = f(x_{n,m})$$

and for $n = 0, ..., N,$ *and* $m = 0, M$ *or* $n = 0, N,$ *and* $m = 1, ..., M-1$*:*

$$u_{n,m} = g(x_{n,m}).$$

Solve the linear system with a conjugate gradient.