Université Paris-Dauphine
EDDIMO
DFR Mathématiques de la Décision

N° attribué par la bibliothèque

Homogénéisation de lois de conservation scalaires et d'équations de transport

THÈSE

pour l'obtention du titre de

DOCTEUR EN SCIENCES
Spécialité Mathématiques

(Arrêté du 7 Août 2006)

présentée et soutenue publiquement le 08 octobre 2007 par

Anne-Laure DALIBARD (ép. ROUX)

Jury :

Directeur de thèse : **Pierre-Louis LIONS**
Professeur au Collège de France
Rapporteurs : **Benoît PERTHAME**
Professeur à l'Université Pierre et Marie Curie
Denis SERRE
Professeur à l'ENS Lyon
Examinateurs : **Jean DOLBEAULT**
Directeur de recherche CNRS à l'Université Paris-Dauphine
Thierry GOUDON
Professeur à l'Université des Sciences et Technologies de Lille
Laure SAINT-RAYMOND
Professeur à l'ENS Ulm

CEREMADE - UMR 7534
Université Paris-Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16

Résumé. Cette thèse est consacrée à l'étude du comportement asymptotique de solutions d'une classe d'équations aux dérivées partielles avec des coefficients fortement oscillants. Dans un premier temps, on s'intéresse à une famille d'équations non linéaires, des lois de conservation scalaires hétérogènes, qui interviennent dans divers problèmes de la mécanique des fluides ou de l'électromagnétisme non linéaire. On suppose que le flux de cette équation est périodique en espace, et que la période des oscillations tend vers zéro. On identifie alors les profils asymptotiques microscopique et macroscopique de la solution, et on démontre un résultat de convergence forte ; en particulier, on montre que lorsque la condition initiale ne suit pas le profil microscopique dicté par l'équation, il se forme une couche initiale en temps durant laquelle les solutions s'adaptent à celui-ci. Dans un second temps, on considère une équation de transport linéaire, qui modélise l'évolution de la densité d'un ensemble de particules chargées dans un potentiel électrique aléatoire et très oscillant. On établit l'apparition d'oscillations microscopiques en temps et en espace dans la densité, en réponse à l'excitation par le potentiel électrique. On donne également des formules explicites pour l'opérateur de transport homogénéisé lorsque la dimension de l'espace est égale à un.

Mots clés. Homogénéisation. Loi de conservation scalaire. Formulation cinétique. Équation de transport. Équation parabolique.

Abstract. In this thesis, we study the asymptotic behavior of solutions of a class of partial differential equations with strongly oscillating coefficients. First, we focus on a family of nonlinear evolution equations, namely parabolic scalar conservation laws. These equations are encountered in various problems of fluid mechanics and nonlinear electromagnetism. The flux is assumed to be periodic with respect to the space variable, and the period of the oscillations goes to zero. The asymptotic profiles in the microscopic and macroscopic variables are first identified. Then, we prove a result of strong convergence; in particular, when the initial data does not match the microscopic outline dictated by the equation, it is shown that there is an initial layer in time during which the solution adapts itself to this profile. The other equation studied in this thesis is a linear transport equation, modeling the evolution of the density of charged particles in a highly oscillating random electric potential. It is proved that the density has fast oscillations in time and space, as a response to the excitation by the electric potential. We also derive explicit formulas for the homogenized transport operator when the space dimension is equal to one.

Keywords. Homogenization. Scalar conservation law. Kinetic formulation. Transport equation. Parabolic equation.

Remerciements

Tout d'abord, je souhaite remercier profondément Pierre-Louis Lions, mon directeur de thèse, d'avoir accepté de superviser mes premiers pas dans la recherche, et de m'avoir proposé plusieurs sujets d'étude tous plus stimulants les uns que les autres. Être son élève fut un honneur et un plaisir, et je suis très reconnaissante de la disponibilité sans faille dont il a fait preuve en dépit de son emploi du temps chargé. Il va sans dire que j'ai beaucoup appris à ses côtés, et sa curiosité scientifique restera pour moi un exemple.

J'adresse tous mes remerciements à Denis Serre, qui a accepté d'être rapporteur de cette thèse, pour le sérieux avec lequel il s'est acquitté de cette tâche et pour son intérêt pour mes travaux. Je suis également très redevable envers Benoît Perthame pour son travail de rapporteur, ainsi que pour les nombreuses discussions que nous avons eues autour des formulations cinétiques.

Je remercie aussi Thierry Goudon d'avoir accepté de faire partie de mon jury de thèse.

Merci à Jean Dolbeault de m'avoir prodigué ses conseils précieux lors de la présoutenance, et d'être membre de ce jury de thèse.

Je suis très reconnaissante à Laure Saint-Raymond, qui m'a proposé des sujets de recherche nouveaux - mais pas moins captivants ! - et qui m'a accueillie avec gentillesse et bienveillance dans son bureau à l'ENS. Le travail que nous avons amorcé s'annonce fructueux, et en raison du rôle de transition qu'elle assure, dans le passage de mon statut de doctorante à celui de future chercheuse, je suis heureuse et honorée de la compter parmi les membres de mon jury.

Je voudrais profiter de cette occasion pour remercier les membres du CEREMADE, que j'ai eu la joie de côtoyer pendant trois ans, dans une atmosphère de travail stimulante et conviviale. En particulier, je suis redevable envers l'ensemble des secrétaires du laboratoire pour leur patience face à mon incompétence administrative ! Je garde aussi de très bons souvenirs des moments partagés avec tous les doctorants et ATER. Je remercie en particulier les membres "de C618", qui furent souvent de bon conseil, et qui ont su rapidement me mettre à l'aise lors de mon arrivée au labo, malgré ma timidité et mes rougissements incessants...

Je souhaite enfin exprimer toute ma gratitude à mes parents, dont le soutien et les conseils ont été un moteur considérable lors de mes études, et le sont encore aujourd'hui. Je remercie aussi mes frères, mes beaux-parents et mes amis pour leur présence tout au long de la préparation de cette thèse (et même avant !), avec une mention spéciale à tous les non-mathématiciens qui ont eu le courage de lire ce manuscrit.

Et bien sûr, un très grand merci à Ced...

Table des matières

Chapitre 1

Introduction

1.1 Présentation générale

On s'intéresse ici à des équations aux dérivées partielles qui traduisent la conservation locale, et *a fortiori* globale d'une certaine grandeur physique (quantité de mouvement ou énergie d'une particule dans un fluide, densité d'un gaz, etc.) Dans les premiers chapitres de cette thèse, on considèrera des lois de conservation scalaires, équations d'évolution non linéaires qui interviennent dans des modèles de fluides compressibles, d'électromagnétisme non linéaire, ou encore de mouvements cellulaires. Le dernier chapitre, lui, est consacré à une équation de transport linéaire, qui régit l'évolution d'une densité de particules chargées soumises à un champ électrique. La particularité de notre étude vient de ce que le milieu dans lequel on considère l'évolution temporelle de cette grandeur comporte de fortes hétérogénéités. Notre but est de décrire l'évolution du système lorsque la taille des hétérogénéités tend vers 0 : en particulier, on aimerait parvenir à exhiber un nouveau milieu dit « homogénéisé », ainsi que les équations de conservation ou de transport de la grandeur étudiée dans ce milieu.

Ce type d'étude s'inscrit dans le cadre de la théorie mathématique de l'homogénéisation ; cette théorie s'est essentiellement développée ces trente dernières années, dans le but de décrire mathématiquement les propriétés des matériaux composites, c'est-à-dire constitués d'au moins deux substances différentes intimement mêlées. L'intérêt pour de telles structures s'explique par les propriétés physiques macroscopiques particulières qu'elles présentent : par exemple, la conduction de la chaleur ou du courant électrique s'effectueront différemment - et parfois « mieux » que dans chacun des deux constituants homogènes pris séparément - en raison de leur juxtaposition au niveau microscopique. De nombreuses applications industrielles tirent parti de ces propriétés améliorées, par exemple la supraconductivité des composites multifilamentaires. Le but de la théorie de l'homogénéisation est précisément de décrire les propriétés moyennes des matériaux au niveau macroscopique en tenant compte de leur arrangement microscopique.

Du point de vue mathématique, la description au niveau macroscopique de grandeurs possédant des fluctuations à une échelle microscopique passe par l'introduction de fonctions du type

$$f\left(x, \frac{x}{\varepsilon}\right).$$

Ci-dessus, l'échelle macroscopique, qui correspond à l'échelle d'observation, est représentée par la variable d'espace x. L'échelle microscopique est l'échelle caractéristique du milieu : elle correspond à la taille des hétérogénéités ou des fluctuations. Le rapport entre ces deux échelles typiques est noté ε, où ε est un « petit »paramètre, et la variable qui décrit les phénomènes à l'échelle microscopique est donc x/ε.

Des hypothèses de modélisation supplémentaires sont ensuite nécessaires pour préciser l'allure de la fonction $f(x, y)$. Le cadre le plus simple est celui des fonctions périodiques en la variable microscopique y, ce qui revient à supposer que les hétérogénéités du milieu sont réparties de façon périodique. Ce postulat est plus ou moins légitime d'un point de vue physique suivant les systèmes étudiés, mais il simplifie en général considérablement leur analyse mathématique. Les chapitres 2, 3, 4, 5 correspondant respectivement aux articles de recherche [14, 19, 18, 17], sont des exemples d'homogénéisation dans un cadre périodique.

De façon peut-être plus réaliste, on peut imaginer que la répartition des hétérogénéités est aléatoire, mais pas nécessairement périodique. On suppose également que la loi de répartition est stationnaire, c'est-à-dire qu'elle possède des propriétés d'invariance par translation. Le formalisme correspondant est le suivant : on se donne un espace de probabilités (Ω, Σ, P), sur lequel agit un groupe de transformations $\tau_y, y \in \mathbb{R}^N$. La mesure P est invariante par le groupe de transformations $(\tau_y)_{y\in\mathbb{R}^N}$. Une fonction $u \in L^\infty(\mathbb{R}^N \times \Omega)$ est dite stationnaire si

$$u(y+z,\omega) = u(y,\tau_z\omega) \quad \text{p.p.}$$

Si on suppose de surcroît que le groupe de transformations est ergodique pour la mesure P, ce qui signifie

$$\forall A \in \Sigma \quad \left(\tau_y A = A\ \forall y \in \mathbb{R}^N\right) \Rightarrow P(A) = 0 \text{ ou } P(A) = 1,$$

alors l'environnement est dit « stationnaire ergodique ». Le chapitre 6 est un exemple de problème d'homogénéisation dans un cadre stationnaire ergodique.

Introduisons à présent les espaces fonctionnels qui seront utilisés dans ce chapitre : tout d'abord, on note Y le tore $[0,1]^N$, et $\mathcal{C}^\infty_{\text{per}}(Y)$ l'espace des fonctions Y-périodiques dans $\mathcal{C}^\infty(\mathbb{R}^N)$. Ensuite, on définit

$$\begin{aligned}
H^1_{\text{per}}(Y) &:= \overline{\mathcal{C}^\infty_{\text{per}}(Y)}^{H^1(Y)},\ ||\cdot||_{H^1_{\text{per}}(Y)} = ||\cdot||_{H^1(Y)},\\
W^{k,\infty}_{\text{per}}(Y\times\mathbb{R}) &:= \overline{\mathcal{C}^\infty_{\text{per}}(Y\times\mathbb{R})}^{W^{k,\infty}(Y\times\mathbb{R})}, \quad k\in\mathbb{N},\\
W^{1,\infty}_{\text{per,loc}}(Y\times\mathbb{R}) &:= \{u = u(y,v) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^{N+1}), u \text{ est } Y-\text{périodique en } y\},\\
\mathcal{C}_{\text{per}}(Y) &= \{u \in \mathcal{C}(\mathbb{R}^N);\ u \text{ est } Y-\text{périodique}\},\\
\mathcal{D}_{\text{per}}(Y\times\mathbb{R}^m) &= \{u \in \mathcal{C}^\infty(\mathbb{R}^N_y \times \mathbb{R}^m_v);\ u \text{ est } Y-\text{périodique en } y,\\
&\qquad \text{et } \exists R>0,\ \forall v\in\mathbb{R}^m,\ |v|\geq R,\ u(y,v)=0\}.
\end{aligned}$$

On note également, pour v fonction Y-périodique,

$$\langle v\rangle_Y := \int_Y v(y)\,dy.$$

Dans la partie suivante, nous décrirons les problématiques majeures de cette thèse, ainsi que le contexte dans lequel s'inscrit ce travail. Puis dans les deux dernières parties, nous expliquerons quels sont les résultats principaux de ce mémoire.

1.2 Résultats antérieurs et problématiques

L'objectif initial de ce travail était l'étude de l'homogénéisation de lois de conservation scalaires hyperboliques, c'est-à-dire la description du comportement asymptotique de la famille $(u^\varepsilon)_{\varepsilon>0}$ des solutions de l'équation

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) = 0 \quad t\geq 0,\ x\in\mathbb{R}^N, \tag{1.1}$$

$$u^\varepsilon(t=0) = u_0\left(x, \frac{x}{\varepsilon}\right). \tag{1.2}$$

Dans l'équation précédente, on suppose que le flux A appartient à $W^{1,\infty}_{\text{per,loc}}(Y \times \mathbb{R})^N$, et que la condition initiale $u_0 = u_0(x, y)$ appartient à $L^\infty(\mathbb{R}^N_x \times Y)$.

Ce type d'étude s'apparente à l'homogénéisation d'équations de transport, un thème qui a reçu une attention considérable ces dernières années. On pourra par exemple se référer à l'article de revue de F. Castella, P. Degond et T. Goudon [11], et aux références de cet article. Les premières contributions sur le sujet remontent aux travaux de Y. Amirat, K. Hamdache et A. Ziani [6], de Weinan E [24], et de Thomas Y. Hou et Xue Xin [42]. Plus récemment, ce type de phénomène a également été étudié par R. Alexandre [1, 2], P.-E. Jabin et A. Tzavaras [43], T. Goudon et F. Poupaud [39, 38]. Pour des variantes plus « cinétiques », on pourra également consulter [23, 35, 36, 37, 33, 54].

En ce qui concerne l'équation (1.1), deux types de résultats antérieurs étaient connus : les premiers se limitaient à une étude en dimension un, de façon à pouvoir mener des calculs explicites sur les problèmes microscopiques (voir [49, 27, 25, 4]), tandis que les seconds étaient valables en dimension quelconque, mais avec une condition de structure de type « produit à divergence nulle » sur le flux (voir [24, 43]). Les résultats obtenus sous ces deux types d'hypothèses sont assez dissemblables.

Rappelons tout d'abord le résultat principal de [24] :

Proposition 1.2.1. *Soit $a \in \mathcal{C}^1_{per}(Y)^N$ tel que $\text{div}_y a = 0$. On note*

$$\mathbb{K} := \{\varphi \in L^2(Y), \text{div}_y(a\varphi) = 0\},$$

et P la projection orthogonale sur $\mathbb{K}$ pour le produit scalaire dans $L^2(Y)$. On note $\tilde{a} := P(a)$.

Soit $f \in \mathcal{C}^1(\mathbb{R})$ telle que $f'(v) \neq 0$ pour tout $v \in \mathbb{R}$, et soit $u_0 \in L^\infty(\mathbb{R}^N \times Y)$ telle que $u_0(x, \cdot) \in \mathbb{K}$ p.p.

Pour $\varepsilon > 0$, on considère la solution entropique de l'équation de transport non linéaire

$$\partial_t u^\varepsilon + \text{div}_x \left(a\left(\frac{x}{\varepsilon}\right) f(u^\varepsilon) \right) = 0, \tag{1.3}$$

$$u^\varepsilon(t = 0, x) = u_0\left(x, \frac{x}{\varepsilon}\right). \tag{1.4}$$

Alors

$$u^\varepsilon(t, x) - u\left(t, x, \frac{x}{\varepsilon}\right) \to 0 \quad \textit{dans } L^1_{loc},$$

où $u \in \mathcal{C}([0, \infty), L^1_{loc}(\mathbb{R}^N \times Y)) \cap L^\infty([0, \infty) \times \mathbb{R}^N \times Y)$ est la solution entropique de

$$\partial_t u + \text{div}_x (\tilde{a}(y) f(u)) = 0, \tag{1.5}$$

$$u(t = 0, x, y) = u_0(x, y). \tag{1.6}$$

De plus, $u(t, x) \in \mathbb{K}$ presque partout.

Expliquons à présent quels sont les résultats génériques en dimension un d'espace : tout d'abord, il est possible lorsque $N = 1$ d'utiliser l'équivalence avec les équations de Hamilton-Jacobi et donc les résultats de P.L. Lions, G. Papanicolaou et S.R.S. Varadhan dans [49] ; précisément, si on considère la solution v^ε de l'équation de Hamilton-Jacobi

$$\begin{cases} \partial_t v^\varepsilon + A\left(\frac{x}{\varepsilon}, \partial_x v^\varepsilon\right) = 0, \\ v^\varepsilon(t=0) = v_0(x), \end{cases} \tag{1.7}$$

alors $u^\varepsilon = \partial_x v^\varepsilon$ vérifie une loi de conservation de type (1.1). On rappelle donc le résultat d'homogénéisation montré dans [49] :

Proposition 1.2.2. *Soit $A \in \mathcal{C}_{per}(\mathbb{R}^N_x \times \mathbb{R}^N_p)$. On suppose que*

$$A(x,p) \to \infty \quad \textit{lorsque } p \to \infty,$$

uniformément pour $x \in \mathbb{R}^N$.

On a alors les résultats suivants :

1. *Existence et unicité de l'hamiltonien homogénéisé :*
   Pour tout $p \in \mathbb{R}$, il existe un unique réel λ (que l'on note par la suite $\bar{A}(p)$) tel qu'il existe une solution de viscosité $v \in \mathcal{C}_{per}(\mathbb{R}^N)$ de l'équation

   $$A(y, p + \nabla_y v(y)) = \lambda, \quad y \in \mathbb{R}^N.$$

   De plus, la fonction $p \mapsto \bar{A}(p)$ est continue.
2. *Convergence de v^ε :*
   Pour tout $\varepsilon > 0$, on note $v^\varepsilon = v^\varepsilon(t,x)$ l'unique solution de viscosité de l'équation

   $$\partial_t v^\varepsilon + A\left(\frac{x}{\varepsilon}, \nabla_x v^\varepsilon\right) = 0,$$
   $$v^\varepsilon(t=0,x) = v_0(x).$$

   Alors pour tout $T > 0$, v^ε converge uniformément sur $[0,T] \times \mathbb{R}^N$ vers la solution de viscosité v de l'équation

   $$\partial_t v + \bar{A}(\nabla_x v) = 0,$$
   $$v(t=0,x) = v_0(x).$$

3. *Application à l'homogénéisation de lois de conservation scalaires :*
   Soit u^ε la solution entropique de la loi de conservation (1.1), avec $u^\varepsilon(t=0) = u_0(x)$. Alors lorsque ε tend vers 0, on a

   $$u^\varepsilon \rightharpoonup \bar{u} \quad w^* - L^\infty,$$

   où $\bar{u}$ est l'unique solution entropique de

   $$\partial_t \bar{u} + \mathrm{div}_x \bar{A}(\bar{u}) = 0$$
   $$\bar{u}(t=0,x) = u_0(x).$$

Les articles de Weinan E [25] et Weinan E et Denis Serre [27] traitent également d'un cas particulier de (1.1) avec un flux A de la forme

$$A(y,p) = f(p) - V(y), \tag{1.8}$$

avec f strictement monotone et continue sur $\mathbb{R}$ telle que $\lim_{\pm\infty} f = \pm\infty$, V de moyenne nulle. On note g la fonction réciproque de f sur $\mathbb{R}$. Soulignons que ce type flux ne vérifie pas les hypothèses de la proposition précédente, puisque l'on n'a pas

$$f(v) \to +\infty \quad \text{lorsque } |v| \to \infty.$$

Néanmoins, dans [4], Debora Amadori démontre le même résultat avec un flux A de la forme (1.8) et qui vérifie les hypothèses de la proposition 1.2.2.

Le résultat démontré dans [27] est le suivant :

Proposition 1.2.3. *On suppose que le flux A est donné par* (1.8) *Soit $\bar{A} : \mathbb{R} \to \mathbb{R}$ la fonction définie par l'identité*

$$p = \left\langle g\left(\bar{A}(p) + V(\cdot)\right)\right\rangle, \quad \forall p \in \mathbb{R},$$

où g est l'application réciproque de f sur $\mathbb{R}$.

On suppose que la condition initiale est bien préparée, c'est-à-dire

$$\partial_y \left[A(y, u_0(x,y))\right] = 0. \tag{1.9}$$

On pose

$$u(t,x,y) = g(V(y) + \bar{A}(\bar{u})),$$

où $\bar{u}$ est l'unique solution entropique de

$$\begin{aligned} &\partial_t \bar{u} + \mathrm{div}_x \bar{A}(\bar{u}) = 0 \\ &\bar{u}(t=0,x) = \langle u_0(x,\cdot)\rangle . \end{aligned}$$

Alors

$$u^\varepsilon(t,x) - u\left(t, x, \frac{x}{\varepsilon}\right) \to 0$$

dans L^1_{loc}.

Les résultats des propositions 1.2.1 et 1.2.3 sont de nature extrêmement différente. En effet, dans le cas de la dimension un, il est possible de définir un opérateur homogénéisé, ce qui n'est pas le cas en général dès que la dimension est supérieure ou égale à deux, comme le montre la proposition 1.2.1 : si l'on pose $\bar{u} = \langle u \rangle$, où u est l'unique solution entropique de (1.5), alors $\bar{u}$ ne vérifie aucune équation « remarquable ». Autrement dit, il est impossible dans ce cas de définir un problème d'évolution bien posé, ne dépendant que du flux A, et dont $\bar{u}$ soit solution. Ainsi, dès que $N \geq 2$, le problème limite ne peut se découpler en une équation d'évolution macroscopique et une équation régissant le comportement microscopique. Cette absence de problème homogénéisé vient de la forte non-unicité des solutions du problème microscopique, dit « de la cellule »

$$\mathrm{div}_y A(y,u) = 0,$$

et du fait que l'on ne peut pas intégrer facilement cette équation dès que $N \geq 2$. Néanmoins, il convient de souligner qu'il y a unicité des solutions du système limite (1.5), en dépit de la non-unicité des solutions du problème de la cellule ; mais l'unicité vient ici d'une propriété de contraction pour l'équation d'évolution (1.5). De plus, cette équation d'évolution mêle les variables macroscopiques t et x, et la variable microscopique y. L'effet principal de la non-unicité pour les solutions du problème de la cellule est donc bien l'absence de notion de problème homogénéisé, plutôt que la non-unicité des solutions du système limite. Ce point sera crucial lorsque nous aborderons les résultats d'homogénéisation pour des flux quelconques.

Le but de ce travail était d'obtenir le même type de résultat que ceux des propositions 1.2.1 et 1.2.3 pour des flux arbitraires, en dimension quelconque, sans hypothèse de structure du type $A(y,\xi) = a(y)f(\xi)$, et plus généralement, sans condition de divergence nulle sur le flux A. Cette question soulève d'emblée une première difficulté, l'obtention de bornes *a priori* uniformes en ε sur la suite u^ε. En effet, si le flux A n'est pas à divergence nulle, les bornes usuelles dans L^∞ ou L^1_{loc}, calculées par exemple par Kruzkhov (voir [69, 70]), sont d'ordre $1/\varepsilon$, et ne sont par conséquent d'aucune utilité pour les passages à la limite. Avant de s'atteler à l'homogénéisation de l'équation (1.1), il semblait donc nécessaire de mieux comprendre la structure de cette équation, et de construire des outils adaptés à son étude. À cette fin, les premiers travaux de cette thèse ont porté sur une équation inspirée de (1.1), mais dont l'étude était facilitée par la présence d'un terme de viscosité évanescente :

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) - \varepsilon \Delta_x u^\varepsilon = 0 \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{1.10}$$

$$u^\varepsilon(t=0) = u_0\left(x, \frac{x}{\varepsilon}\right). \tag{1.11}$$

Avant tout, quelques remarques sur l'échelle choisie pour la viscosité dans l'équation (1.10) s'imposent : en effet, le terme de diffusion est du même ordre de grandeur que la taille des hétérogénéités, soit ε. Comme on le verra, cela entraîne que la diffusion agit principalement au niveau microscopique. D'un point de vue mathématique, l'introduction du terme de viscosité est artificielle, et a pour but de simplifier l'étude des problèmes microscopiques ; mais dans la mesure où la viscosité n'a aucun effet à une échelle macroscopique, les preuves d'homogénéisation à proprement parler se rapprochent de celles mises en œuvre pour des lois de conservation ou des équations de transport hyperboliques, comme (1.1) ou (1.3).

Nous verrons qu'il est possible pour l'équation (1.10) de mener une étude exhaustive du comportement asymptotique de la suite u^ε ; les résultats d'homogénéisation que nous démontrerons dans ce contexte se rapprochent sans doute plus des propositions 1.2.2 et 1.2.3 que de la proposition 1.2.1. En effet, sous certaines conditions sur le flux A et la condition initiale, nous montrerons qu'il existe une fonction $v : Y \times \mathbb{R} \to \mathbb{R}$, unique solution d'une équation microscopique, telle que

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{dans } L^1_{\text{loc}}.$$

De plus, la fonction $\bar{u}$ peut être identifiée de façon univoque comme l'unique solution d'un problème homogénéisé. En ce sens, ce résultat rappelle celui de la proposition 1.2.3.

Dans la partie suivante de cette introduction, ainsi que dans les chapitres 2, 3, et 4 de ce mémoire, nous expliquerons les principales étapes du processus d'homogénéisation de l'équation (1.10). En particulier, nous montrerons l'existence de solutions stationnaires bornées de l'équation (1.10), ce qui permettra d'une part d'obtenir des bornes *a priori* dans L^∞ pour la famille de solutions (u^ε), et d'autre part, de construire une formulation cinétique pour l'équation (1.10), basée sur ces solutions stationnaires. Soulignons que cette notion de formulation cinétique peut être transposée à l'équation (1.1) ; cette construction sera la clé du processus d'homogénéisation de l'équation (1.1). Ainsi, l'étude du problème simplifié (1.10) a abouti à la mise en place de techniques adaptées aux problèmes d'homogénéisation de lois de conservation scalaires ; par la suite, ces méthodes ont pu être appliquées avec succès dans le cadre de l'équation (1.1).

Enfin, la présence du terme de viscosité dans l'équation (1.10) a permis d'aborder de nouvelles questions d'homogénéisation ; en particulier, nous mettrons en évidence un phénomène de couche initiale pour l'équation (1.10). Ce type de phénomène se produit lorsque la condition initiale u_0 n'est pas adaptée au profil microscopique dicté par l'équation, c'est-à-dire lorsque l'équation (1.9) n'est pas vérifiée par exemple, dans le cas de l'équation (1.1). Dans le cas hyperbolique, il est certainement très difficile de traiter en toute généralité ce type de question. Mais dans le modèle parabolique choisi ici, on peut montrer, au moyen d'estimations fines, un résultat d'homogénéisation pour des données mal préparées.

Les techniques d'homogénéisation développées pour l'équation (1.10) ont ensuite permis d'attaquer l'homogénéisation de l'équation (1.1) sous un angle nouveau, et ainsi de généraliser significativement les résultats de la proposition 1.2.1. En particulier, nous verrons que sans restriction de structure sur le flux A, il est possible de définir pour l'équation (1.1) une notion de problème limite, qui se compose d'une équation microscopique et d'une équation d'évolution mêlant les variables macroscopiques et microscopiques. De ce point de vue, le résultat présenté dans cette thèse se rapproche fortement de la proposition 1.2.1. La différence majeure réside dans le fait que le problème limite est en général un problème cinétique, qui ne peut se réduire à une loi de conservation. Nous expliquerons cela plus en détail par la suite. Le résultat de la proposition 1.2.1 est donc un cas particulier pour lequel la forme du problème limite est assez simple.

Dans le dernier chapitre de ce mémoire, nous aborderons également un problème d'homogénéisation d'équations de transport linéaires dans un cadre stationnaire ergodique ; cette étude est complètement indépendante de celle des lois de conservation scalaires, bien qu'il s'agisse toujours de caractériser le comportement asymptotique de solutions d'équations d'évolution avec des coefficients fortement oscillants. Considérons la solution f^ε de l'équation de transport

$$\begin{cases} \partial_t f^\varepsilon(t,x,\xi,\omega) + \xi\cdot\nabla_x f^\varepsilon(t,x,\xi,\omega) - \dfrac{1}{\varepsilon}\nabla_y u\left(\dfrac{x}{\varepsilon},\omega\right)\cdot\nabla_\xi f^\varepsilon(t,x,\xi,\omega) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R}^N, \xi\in\mathbb{R}^N, \omega\in\Omega, \\ f^\varepsilon(t=0,x,\xi,\omega) = f_0\left(x,\dfrac{x}{\varepsilon},\xi,\omega\right). \end{cases} \tag{1.12}$$

Ici, (Ω, P) est un espace de probabilités muni d'un groupe ergodique de transformations invariantes $(\tau_y)_{y\in\mathbb{R}^N}$. La fonction $f_0 = f_0(x, y, \xi, \omega)$ est stationnaire en (y, ω) presque partout en x, ξ, et la fonction $u \in L^\infty(\mathbb{R}^N_y \times \Omega)$ est stationnaire. Cette équation décrit l'évolution de la densité d'un ensemble de particules chargées soumises à un champ électrique aléatoire fortement oscillant, dérivant du potentiel $u(x/\varepsilon, \omega)$.

Cette équation avait été étudiée, dans un cadre périodique, par Emmanuel Frénod et Kamel Hamdache dans [33]. L'enjeu était donc ici de généraliser les résultats de ces auteurs à un cadre stationnaire ; dans un premier temps, il est apparu préférable de ne pas tenter d'utiliser les mêmes techniques que dans le cadre périodique, afin de s'affranchir des problèmes de compacité inhérents au cadre stationnaire. Ce changement de stratégie s'est révélé fructueux, puisque les résultats obtenus dans l'article [16], qui seront décrits au chapitre 6, sont plus fins que ceux de [33]. En particulier, nous verrons qu'il est possible de démontrer un résultat de convergence forte, tandis que le résultat de [33] ne donnait que la convergence faible de la suite f^ε. En effet, nous montrerons qu'il existe une fonction $f = f(t, x; \tau, y, \xi, \omega)$ telle que lorsque $\varepsilon \to 0$

$$f^\varepsilon(t, x, \xi, \omega) - f\left(t, x; \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) \to 0 \quad \text{dans } L^1_{\text{loc}}.$$

Les oscillations microscopiques en espace donnent ainsi naissance à des oscillations microscopiques en temps, qui ne disparaissent pas lorsque $t/\varepsilon \to \infty$.

Dans ce chapitre d'introduction, la troisième partie est consacrée à l'équation (1.10), et la quatrième à l'équation (1.12). Pour chaque modèle, on tâchera d'exposer dans les grandes lignes les principaux résultats et les méthodes de preuve.

1.3 Homogénéisation de lois de conservation scalaires

Dans cette partie, on décrit brièvement les résultats qui ont trait à l'homogénéisation des équations (1.1) et (1.10). Certaines des preuves de ce mémoire de thèse sont inspirées de techniques de [24, 27], ou reposent sur des techniques introduites dans [24] ; néanmoins, l'analyse de l'équation (1.10) est facilitée par la présence du terme de viscosité, ce qui permet une meilleure caractérisation des problèmes microscopiques. Ainsi, il sera possible de découpler les échelles macroscopiques et microscopiques des problèmes limites pour l'équation (1.10), tandis que dans le problème (1.5), la moyenne de u ne vérifie pas d'équation remarquable. En d'autres termes, il n'y a pas véritablement pour (1.3)-(1.4) de problème homogénéisé. Bien évidemment, il n'y a donc pas non plus de problème homogénéisé pour l'équation (1.1), puisque (1.3) en est un cas particulier.

1.3.1 Présentation formelle

Afin de trouver la forme des problèmes limites pour (1.1), (1.10), on postule un développement asymptotique multi-échelles en puissances de ε, appelé « Ansatz ». Ce type de méthode est classique en théorie de l'homogénéisation ; on pourra par exemple se référer à [9, 44, 12].

Ici, on commence par prendre un développement du type

$$u^\varepsilon(t,x) = u^0\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t,x,\frac{x}{\varepsilon}\right) + \cdots, \tag{1.13}$$

où chaque terme $u^i(t,x,y)$ est périodique en y.

Autant que possible, nous mènerons simultanément les calculs pour l'équation (1.1) et pour l'équation (1.10), de façon à souligner les différences entre les deux types de comportements asymptotiques. Afin d'éviter les confusions, nous noterons u^i les termes du développement asymptotique pour la solution de (1.10), et v^i ceux du développement pour la solution de (1.1).

On insère le membre de droite de (1.13) dans l'équation (1.10) (respectivement (1.1)), on effectue un développement limité dans lequel on identifie les puissances de ε. On obtient ainsi une cascade d'équations sur les termes successifs du développement (1.13). Le but de ce calcul est d'identifier les fonctions u^0 et v^0.

Dans le cas de l'équation (1.10), on trouve, pour le terme d'ordre ε^{-1}, une équation elliptique, dite « équation de la cellule », qui régit le comportement microscopique de u^0 :

$$-\Delta_y u^0 + \mathrm{div}_y A(y, u^0) = 0.$$

On remarque que les variables t, x sont des paramètres de l'équation précédente ; un résultat d'existence et d'unicité pour cette équation elliptique, qui sera énoncé et démontré plus tard, entraîne que

$$u^0(t,x,y) = v(y, \bar{u}(t,x)), \tag{1.14}$$

où $\bar{u}(t,x) = \langle u^0(t,x,\cdot)\rangle$ et $v = v(y,p) \in H^1_{\mathrm{per}}(Y),\ y \in Y,\ p \in \mathbb{R}$ est solution de l'équation elliptique

$$-\Delta_y v(y,p) + \mathrm{div}_y A(y, v(y,p)) = 0, \quad \langle v(\cdot,p)\rangle = p. \tag{1.15}$$

Admettons provisoirement l'existence et l'unicité des solutions de l'équation (1.15) ; le profil microscopique (c'est-à-dire la dépendance en y) de la fonction u^0 est alors complètement identifié grâce à la formule (1.14). Il ne reste donc plus qu'à trouver une équation (bien posée) sur $\bar{u}$ pour définir la fonction u^0 de façon univoque. L'équation sur $\bar{u}$ (encore appelée « problème homogénéisé ») est obtenue en calculant le coefficient en facteur de ε^0 dans le développement asymptotique ; mais avant cela, écrivons l'équation de la cellule pour le problème hyperbolique (1.1).

Dans le cas de l'équation (1.1), on trouve que v^0 vérifie l'équation

$$\mathrm{div}_y A(y, v^0) = 0.$$

La situation est alors beaucoup plus compliquée : en effet, les solutions entropiques d'une telle équation ne sont pas uniques en général, comme le montrent des exemples en dimension un (voir [49] et les calculs dans le cas où $A(y,v) = |v|^2 - V(y)$). Par ailleurs, l'existence de solutions de cette équation est un problème ouvert dès que $N \geq 2$ et que le flux n'est pas à divergence nulle. Notons que si le flux est à divergence nulle, les constantes sont solutions de l'équation ci-dessus, mais en général, il peut y avoir d'autres solutions. Dans le cas hyperbolique, l'équation de la cellule ne suffit donc pas à caractériser le comportement microscopique de v^0. Nous ne présenterons pas ici le calcul des termes suivants du développement, car ceux-ci n'apportent pas véritablement de lumière sur le comportement macroscopique de v^0. Il apparaît donc clairement ici que le cas hyperbolique est nettement plus difficile à traiter que le cas parabolique avec viscosité évanescente.

Retournons à présent au cas de l'équation (1.10), et calculons le terme suivant du développement. Le terme d'ordre ε^0 donne le comportement macroscopique de u^0 ; précisément, on trouve

$$\frac{\partial u^0}{\partial t} + \sum_{i=1}^{N} \frac{\partial A_i(y,u^0)}{\partial x_i} = \Delta_y u^1 + 2\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i \partial y_i} u^0 - \sum_{i=1}^{N} \frac{\partial}{\partial y_i}\left(\frac{\partial A_i}{\partial v}(y,u^0)u^1\right). \quad (1.16)$$

En prenant la moyenne de cette équation sur une cellule, on fait disparaître tous les termes qui comportent des dérivées microscopiques ; de plus, l'identité (1.14) donne finalement l'équation sur $\bar{u}$:

$$\begin{cases} \partial_t \bar{u} + \mathrm{div}_x \bar{A}(\bar{u}) = 0, \\ \bar{u}(t=0,x) = \langle u_0(x,\cdot)\rangle , \end{cases} \quad (1.17)$$

où le flux $\bar{A}$ est donné par

$$\bar{A}(p) = \langle A(\cdot, v(\cdot,p))\rangle , \quad p \in \mathbb{R}.$$

Le terme u^0 est ainsi parfaitement identifié ; l'équation (1.16) permet ensuite, si besoin est, de connaître le profil microscopique de u^1. Néanmoins, ce procédé soulève une question non triviale : que se passe-t-il lorsque l'identité (1.14) n'est pas vérifiée à l'instant $t = 0$? En effet, à $t = 0$, l'Ansatz (1.13) donne

$$u^0(t=0,x,y) = u_0(x,y),$$

et il se peut que $u_0(x,y)$ ne soit pas de la forme $v(y, \bar{u}_0(x))$. On parle en ce cas de donnée initiale « mal préparée », et on s'attend alors à ce qu'il se forme une couche initiale durant laquelle la solutions u^ε de (1.10) s'adapte au profil microscopique dicté par l'équation (1.10), c'est-à-dire (1.14). Des oscillations microscopiques en temps se forment, et la contradiction avec l'Ansatz (1.13) est levée en remplaçant (1.13) par

$$u^\varepsilon(t,x) = u^0\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t,x,\frac{x}{\varepsilon}\right) + \cdots. \quad (1.18)$$

Le terme d'ordre ε^{-1} donne alors une équation d'évolution parabolique au niveau microscopique (c'est-à-dire dans les variables τ, y) sur $u^0 = u^0(t,x,\tau,y)$:

$$\partial_\tau u^0 + \mathrm{div}_y A(y,u^0) - \Delta_y u^0 = 0. \quad (1.19)$$

On montrera que la parabolicité de cette équation force la convergence en temps long vers un état stationnaire, c'est-à-dire une solution de l'équation elliptique (1.15). Ainsi, après un laps de temps de l'ordre de quelques ε, tout se passe comme si la condition initiale avait été bien préparée, et l'Ansatz (1.13) est de nouveau valide.

Le chapitre 4 est entièrement consacré à cette question, et nous ne reviendrons pas dessus dans cette introduction.

Notons que les résultats de Weinan E dans [24, 25], et Weinan E et Denis Serre dans [27] sont valables à condition que les données initiales soient bien préparées. Dans le cas hyperbolique, le traitement des couches initiales dans le cas de données bien préparées semble beaucoup plus compliqué ; des hypothèses de non linéarité sur le flux sont certainement nécessaires. En effet, si l'équation est linéaire, les oscillations ont tendance à se propager sans se dégrader, et il n'y a donc pas d'espoir d'arriver au même type de résultat. À cet égard, signalons que le dernier chapitre de cette thèse, qui porte justement sur l'homogénéisation d'une équation de transport linéaire, met en évidence un phénomène d'oscillations microscopiques en temps dans un cas de données mal préparées ; ces oscillations microscopiques sont propagées par l'équation, et ne disparaissent pas lorsque la variable microscopique de temps tend vers l'infini. Dans le cas de flux non linéaires, peu de résultats sont connus ; les premières études remontent aux travaux de Peter Lax dans [48], dans le cas d'un flux homogène strictement convexe en dimension un, avec une condition initiale oscillante. Ce résultat a ensuite été redémontré, et généralisé au cas de la dimension deux, par Bjorn Engquist et Weinan E dans [28]. Un résultat analogue (en dimension un) est également démontré par Constantine Dafermos dans [13]. Enfin, des travaux de Denis Serre dans [65], et de Debora Amadori et Denis Serre dans [5], toujours dans le cas de la dimension un, traitent de problèmes analogues.

Dans les sous-parties suivantes, on donne les principaux résultats ayant trait à l'homogénéisation de (1.10) et (1.1).

1.3.2 Résultats dans le cas parabolique avec viscosité évanescente

Dans cette sous-partie, on commence par énoncer un résultat d'existence et d'unicité pour l'équation (1.15), et une description du comportement en temps grand des solutions de (1.19), avec quelques éléments de preuve. Enfin, on donne le résultat d'homogénéisation pour les solutions de (1.10), pour lequel on donne un schéma de démonstration dans la sous-partie 1.3.4.

A Problème de la cellule

Théorème 1. *Soit* $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$.

On pose

$$a_i(y,p) = \frac{\partial A_i}{\partial p}(y,p), \quad 1 \le i \le N, \quad a_{N+1}(y,p) = -\mathrm{div}_y A(y,p).$$

On suppose qu'il existe des réels $C_0 > 0$, $m \in [0,\infty)$, $n \in [0, \frac{N+2}{N-2})$ *si* $N \ge 3$, *tels que pour tout* $(y,p) \in Y \times \mathbb{R}$

$$|a_i(y,p)| \le C_0 \left(1 + |p|^m\right) \quad 1 \le i \le N, \tag{1.20}$$

$$|a_{N+1}(y,p)| \le C_0 \left(1 + |p|^n\right). \tag{1.21}$$

Alors pour tout $p \in \mathbb{R}$*, il existe au plus une solution* $v(\cdot, p) \in H^1_{per}(Y)$ *de l'équation* (1.15).

Supposons en plus de (1.20),(1.21) *que l'une au moins des conditions suivantes est vérifiée :*

$$m = 0 \tag{1.22}$$

$$\textit{ou bien} \quad 0 \leq n < 1 \tag{1.23}$$

$$\textit{ou bien} \quad n < \min\left(\frac{N+2}{N}, 2\right) \ \textit{et} \ \exists p_0 \in \mathbb{R},\ \forall y \in Y \ a_{N+1}(y, p_0) = 0. \tag{1.24}$$

Alors pour tout $p \in \mathbb{R}$*, il existe une (unique) solution* $v(\cdot, p) \in H^1_{per}(Y)$ *de* (1.15).

En outre, pour tout $p \in \mathbb{R}$*,* $v(\cdot, p)$ *appartient à* $W^{2,q}_{per}(Y)$ *pour tout* $1 < q < +\infty$ *et vérifie l'estimation suivante pour tout* $R > 0$ *:*

$$||v(\cdot, p)||_{W^{2,q}(Y)} \leq C \quad \forall p \in \mathbb{R}, |p| \leq R, \tag{1.25}$$

et la constante C *ne dépend que de* N*,* Y*,* C_0*,* m*,* n*,* q *et* R*.*

Enfin, la famille $(v(y,p))p \in \mathbb{R}$ *est croissante en* p *:*

$$\forall p, p' \in \mathbb{R},\ \forall y \in Y, \quad p < p' \Rightarrow v(y,p) < v(y,p'). \tag{1.26}$$

Donnons à présent quelques éléments pour la preuve du théorème 1 . L'estimation (1.25) découle de résultats de régularité combinés avec un argument de « bootstrap ». L'unicité est une conséquence d'un principe du maximum fort pour les équations elliptiques. Le point le plus problématique, et dont la résolution n'est pas encore entièrement satisfaisante, réside dans l'obtention d'estimations *a priori* dans $H^1_{\text{per}}(Y)$ pour l'équation (1.15), en vue de la preuve de l'existence de solutions. Les estimations *a priori* peuvent être obtenues de trois façons différentes, ce qui explique la présence des trois régimes d'hypothèses (1.22)-(1.24). Dans le cas où l'hypothèse (1.22) est vérifiée, il suffit de linéariser l'équation (1.15) en écrivant

$$A(y, v(y,p)) = A(y, 0) + v(y,p)b(y),$$

avec $b \in L^\infty(Y)^N$ grâce à l'hypothèse (1.22). Les estimations dans $H^1_{\text{per}}(Y)$ sont alors obtenues par des méthodes classiques.

Dans le cas où l'hypothèse (1.23) (ou l'hypothèse (1.24)) est vérifiée, on multiplie l'équation (1.15) par v et on intègre par parties ; en utilisant (1.21), et en posant

$$B_i(y, w) = \int_0^w A_i(y, p+r)\, dr \qquad \text{pour } 1 \leq i \leq N,$$
$$\tilde{u}(y,p) = v(y,p) - p \in H^1_{\text{per}}(Y),\ \langle \tilde{u} \rangle = 0,$$

on obtient successivement

$$\begin{aligned}
\int_Y |\nabla \tilde{u}|^2 \, dy &= \int_Y A(y, p+\tilde{u}) \cdot \nabla \tilde{u} \, dy \qquad (1.27)\\
&= \underbrace{\sum_{i=1}^N \int_Y \frac{\partial}{\partial y_i} \left[B_i(y, \tilde{u}(y)) \right] \, dy}_{=0} - \int_Y \sum_{i=1}^N \frac{\partial B_i}{\partial y_i}(y, \tilde{u}(y)) \, dy\\
&= \int_Y \int_0^{\tilde{u}(y)} a_{N+1}(y, p+r) \, dr\\
&\leq C_0 \int_Y \int_0^{\tilde{u}(y)} \left(1 + (|p| + |r|)^n\right) \, dr \, dy\\
||\nabla \tilde{u}||_{L^2(Y)} &\leq C \left((1+|p|)^{\frac{n}{2}} ||\tilde{u}||_{L^1}^{\frac{1}{2}} + ||\tilde{u}||_{L^{n+1}}^{\frac{n+1}{2}} \right). \qquad (1.28)
\end{aligned}$$

Si $n < 1$, on obtient immédiatement une estimation H^1 sur $\tilde{u}$ (et donc sur v) grâce à l'inégalité de Poincaré-Wirtinger. Il ne reste donc plus qu'à examiner le cas de l'hypothèse (1.24) ; en ce cas, on a $v(y, p_0) = p_0$ d'après le résultat d'unicité. De plus, d'après (1.26),

$$||v(y,p) - v(y,p_0)||_{L^1(Y)} = \operatorname{sgn}(p - p_0) \int_Y \left[v(\cdot, p) - v(\cdot, p_0) \right] = |p - p_0|,$$

et donc

$$||v(\cdot, p)||_{L^1(Y)} \leq (|p_0| + |p - p_0|) \, .$$

On a ainsi obtenu une estimation *a priori* L^1 sur v. Pour des exposants n suffisamment « petits », on interpole ensuite L^{n+1} entre L^1 et L^{q_0} où

$$q_0 = \frac{2N}{N-2} \quad \text{si } N > 2, \quad q_0 \text{ quelconque sinon.}$$

L'exposant critique dans (1.24) est exactement celui nécessaire pour obtenir une estimation H^1 de cette façon (on rappelle l'injection de Sobolev $H^1_{\text{per}}(Y) \subset L^{q_0}(Y)$). □

B Homogénéisation

Théorème 2. *Soit* $u_0 \in L^1_{loc}(\mathbb{R}^N; \mathcal{C}_{per}(Y))$.

On suppose qu'il existe des constantes $\beta_1, \beta_2 \in \mathbb{R}$ *telles que*

$$v(y, \beta_1) \leq u_0(x, y) \leq v(y, \beta_2) \quad p.p.\ x \in \mathbb{R}^N, y \in Y. \qquad (1.29)$$

On suppose toujours que le flux $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$ *vérifie* (1.20),(1.21), *ainsi que* (1.22),(1.23) *ou* (1.24), *et que* $\frac{\partial a_i}{\partial y_j} \in L^\infty_{loc}(Y \times \mathbb{R})$, $\partial_v a_i(y, \cdot) \in \mathcal{C}(\mathbb{R})$ *pour presque tout* $y \in Y$ *et* $1 \leq i \leq N+1$, $1 \leq j \leq N$.

Alors pour tout $T > 0$, $R > 0$

$$\left|\left| u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \right|\right|_{L^1((0,T) \times B_R)} \to 0 \quad \textit{lorsque } \varepsilon \to 0,$$

où $\bar{u}$ est la solution du problème homogénéisé (1.17) *avec pour donnée initiale*

$$\bar{u}(t=0,x) = \bar{u}_0(x) = \langle u_0(x,\cdot)\rangle \,.$$

Ce théorème est valable pour des données initiales bien préparées ou mal préparées. Nous donnerons dans la partie 1.3.4 quelques éléments de preuve pour les résultats d'homogénisation, pour le cas parabolique et pour le cas hyperbolique simultanément, dans le cas de données bien préparées.

1.3.3 Résultats dans le cas hyperbolique

Dans cette partie, nous nous contenterons de donner des résultats dans le cas d'un flux à divergence nulle, afin de simplifier la présentation. Toutefois, tous ces résultats peuvent être généralisés à des flux quelconques. On renvoie au chapitre 5 pour des énoncés précis dans le cas où le flux n'est pas à divergence nulle.

On commence par donner une notion de problème limite dans le cas hyperbolique. Comme nous l'avons déjà souligné, le problème limite est un problème cinétique par essence ; celui-ci est à rapprocher de la formulation cinétique pour les lois de conservation scalaires, mais ne peut cependant pas se réduire à une équation de type loi de conservation. On donne ensuite un résultat d'existence et d'unicité pour ce problème limite, et enfin on donne un résultat de convergence pour la famille u^ε de solutions de (1.1).

On introduit la fonction $\chi : \mathbb{R}^2 \to \{-1, 1, 0\}$ définie par

$$\chi(v,u) := \begin{cases} 1 & \text{si } 0 < v < u, \\ -1 & \text{si } u < v < 0, \\ 0 & \quad\text{sinon.} \end{cases}$$

Définition 1.3.1 (Solutions du problème limite)**.** *Soit $f \in L^\infty([0,\infty), L^1(\mathbb{R}^N \times Y \times \mathbb{R}))$, $u_0 \in L^1 \cap L^\infty(\mathbb{R}^N \times Y)$. On dit que f est une solution cinétique généralisée du problème limite, avec donnée initiale $\chi(\xi, u_0)$, s'il existe une distribution $\mathcal{M} \in \mathcal{D}'_{per}([0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R})$ telle que les propriétés suivantes soient vérifiées :*

1. *Support compact en ξ : il existe une constante $K > 0$ telle que*

$$\text{Supp} f, \text{Supp}\mathcal{M} \subset [0,\infty) \times \mathbb{R}^N \times Y \times [-K, K]; \tag{1.30}$$

2. *Équation microscopique : f est solution au sens des distributions de l'équation*

$$\text{div}_y(a(y,\xi) f(t,x,y,\xi)) = 0. \tag{1.31}$$

3. *Équation d'évolution : le couple $(f, \mathcal{M})$ est solution au sens des distributions de l'équation*

$$\begin{cases} \partial_t f + a(y,\xi)\cdot\nabla_x f = \mathcal{M}, \\ f(t=0,x,y,\xi) = \chi(\xi, u_0(x,y)) =: f_0(x,y,\xi); \end{cases} \tag{1.32}$$

Autrement dit, pour toute fonction test $\phi \in \mathcal{D}_{per}([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$*, on a*

$$\int_0^\infty\int_{\mathbb{R}^N\times Y\times\mathbb{R}} f(t,x,y,\xi)\left\{\partial_t\phi(t,x,y,\xi)+a(y,\xi)\cdot\nabla_x\phi(t,x,y,\xi)\right\}dt\,dx\,dy\,d\xi =$$
$$= -\langle\phi,\mathcal{M}\rangle_{\mathcal{D},\mathcal{D}'} - \int_{\mathbb{R}^N\times Y\times\mathbb{R}}\chi(\xi,u_0(x,y))\phi(t=0,x,y,\xi)\,dx\,dy\,d\xi.$$

4. *Contraintes sur* f *: il existe une mesure positive* $\nu \in M^1_{per}([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ *telle que*

$$\partial_\xi f = \delta(\xi) - \nu, \tag{1.33}$$
$$\operatorname{sgn}(\xi)f(t,x,y,\xi) = |f(t,x,y,\xi)| \le 1 \quad p.p. \tag{1.34}$$
$$\frac{1}{\tau}\int_0^\tau ||f(s)-f_0||_{L^2(\mathbb{R}^N\times Y\times\mathbb{R}}\,ds \underset{\tau\to 0}{\longrightarrow} 0. \tag{1.35}$$

5. *Contrainte sur* $\mathcal{M}$ *: pour toute fonction* $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^N$ *telle que* $\varphi \ge 0$*, la distribution* $\mathcal{M} *_{t,x} \varphi$ *appartient à* $\mathcal{C}([0,\infty)\times\mathbb{R}^N, L^2(Y\times\mathbb{R}))$*, et de plus*

$$\left\{\begin{array}{l}\int_{Y\times\mathbb{R}}(\mathcal{M} *_{t,x}\varphi)(t,x,\cdot)\,\psi \le 0,\\ \forall\psi\in L^\infty_{loc}(Y\times\mathbb{R}),\ \operatorname{div}_y(a\psi)=0,\ \text{et } \partial_\xi\psi\ge 0.\end{array}\right. \tag{1.36}$$

L'existence et l'unicité de solutions du système limite est assurée par le théorème suivant :

Théorème 3. *Soit* $A \in W^{2,\infty}_{per,loc}(Y\times\mathbb{R})$ *tel que* $\operatorname{div}_y A = 0$.

1. *Existence : soit* $u_0 \in L^\infty(\mathbb{R}^N\times Y)\cap L^1(\mathbb{R}^N,\mathcal{C}_{pér}(Y))$ *telle que l'équation suivante soit vérifiée au sens des distributions :*

$$\operatorname{div}_y\left(a(y,\xi)\chi(\xi,u_0(x,y))\right) = 0.$$

Alors il existe une fonction f*, solution cinétique généralisée du problème limite, avec comme donnée initiale* $\chi(\xi,u_0)$*.*

2. *« Rigidité » : soit* $u_0 \in L^\infty\cap L^1(\mathbb{R}^N\times Y)$*, et soit* $f \in L^\infty([0,\infty),L^1(\mathbb{R}^N\times Y\times\mathbb{R}))$ *une solution cinétique généralisée du problème limite, avec comme donnée initiale* $\chi(\xi,u_0)$*. Alors il existe une fonction* $u \in L^\infty([0,\infty);L^1(\mathbb{R}^N\times Y))\cap L^\infty([0,\infty)\times\mathbb{R}^N\times Y)$ *telle que*

$$f(t,x,y,\xi) = \chi(\xi,u(t,x,y)) \quad \textit{presque partout.}$$

3. *Unicité et principe de contraction : soit* $u_0, v_0 \in L^\infty\cap L^1(\mathbb{R}^N\times Y)$*, et soit* f,g *deux solutions cinétiques généralisées du problème limite, avec comme données initiales* $\chi(\xi,u_0)$ *et* $\chi(\xi,v_0)$ *respectivement. Alors pour tout* $t>0$*, on a*

$$||f(t)-g(t)||_{L^1(\mathbb{R}^N\times Y\times\mathbb{R})} \le ||u_0-v_0||_{L^1(\mathbb{R}^N\times Y)}. \tag{1.37}$$

Par conséquent, pour tout $u_0 \in L^\infty\cap L^1(\mathbb{R}^N,\mathcal{C}_{pér}(Y))$*, il existe une unique fonction* $f \in L^\infty([0,\infty),L^1(\mathbb{R}^N\times Y\times\mathbb{R}))$ *solution cinétique généralisée du problème limite.*

Ce théorème sera démontré en détail au chapitre 5.

Passons à présent au résultat d'homogénéisation :

Théorème 4. *Soit* $A \in W^{2,\infty}_{per,loc}(Y \times \mathbb{R})$*. On suppose que*

$$\mathrm{div}_y A(y,\xi) = 0 \quad \forall (y,\xi) \in \mathbb{R}^{N+1},$$

et on suppose que la condition initiale u_0 *vérifie les conditions suivantes :*

$$u_0 \in L^\infty(\mathbb{R}^N \times Y), \tag{1.38}$$

$$u_0 \in L^1(\mathbb{R}^N, \mathcal{C}_{per}(Y)), \tag{1.39}$$

$$\mathrm{div}_y \left(a(y,\xi)\chi(\xi, u_0(x,y))\right) = 0 \quad \text{dans } \mathcal{D}'. \tag{1.40}$$

Soit $f = \chi(\xi, u)$ *l'unique solution cinétique généralisée du problème limite, avec comme donnée initiale* $\chi(\xi, u_0)$*. Alors, lorsque* ε *tend vers zéro, on a*

$$\chi(\xi, u^\varepsilon(t,x)) \overset{2\text{ éch.}}{\rightharpoonup} \chi(\xi, u(t,x,y)). \tag{1.41}$$

Par conséquent, pour tout noyau régularisant φ^δ *de la forme*

$$\varphi^\delta(x) = \frac{1}{\delta^N}\varphi\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}^N,\ \delta > 0,$$

avec $\varphi \in \mathcal{D}(\mathbb{R}^N)$*,* $\int \varphi = 1$*,* $0 \le \varphi \le 1$*, et pour tout compact* $K \subset [0,\infty) \times \mathbb{R}^N$*, on a la convergence suivante :*

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left|\left| u^\varepsilon(t,x) - u *_x \varphi^\delta\left(t, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(K)} = 0. \tag{1.42}$$

1.3.4 Techniques d'homogénéisation pour les lois de conservation scalaires

Dans cette partie, on donne une méthode pour démontrer les résultat d'homogénéisation des théorèmes 2 et 4. Cette technique avait initialement été mise en œuvre dans l'article [19] pour l'homogénéisation de l'équation (1.10), et est décrite en détail dans le chapitre 3. Appliquée ensuite à l'équation (1.1), elle a permis de définir une notion de problème limite dans le cas hyperbolique (voir 1.3.1) et de donner un résultat de convergence. L'idée maîtresse est d'écrire une formulation cinétique adaptée à l'équation que l'on désire homogénéiser, et de passer ensuite à la limite à deux échelles dans l'équation de formulation cinétique. Elle se distingue donc des preuves d'homogénéisation mises en œuvre par Weinan E dans [24] et par Weinan E et Denis Serre dans [27], puisque les preuves de ces deux articles reposent sur l'utilisation des mesures d'Young à deux échelles, une notion introduite par Weinan E dans [24]. L'utilisation de formulations cinétiques plutôt que des mesures d'Young permet d'écrire des preuves plus simples et plus souples, et il est possible que cette technique puisse être transposée à d'autres équations d'évolution admettant des formulations cinétiques, dans le but d'étudier des phénomènes d'oscillations.

Dans toute cette partie, on se concentrera sur le cas de données bien préparées, et on traitera simultanément l'homogénéisation de l'équation (1.1) et celle de l'équation (1.10). Afin de ne pas compliquer la présentation, on n'explique pas ici l'obtention des formulations cinétiques pour les équations (1.1) et (1.10), et on renvoie à l'article [15] pour la formulation cinétique de (1.1), et à l'article [19] (ou au chapitre 3) pour celle de (1.10).

Le résultat est le suivant : soit u^ε (resp. v^ε) la solution entropique de l'équation (1.1) (resp. (1.10)), avec pour données initiales

$$\begin{aligned} u^\varepsilon(t=0,x) &= u_0\left(x,\frac{x}{\varepsilon}\right), \\ v^\varepsilon(t=0,x) &= v\left(\frac{x}{\varepsilon},\bar{v}_0(x)\right), \quad \bar{v}_0 \in L^\infty(\mathbb{R}^N). \end{aligned}$$

On suppose de surcroît que u_0 vérifie (1.38)-(1.40). On définit, pour $\varepsilon > 0$, les fonctions

$$\begin{aligned} f^\varepsilon(t,x,p) &:= \chi(p,u^\varepsilon(t,x)), \\ g^\varepsilon(t,x,p) &:= \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<v^\varepsilon(t,x)}, \\ \tilde{g}^\varepsilon(t,x,p) &:= g^\varepsilon(t,x,p)\frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon},p\right). \end{aligned}$$

Alors il existe des mesures positives $m^\varepsilon(t,x,p)$, $\mu^\varepsilon(t,x,p)$ telles que les équations suivantes soient vérifiées au sens des distributions :

$$\frac{\partial}{\partial t}f^\varepsilon + \frac{\partial}{\partial x_i}\left(a_i\left(\frac{x}{\varepsilon},p\right)f^\varepsilon(t,x,p)\right) = \frac{\partial m^\varepsilon}{\partial p} \tag{1.43}$$

$$f^\varepsilon(t=0,x,p) = \chi\left(p,u_0\left(x,\frac{x}{\varepsilon}\right)\right), \tag{1.44}$$

$$\frac{\partial}{\partial t}\tilde{g}^\varepsilon + \frac{\partial}{\partial x_i}\left(a_i\left(\frac{x}{\varepsilon},v\left(\frac{x}{\varepsilon},p\right)\right)\tilde{g}^\varepsilon\right) - \varepsilon\Delta_x\tilde{g}^\varepsilon = \frac{\partial \mu^\varepsilon}{\partial p}, \tag{1.45}$$

$$\tilde{g}^\varepsilon(t=0,x,p) = \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<v\left(\frac{x}{\varepsilon},\bar{v}_0(x)\right)} = \mathbf{1}_{p<\bar{u}_0(x)}. \tag{1.46}$$

Dans la dernière ligne, on a utilisé le fait que la condition initiale pour v^ε est bien préparée.

L'idée est de traduire les résultats de convergence forte énoncés dans les théorèmes 2, 4, par un résultat de convergence à deux échelles (donc de convergence faible) pour les suites f^ε, g^ε. Rappelons tout d'abord la notion de convergence à deux échelles, formalisée par Grégoire Allaire dans [3], d'après une idée de Gabriel N'Guetseng (voir [56]) :

Proposition 1.3.1. *Soit $\{w^\varepsilon\}_{\varepsilon>0}$ une famille bornée de $L^2(\Omega)$, où Ω est un ouvert quelconque de $\mathbb{R}^N$. Alors il existe une suite $(\varepsilon_n)_{n\in\mathbb{N}}$ de nombres strictement positifs tendant vers 0, ainsi qu'une fonction $v^0 \in L^2(\Omega\times Y)$, telles que*

$$\int_\Omega \psi\left(x,\frac{x}{\varepsilon_n}\right)w^{\varepsilon_n}(x)\,dx \to \int_{\Omega\times Y}\psi(x,y)v^0(x,y)\,dx\,dy$$

pour tout $\psi \in \mathcal{C}_{per}(Y,L^2(\Omega))$.

On dit alors que la suite $\{w^{\varepsilon_n}\}_{n\in\mathbb{N}}$ converge à deux échelles vers v_0.

Ce concept est aisément généralisable à des familles de fonctions bornées dans $L^\infty((0,\infty)\times\mathbb{R}^N_x\times\mathbb{R})$; les fonctions test oscillantes considérées seront alors du type

$$\psi\left(t,x,\frac{x}{\varepsilon},p\right),$$

avec $\psi\in\mathcal{D}_{\text{per}}([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R}_p)$. Il existe donc deux suites $(\varepsilon_n)_{n\in\mathbb{N}}$ et $(\varepsilon'_n)_{n\in\mathbb{N}}$, tendant vers 0, ainsi que des fonctions $f=f(t,x,y,p)$, $g=g(t,x,y,p)\in L^\infty((0,\infty)\times\mathbb{R}^N_x\times Y\times\mathbb{R})$, telles que la suite f^{ε_n} (resp. la suite $g^{\varepsilon'_n}$) converge à deux échelles vers f (resp. g). Par souci de simplicité, on note f^ε, g^ε les suites f^{ε_n}, $g^{\varepsilon'_n}$.

On peut alors démontrer que le résultat de convergence forte du théorème 2 est équivalent à

$$g^\varepsilon \overset{2\text{ éch.}}{\rightharpoonup} \mathbf{1}_{v(y,p)<v(y,\bar u(t,x)}=\mathbf{1}_{p<\bar u(t,x)},$$

et que le résultat de convergence forte du théorème 4 est équivalent à

$$f^\varepsilon \overset{2\text{ éch.}}{\rightharpoonup} \chi(p,u(t,x,y)).$$

Le but est à présent de passer à la limite dans les équations (1.43), (1.45), afin de montrer que f est une solution cinétique généralisée du système limite (1.32), et que $g=\mathbf{1}_{p<\bar u}$, où $\bar u$ est l'unique solution entropique du problème homogénéisé (1.17).

Pour cela, on commence par démontrer que f et g vérifient les conditions suivantes :

$$\text{sgn}(f)=|f|\leq 1\quad\text{p.p.},\tag{1.47}$$
$$f(t,x,y,p)=0\quad\text{si }|p|>||u_0||_\infty,\tag{1.48}$$
$$\partial_p f=\delta(p)-\nu_1(t,x,y,p),\quad \nu_1\text{ mesure positive.}\tag{1.49}$$

et

$$0\leq g\leq 1\quad\text{p.p.},\tag{1.50}$$
$$g(t,x,y,p)=0\quad\text{si }p>\sup\bar u_0,\tag{1.51}$$
$$g(t,x,y,p)=1\quad\text{si }p<\inf\bar u_0,\tag{1.52}$$
$$\partial_p g=-\nu_2(t,x,y,p),\quad \nu_2\text{ mesure positive.}\tag{1.53}$$

Ces propriétés sont obtenues par passage à la limite faible dans des propriétés analogues vérifiées par f^ε et g^ε ; elles sont démontrées en détail aux chapitres 3 et 5.

Ensuite, on étudie les profils microscopiques de f et g. À cette fin, on choisit des fonctions tests du type $\varphi^\varepsilon=\varepsilon\psi(t,x,x/\varepsilon,p)$ dans les équations (1.43) et (1.45), et on passe à la limite à deux échelles. Grâce aux bornes sur m^ε, μ^ε et f^ε, g^ε, tous les termes qui comportent au moins une dérivée macroscopique de ψ (c'est-à-dire, en t, x, ou p), tendent vers 0 quand $\varepsilon\to 0$, et il ne reste que les termes qui ne comportent que des dérivées microscopiques (en y) de ψ. On obtient ainsi

$$\text{div}_y\left[a(y,p)f\right]=0,$$

ce qui signifie que f est solution de l'équation microscopique (1.31), et

$$-\Delta_y\left[\frac{\partial v}{\partial p}(y,p)g\right]+\text{div}_y\left[a(y,v(y,p))\frac{\partial v}{\partial p}(y,p)g\right]=0.$$

L'équation sur f n'est pas suffisante pour déterminer de façon univoque le profil microscopique de f. En revanche, on peut montrer que $\partial v/\partial p$ est une mesure positive, solution de l'équation

$$-\Delta_y \left[\frac{\partial v}{\partial p}(y,p)\right] + \mathrm{div}_y \left[a(y, v(y,p))\frac{\partial v}{\partial p}(y,p)\right] = 0.$$

En utilisant le théorème de Krein-Rutman, on en déduit que la fonction que g est indépendante de y. Donc $g = g(t,x,p)$.

Enfin, on montre que f et g sont solutions des équations cinétiques *ad hoc* ; commençons par le cas de g, qui est un peu plus simple. On prend à présent pour l'équation (1.45) des fonctions tests qui n'ont plus d'oscillations en x/ε ; autrement dit, on passe à la limite faible dans l'équation (1.45). À extraction d'une sous-suite près, il existe une mesure positive $m(t,x,p)$ telle que μ^ε converge faiblement dans M^1 vers m, et g est donc solution de l'équation d'évolution

$$\frac{\partial f}{\partial t} + \bar{a}(p)\cdot\nabla_x f = \frac{\partial m}{\partial p}.$$

On a utilisé les propriétés

$$\int_Y v_p(y,p)\,dy = \frac{d}{dp}\int_Y v(y,p)\,dy = 1,$$
$$\int_Y a(y,v(y,p))v_p(y,p)\,dy = \frac{d}{dp}\int_Y A(y,v(y,p))\,dy = \frac{d}{dp}\bar{A}(p) = \bar{a}(p).$$

Cette équation, combinée avec les propriétés (1.47)-(1.49), entraîne grâce à un théorème de rigidité dû à Benoît Perthame (voir [58, 59]) que f est une fonction indicatrice :

$$f = \mathbf{1}_{p<\bar{u}(t,x,)},$$

où $\bar{u} \in \mathcal{C}([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^\infty([0,\infty)\times\mathbb{R}^N)$ est l'unique solution entropique de (1.17). Le résultat est donc démontré pour g.

Pour f, la preuve est un peu moins simple, car il faut montrer que la distribution

$$\mathcal{M} := \frac{\partial f}{\partial t} + a(y,p)\cdot\nabla_x f$$

vérifie les hypothèses de la définition 1.3.1 ; or le lien entre régularisation par convolution et convergence à deux échelles est assez compliqué, et il faut donc utiliser des fonctions tests un peu particulières. On renvoie au chapitre 5 pour la construction de ces fonctions tests.

Ici, nous allons nous contenter de donner une preuve formelle du résultat, en partant du postulat suivant : pour $\psi \in \mathcal{D}([0,\infty)\times\mathbb{R}^N)$, $\psi \geq 0$, et pour $\phi \in L^\infty_{\mathrm{loc}}(Y\times\mathbb{R}_p)$ telle que $\partial_p\phi \geq 0$, $\mathrm{div}_y(a\phi) = 0$, on a

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}(t,x,y,p)\psi(t,x)\phi(y,p)\,dt\,dx\,dy\,dp\\ =\ &\lim_{\varepsilon\to 0}\int_0^\infty\int_{\mathbb{R}^{N+1}} \left\{\frac{\partial f^\varepsilon}{\partial t} + a\left(\frac{x}{\varepsilon},p\right)\cdot\nabla_x f^\varepsilon\right\}\psi(t,x)\phi\left(\frac{x}{\varepsilon},p\right)\,dt\,dx\,dp.\end{aligned}$$

Cette égalité n'a pas vraiment de sens telle qu'elle est écrite ici, mais on peut rendre cette formulation rigoureuse avec des régularisations ; on utilise ici le fait que $\operatorname{div}(a\phi) = 0$. En utilisant l'équation sur f^ε, on en déduit que

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \mathcal{M}(t,x,y,p)\psi(t,x)\phi(y,p)\,dt\,dx\,dy\,dp \\
= &\lim_{\varepsilon\to 0} \int_0^\infty \int_{\mathbb{R}^{N+1}} \frac{\partial m^\varepsilon}{\partial p}\psi(t,x)\phi\left(\frac{x}{\varepsilon},p\right)\,dt\,dx\,dp \\
= &-\lim_{\varepsilon\to 0} \int_0^\infty \int_{\mathbb{R}^{N+1}} m^\varepsilon \psi(t,x)\partial_p\phi\left(\frac{x}{\varepsilon},p\right)\,dt\,dx\,dp \\
\leq &\ 0.
\end{aligned}$$

Ci-dessus, on a utilisé les inégalités $m^\varepsilon \geq 0$, $\partial_p\phi \geq 0$, $\psi \geq 0$. Donc $\mathcal{M}$ vérifie la condition de la définition 1.3.1, et f est une solution cinétique généralisée du problème limite. D'après le théorème 3, on en déduit qu'il existe une fonction u telle que $f = \chi(p,u)$, ce qui entraîne le résultat du théorème 4.

1.4 Homogénéisation d'une équation de transport linéaire

Cette partie est consacrée à l'étude de la famille $\{f^\varepsilon\}_{\varepsilon>0}$ de solutions de (1.12). Afin de simplifier la présentation, on se limite dans cette introduction au cas périodique, c'est-à-dire que l'on suppose que le potentiel u appartient à $\mathcal{C}^2_{\text{pér}}(\mathbb{R}^N)$, et que la condition initiale f_0 appatient à $L^1_{\text{loc,pér}}(\mathbb{R}^N_x \times Y \times \mathbb{R}^N_\xi)$. Néanmoins, tous les résultats de cette partie peuvent être généralisés au cas stationnaire, comme cela sera expliqué en détail au chapitre 6. Comme pour l'étude des équations (1.1), (1.10), on commence par postuler un développement formel en puissances de ε afin de trouver le système limite, puis on donne un résultat de convergence forte du type

$$f^\varepsilon - f\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right) \to 0$$

fortement dans L^1_{loc}, avec f solution du système limite.

Ce type de comportement asymptotique avait été étudié par Emmanuel Frénod et Kamel Hamdache dans [33] dans un cadre périodique ; dans cet article, les auteurs démontraient un résultat de convergence faible pour la famille $\{f^\varepsilon\}_{\varepsilon>0}$. Le même type de système avait également été étudié par Radjesvarane Alexandre [2], ou encore par K. Hamdache, Y. Amirat et A. Ziani [6]. Plus récemment, Laurent Dumas et François Golse [23] ont étudié l'homogénéisation d'une équation de transport avec des termes d'absorption et de « scattering » (voir aussi [35]) ; on pourra également consulter les méthodes exposées dans l'article de revue de F. Castella, P. Degond et T. Goudon [11].

1.4.1 Étude formelle

On s'intéresse au comportement asymptotique de la famille $\{f^\varepsilon\}_{\varepsilon>0}$ de solutions de (1.12). Comme précédemment, on commence par postuler un développement asymptotique en puissances de ε, dont le but est de nous permettre de deviner les problèmes limites. On suppose que

$$f^\varepsilon(t,x,\xi) = f^0\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon f^1\left(t,x,\frac{x}{\varepsilon}\right) + \cdots .$$

On insère ce développement dans l'équation et on identifie les puissances de ε. On obtient alors l'équation microscopique suivante :

$$\xi \cdot \nabla_y f^0(y,\xi,\omega) - \nabla_y u(y,\omega) \cdot \nabla_\xi f^0(y,\xi,\omega) = 0 \tag{1.54}$$

Cela suggère la définition suivante :

Définition 1.4.1. *On note $\mathbb{K}$, et on appelle « espace de contraintes », l'espace vectoriel suivant :*

$$\mathbb{K} := \{f \in L^1_{loc,per}(\mathbb{R}^N_\xi \times Y, L^1(\Omega));\ f \text{ vérifie } (1.54) \text{ dans } \mathcal{D}'_{per}(Y \times \mathbb{R}^N_\xi) \text{ p.s. en } \omega\}.$$

On note $\mathbb{P}$ la projection sur $\mathbb{K}$; pour $f \in L^1_{loc}(\mathbb{R}^N_\xi \times Y, L^1(\Omega))$ stationnaire quelconque, la fonction $\mathbb{P}(f)$ est caractérisée par les propriétés
(i) $\mathbb{P}(f) \in \mathbb{K}$;
(ii) Pour toute fonction $g \in L^\infty(Y \times \mathbb{R}^N_\xi \times \Omega) \cap \mathbb{K}$, g de support compact en ξ, on a

$$\int_{\mathbb{R}^N \times \Omega} (\mathbb{P}(f) - f)(y,\xi,\omega)\, g(y,\xi,\omega)\, d\xi\, dP(\omega) = 0 \quad \text{p.p. } y \in \mathbb{R}^N.$$

On définit enfin

$$\mathbb{K}^\perp := \{f \in L^1_{loc,per}(\mathbb{R}^N_\xi \times Y, L^1(\Omega)); \exists g \in L^1_{loc,per}(\mathbb{R}^N_\xi \times Y, L^1(\Omega)), \quad f = \mathbb{P}(g) - g\}.$$

Comme dans la partie précédente, il se peut que l'équation (1.54) ne soit pas satisfaite par la condition initiale f_0. Comme l'équation (1.12) est linéaire, il est utile d'écrire f_0 sous la forme

$$f_0 = \mathbb{P}(f_0) + [f_0 - \mathbb{P}(f_0)] \ ;$$

on pose $g_0 = \mathbb{P}(f_0)$, $h_0 = f_0 - \mathbb{P}(f_0)$, et on note g^ε, h^ε les solutions de l'équation (1.12) avec comme conditions initiales

$$g_0\left(x,\frac{x}{\varepsilon},\xi\right), \quad h_0\left(x,\frac{x}{\varepsilon},\xi\right)$$

respectivement. Alors on a $f^\varepsilon = g^\varepsilon + h^\varepsilon$, et $g_0 \in \mathbb{K}$, $h_0 \in \mathbb{K}^\perp$, de sorte que la condition initiale pour g^ε est bien préparée, et celle pour h^ε est mal préparée. On s'attend donc à observer des oscillations temporelles rapides dans h^ε, mais pas dans g^ε.

Pour h^ε, on postule un développement du type

$$h^\varepsilon(t,x,\xi) = h^0\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon},\right) + \varepsilon h^1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon},\right) + \cdots ,$$

et on trouve que

$$\partial_\tau h^0 + \xi \cdot \nabla_y h^0 - \nabla_y u \cdot \nabla_\xi h^0 = 0. \tag{1.55}$$

Cette équation régit donc le comportement microscopique en temps de h^ε ; contrairement au cas parabolique étudié précédemment, cette équation préserve toutes les normes L^p, donc on ne s'attend pas à ce que

$$h^0(\tau) \to 0 \quad \text{dans } L^1_{\text{loc}}$$

lorsque $\tau \to \infty$. Il n'y a donc pas de disparition des oscillations temporelles microscopiques dues à la mauvaise préparation de la condition initiale.

Passons à présent aux équations macroscopiques : pour g^ε, en postulant toujours un développement du type

$$g^\varepsilon(t,x,\xi) = g^0\left(t, x, \frac{x}{\varepsilon}\right) + \varepsilon g^1\left(t, x, \frac{x}{\varepsilon}\right) + \cdots,$$

on obtient

$$\partial_t g^0 + \xi \cdot \nabla_x g^0 + \xi \cdot \nabla_y g^1 - \nabla_y u \cdot \nabla_y g^1 = 0.$$

Projetons à présent cette équation sur $\mathbb{K}$; on rappelle que grâce à la première étape, $g^0 \in \mathbb{K}$ presque partout. D'autre part, la projection $\mathbb{P}$ commute avec les dérivées en t et en x. Par conséquent,

$$\mathbb{P}(\xi \cdot \nabla_x g^0) = \mathbb{P}(\xi) \cdot \nabla_x g^0.$$

Enfin on vérifie (formellement) que

$$\left\{\xi \cdot \nabla_y g^1 - \nabla_y u \cdot \nabla_y g^1\right\} \in \mathbb{K}^\perp.$$

On en déduit donc que

$$\partial_t g^0 + \xi^\sharp(y,\xi) \cdot \nabla_x g^0 = 0 \tag{1.56}$$

où

$$\xi^\sharp(y,\xi) := \mathbb{P}((y,\xi) \mapsto \xi).$$

Nous verrons plus tard que h^0 vérifie la même équation de transport ; néanmoins, il semble difficile de déduire cette équation de transport (pour h^0) du développement formel, et nous ne calculons donc pas le terme suivant du développement pour h^0.

1.4.2 Résultats

Dans [33], Emmanuel Frénod et Kamel Hamdache démontrent que la solution f^ε de l'équation (1.12) converge à deux échelles vers une fonction $f = f(t,x,y,\xi)$ qui vérifie les équations (1.54), (1.56), avec $f(t=0,x,y,\xi) = \mathbb{P}(f_0)(x,y,\xi)$. Ici, nous allons démontrer un résultat de convergence forte, mettant en évidence un phénomène d'oscillations microscopiques en temps, et apportant ainsi une vision plus raffinée du comportement des solutions. Le résultat est le suivant :

Théorème 5. *Soit* $f_0 \in L^1_{loc,pér}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times Y)$, *et soit* $f^\varepsilon = f^\varepsilon(t,x,\xi)$ *la solution de* (1.12).

Alors il existe deux fonctions périodiques $g = g(t,x,y,\xi)$ *et* $h = h(t,x;\tau,y,\xi)$, *ainsi qu'une famille* $\{r^\varepsilon(t,x,\xi)\}_{\varepsilon>0}$ *telles que*

$$f^\varepsilon(t,x,\xi) = g\left(t,x,\frac{x}{\varepsilon},\xi\right) + h\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right) + r^\varepsilon(t,x,\xi);$$

les fonctions g, h *et la famille* $\{r^\varepsilon\}_{\varepsilon>0}$ *vérifient les propriétés suivantes :*

- *Convergence vers 0 du terme de reste :* $||r^\varepsilon||_{L^1_{loc}((0,\infty)\times\mathbb{R}^N_x\times\mathbb{R}^N_\xi)} \to 0$ *lorsque* $\varepsilon \to 0$ *;*
- *Contrainte sur* g *:* $g \in L^\infty_{loc}((0,\infty); L^1_{loc,pér}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times Y;))$, *et* $g(t,x) \in \mathbb{K}$ *pour presque tout* $t \geq 0$, $x \in \mathbb{R}^N$ *;*
- *Contraintes sur* h *:* $h \in L^\infty_{loc}([0,\infty) \times [0,\infty), L^1_{loc,pér}(\mathbb{R}^N_x \times Y \times \mathbb{R}^N_\xi))$. *De plus,* $h(t,x;\tau,\cdot) \in \mathbb{K}^\perp$ *pour presque tout* $(t,x,\tau) \in (0,\infty) \times \mathbb{R}^N \times (0,\infty)$, *et* h *est solution de l'équation* (1.55).
  De plus, pour tout $T > 0$

$$\left|\left|\int_0^T h\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right)\,dt\right|\right|_{L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi,)} \to 0 \quad \textit{lorsque } \varepsilon \to 0.$$

- *Équation d'évolution macroscopique pour* g *et* h *:* g *et* h *vérifient l'équation* (1.56), *avec les conditions initiales :*

$$g(t=0,x,y,\xi) = \mathbb{P}(f_0)(x,y,\xi),$$
$$h(t=0,x;\tau=0,y,\xi) = [f_0 - \mathbb{P}(f_0)](x,y,\xi).$$

Rappelons que ce résultat reste vrai dans un cadre stationnaire ergodique (voir le chapitre 6) ; l'hypothèse de périodicité n'est faite ici que pour simplifier la présentation.

La méthode de la preuve est fondée sur la remarque suivante : pour $t > 0$, $x,\xi \in \mathbb{R}^N$, $\omega \in \Omega$, on a

$$f^\varepsilon(t,x,\xi,\omega) = f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right),$$

où (Y,Ξ) est le système Hamiltonien associé à $H(y,\xi,\omega) = 1/2|\xi|^2 + u(y,\omega)$:

$$\begin{cases} \dot{Y}(t,y,\xi,\omega) = -\Xi(t,y,\xi,\omega), \quad t>0 \\ \dot{\Xi}(t,y,\xi,\omega) = \nabla_y u(Y(t,y,\xi,\omega),\omega), \quad t>0 \\ Y(t=0,y,\xi,\omega) = y,\ \Xi(t=0,y,\xi,\omega) = \xi, \quad (y,\xi,\omega) \in \mathbb{R}^{2N} \times \Omega. \end{cases} \tag{1.57}$$

Ainsi, l'étude du comportement asymptotique de f^ε mène naturellement à celle de la limite en temps grand du système (Y,Ξ). La sous-partie 1.4.3 est consacrée à l'analyse du système (Y,Ξ), et on donne une ébauche de la preuve du théorème 5 dans la sous-partie 1.4.4. Enfin, la dernière sous-partie traite du cas intégrable, c'est-à-dire du cas où $N = 1$.

1.4.3 Étude d'un système hamiltonien

On étudie ici la limite en temps grand du système hamiltonien (Y, Ξ), solution de (1.57) avec u fonction périodique de période $\mathbb{T} = [0,1]^N$. On vérifie aisément que pour tout $y \in \mathbb{T}$, $\xi \in \mathbb{R}^N$, et $k \in \mathbb{Z}^N$, $(Y,\Xi)(\cdot, y+k, \xi)$ et $(Y(\cdot,y,\xi)+k, \Xi(\cdot, y\xi))$ satisfont le même système d'équations différentielles et la même condition initiale. Par conséquent,

$$Y(t,y,\xi) + k = Y(t, y+k, \xi) \quad \forall t \geq 0,\ (y,\xi) \in \mathbb{R}^{2N},\ k \in \mathbb{Z}^N.$$

On peut donc considérer (Y, Ξ) comme un système dynamique sur $\mathbb{T} \times \mathbb{R}^N$, en introduisant le semi-groupe

$$T_t : (y,\xi) \in Y \times \mathbb{R}^N \mapsto \left(\tilde{Y}(t,y,\xi), \Xi(t,y,\xi)\right)$$

où $\tilde{Y}(t,y,\xi) \in \mathbb{T}$ et $\tilde{Y}(t,y,\xi) - Y(t,y,\xi) \in \mathbb{Z}^N$. D'après le théorème de Liouville, le semi-groupe T_t préserve la mesure de Lebesgue sur $\mathbb{T} \times \mathbb{R}^N$.

Par ailleurs, on rappelle que le hamiltonien $H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y)$ est constant le long des trajectoires du système hamiltonien (Y, Ξ). Sans perte de généralité, on choisit $u \in \mathcal{C}^2_{\text{per}}(\mathbb{T})$ de telle sorte que

$$\inf_u = 0, \sup u = u_{\max} > 0.$$

Alors pour tout $c > 0$, la mesure

$$dm_c(y,\xi) = \mathbf{1}_{H(y,\xi)<c}\, dy\, d\xi$$

est invariante par le semi-groupe T_t, et

$$m_c(\mathbb{T} \times \mathbb{R}^N) \leq C(2c)^{\frac{N}{2}} < +\infty,$$

où la constante C ne dépend que de N.

On déduit alors du théorème ergodique de Birkhoff (voir par exemple [66, 41]) le résultat suivant :

Proposition 1.4.1. *1. Soit $c > 0$ quelconque, et soit $f \in L^1(\mathbb{T}\times\mathbb{R}^N, dm_c)$. Alors il existe une fonction $\bar{f} \in L^1(\mathbb{T} \times \mathbb{R}^N, dm_c)$ telle que lorsque $\theta \to \infty$,*

$$\frac{1}{\theta}\int_0^\theta f\left(Y(t,y,\xi), \Xi(t,y,\xi)\right)\, dt \to \bar{f}(y,\xi), \quad \text{p.p. et dans } L^1(dm_c).$$

De plus, $\bar{f}$ est invariante par le flot hamiltonien (Y, Ξ) et

$$\int_{\mathbb{T}\times\mathbb{R}^N} f dm_c = \int_{\mathbb{T}\times\mathbb{R}^N} \bar{f} dm_c.$$

2. Soit $f \in L^1_{per,loc}(Y \times \mathbb{R}^N_\xi)$. Alors, lorsque $\theta \to \infty$

$$\frac{1}{\theta}\int_0^\theta f\left(Y(t,y,\xi), \Xi(t,y,\xi)\right)\, dt \to \mathbb{P}(f)(y,\xi), \quad \text{p.p. et dans } L^1(dm_c),$$

où $\mathbb{P}$ est la projection sur $\mathbb{K}$ (voir la définition 1.4.1).

On conclut cette sous-partie par un lien entre le système dynamique (1.57) et l'équation (1.55). Soit $\phi_0 \in L^1_{\text{loc}}(\mathbb{T} \times \mathbb{R})$, et soit

$$\phi(t,y,\xi) = \phi_0\left(Y(t,y,\xi), \Xi(t,y,\xi)\right) \quad t \geq 0,\ y \in \mathbb{T},\ \xi \in \mathbb{R}.$$

Alors ϕ est l'unique solution de l'équation (1.55) avec la condition initiale ϕ_0.

1.4.4 Éléments de preuve

On donne ici les principales idées de la preuve du théorème 5. Le principe de base est d'étudier séparément les données initiales qui ne dépendent que de la variable macroscopique x et celles qui ne dépendent que des variables microscopiques y, ξ. Une fois ces deux cas particuliers résolus, le cas général découle de la linéarité de l'équation et d'un résultat de densité ; on n'expliquera pas ce dernier point ici.

Premier cas : $f_0 = f_0(y,\xi) \in L^1_{\text{loc}}(\mathbb{R}^N_\xi; L^\infty(\mathbb{T}))$

On pose alors

$$\begin{gathered} f = \mathbb{P}(f_0), \\ g = g(\tau, y, \xi) = [f_0 - \mathbb{P}(f_0)]\,(Y(\tau,y,\xi), \Xi(\tau,y,\xi)). \end{gathered}$$

Il est alors immédiat que $f \in \mathbb{K}$ et g vérifie l'équation d'évolution (1.55) d'après la remarque à la fin de la sous-partie précédente. De plus, $f_0 - \mathbb{P}(f_0) \in \mathbb{K}^\perp$, et donc $g \in \mathbb{K}^\perp$ car pour tout $\tau \geq 0$,

$$[\mathbb{P}(\varphi)]\,(Y(\tau,\cdot),\Xi(\tau,\cdot)) = \mathbb{P}\left[\varphi(Y(\tau,\cdot),\Xi(\tau,\cdot))\right].$$

Donc f et g vérifient les équations du théorème 5. De plus,

$$\begin{aligned} & f^\varepsilon(t,x,\xi) - f\left(\frac{x}{\varepsilon},\xi\right) - g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right) \\ = \;\; & f^\varepsilon(t,x,\xi) - f_0\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right), \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right)\right) = 0, \end{aligned}$$

et le théorème est donc vrai avec $r^\varepsilon = 0$.

Deuxième cas : $f_0 = f_0(x) \in W^{1,\infty}(\mathbb{R}^N_x)$

Dans ce cas, on a, pour $t > 0,\ x, \xi \in \mathbb{R}^N,\ \varepsilon > 0$

$$f^\varepsilon(t,x,\xi) = f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right)\right),$$

et le résultat repose alors sur le lemme suivant, qui est au cœur du résultat de convergence forte :

Lemma 1.4.1. *Soit $T > 0$, quelconque. Alors lorsque $\varepsilon \to 0$,*

$$\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right) \to 0$$

dans $L^\infty((0,T), L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi))$.

Démonstration. On écrit, pour $t > 0$ quelconque

$$\begin{aligned} \varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right) &= \varepsilon\left\{Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right) - \frac{x}{\varepsilon} + \frac{t}{\varepsilon}\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right)\right\} \\ &= \varepsilon\left\{\int_0^{\frac{t}{\varepsilon}} \dot{Y}\left(\tau,\frac{x}{\varepsilon},\xi\right)\,d\tau + \frac{t}{\varepsilon}\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right)\right\} \\ &= -t\left\{\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \Xi\left(\tau,\frac{x}{\varepsilon},\xi\right)\,d\tau - \xi^\sharp\left(\frac{x}{\varepsilon},\xi\right)\right\} \end{aligned}$$

On prend $R, c > 0$ quelconques, et $0 < \alpha < T$ quelconque. Posons $c' = c^2/2 + u_{\max}$. Alors si $t > \alpha$,

$$\int_{x\leq R}\int_{\xi\leq c}\left|\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}}\Xi\left(\tau,\frac{x}{\varepsilon},\xi\right)\,d\tau-\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right)\right|\,dx\,d\xi\leq$$

$$\leq CR^N\sup_{s\geq\alpha}\left|\left|\frac{\varepsilon}{s}\int_0^{\frac{s}{\varepsilon}}\Xi\left(\tau,\cdot\right)\,d\tau-\xi^\sharp(\cdot)\right|\right|_{L^1(Y\times\mathbb{R}^N_\xi,dm_{c'})},$$

et la constante C ne dépend que de N. Le majorant de l'inégalité ci-dessus tend vers 0 pour tout $\alpha > 0$; il ne reste plus qu'à voir ce qui se passe pour les temps petits.

Si $|\xi| \leq c$, alors

$$H(y,\xi) = H(Y(\tau,y,\xi),\Xi(\tau,y,\xi)) \leq c' \quad \forall y\in Y.$$

On en déduit que

$$\Xi(\tau,y,\xi)\leq\sqrt{c^2+2u_{\max}}\quad\forall\tau>0,\ \forall y\in Y,$$
$$\xi^\sharp(y,\xi)\leq\sqrt{c^2+2u_{\max}}\quad\forall y\in Y.$$

Donc si $\xi \leq c$, pour tout $\varepsilon > 0$, $t > 0$, $x \in \mathbb{R}^N$, on a

$$\left|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right)-x+t\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right)\right|\leq 2\sqrt{c^2+2u_{\max}}t.$$

On obtient ainsi une majoration du type

$$\left|\left|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right)-x+t\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right)\right|\right|_{L^\infty((0,T),L^1(B_R\times B_c))}\leq C_{R,c}\left(\alpha+\gamma\left(\frac{\varepsilon}{\alpha}\right)\right)\quad\forall\alpha,\varepsilon>0$$

avec $\lim_0^+\gamma = 0$. Cela entraîne le résultat annoncé.

□

Revenons à présent à la preuve du théorème 5. Soit f la solution de l'équation de transport (1.56) avec $f(t=0) = f_0$. Alors on a

$$f^\varepsilon(t,x,\xi)-f\left(t,x;\frac{x}{\varepsilon},\xi\right)\;=\;f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right)\right)-f_0\left(x-t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)$$

$$\left|f^\varepsilon(t,x,\xi)-f\left(t,x;\frac{x}{\varepsilon},\xi\right)\right|\;\leq\;||f_0||_{W^{1,\infty}}\left|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi\right)-x+t\xi^\sharp\left(\frac{x}{\varepsilon},\xi\right)\right|.$$

Le théorème est alors une conséquence facile du lemme 1.4.1, avec $g = 0$ et

$$r^\varepsilon(t,x,\xi)=f^\varepsilon(t,x,\xi)-f\left(t,x;\frac{x}{\varepsilon},\xi\right).$$

Troisième cas : f_0 quelconque.

Ce cas découle facilement des étapes précédentes, grâce à la linéarité de l'équation de transport et au résultat de convergence forte du lemme 1.4.1. On commence par considérer des conditions initiales du type

$$f_0(x,y,\xi) = a(x)b(y,\xi),$$

puis on généralise le résultat à des conditions initiales quelconques grâce à une propriété de contraction de l'équation de transport. Les détails sont expliqués au chapitre 6.

1.4.5 Le cas intégrable

Dans cette dernière sous-partie, on donne quelques formules explicites pour la projection $\mathbb{P}$ dans le cas où $N = 1$, ou plus généralement, dans le cas où

$$u(y) = \sum_{i=1}^{N} u_i(y_i), \quad y = (y_1, \cdots, y_N) \in [0,1]^N,$$

et chaque $u_i : \mathbb{R} \to \mathbb{R}$ $(1 \le i \le N)$ est une fonction $\mathcal{C}^2$, périodique de période 1.

Dans ces conditions, le système (Y, Ξ) est intégrable, et chaque système (Y_i, Ξ_i) est un système hamiltonien en dimension 1, dérivant de l'Hamiltonien

$$H_i(y_i, \xi_i) = \frac{1}{2}|\xi_i|^2 + u_i(y_i), \quad y_i \in [0,1],\ \xi_i \in \mathbb{R}.$$

Dans ce cas, on peut donner des formules explicites pour $\xi^\sharp$ ainsi que pour la projection $\mathbb{P}$ grâce aux formules du paragraphe 1.4.3. Pour cela, on commence par étudier le cas où $N = 1$, puis on étend ces résultats au cas où N est quelconque, lorsque cela est possible. On donne ici sans démonstration les résultats obtenus dans ce cadre, et on renvoie au chapitre 6 pour les calculs explicites menant à ces expressions.

Commençons par l'expression de la projection $\mathbb{P}$ en dimension un : si $f \in L^1_{\text{per,loc}}([0,1] \times \mathbb{R})$, alors

$$\mathbb{P}(f)(y,\xi) = \bar{f}(\text{sgn}(\xi), H(y,\xi))$$

où

$$\bar{f}(\eta, \mathcal{E}) := \frac{\left\langle \mathbf{1}_{u<\mathcal{E}} \left[f\left(\cdot, \sqrt{2\left(\mathcal{E}-u\right)}\right) + f\left(\cdot, -\sqrt{2\left(\mathcal{E}-u\right)}\right)\right] \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle}{2\left\langle \mathbf{1}_{u<\mathcal{E}} \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle}$$

si $\mathcal{E} < u_{\max}$, et

$$\bar{f}(\eta, \mathcal{E}) := \frac{\left\langle f\left(\cdot, \eta\sqrt{2\left(\mathcal{E}-u\right)}\ \right) \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle}{\left\langle \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle}$$

si $\mathcal{E} > u_{\max}$.

En prenant $f(y,\xi) = \xi$, on en déduit que

$$\xi^\sharp(y,\xi) = \text{sgn}(\xi)\bar{\xi}(H(y,\xi)),$$

où

$$\bar{\xi}(\mathcal{E}) = \begin{cases} 0 & \text{si } \mathcal{E} < u_{\max}, \\ \dfrac{1}{\left\langle \frac{1}{\sqrt{2(\mathcal{E}-u)}} \right\rangle} & \text{si } \mathcal{E} > u_{\max}. \end{cases}$$

Cette dernière formule reste vraie en dimension quelconque, puisque la i−ème coordonnée de $\xi^\sharp$ ne dépend que du système hamiltonien monodimensionnel (Y_i, Ξ_i).

1.5 Perspectives et problèmes ouverts

En ce qui concerne l'homogénéisation de lois de conservation scalaires hyperboliques dans un cadre périodique, deux points essentiels restent ouverts : l'existence de solutions du problème de la cellule dans le cas hyperbolique, et le comportement de la famille u^ε dans le cas de données initiales mal préparées. Pour ces deux points, des hypothèses de non linéarité sur le flux semblent nécessaires pour que les résultats démontrés dans le cas parabolique restent vrais, et la non linéarité interviendrait d'ailleurs pour la même raison : dans les deux cas, il s'agirait d'obtenir un résultat de compacité pour une équation hyperbolique (équation de la cellule ou équation d'évolution microscopique en temps). L'existence de solutions du problème de la cellule pourrait également être d'une aide précieuse pour l'étude du phénomène de couche initiale. En effet, il est possible que l'écriture d'une formulation cinétique adaptée à l'équation d'évolution simplifie l'étude de la compacité des solutions de la dite équation. Mais la question du comportement en temps long des solutions de lois de conservation scalaires semble difficile, et à ce jour, peu de résultats sont connus dès que la dimension de l'espace est supérieure ou égale à deux. Quant à l'existence de solutions du problème de la cellule, précisons que la difficulté principale est d'obtenir des estimations *a priori* dans L^∞ sur les solutions ; mais dès que la dimension est supérieure ou égale à deux, l'opérateur de divergence se prête peu aux calculs explicites, et la méthode utilisée en dimension un ne semble pas permettre l'obtention de ces bornes *a priori*.

Il serait également intéressant d'étudier l'homogénéisation de lois de conservation scalaires dans un cadre stochastique. Cette question est aujourd'hui complètement ouverte, sauf en dimension un, où des résultats de convergence faible peuvent être déduits de l'homogénéisation d'équations de Hamilton-Jacobi.

D'autre part, il est vraisemblable que l'on puisse généraliser le résultat d'homogénéisation sur les équations de transport linéaires, exposé dans la dernière partie de cette introduction, en prenant des hypothèses moins restrictives sur le potentiel u. On pourrait par exemple imaginer de travailler avec des fonctions u sous-linéaires (dans le cadre stationnaire). Le cas intégrable (c'est-à-dire, le cas où $N = 1$) donne également naissance à des problèmes ouverts originaux de théorie des systèmes dynamiques, détaillés dans le chapitre 6.

Organisation du mémoire :

Les chapitres 2 et 3 (parties 3.1 à 3.4) correspondent aux articles de recherche [14, 19] respectivement, et traitent de l'homogénéisation de l'équation (1.10) dans le cas de données bien préparées ; seule la cinquième et dernière partie du chapitre 3 (Appendice) ne figure pas dans l'article [19], et n'a pas fait l'objet d'une publication antérieure. Cette partie généralise les résultats des chapitres 2 et 3 à des flux possédant une dépendance macroscopique explicite. Le chapitre 4 est issu de l'article [18], et apporte une preuve d'homogénéisation pour l'équation (1.10) dans le cas de données mal préparées. Le chapitre 5 traite de l'homogénéisation de lois de conservation scalaires hyperboliques. Enfin, le chapitre 6 correspond à l'article [16], et est dédié à l'homogénéisation de l'équation (1.12).

Une bibliographie générale se trouve à la fin du manuscrit.

Chapitre 2

Homogénéisation d'une loi de conservation scalaire avec viscosité évanescente. Partie I : données bien-préparées

On étudie ici la limite quand $\varepsilon \to 0$ des solutions de l'équation

$$\partial_t u^\varepsilon + \operatorname{div}_x \left[A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\right] - \varepsilon \Delta_x u^\varepsilon = 0.$$

Après avoir identifié le problème homogénéisé grâce à un développement asymptotique, on montre que u^ε se comporte dans L^2_{loc} comme $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$ lorsque $\varepsilon \to 0$, où v est la solution d'un problème de la cellule et $\bar{u}$ celle du problème homogénéisé. La preuve utilise les mesures d'Young à deux échelles, une généralisation des mesures d'Young adaptée aux problèmes d'homogénéisation à deux échelles.

Ce chapitre a fait l'objet d'une publication dans le Journal de Mathématiques Pures et Appliquées (volume **86**, 2006).

2.1 Introduction

This paper is devoted to the analysis of the behavior as $\varepsilon \to 0$ of the solutions $u^\varepsilon \in L^\infty_{\text{loc}}([0,\infty)\times\mathbb{R}^N)\cap\mathcal{C}([0,\infty),L^1_{\text{loc}}(\mathbb{R}^N))\cap L^2_{\text{loc}}([0,\infty),H^1_{\text{loc}}(\mathbb{R}^N))$ of the parabolic scalar conservation law :

$$\frac{\partial u^\varepsilon}{\partial t}(t,x)+\sum_{i=1}^N \frac{\partial}{\partial x_i}A_i\left(\frac{x}{\varepsilon},u^\varepsilon(t,x)\right)-\varepsilon\Delta u^\varepsilon=0 \quad t\geq 0,\ x\in\mathbb{R}^N, \tag{2.1}$$

$$u^\varepsilon(t=0)=u_0\left(x,\frac{x}{\varepsilon}\right). \tag{2.2}$$

The functions $A_i=A_i(y,v)$ $(y\in Y,\ v\in\mathbb{R})$ are assumed to be Y-periodic, where $Y=\Pi_{i=1}^N(0,T_i)$ is the unit cell, and $u_0\in L^\infty(\mathbb{R}^N\times Y)$ is also Y-periodic (in fact, a little more regularity is necessary in order to ensure that $u_0\left(x,\frac{x}{\varepsilon}\right)$ is measurable, see for instance section 5 in [3]).

Our goal is to derive the homogenized problem, i.e. to show that there exists a function $u^0=u^0(t,x,y)$ such that as $\varepsilon\to 0$,

$$u^\varepsilon(t,x)\to u^0(t,x,y)$$

(the precise meaning of the above convergence will be made clear later on) and to find the equations solved by u^0. The homogenized operator can be computed by means of a formal double-scale expansion (see [9]), as we shall see in the second section; our main result is that the y-average of u^0 is the solution of a hyperbolic scalar conservation law, the flux of which can be computed in terms of A and of the solution of a quasilinear elliptic cell problem.

Notice that the viscosity has the same order of magnitude than the size of the heterogeneities, characterized by the small parameter ε; hence, the problem we study in this article is closer to the homogenization of conservation laws and transport equations than to the homogenization of parabolic equations in which the viscosity is of order 1; therefore, the technique we shall use for the proof is inspired from the one developed by W. E and D. Serre in [27] (see also [24], [25], and [49] for an equivalent formulation using Hamilton-Jacobi equations) for the homogenization of a one-dimensional conservation law. From a mathematical point of view, the role of the viscosity here is to simplify the analysis of the cell problem, but it is not fundamental in the convergence proof. Speaking in more physical terms, we will see that viscosity has an effect at a microscopic level only. This is obvious when looking at the homogenized problem : the cell equation, which rules the microscopic behavior of u^0, remains elliptic, while the viscosity vanishes from the macroscopic evolution equation, which is a hyperbolic conservation law.

The proof of our main result relies on the use of two-scale convergence, which was introduced by Allaire in [3], following an idea of Nguetseng (see [56]). The fundamental idea of Allaire and Nguetseng is to try and justify the formal two-scale expansions

$$u^\varepsilon(x)=u^0\left(x,\frac{x}{\varepsilon}\right)+\varepsilon u^1\left(x,\frac{x}{\varepsilon}\right)+\cdots$$

widely used in homogenization theory by expressing u^0 as a particular weak limit : precisely, let us recall the basic result of two-scale convergence (see [3]) :

Proposition 2.1.1. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of $L^2(\Omega)$, where Ω is an open set of $\mathbb{R}^N$. Then as $\varepsilon \to 0$, there exists a subsequence, still denoted by ε, and $u^0 \in L^2(\Omega \times Y)$, such that*

$$\int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) u^\varepsilon(x)\, dx \to \int_{\Omega\times Y} \psi(x,y) u^0(x,y)\, dx\, dy$$

for all $\psi \in \mathcal{C}_{per}(Y, L^2(\Omega))$

Two-scale convergence is thus based on an appropriate choice of oscillating test functions (see also [29] for a variant of this method applied to Hamilton-Jacobi equations, and [12] for an exposition of Tartar's method of oscillating test functions). Unfortunately, we will not be able to use this theorem in the form given by Allaire because of the non-linearity of equation (2.1); instead, we will need two-scale Young measures, a tool introduced by Weinan E in [24] which handles non-linearities and in which the information contained in two-scale limits is included. We will give more details about two-scale Young measures and their properties in the third section.

Throughout this article, we use the notation

$$\langle v \rangle_Y := \frac{1}{|Y|}\int_Y v(y)\, dy,$$

and we will work in the following functional spaces : if $\mathcal{C}^\infty_{\text{per}}(Y)$ denotes the space of Y-periodic functions in $\mathcal{C}^\infty(\mathbb{R}^N)$, then :

$$\begin{aligned}
H^1_{\text{per}}(Y) &:= \overline{\mathcal{C}^\infty_{\text{per}}(Y)}^{H^1(Y)},\ ||\cdot||_{H^1_{\text{per}}(Y)} = ||\cdot||_{H^1(Y)},\\
V &:= \{v \in H^1_{\text{per}}(Y), \langle v\rangle_Y = 0\},\ ||v||_V = ||\nabla v||_{L^2(Y)}\\
\mathcal{C}^\infty_{\text{per}}(Y\times\mathbb{R}) &:= \{f = f(y,v) \in \mathcal{C}^\infty(\mathbb{R}^N\times\mathbb{R}); f \text{ is } Y-\text{periodic in } y\},\\
W^{k,\infty}_{\text{per}}(Y\times\mathbb{R}) &:= \overline{\mathcal{C}^\infty_{\text{per}}(Y\times\mathbb{R})}^{W^{k,\infty}(Y\times\mathbb{R})},\quad k\in\mathbb{N},\\
W^{1,\infty}_{\text{per,loc}}(Y\times\mathbb{R}) &:= \{u = u(y,v)\in W^{1,\infty}_{\text{loc}}(\mathbb{R}^{N+1}), u \text{ is } Y-\text{periodic in } y\},\\
K_{\text{per}} &:= \{v(x,y)\in\mathcal{C}^\infty(\mathbb{R}^N\times Y); v \text{ is } Y-\text{periodic in } y\\
&\qquad\qquad \text{and has compact support in } x\},\\
J_{\text{per}} &:= \{v(t,x,y)\in\mathcal{C}^\infty([0,+\infty)\times\mathbb{R}^N\times Y); v\ Y-\text{periodic in } y\\
&\qquad\qquad \text{and has compact support in } t,x\}.
\end{aligned}$$

Thanks to the Poincaré-Wirtinger inequality, the norm on V is equivalent to the H^1 norm.

We will often use the following notations:

$$a_i(y,v) := \frac{\partial A_i(y,v)}{\partial v}\quad (1\le i\le N),\quad a_{N+1}(y,v) := -\sum_{i=1}^N \frac{\partial A_i(y,v)}{\partial y_i}.$$

The organization of the paper is as follows : in the next subsection, we state our main results, which consist in two theorems : theorem 1 states the existence and uniqueness of solutions of the cell problem, and theorem 2 gives the strong convergence of the sequence u^ε in case of well-prepared initial data. In the next section we derive the homogenized problem thanks to formal double-scale expansions, and we perform the analysis of the cell problem (2.8). In the third and last section, we give two proofs of theorem 2, the first one using the L^1 contraction principle for equation (2.1), but requiring very strong regularity assumptions, and the second one using two-scale Young measures.

2.1.1 Main results

Theorem 1. *Let $A \in W^{1,\infty}_{per,loc}(Y\times\mathbb{R})^N$. Assume that there exist $C_0 > 0$, $m \in [0,\infty)$, $n \in [0, \frac{N+2}{N-2})$ when $N \geq 3$, such that for all $(y,p) \in Y \times \mathbb{R}$*

$$|a_i(y,p)| \leq C_0\left(1+|p|^m\right) \quad \forall\, 1 \leq i \leq N, \tag{2.3}$$
$$|a_{N+1}(y,p)| \leq C_0\left(1+|p|^n\right). \tag{2.4}$$

Assume as well that one of the following conditions holds:

$$m = 0 \tag{2.5}$$
$$or \quad 0 \leq n < 1 \tag{2.6}$$
$$or \quad n < \frac{N+2}{N} \quad and\ \exists p_0 \in \mathbb{R},\ \forall y \in Y\ a_{N+1}(y,p_0) = 0. \tag{2.7}$$

Then for all $p \in \mathbb{R}$, there exists a unique solution $\tilde{u} \in V$ of the cell problem

$$-\Delta_y \tilde{u} + \mathrm{div}_y A(y, p+\tilde{u}) = 0; \tag{2.8}$$

For all $p \in \mathbb{R}$, $\tilde{u}(\cdot,p)$ belongs to $W^{2,q}_{per}(Y)$ for all $1 < q < +\infty$ and satisfies the following a priori estimate for all $R > 0$

$$||\tilde{u}(\cdot,p)||_{W^{2,q}(Y)} \leq C \quad \forall p \in \mathbb{R}, |p| \leq R, \tag{2.9}$$

for some constant C depending only on N, Y, C_0, m, n, q and R.

Theorem 2. *Assume that $A \in W^{1,\infty}_{per,loc}(Y\times\mathbb{R})^N$ satisfies the assumptions of theorem 1, and that $\frac{\partial a_i}{\partial y_j} \in L^1_{loc}(Y\times\mathbb{R})$, $\frac{\partial a_i}{\partial v} \in L^1_{loc}(Y\times\mathbb{R})$ for $1 \leq i \leq N+1$, $1 \leq j \leq N$.*

Let $p \in \mathbb{R}$, and let $\tilde{u}$ be the unique solution in V of the cell problem (2.8).

Let

$$\bar{A}_i(p) := \frac{1}{|Y|}\int_Y A_i(y, p+\tilde{u}(y,p))\, dy. \tag{2.10}$$

Assume also that u_0 is "well-prepared", i.e. satisfies

$$u_0(x,y) = v(y, \bar{u}_0(x)) \tag{2.11}$$

for some $\bar{u}_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$.

Then as ε goes to 0,

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0 \quad in\ L^2_{loc}([0,\infty)\times\mathbb{R}^N),$$

where $\bar{u} = \bar{u}(t,x) \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty) \times \mathbb{R}^N)$ *is the unique entropy solution of the hyperbolic scalar conservation law*

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \displaystyle\sum_{i=1}^N \frac{\partial \bar{A}_i(\bar{u}(t,x))}{\partial x_i} = 0, \\ \bar{u}(t=0,x) = \bar{u}_0(x) \in L^1 \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{2.12}$$

Remark 2.1.1. *Notice that in general, the null function is not a solution of* (2.1)*, unless we make the additional hypothesis* $a_{N+1}(y,0) = 0$ *for all* $y \in Y$*. Therefore, in general there are no global* L^1 *bounds on the solutions of* (2.1)*, even if* $u_0\left(x, \frac{x}{\varepsilon}\right) \in L^1(\mathbb{R}^N)$*. Moreover, slightly stronger assumptions on the flux* A *are required in general in order to ensure the existence of solutions of* (2.1)*, e.g.* $A \in W^{2,\infty}_{per}(Y \times \mathbb{R})$*. The hypothesis* $\frac{\partial a_i}{\partial y_j}, \frac{\partial a_i}{\partial v} \in L^1_{loc}(Y \times \mathbb{R})$ *is necessary so that the* L^1 *contraction principle holds.*

Remark 2.1.2. *Assumption* (2.11) *means that the initial data is already adapted to the microstructure; if it is not, i.e. if it cannot be written in the form*

$$u_0(x,y) = v\left(y, \bar{u}_0(x)\right),$$

then it is expected that there will be an initial layer of order ε *during which the solution will adjust itself to the microstructure; this problem is not addressed here, and will be delt with in a future article.*

2.2 Formal computation of the homogenized problem

In order to compute the effective equations which rule the system in the limit $\varepsilon \to 0$, we use double scale asymptotic expansions (see [9] for a general presentation of this technique): assume that u^ε satisfies the following Ansatz :

$$u^\varepsilon(t,x) = u^0\left(t, x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t, x, \frac{x}{\varepsilon}\right) + \cdots .$$

Inserting this expansion in equation (2.1) and identifying the powers of ε, we derive the following equations on u^0, u^1 :
Order ε^{-1} *:*

$$-\Delta_y u^0(t,x,y) + \mathrm{div}_y A(y, u^0(t,x,y)) = 0; \tag{2.13}$$

Order ε^0 *:*

$$\frac{\partial u^0}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i}\left[A_i(y,u^0)\right] - 2\Delta_{xy} u^0 - \Delta_y u^1 + \sum_{i=1}^N \frac{\partial}{\partial y_i}\left[u^1 a_i(y,u^0)\right] = 0. \tag{2.14}$$

(2.13) leads us to write u^0 in the form

$$u^0(t,x,y) = \bar{u}(t,x) + \tilde{u}(y, \bar{u}(t,x)),$$

where $\bar{u}(t,x) := \langle u^0(t,x,\cdot)\rangle_Y$ and $\tilde{u} = \tilde{u}(y,p),\ y \in Y, p \in \mathbb{R}$ satisfies the so-called cell equation

$$-\Delta_y \tilde{u} + \text{div} A(y, p + \tilde{u}(y,p)) = 0$$

together with the condition $\langle \tilde{u}\rangle_Y = 0$ for all p. Then, averaging (2.14) with respect to y yields the evolution equation on $\bar{u}$:

$$\frac{\partial \bar{u}}{\partial t} + \sum_{i=1}^N \frac{\partial \bar{A}_i(\bar{u})}{\partial x_i} = 0,$$

where the homogenized flux $\bar{A}_i$ can be computed thanks to the formula

$$\bar{A}_i(p) := \langle A(\cdot, p + \tilde{u}(\cdot,p))\rangle_Y .$$

The ε^0 term also allows us to derive the equation on u^1 :

$$-\Delta_y u^1 + \sum_{i=1}^N \frac{\partial}{\partial y_i}\left[u^1 a_i(y,u^0)\right] = 2\Delta_{xy} u^0 - \left\{\frac{\partial u^0}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i}\left[A_i(y,u^0)\right]\right\}.$$

Unfortunately, these calculations are entirely formal, and must be justified rigorously. In the following subsections, we will show that the homogenized equations computed above have solutions, and in the third section, we shall prove the convergence of u^ε to the solution of the homogenized problem.

2.2.1 Cell problem

This subsection is devoted to the proof of theorem 1. In fact, more general results can be proved, which we state in the following lemmas.

Lemma 2.2.1. *Assume $A \in W^{1,\infty}_{per,\ loc}(Y \times \mathbb{R})$ satisfies (2.3),(2.4) with $m \geq 0$ arbitrary, $n \in [0, \frac{N+2}{N-2})$ when $N > 2$ (if $N \leq 2$, there is no restriction on n).*

1. *Regularity: If $\tilde{u} \in H^1_{per}(Y)$ is a solution of (2.8) for some $p \in \mathbb{R}$, then $\tilde{u} \in W^{2,q}(Y)$ for all $1 < q < +\infty$, and the following estimate holds: for all $R > 0$, there exists a constant $C > 0$ depending only on R, q, N, Y, m, n and C_0, and a constant M depending only on q, m, n and N such that*

$$||\tilde{u}(\cdot,p)||_{W^{2,q}(Y)} \leq C(1 + ||\tilde{u}||_{H^1(Y)})^M \quad \forall p \in [-R,R]. \qquad (2.15)$$

2. *Uniqueness and monotony: for all $p \in \mathbb{R}$, there exists at most one solution $\tilde{u}(y,p) \in V$ of (2.8). Moreover, if $\tilde{u}(y,p)$ and $\tilde{u}(y,p')$ are two solutions of (2.8) with $p \geq p'$, then setting $v(y,p) := p + \tilde{u}(y,p)$ we have*

$$v(y,p) \geq v(y,p') \quad a.e.\ on\ Y.$$

3. *p-derivative : assume that there exists a solution of* (2.8) *for all* $p \in \mathbb{R}$ *and that*

$$K_R := \sup_{|p| \leq R} ||\tilde{u}(\cdot, p)||_{H^1(Y)} < +\infty \quad \forall R > 0.$$

Then for all $p \in \mathbb{R}$, $\frac{\partial \tilde{u}}{\partial p}(\cdot, p) \in H^1_{per}(Y)$ *and for all* $R > 0$ *there exists* $C = C(R, N, Y, C_0, m, n, K_R)$ *such that*

$$\left|\left|\frac{\partial \tilde{u}}{\partial p}\right|\right|_{L^\infty((-R,R),H^1_{per}(Y))} \leq C. \tag{2.16}$$

Moreover, if $a(y, \cdot) \in \mathcal{C}(\mathbb{R})$ *for a.e.* y, *then* $\frac{\partial v}{\partial p} \in H^1_{per}(Y)$ *is the unique solution of*

$$-\Delta_y \frac{\partial v}{\partial p} + \mathrm{div}_y \left(a(y, v(y,p)) \frac{\partial v}{\partial p} \right) = 0 \tag{2.17}$$

under the constraint $\left\langle \frac{\partial v}{\partial p} \right\rangle_Y = 1$.

The Krein-Rutman theorem ensures that

$$\frac{\partial v}{\partial p}(y, p) > 0 \qquad \text{for a.e. } (y, p) \in Y \times \mathbb{R}. \tag{2.18}$$

If additionnally $a_i \in L^\infty(Y \times \mathbb{R})$ *for* $1 \leq i \leq N$ *(i.e.* $m = 0$*), then there exists* $\alpha > 0$ *depending only on* N, Y, *and* $\max_{1 \leq i \leq N} ||a_i||_\infty$ *such that*

$$\frac{\partial v(y,p)}{\partial p} \geq \alpha > 0 \quad \forall y \in Y \ \forall p \in \mathbb{R}. \tag{2.19}$$

Hence

$$\inf_Y v(y, p) \to +\infty \quad \text{as } p \to +\infty, \tag{2.20}$$

$$\sup_Y v(y, p) \to -\infty \quad \text{as } p \to -\infty. \tag{2.21}$$

We now state the existence result:

Lemma 2.2.2. *Assume* $A \in W^{1,\infty}_{per,\ loc}(Y \times \mathbb{R})$ *satisfies* (2.3),(2.4) *with* m *and* n *satisfying one of the three conditions* (2.5), (2.6) *or* (2.7). *Then there exists a (unique) solution of* (2.8) *for all* $p \in \mathbb{R}$, *and it satisfies the following a priori estimate*

$$||\tilde{u}(p)||_{H^1} \leq C_R \quad \forall p \in [-R, R], \tag{2.22}$$

where C_R *depends on*

1. N, Y, *and* C_0 *when* (2.5) *is satisfied;*

2. N, Y, C_0, R *and* n *when* (2.6) *is satisfied;*

3. N, Y, C_0, R, n and p_0 when (2.7) is satisfied.

Remark 2.2.1. *Hypothesis (2.7) can be slightly relaxed : in fact, we only need that for all $\lambda \in [0,1]$, there exists $p_\lambda \in \mathbb{R}$ and $u_\lambda \in V$ such that*

$$-\Delta_y u_\lambda + \lambda \mathrm{div}_y A(y, p_\lambda + u_\lambda) = 0$$

and $\sup_{\lambda\in[0,1]}(|p_\lambda| + ||u_\lambda||_{L^1(Y)}) < +\infty$.

In that case, the constant C_R in the a priori estimate (2.22) depends on N, Y, C_0, R, n and $\sup_{\lambda\in[0,1]}(|p_\lambda| + ||u_\lambda||_{L^1(Y)})$.

If $a_{N+1}(y, p_0) \equiv 0$, we can take $p_\lambda = p_0$ for all $\lambda \in [0,1]$, and $u_\lambda \equiv 0$.

We will need the following lemma, of which we skip the proof :

Lemma 2.2.3. *Let $b \in L^\infty(Y)^N$, $\alpha > 0$, $f \in L^2(Y)$. Let $m \in H^1_{per}(Y)$ be a solution of*

$$-\Delta_y m + \mathrm{div}_y(bm) = f$$

such that $\left|\int_Y m\right| \leq \alpha$.

There exists a positive constant C, depending only on N, Y, $||b||_{L^\infty(Y)^N}$, $||f||_{L^2(Y)}$ and α, s.t.

$$||m||_{H^1(Y)} \leq C.$$

Proof of Lemma 2.2.1.

- *First step : A priori* estimates :

Multiplying equation (2.8) by $|\tilde{u}|^{q-1}\tilde{u}$, for some $q \geq 1$, we see that if $\tilde{u} \in V \cap L^{n+q}$ is a solution of (2.8), then $\tilde{u}$ satisfies

$$q\int_Y |\nabla\tilde{u}|^2|\tilde{u}|^{q-1}\,dy = q\int_Y |\tilde{u}|^{q-1}A(y, p+\tilde{u})\cdot\nabla\tilde{u}\,dy;$$

set

$$B_i(y,w) = \int_0^w |r|^{q-1}A_i(y, p+r)\,dr \qquad \text{for } 1 \leq i \leq N.$$

Then using hypothesis (2.4)

$$\begin{aligned}
q\int_Y |\nabla\tilde{u}|^2|\tilde{u}|^{q-1}\,dy &= q\sum_{i=1}^N \underbrace{\int_Y \frac{\partial}{\partial y_i}\left[B_i(y,\tilde{u}(y))\right]\,dy}_{=0} - q\int_Y \sum_{i=1}^N \frac{\partial B_i}{\partial y_i}(y,\tilde{u}(y))\,dy \\
&= q\int_Y\int_0^{\tilde{u}(y)} |r|^{q-1}a_{N+1}(y,p+r)\,dr \\
&\leq qC_0\int_Y\int_0^{\tilde{u}(y)} |r|^{q-1}\left(1 + (|p|+|r|)^n\right)\,dr\,dy \\
\left|\left|\nabla\left(\tilde{u}^{\frac{q+1}{2}}\right)\right|\right|_{L^2(Y)} &\leq C\left((1+|p|)^{\frac{n}{2}}||\tilde{u}||^{\frac{q}{2}}_{L^q} + ||\tilde{u}||^{\frac{n+q}{2}}_{L^{n+q}}\right),
\end{aligned} \tag{2.23}$$

for all $q \geq 1$ and for some constant C depending only on N, n, Y, C_0 and q.

- *Second step :* $\tilde{u} \in \cap_{1 \le r < +\infty} L^r(Y)$:

Let $R > 0$ arbitrary, and let $p \in [-R, R]$, $n_0 = \max(1, n)$. According to the a priori estimate (2.23), there exists a constant C_R depending only on R, N, n, Y, C_0 and q such that if $\tilde{u} \in V \cap L^{q+n_0}(Y)$ is a solution of (2.8)

$$||\tilde{u}^{\frac{q+1}{2}}||_{H^1} \le C_R \left(1 + ||\tilde{u}||_{L^{q+n_0}}\right)^{\frac{q+n_0}{2}}$$

H^1 is imbedded in $L^{\frac{2N}{N-2}}(Y)$ for $N > 2$, and in $L^r(Y)$ for $N \le 2$, $1 \le r < +\infty$ arbitrary. Hence if $\tilde{u} \in V$ is a solution of (2.8)

$$\tilde{u} \in \cap_{1 \le r < +\infty} L^r(Y) \quad \text{if } N \le 2,$$
$$\tilde{u} \in L^{q+n_0}(Y) \Rightarrow \tilde{u} \in L^{\frac{(q+1)N}{N-2}}(Y) \quad \forall q \in [1, +\infty) \text{ if } N > 2.$$

When $N > 2$, define the sequence $(q_k)_{k \ge 1}$ by

$$q_1 = 1, \quad q_{k+1} + n_0 = (q_k + 1)\frac{N}{N-2}, \quad k \ge 1.$$

Then it is easily checked that since $n < \frac{N+2}{N-2}$, $q_k \ge 1$ for all $k \ge 1$ and

$$u \in L^{q_k + n}(Y) \quad \forall k \ge 1,$$
$$q_k \to +\infty \quad \text{as } k \to \infty.$$

Moreover,

$$||\tilde{u}||_{L^{1+n_0}} \le C||\tilde{u}||_{L^{\frac{2N}{N-2}}} \le C||\tilde{u}||_{H^1}$$

where the constant C depends only on N, Y, and n.

In all cases, $\tilde{u} \in \cap_{1 \le r < +\infty} L^r(Y)$. And for all $r \ge 2$, there exists a constant C_R depending only on R, r, N, n, C_0 and Y, and a constant M depending only on r, n and N such that for all $p \in [-R, R]$, for all solutions $\tilde{u} \in V$ of (2.8)

$$||\tilde{u}||_{L^r} \le C_R \left(1 + ||\tilde{u}||_{H^1}\right)^M. \tag{2.24}$$

- *Third step :* $W^{2,r}$ estimates :

Let $R > 0$, and let $p \in \mathbb{R}$, $|p| \le R$; let $\tilde{u}$ be a solution of (2.8) for the parameter p.

Since $\tilde{u} \in H^1_{\text{per}}(Y)$, the chain rule allows us to write

$$-\Delta_y \tilde{u} = a_{N+1}(y, p + \tilde{u}(y)) - a(y, p + \tilde{u}(y)) \cdot \nabla_y \tilde{u}. \tag{2.25}$$

In the above equation, $a_{N+1}(y, p + \tilde{u}(y))$, $a(y, p + \tilde{u}(y))$ belong to $L^r(Y)$ for all $r \in [1, +\infty)$, and $\nabla_y \tilde{u} \in L^2(Y)$.

Hence the right-hand side belongs to $L^q(Y)$ for all $1 < q < 2$, with locally uniform bounds in p. Using interior regularity results for elliptic equations (see [34],[47]) combined with the periodicity, it can be proved that $\tilde{u} \in W^{2,q}(Y)$ for all $q < 2$ and

$$||\tilde{u}||_{W^{2,q}(Y)} \le C(1 + ||\tilde{u}||_{H^1})^M, \tag{2.26}$$

for a constant C depending only on C_0, m, n, N, Y, q, and R and a constant M depending only on m, n, N and q.

Next, Sobolev imbeddings entail that $\nabla\tilde{u} \in L^\infty_{\text{loc}}(\mathbb{R}, L^q(Y))$ for all $q > 1$ such that $\frac{1}{q} > \frac{1}{2} - \frac{1}{N}$ and we can repeat the same argument as above replacing 2 by $\frac{2N}{N-2}$ (if $N > 2$).

More precisely, let us define the sequence q_k by

$$\frac{1}{q_k} = \frac{1}{2} - \frac{k}{N} \quad \text{if } k < \frac{N}{2};$$

then it is easily checked using the above method that

$$\tilde{u} \in W^{1,q}(Y)\ \forall q \in (1, q_k) \Rightarrow \tilde{u} \in W^{1,q}(Y)\ \forall q \in (1, q_{k+1}),$$

as long as $k + 1 < \frac{N}{2}$, and with bounds of the type (2.26).

By induction, $\tilde{u} \in W^{1,q}(Y)$ for all $1 < q < q_{k_0}$, where k_0 is the integer defined by

$$k_0 < \frac{N}{2} \leq k_0 + 1.$$

Then $q_{k_0} \geq N$; consequently, $\tilde{u} \in W^{2,q}(Y)$ for all $q < N$, and thus $\tilde{u} \in W^{1,r}(Y)$ for all $r \geq 1$. Plugging this result once more into (2.25) yields $\tilde{u} \in W^{2,r}$ for all $r \geq 1$, with bounds of the type (2.26). Hence (2.15) is proved.

- *Fourth step :* Uniqueness and monotony of solutions of (2.8) :

If $\tilde{u}_1$ and $\tilde{u}_2$ are two solutions of (2.8) for parameters p_1, p_2, then $w_{p_1,p_2} := (p_1 + \tilde{u}_1) - (p_2 + \tilde{u}_2) \in V$ satisfies an elliptic equation

$$-\Delta_y w_{p_1,p_2} + \text{div}_y(b_{p_1,p_2} w_{p_1,p_2}) = 0,$$

where

$$b_{p_1,p_2}(y) = \int_0^1 a(y, (1-\tau)v(y,p_1) + \tau v(y,p_2))\, d\tau.$$

Thanks to the regularity result we have just shown, $b_{p_1,p_2} \in L^\infty(Y)^N$ for all $p_1, p_2 \in \mathbb{R}$. And for all $R > 0$, there exists a constant C depending on N, Y, C_0, m, n, R, $||\tilde{u}(p_1)||_{H^1}$, $||\tilde{u}(p_2)||_{H^1}$, such that

$$||b_{p_1,p_2}||_{L^\infty(Y)^N} \leq C \quad \forall p_1, p_2 \in [-R, R].$$

The uniqueness and the monotony follow from the following lemma

Lemma 2.2.4. *Let $b \in L^\infty(Y)^N$, and let $v \in H^1_{per}(Y)$ be a solution of the linear elliptic equation*

$$-\Delta_y v + \text{div}_y(bv) = 0. \tag{2.27}$$

There exists a positive probability measure $m \in M^1_{per}(Y) = \mathcal{C}_{per}(Y)'$ and a constant $c \in \mathbb{R}$ such that $v = cm$. In particular, if $\langle v\rangle_Y = 0$, then $v = 0$.

We postpone the proof of the lemma.

Hence, since $\langle w_{p_1,p_2}\rangle_Y = (p_1 - p_2)$, we deduce that $w_{p_1,p_2} = (p_1 - p_2)m_{p_1,p_2}$, with m_{p_1,p_2} a positive measure on Y. If $p_1 = p_2$, then $w_{p_1,p_2} = 0$, and the uniqueness is proved. If $p_1 > p_2$, then

$$v(y, p_1) > v(y, p_2) \quad \forall y \in Y.$$

As a consequence, we deduce

$$||v(y, p_1) - v(y, p_2)||_{L^1(Y)} = \int_Y (v(y, p_1) - v(y, p_2))\ dy = |Y|(p_1 - p_2)$$

- *Fifth step :* p-derivative:

Now, $m_{p_1,p_2}(y) = \frac{v(y,p_1)-v(y,p_2)}{p_1-p_2}$ is a positive measure on Y for $p_1 \neq p_2$, $p_1, p_2 \in \mathbb{R}$, and m_{p_1,p_2} satisfies

$$-\Delta_y m_{p_1,p_2} + \mathrm{div}_y(b_{p_1,p_2} m_{p_1,p_2}) = 0, \quad \langle m_{p_1,p_2}\rangle = 1. \tag{2.28}$$

Assume that

$$K_R := \sup_{|p|\leq R} ||\tilde{u}(p)||_{H^1(Y)} < +\infty \quad \forall R > 0.$$

Then for all $R > 0$ there exists a constant $C_R > 0$ depending only on N, Y, n, m, C_0, R and K_R such that

$$||b_{p_1,p_2}||_{L^\infty(Y)^N} \leq C_R \quad \forall p_1, p_2 \in [-R, R].$$

Hence, using lemma 2.2.3, there exists a positive constant C, depending only on R, C_0, m, n, N, Y, K_R such that

$$||m_{p_1,p_2}||_{H^1(Y)} \leq C \quad \forall (p_1, p_2) \in \mathbb{R}^2,\ p_1 \neq p_2, |p_1|, |p_2| \leq R.$$

Let $p_n \to p_0$, $p_0 \in [-R, R]$. Extracting a subsequence, $m_{p_0,p_n}(\cdot)$ converges weakly in $H^1_{\mathrm{per}}(Y)$, strongly in $L^2(Y)$ to $\frac{\partial v}{\partial p}(y, p_0)$ and b_{p_n,p_0} converges to $a(y, v(y, p_0))$. Passing to the limit in equation (2.28) leads to equation (2.17). *A priori* estimates are obtained using lemma 2.2.3, and eventually, lemma 2.2.4 entails that $\frac{1}{|Y|}\frac{\partial v}{\partial p}(y, p_0)$ is a positive probability measure on Y for all p_0.

- *Sixth step :* Proof of (2.20), (2.21) :

When $m = 0$, $\frac{\partial v}{\partial p}$ satisfies (2.17) with

$$||a(y, v(y, p))||_{L^\infty(Y)^N} \leq 2C_0 \quad \forall p \in \mathbb{R}.$$

According to the Harnack inequality (see for instance [34]) combined with the periodicity, there exists a constant C depending only on C_0, N, and Y such that

$$\sup_Y w \leq C \inf_Y w.$$

Since $\int_Y w = |Y|$, $\sup_Y w \geq 1$. Hence there exists a positive constant α, depending only on C_0, N, and Y, such that

$$\frac{\partial v(y, p)}{\partial p} \geq \alpha$$

and (2.20), (2.21) are proved. □

Proof of Lemma 2.2.2. Let us define the operator $T : u \in V \mapsto v \in V$ where $v = T(u) \in V$ is the unique solution of the elliptic equation

$$-\Delta_y v = -\mathrm{div}_y A(y, p + u(y)).$$

Fixed points of T are solutions of (2.8), and T is a continuous compact operator.

We want to apply Schaefer's fixed point theorem, and thus prove that

$$\{u \in V; \exists \lambda \in [0,1],\ u = \lambda T(u)\}$$

is bounded. In the sequel, we take $u \in V$, $\lambda \in [0,1]$ such that $u = \lambda T(u)$, and we try and derive a bound on u.

We begin with the case $m = 0$. In that case, u satisfies

$$-\Delta_y u + \mathrm{div}(bu) = \lambda a_{N+1}(y, 0),$$

where

$$b(y) = \lambda \int_0^1 a(y, t(p+u))\, dt.$$

Hence $b \in L^\infty(Y)^N$ and

$$||b||_{L^\infty(Y)^N} \le ||a||_{L^\infty(Y\times\mathbb{R})^N} \le 2C_0,$$
$$||\lambda a_{N+1}(y,0)||_{L^2(Y)} \le C_0 |Y|^{\frac{1}{2}}.$$

Thus according to lemma 2.2.3, there exists a constant C depending only on N, Y and C_0, such that

$$||u||_{H^1(Y)} \le C$$

and the estimate is proved.

When either (2.6) or (2.7) are satisfied, for all $u \in V$ such that $u = \lambda T(u)$, the a priori estimate (2.23) with $q = 1$ and changing A into λA yields

$$||u||_{H^1(Y)} \le C \left((1+|p|)^{\frac{n}{2}} ||u||_{L^1}^{\frac{1}{2}} + ||u||_{L^{n+1}}^{\frac{n+1}{2}} \right) \tag{2.29}$$

for some constant C depending only on N, n, C_0 and Y.

If $n < 1$, then it is easily seen that this inequality leads to an H^1 a priori estimate, and thus to the existence of solutions of (2.8). Hence, we now focus on the case $n \ge 1$.

Since $n + 1 < \frac{2N+2}{N} \le \frac{2N}{N-2}$ (if $N > 2$), we can interpolate L^{n+1} between L^1 and $L^{\frac{2N}{N-2}}$: let $\theta \in (0,1]$ such that

$$\frac{1}{n+1} = \frac{\theta}{1} + \frac{1-\theta}{q_0}$$

where $q_0 := \frac{2N}{N-2}$. Then

$$\begin{aligned} ||u||_{L^{n+1}}^{\frac{n+1}{2}} &\le ||u||_{L^1}^{\frac{(n+1)\theta}{2}} ||u||_{L^{q_0}}^{\frac{(n+1)(1-\theta)}{2}} \\ &\le ||u||_{L^1}^{\frac{(n+1)\theta}{2}} ||u||_{H^1}^{\frac{(n+1)(1-\theta)}{2}}. \end{aligned}$$

It is easily checked that $n < \frac{N+2}{N}$ if and only if $\frac{(n+1)(1-\theta)}{2} < 1$. The whole problem thus reduces to find L^1 estimates for the solutions of (2.8). This is quite easy if hypothesis (2.7) is satisfied. Indeed, in that case, $\tilde{u}(y, p_0) \equiv 0$ is a special solution of (2.8) for $p = p_0$ and for the flux λA; hence, according to lemma 2.2.1,

$$||u||_{L^1} = ||u - \tilde{u}(p_0)||_{L^1} \leq |p - p_0||Y| + ||(p + u) - (p_0 + \tilde{u}(p_0))||_{L^1} \leq 2|p - p_0||Y|.$$

Plugging these estimates into (2.29) yields

$$||u||_{H^1} \leq C_R \left(1 + ||u||_{H^1}^{\frac{(n+1)(1-\theta)}{2}}\right) \tag{2.30}$$

for all p such that $|p| \leq R$, where the constant C_R depends only on N, Y, n, C_0, p_0 and R. Since $\frac{(n+1)(1-\theta)}{2} < 1$, $||u||_{H^1}$ is bounded by a constant depending on the same parameters as C_R. Hence the a priori estimate is proved and solutions of (2.8) exist for all $p \in \mathbb{R}$.

□

Proof of Lemma 2.2.4. The constant function equal to 1 on Y, denoted by $\bar{1}$, is a solution of the dual problem

$$-\Delta_y \bar{1} - b(y) \cdot \nabla_y \bar{1} = 0. \tag{2.31}$$

We want to prove, using the strong form of the Krein-Rutman theorem, that there exists a constant $c \in \mathbb{R}$ such that $w = cm$, where $m > 0$ is a solution of (2.27). Indeed, in that case $c = 0$ necessarily since $\langle w \rangle_Y = 0$ and thus $w = 0$.

Let us introduce the operator $F : u \in L^2(Y) \mapsto v \in H$ where $v = F(u)$ is the unique solution of the equation

$$-\Delta v - b \cdot \nabla v + \alpha v = \alpha u,$$

and α is a positive constant chosen so that the bilinear form associated to F is coercive (e.g. $\alpha = \frac{||b||_\infty^2}{2} + \frac{1}{2}$). With that choice of α F is a strictly positive operator.

Next, using once again interior regularity results for linear elliptic equations combined with the periodicity, we show that F maps $L^q(Y)$ into $W^{2,q}_{\text{per}}(Y)$ for all $q \geq 2$. Hence, the restriction of F to $\mathcal{C}_{\text{per}}(\overline{Y})$, still denoted by F, is a compact operator from $\mathcal{C}_{\text{per}}(\overline{Y})$ into itself.

The last step consists in using the maximum principle: if $u \in \mathcal{C}_{\text{per}}(\overline{Y})$, $u \geq 0$, $u \neq 0$ and $v = F(u)$, then $v(y) > 0$ for all $y \in \overline{Y}$ (see for instance [61]; the maximum principle is in general proved for classical solutions of elliptic equations with regular coefficients. However the proofs remain unchanged for weak solutions and $b \in L^\infty$ provided the following property holds true for any $\gamma > 0$:

$$u \in L^2(Y), u \geq \gamma \quad \text{a.e.} \Rightarrow v = F(u) \geq \gamma.$$

This property can be proved by approximating b in L^q for $1 < q < \infty$ by a sequence $b_n \in \mathcal{C}^\mu(\overline{Y})$ for some $\mu \in (0, 1)$.)

Hence, $F : \mathcal{C}_{\text{per}}(\overline{Y}) \to \mathcal{C}_{\text{per}}(\overline{Y})$ is a strongly positive operator.

We conclude by using the strong form of the Krein-Rutman theorem (see [20], [45]): since $F(\bar{1}) = \bar{1}$, the spectral radius of F is equal to 1 and 1 is a simple eigenvalue of F^*, the adjoint of F, with a positive eigenvector. Let $m \in M^1_{\text{per}}(Y) = \mathcal{C}_{\text{per}}(Y)'$ be the unique positive invariant measure such that $\langle m\rangle_Y = 1$ and $F^*(m) = m$. Since $v \in H^1_{\text{per}}(Y) \subset M^1_{\text{per}}(Y)$ solves (2.27), $F^*(v) = v$; thus, there exists $c \in \mathbb{R}$ such that $v = cm$. If $\langle v\rangle_Y = 0$, then $c = 0$ and $v = 0$, which completes the proof of the lemma.

□

Remark 2.2.2. *This lemma can be generalized without any difficulty to the case $b \in L^q$ for some $q > N$ using the inequality*

$$\left|\int_Y vb\cdot\nabla v\right| \le C||b||_{L^q}||v||_{L^2}^{1-\frac{N}{q}}||\nabla v||_{L^2}^{1+\frac{N}{q}}$$

where C is a constant depending only on N and Y.

Remark 2.2.3. *Let us point out that the techniques we have used in order to find a priori bounds on the solutions of the cell problem rely strongly on the ellipticity of equation* (2.8). *In particular, when the viscosity is equal to 0 in equation* (2.1)*, the cell problem becomes*

$$\text{div}_y A(y, p + \tilde{u}(y)) = 0,$$

and we have no clue how to derive a priori bounds on the solutions of the above equation in general. The few cases in which we are able to prove such bounds suggest strongly that the flux A should be nonlinear. However, it is an open problem how to treat such an equation in general, and which hypotheses should be expected on the flux. We will come back on these questions in a future paper.

Before going any further in the multi-scale analysis of problem (2.1), let us mention a few examples in which hypotheses (2.5), (2.6), and (2.7) seem "natural".

Take $A(y, v) = b(y)f(v)$, where $b \in W^{1,\infty}(Y)^N$ has values in $\mathbb{R}^N$, $f \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ is scalar.

If $\text{div}_y b \equiv 0$ on Y, then constants are solutions of equation (2.8). Lemma 2.2.1 asserts that there are no other solutions as long as f has polynomial growth. Notice that in that case, hypothesis (2.7) is satisfied.

Let us study now the less trivial case $b(y) = \nabla_y\phi(y)$, where $\phi \in \mathcal{C}^1_{\text{per}}(Y)$. Assume that f does not vanish on $\mathbb{R}$ (otherwise we are in case (2.7)); without loss of generality, we can assume that

$$f(v) > 0 \quad \forall v \in R.$$

We can thus define

$$H(v) := \int_0^v \frac{1}{f(w)}\,dw \quad \forall w \in \mathbb{R}.$$

It is obvious that any solution of

$$-\nabla_y u = -\nabla_y\phi(y)f(p + u) \tag{2.32}$$

is a solution of (2.8); hence, we search for particular solutions of (2.8) which satisfy (2.32).

(2.32) is equivalent to

$$\nabla_y H(p+u) = \nabla \phi,$$

and thus to

$$H(p+u) = \phi + \text{cst}.$$

Thus we deduce that solutions of (2.32) exist if and only if

$$H(+\infty) - H(-\infty) = \int_{\mathbb{R}} \frac{1}{f} > \text{osc}\phi. \tag{2.33}$$

In particular, this is always satisfied when $|f(v)| \leq C_0(1+|v|)^n$ for some $n<1$ (i.e. when (2.6) holds) since in that case

$$H(+\infty) - H(-\infty) = \int_{\mathbb{R}} \frac{1}{f} = +\infty.$$

Assume that (2.33) is satisfied; notice that $H \in \mathcal{C}^1(\mathbb{R})$ and $H' = \frac{1}{f}$ does not vanish on $\mathbb{R}$. Hence H is a $\mathcal{C}^1$ diffeomorphism from $\mathbb{R}$ to $(H(-\infty), H(+\infty)) =: (\alpha, \beta)$. We denote by $H^{-1} : (\alpha,\beta) \to \mathbb{R}$ its reciprocal application. Let

$$c_+ := \beta - \max \phi,$$
$$c_- := \alpha - \min \phi.$$

Then for all $c \in (c_-, c_+)$ we can define

$$v_c := H^{-1}(\phi + c), \quad u_c := v_c - \langle v_c \rangle \tag{2.34}$$

and u_c is a solution of (2.8) for all $c \in (c_-, c_+)$. Hence, when (2.33) is satisfied, we have found special solutions of (2.8). If $c_\pm = \pm\infty$, then we have found solutions for all values of the parameter p in (2.8). If $|f(v)| \leq C_0(1+|v|)^n$ with $n < \frac{N+2}{N}$, then we deduce that there exist solutions of (2.8) for all values of p as well thanks to lemma 2.2.2 and the remark following the lemma (changing A into λA is equivalent to changing ϕ into $\lambda\phi$).

Reciprocally, let us prove that (2.33) is a necessary condition for solutions of (2.8) to exist at all when $n < \frac{N+2}{N-2}$. Let $u_0 \in V$ be a solution of (2.8) for the parameter $p_0 \in \mathbb{R}$, and let $v_0 := p_0 + u_0$. According to lemma (2.2.1), $v_0 \in L^\infty(Y)$. Hence we can change the function f for values of v larger than $||v_0||_{L^\infty}$ so that the function $\tilde{f}$ thus obtained satisfies (2.33) and

$$\tilde{f}(v_0(y)) = f(v_0(y)) \quad \forall y \in Y.$$

We can even choose $\tilde{f}$ so that

$$\int_0^\infty \frac{1}{\tilde{f}} = \int_{-\infty}^0 \frac{1}{\tilde{f}} = +\infty.$$

In that case, we have proved that there exist solutions of (2.8) for the flux $\nabla\phi(y)\tilde{f}(v)$ for all values of the parameter p. Let u_{c_0} be the solution for the parameter p_0, $v_{c_0} := u_{c_0} + p_0$. Then

$$-\Delta_y v_{c_0} + \mathrm{div}_y\left(\nabla\phi(y)\tilde{f}(v_{c_0}(y))\right) = 0,$$
$$-\Delta_y v_0 + \mathrm{div}_y\left(\nabla\phi(y)\tilde{f}(v_0(y))\right) = -\Delta_y v_0 + \mathrm{div}_y\left(\nabla\phi(y)f(v_0(y))\right) = 0,$$

and by uniqueness of the solutions of (2.8) for the flux $\nabla\phi(y)\tilde{f}(v)$, $v_0 = v_{c_0}$. Consequently,

$$v_0 = \tilde{H}^{-1}(\phi + c_0) = H^{-1}(\phi + c_0)$$

and (2.33) is satisfied. Moreover, we have proved that all solutions of (2.8) can be written in the form (2.34).

Now, let us explain why condition (2.7) is optimal, to a certain extent. Take $f(v) = (1 + |v|^2)^{\frac{n}{2}}$ for some $n > \frac{N+2}{N}$. Then $\alpha = -\beta \in \mathbb{R}$. Assume that (2.33) is satisfied. In order to simplify our analysis, we assume as well that ϕ attains its minimum in a unique point y_0 in the interior of Y.

We define

$$v_- := H^{-1}(\phi + c_-);$$

$v_-(y)$ is finite for all $y \neq y_0$. Moreover, if $u \in V$ is a solution of (2.8) for the parameter p, then u can be written in the form (2.34). Thus there exists $c \in (c_-, c_+)$ such that $u + p = v_c$ and necessarily

$$p + u > v_-.$$

Hence, if we can prove that $v_- \in L^1(Y)$, we will be able to derive a lower bound on the admissible values of p so that there exists a solution of (2.8) for the parameter p. In other words, there will be no solution for $p < \langle v_- \rangle$.

Let us prove that $v_- \in L^1(Y)$: there exists a constant $c \geq 1$ such that for y in a neighbourhood V_0 of y_0

$$\frac{1}{c}|y - y_0|^2 \leq \phi(y) - \phi(y_0) \leq c|y - y_0|^2.$$

Hence

$$\frac{1}{c}|y - y_0|^2 \leq H(v_-) - \alpha \leq c|y - y_0|^2.$$

On the other hand, there exists a constant C depending only on n such that for all $A \geq 1$,

$$\frac{1}{C}\frac{1}{A^{n-1}} \leq \int_A^\infty \frac{1}{f(v)}\,dv \leq C\frac{1}{A^{n-1}}.$$

Choose V_0 such that $v_-(y) \leq -1$ in V_0. In V_0,

$$\frac{1}{C}\frac{1}{|v_-|^{n-1}} \leq H(v_-) - \alpha = \int_{-\infty}^{v_-} \frac{1}{f(v)}\,dv \leq C\frac{1}{|v_-|^{n-1}}.$$

Thus, there exists a constant C such that for all $y \in Y_0$

$$|v_-| \leq \frac{C}{|y - y_0|^{\frac{2}{n-1}}}.$$

If $n > \frac{N+2}{N}$, then $\frac{2}{n-1} < N$ and the singularity in y_0 is integrable: $v_- \in L^1(Y)$.
Let us gather our results in the following

Lemma 2.2.5. *Let $A(y,v) = \nabla\phi(y) f(v)$, with $f(v) > 0$ for all $v \in \mathbb{R}$. Assume that*

$$f(v) \leq C_0(1+|v|)^n \quad \text{with } n < \frac{N+2}{N-2}.$$

Then

1. *There exist solutions of* (2.8) *for some values (but possibly not all) of the parameter p if and only if*
$$\int_{\mathbb{R}} \frac{1}{f} > \mathrm{osc}\phi.$$
2. *If the above inequality is satisfied and $f(v) \leq C_0(1+|v|)^n$ with $n < \frac{N+2}{N}$ then there exist solutions of* (2.8) *for all values of the parameter p.*
3. *If*
$$\int_0^\infty \frac{1}{f} = \int_{-\infty}^0 \frac{1}{f} = +\infty$$
then there exist solutions of (2.8) *for all values of $p \in \mathbb{R}$.*
4. *If $|f(v)| = (1+|v|^2)^{\frac{n}{2}}$ with $n > \frac{N+2}{N}$, then there exists $\phi \in \mathcal{C}^1_{per}(Y)$ and $p_-, p_+ \in \mathbb{R}$ such that there are no solutions of* (2.8) *for $p < p_-$ or $p > p_+$.*

The second point in the above lemma is the analogue of condition (2.7), and the third one of (2.6) (or (2.5): if f is uniformly Lipschitz, then it satisfies $f(v) \leq C_0(1+|v|)$, and thus the condition in the third point of the lemma is satisfied). Hence this example somehow explains the different conditions which are required for existence, and enlights the various regimes which can occur. However, hypotheses (2.5), (2.6) and (2.7) do not cover all the cases in which the existence holds, even in this rather simplified problem. A more general and more thorough existence theory remains to be accomplished.

As a conclusion to this subsection, let us also mention that the above example also provides cases when the convergences (2.20), (2.21) do not hold. Indeed, assume that $\alpha, \beta \in \mathbb{R}$ and that solutions of (2.8) (or, equivalently, of (2.32)) exist for all $p \in \mathbb{R}$. Then

$$\lim_{p\to+\infty} \inf_Y v(y,p) = \lim_{c\to c_+} H^{-1}(\inf\phi + c) = H^{-1}(\beta - \mathrm{osc}\phi) < +\infty$$

and similarly $\lim_{p\to-\infty} \sup_Y v(y,p) > -\infty$.

2.2.2 Evolution equation and first order corrector

Once $\tilde{u}$ is rigourously defined, we can compute the homogenized flux

$$\bar{A}(p) := \langle A(\cdot, p + \tilde{u}(\cdot, p))\rangle \tag{2.35}$$

Define also for $1 \leq i \leq N$

$$\bar{a}_i(p) := \frac{\partial \bar{A}_i(p)}{\partial p} = \left\langle \frac{\partial v(\cdot,p)}{\partial p}\, a_i(\cdot, v(\cdot,p)) \right\rangle.$$

Then according to the results of the preceding subsection, $\bar{a}_i \in L^\infty_{\text{loc}}(\mathbb{R})$.

$\bar{u}$ can thus be defined as the unique entropy solution in $\mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty)\times\mathbb{R}^N)$ of the scalar conservation law (see for instance [59] for a complete theory of existence and uniqueness)

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \displaystyle\sum_{i=1}^N \frac{\partial \bar{A}_i(\bar{u})}{\partial x_i} = 0, \quad t \geq 0,\ x \in \mathbb{R}^N \\ \bar{u}(t=0,x) = \bar{u}_0(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{2.36}$$

On sets of $[0,\infty)\times\mathbb{R}^N$ on which $\bar{u}$ is regular (say $W^{1,1}$), one can define the first order corrector u^1 by

$$-\Delta_y u^1 + \sum_{i=1}^N \frac{\partial}{\partial y_i}\left[u^1 a_i(y,u^0)\right] = 2\Delta_{xy}u^0 - \left\{\frac{\partial u^0}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i}\left[A_i(y,u^0)\right]\right\}. \tag{2.37}$$

(t,x) are parameters; since the right hand side has mean zero, and the only solutions of the adjoint equation

$$-\Delta_y w - a_i(y,u^0)\cdot\nabla_y w = 0$$

are constants, one can apply the Riesz-Fredholm theory to show that solutions of (2.37) exist, and are unique up to solutions of the homogeneous equation

$$-\Delta_y w + \sum_{i=1}^N \frac{\partial}{\partial y_i}\left[w a_i(y,u^0)\right] = 0.$$

Comparing the above equation to (2.17), and recalling the results of the proof of lemma 2.2.1, we see that the solutions of the homogeneous equation can be written $w(t,x,y) = c(t,x)\frac{\partial v}{\partial p}(y,\bar{u}(t,x))$. In particular, u^1 is unequivocally defined under the condition

$$\int_Y u^1(t,x,y)\frac{\partial v}{\partial p}(t,x,y)\,dy = 0 \quad \text{a.e. } (t,x)\in[0,\infty)\times\mathbb{R}^N.$$

Pushing the calculations a little further, we write u^1 in the slightly more sympathetic form

$$u^1(t,x,y) = \sum_{i=1}^N \frac{\partial \bar{u}(t,x)}{\partial x_i}\chi_i(y,\bar{u}(t,x)),$$

where $\chi_i(\cdot,p) \in H^1_{\text{per}}(Y)\ \forall p\in\mathbb{R}$ solves the elliptic equation :

$$\begin{aligned} -\Delta_y\chi_i + \sum_{j=1}^N \frac{\partial}{\partial y_j}\left(a_j(y,v(y,p))\chi_i\right) &= 2\frac{\partial^2 v(y,p)}{\partial y_i\,\partial p} + \frac{\partial v(y,p)}{\partial p}\frac{\partial}{\partial p}\left\langle A_i(\cdot,v(\cdot,p))\right\rangle_Y \\ &\quad -\frac{\partial}{\partial p}\left(A_i(y,v(y,p))\right). \end{aligned} \tag{2.38}$$

As before, the existence and uniqueness of χ_i follow from the Fredholm alternative provided the condition

$$\int_Y \chi_i \frac{\partial v}{\partial p} = 0 \quad \forall p \in \mathbb{R}$$

holds true.

Let us summarize the results of this subsection in the following

Lemma 2.2.6. *Assume $A \in W^{1,\infty}_{per,\ loc}(Y \times \mathbb{R})$ satisfies (2.3), (2.4). Then there exists a unique entropy solution $\bar{u} \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty) \times \mathbb{R}^N)$ of the hyperbolic scalar conservation law (2.36).*

If $\bar{u} \in W^{1,1}(\mathcal{O})$, where $\mathcal{O} \subset [0,\infty) \times \mathbb{R}^N$, then for $(t,x) \in \mathcal{O}$, there exists a unique $u^1(t,x,\cdot) \in H^1_{per}(Y)$ satisfying (2.37) and the condition

$$\int_Y u^1(t,x,y) \frac{\partial v}{\partial p}(t,x,y)\, dy = 0 \quad a.e.\ (t,x) \in \mathcal{O}.$$

Moreover, u^1 can be written

$$u^1(t,x,y) = \sum_{i=1}^N \frac{\partial \bar{u}(t,x)}{\partial x_i} \chi_i(y, \bar{u}(t,x)),$$

where $\chi_i(\cdot,p) \in H^1_{per}(Y)$ satisfies equation (2.38) $\forall p \in \mathbb{R}$.

In the rest of the article, we set

$$u^0(t,x,y) := \bar{u}(t,x) + \tilde{u}(y, \bar{u}(t,x)) = v(y, \bar{u}(t,x)). \tag{2.39}$$

2.3 Convergence proof

2.3.1 Naive idea using L^1 contraction principle

We are now ready to prove the convergence result announced in theorem 2. A first naive idea consists in computing the equation satisfied by $u^0\left(t,x,\frac{x}{\varepsilon}\right)$, or rather

$$v^\varepsilon(t,x) := u^0\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t,x,\frac{x}{\varepsilon}\right),$$

where u^0 and u^1 were defined in the last section : assuming that $\bar{u}$ and A are regular in order to compute all the necessary derivations, v^ε is a solution of

$$\frac{\partial v^\varepsilon}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, v^\varepsilon\right) - \varepsilon \Delta_x v^\varepsilon = f^\varepsilon,$$

where

$$
\begin{aligned}
f^\varepsilon(t,x) &= \frac{1}{\varepsilon}\left\{\frac{\partial A_i}{\partial y_i}\left(\frac{x}{\varepsilon},u^0+\varepsilon u^1\right)-\frac{\partial A_i}{\partial y_i}\left(\frac{x}{\varepsilon},u^0\right)-\varepsilon u^1\frac{\partial^2 A_i}{\partial y_i\,\partial v}\left(\frac{x}{\varepsilon},u^0\right)\right\}\\
&+ \frac{1}{\varepsilon}\left\{\frac{\partial u^0}{\partial y_i}\left[\frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon},u^0+\varepsilon u^1\right)-\frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon},u^0\right)-\varepsilon u^1\frac{\partial^2 A_i}{\partial v^2}\left(\frac{x}{\varepsilon},u^0\right)\right]\right\}\\
&+ \varepsilon^0\left\{\left(\frac{\partial u^1}{\partial y_i}+\frac{\partial u^0}{\partial x_i}\right)\left[\frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon},u^0+\varepsilon u^1\right)-\frac{\partial A_i}{\partial v}\left(\frac{x}{\varepsilon},u^0\right)\right]\right\}\\
&+ \varepsilon^1\left\{\frac{\partial u^1}{\partial t}+\frac{\partial u^1}{\partial x_i}a_i\left(\frac{x}{\varepsilon},u^0+\varepsilon u^1\right)-\Delta_x u^0-2\Delta_{xy}u^1\right\}\\
&- \varepsilon^2\Delta_x u^1. \qquad (2.40)
\end{aligned}
$$

Assuming that u^ε satisfies (2.11),

$$v^\varepsilon(t=0,x)-u^\varepsilon(t=0,x)=\varepsilon u^1\left(t=0,x,\frac{x}{\varepsilon}\right)=\varepsilon\sum_{i=1}^N\frac{\partial\bar u}{\partial x_i}\chi_i\left(\frac{x}{\varepsilon},\bar u_0(x)\right).$$

We assume that $a_{N+1}(y,0)=0$, so that $u^\varepsilon(t)$ and $v^\varepsilon(t)$ belong to $L^1(\mathbb{R}^N)$ for all $t\geq 0$. Thus, using the L^1 contraction property for equation (2.1) yields :

$$||u^\varepsilon(T)-v^\varepsilon(T)||_{L^1(\mathbb{R}^N)}\leq\varepsilon||u^1\left(t=0,x,\frac{x}{\varepsilon}\right)||_{L^1(\mathbb{R}^N)}+\int_0^T\int_{\mathbb{R}^N}|f^\varepsilon(t,x)|\,dx\,dt.$$

The next step consists in deriving a bound of order ε for f^ε. The calculations are lengthy and fastidious, and require very strong regularity assumptions on $\bar u$ and on the flux A : for instance, in order to upper bound the first two terms in (2.40), which are Taylor expansions, we need to assume $A\in W^{3,\infty}_{\text{per}}(Y\times\mathbb{R})^N$. Eventually, we obtain the following rough estimate :

$$\int_0^T\int_{\mathbb{R}^N}|f^\varepsilon(t,x)|\,dx\,dt\leq C\varepsilon\int_0^T\int_{\mathbb{R}^N}g^\varepsilon(t,x)\,dx\,dt,$$

where

$$g^\varepsilon(t,x):=|\nabla_x\bar u|^2+|\partial_t\bar u||\nabla_x\bar u|+|D^2\bar u|+|\partial_t\nabla_x\bar u|+|D^3\bar u|+|\nabla_x\bar u|^3+|D^2\bar u||\nabla_x\bar u| \qquad (2.41)$$

and C is a constant depending only on N, Y, and the bounds on A.

We do not give the details of the proof here; the main advantage of this method is to give a better understanding of the problem thanks to explicit calculations. The proof we will give of theorem 2 in this article does not require as many calculations, but might seem less intuitive since the convergence is "hidden" behind Young measures.

2.3.2 A few results about two-scale Young measures

Let us first recall a few results about two-scale Young measures : standard Young measures were introduced by Luc Tartar in [68] in the framework of compensated compactness as a tool to study weak limits of non-linear functions. Weinan E in [24] combined Tartar's results with Nguetseng's and Allaire's theory of two-scale convergence (see [3], [56]) and proved the following lemma:

Lemma 2.3.1. *Assume we have a sequence of functions $\{v^\varepsilon\}_{\varepsilon>0}$, with $v^\varepsilon : \mathbb{R}^N \to K$, where K is a compact set of $\mathbb{R}$. Then there exists a subsequence, still denoted by $\{v^\varepsilon\}_{\varepsilon>0}$, and a family of parametrized probability measures $\{\nu_{x,y}(\lambda)\}$ supported in K, which depends measurably on (x,y), and is periodic in y with period Y, such that as $\varepsilon \to 0$,*

$$\int_{\mathbb{R}^N} F(v^\varepsilon(t,x))\psi\left(x, \frac{x}{\varepsilon}\right)\, dx \to \int_{\mathbb{R}^N\times Y} \langle F(\lambda), \nu_{x,y}\rangle\, \psi(x,y)\, dx\, dy \tag{2.42}$$

for all $\psi \in K_{per}$, $F \in \mathcal{C}(K)$. The subsequence does not depend on ψ or F.

$\{\nu_{x,y}(\lambda)\}$ is the two-scale Young measure associated to the sequence v^ε.

For our application, we will need the following straightforward generalization of E's lemma :

Corollary 2.3.1. *Assume we have a sequence of functions $\{v^\varepsilon\}_{\varepsilon>0}$, with $v^\varepsilon : [0,\infty)\times\mathbb{R}^N \to K$, where K is a compact set of $\mathbb{R}$. Then there exists a subsequence, still denoted by $\{v^\varepsilon\}_{\varepsilon>0}$, and a family of parametrized probability measures $\{\nu_{t,x,y}(\lambda)\}$ supported in K, which depends measurably on (t,x,y), and is periodic in y with period Y, such that as $\varepsilon \to 0$,*

$$\int_{[0,\infty)\times\mathbb{R}^N} F\left(\frac{x}{\varepsilon}, v^\varepsilon(t,x)\right)\psi\left(t,x,\frac{x}{\varepsilon}\right)\, dt\, dx \to$$

$$\to \int_{[0,\infty)\times\mathbb{R}^N\times Y} \langle F(y,\lambda), \nu_{t,x,y}\rangle\, \psi(t,x,y)\, dt\, dx\, dy \tag{2.43}$$

for all $\psi \in J_{per}$, $F \in \mathcal{C}_{per}(Y\times K)$. The subsequence does not depend on ψ or F.

We will also use the following lemma, due to Tartar (see [68]):

Lemma 2.3.2. *The two-scale Young measure $\{\nu_{t,x,y}\}$ associated with $\{v^\varepsilon\}_{\varepsilon>0}$ reduces to a family of Dirac measures $\delta_{V(t,x,y)}$ if and only if*

$$\left|\left|v^\varepsilon(t,x) - V\left(t,x,\frac{x}{\varepsilon}\right)\right|\right|_{L^2_{loc}([0,\infty)\times\mathbb{R}^n)} \to 0.$$

We want to apply corollary 2.3.1 to the sequence u^ε of solutions of (2.1). Let us prove that u^ε is bounded. First, recall that $u^\varepsilon(t=0,x) = v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$ with $\bar{u}_0 \in L^\infty(\mathbb{R}^N)$. Thus, setting $C = ||\bar{u}_0||_{L^\infty(\mathbb{R}^N)}$ and recalling (2.18), we have

$$v\left(\frac{x}{\varepsilon}, -C\right) \le u^\varepsilon(t=0,x) \le v\left(\frac{x}{\varepsilon}, C\right) \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Then, notice that for all $p \in \mathbb{R}$, $v\left(\frac{x}{\varepsilon}, p\right)$ is a stationary solution of (2.1) and that the evolution operator associated to (2.1) is order preserving. Hence,

$$v\left(\frac{x}{\varepsilon}, -C\right) \le u^\varepsilon(t,x) \le v\left(\frac{x}{\varepsilon}, C\right) \qquad \text{for a.e. } t,x \in [0,\infty)\times\mathbb{R}^N$$

and

$$||u^\varepsilon||_{L^\infty([0,\infty)\times\mathbb{R}^N)} \le \max\left(||v(\cdot,C)||_{L^\infty(Y)}, ||v(\cdot,-C)||_{L^\infty(Y)}\right) =: k.$$

Thanks to this estimate, we can use corollary 2.3.1 for the sequence u^ε, with $K = [-k,k]$. Let $\nu_{t,x,y}$ be the two-scale Young measure associated to the sequence u^ε. As in [24], [27], [25], the goal is to reduce the family $\{\nu_{t,x,y}\}_{t,x,y}$ to a family of Dirac masses, which will lead to the strong convergence in L^2_{loc}, as announced in theorem 2.

2.3.3 Formulation of the cell problem in terms of Young measures

With this aim in view, we use once again the fact that $v\left(\frac{x}{\varepsilon}, p\right)$ is a stationary solution of (2.1), combined with the L^1 contraction principle for equation (2.1) and we obtain the following inequality:

$$\frac{\partial}{\partial t}\left|u^\varepsilon - v\left(\frac{x}{\varepsilon}, p\right)\right| + \frac{\partial}{\partial x_i}\left[\operatorname{sgn}\left(u^\varepsilon - v\left(\frac{x}{\varepsilon}, p\right)\right)\left(A_i\left(\frac{x}{\varepsilon}, u^\varepsilon\right) - A_i\left(\frac{x}{\varepsilon}, v\left(\frac{x}{\varepsilon}, p\right)\right)\right)\right] - \varepsilon\Delta_x|u^\varepsilon - v\left(\frac{x}{\varepsilon}, p\right)| \le 0 \quad (2.44)$$

In a first step, we multiply (2.44) by positive test functions $\varepsilon\varphi\left(t, x, \frac{x}{\varepsilon}\right)$, where $\varphi \in J_{\text{per}}$, and we pass to the limit as $\varepsilon \to 0$ using corollary 2.3.1 in order to derive information at the microscopic level on the mesure ν. This leads to the inequality (in the sense of distributions on $[0,\infty) \times \mathbb{R}^N \times Y$, for all $p \in \mathbb{R}$) :

$$-\Delta_y \left\langle |\lambda - v(y,p)|, \nu_{t,x,y}\right\rangle + \operatorname{div}_y \left\langle \operatorname{sgn}(\lambda - v(y,p))[A(y,\lambda) - A(y, v(y,p))], \nu_{t,x,y}\right\rangle \le 0.$$

Since the left-hand side has mean zero, the inequality is in fact an equality :

$$\Delta_y \left\langle |\lambda - v(y,p)|, \nu_{t,x,y}\right\rangle + \operatorname{div}_y \left\langle \operatorname{sgn}(\lambda - v(y,p))[A(y,\lambda) - A(y, v(y,p))], \nu_{t,x,y}\right\rangle = 0. \quad (2.45)$$

As we shall see in the sequel of the proof, we need to prove that the quantity

$$\left\langle \operatorname{sgn}(\lambda - v(y,p)), \nu_{t,x,y}\right\rangle$$

is well defined and independant of $y \in Y$. This result can be obtained in a rather simple and straightforward way by deriving equation (2.45) with respect to p; unfortunately, this manipulation is valid if and only if $\nu_{t,x,y}(v(y,p)) = 0$. However, deriving equation (2.45) on the right and on the left yields the following lemma :

Lemma 2.3.3. *We use the convention*

$$\left\langle \operatorname{sgn}(\lambda - \alpha), \nu_{t,x,y}\right\rangle := \nu_{t,x,y}(\lambda > \alpha) - \nu_{t,x,y}(\lambda < \alpha).$$

Then for all $p \in \mathbb{R}$, $\langle \operatorname{sgn}(\lambda - v(y,p)), \nu_{t,x,y}\rangle$ is well defined and is independant of $y \in Y$: there exists $C = C(t,x,p) \in L^\infty([0,\infty) \times \mathbb{R}^N \times \mathbb{R})$ such that

$$\left\langle \operatorname{sgn}(\lambda - v(y,p)), \nu_{t,x,y}\right\rangle = C(t,x,p) \quad \text{for a.e. } (t,x,y) \in [0,\infty) \times \mathbb{R}^N \times Y, \ \forall p \in \mathbb{R}.$$

We postpone the proof of the lemma to subsection 3.5.

2.3.4 Reduction of Young measures

As in [24], [25], [27], we apply DiPerna's method in [22] to reduce the family $\{\nu_{t,x,y}\}_{t,x,y}$ to a family of Dirac masses : we want to prove that

$$\partial_t \int_Y \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y}\right\rangle\, dy + \partial_{x_i} \int_Y \left\langle \eta_i(y, \lambda, u^0(t,x,y)), \nu_{t,x,y}\right\rangle\, dy \le 0, \quad (2.46)$$

where
$$\eta_i(y,\lambda,v) := \operatorname{sgn}(\lambda - v)\,[A_i(y,\lambda) - A_i(y,v)]\,.$$
Indeed, if (2.46) is true, then we multiply (2.46) by $e^{-|x|}$ (recall that u^0 is bounded in L^∞, but not in $L^1(\mathbb{R}^N \times Y)$ in general) and we get
$$\begin{aligned} & \frac{d}{dt}\int_{\mathbb{R}^N\times Y} \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y}\right\rangle e^{-|x|}\,dy\,dx \\ \leq\; & C\int_{\mathbb{R}^N\times Y} \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y}\right\rangle e^{-|x|}\,dy\,dx, \end{aligned}$$
where $C = ||a_i||_{L^\infty(Y\times[-k,k])}$. Hence, by Gronwall's lemma,
$$\begin{aligned} & \int_{\mathbb{R}^N\times Y} \left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y}\right\rangle e^{-|x|}\,dy\,dx \\ \leq\; & e^{Ct}\int_{\mathbb{R}^N\times Y} \langle |\lambda - v(y,\bar{u}_0(x))|, \nu_{t=0,x,y}\rangle\, e^{-|x|}\,dy\,dx. \end{aligned} \tag{2.47}$$
Moreover, since the initial data is well prepared thanks to (2.11),
$$\left\langle |\lambda - u^0(t=0,x,y)|, \nu_{t=0,x,y}\right\rangle = 0. \tag{2.48}$$

Thus, combining (2.47) and (2.48), we obtain
$$\left\langle |\lambda - u^0(t,x,y)|, \nu_{t,x,y}\right\rangle = 0 \quad \text{for a.e. } (t,x,y) \in [0,\infty)\times\mathbb{R}^N\times Y,$$
which entails
$$\nu_{t,x,y} = \delta_{u^0(t,x,y)}. \tag{2.49}$$

(2.46) remains to be proved. Formally, the left-hand side of (2.46) can be split into a sum of two terms :
$$\int_Y \left[\left\langle |\lambda - u^0(t,x,y)|, \partial_t \nu_{t,x,y}\right\rangle + \sum_{i=1}^N \left\langle \eta_i(y,\lambda,u^0(t,x,y)), \partial_{x_i}\nu_{t,x,y}\right\rangle \right] dy \tag{2.50}$$
and
$$\int_Y \left\langle \partial_t|\lambda - u^0(t,x,y)| + \sum_{i=1}^N \partial_{x_i}\eta_i(y,\lambda,u^0(t,x,y)), \nu_{t,x,y}\right\rangle dy. \tag{2.51}$$

First, in order to prove that (2.50) is nonpositive, we multiply (2.44) by nonnegative test functions $\varphi = \varphi(t,x) \in \mathcal{D}([0,\infty)\times\mathbb{R}^N)_+$ and pass to the limit as $\varepsilon \to 0$ using once again corollary 2.3.1. We obtain, in the sense of distributions and for all $p \in \mathbb{R}$,
$$\partial_t \int_Y \langle |\lambda - v(y,p)|, \nu_{t,x,y}\rangle + \sum_{i=1}^N \partial_{x_i}\int_Y \langle \eta_i(y,\lambda,v(y,p)), \nu\rangle \leq 0. \tag{2.52}$$

(2.52) yields
$$\int_Y \left[\langle |\lambda - v(y,p)|, \partial_t \nu_{t,x,y}\rangle + \sum_{i=1}^N \langle \eta_i(y,\lambda,v(y,p)), \partial_{x_i}\nu_{t,x,y}\rangle \right] dy \leq 0$$

for all $p \in \mathbb{R}$. The choice $p = \bar{u}(t, x)$ implies that (2.50) is nonpositive.

Proving that the term (2.51) is nonpositive is a bit more difficult, mainly because if $\bar{u}$ is an entropy solution of (2.36), there is no reason why u^0 should be an entropy solution of the scalar law

$$\frac{\partial u^0}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(y, u^0) = g(t, x, y),$$

where g is a source term with null Y-average (recall that u^1 is defined only on the sets on which $\bar{u}$ is regular; on such sets, it can be proved that u^0 is indeed an entropy solution of such a law).

The idea is to use the results on kinetic formulation of conservation laws (see for instance [59]): if $S \in \mathcal{C}^2(\mathbb{R})$, then

$$\frac{\partial S(\bar{u})}{\partial t} + \sum_{i=1}^{N} \frac{\partial \bar{\eta}_i(\bar{u})}{\partial x_i} = -\int_{\mathbb{R}} S''(p) m(t, x, p)\, dp, \tag{2.53}$$

where m is the entropy defect measure associated to $\bar{u}$, and $\bar{\eta}_i$ is defined by

$$\bar{\eta}_i'(p) = \bar{a}_i(p) S'(p).$$

Set, for $(y, \lambda) \in Y \times \mathbb{R}$,

$$S^{y,\lambda}(p) := |v(y, p) - \lambda|,$$
$$\eta_i^{y,\lambda}(p) := \int_0^p \bar{a}_i(q) \operatorname{sgn}(v(y, q) - \lambda) \frac{\partial v(y, q)}{\partial q}\, dq.$$

Unfortunately $S^{y,\lambda}$ is not $\mathcal{C}^2$: thus, we use (2.53) for $S^{y,\lambda,\delta}(p) := S^{y,\lambda} * \varphi_\delta(p)$, where φ_δ is a standard mollifier, and we let $\delta \to 0$. It can be readily shown that $S^{y,\lambda,\delta}$ (resp. $\eta_i^{y,\lambda,\delta}$) converges to $S^{y,\lambda}$ (resp. $\eta_i^{y,\lambda}$) uniformly on compact sets of $\mathbb{R}$ and uniformly for $(y, \lambda) \in Y \times K$ (recall that $\nu_{t,x,y}$ is supported in K). Thus as $\delta \to 0$, in the sense of distributions on $[0, \infty) \times \mathbb{R}^N$,

$$\int_Y \left\langle \partial_t S^{y,\lambda,\delta}(\bar{u}(t, x)), \nu_{t,x,y} \right\rangle\, dy \rightharpoonup \int_Y \left\langle \partial_t S^{y,\lambda}(\bar{u}(t, x)), \nu_{t,x,y} \right\rangle\, dy,$$

and the same convergence holds for $\partial_{x_i} \eta_i^{y,\lambda,\delta}(\bar{u})$.

On the other hand,

$$S^{y,\lambda,\delta\prime\prime}(p) = \int_{\mathbb{R}} \frac{\partial v}{\partial p}(y, p') \operatorname{sgn}(v(y, p') - \lambda) \varphi_\delta'(p - p')\, dp';$$

using lemma 2.3.3 and the property $\left\langle \frac{\partial v}{\partial p} \right\rangle = 1$ yields

$$\int_Y \left\langle S^{y,\lambda,\delta\prime\prime}(p), \nu_{t,x,y} \right\rangle\, dy = \int_Y \left\langle \left[\int_{\mathbb{R}} \operatorname{sgn}(v(y, p') - \lambda) \varphi_\delta'(p - p')\, dp' \right], \nu_{t,x,y} \right\rangle\, dy.$$

Then, using a regularization of the function signum it can be proved that

$$\int_{\mathbb{R}} \operatorname{sgn}(v(y, p') - \lambda) \varphi_\delta'(p - p')\, dp' \geq 0 \quad \forall y \in Y, \lambda \in \mathbb{R},$$

and consequently

$$-\int_{\mathbb{R}\times Y}\left\langle S^{y,\lambda,\delta''}(p),\nu_{t,x,y}\right\rangle\, m(t,x,p)\,dp\,dy\leq 0.$$

Thus, passing to the limit as $\delta\to 0$, we obtain

$$\int_Y\left\langle \partial_t|\lambda-v(y,\bar u(t,x))|+\sum_{i=1}^N\partial_{x_i}\eta_i^{y,\lambda}(\bar u),\nu_{t,x,y}\right\rangle\,dy\leq 0 \tag{2.54}$$

where the inequality is meant in the sense of distributions.

We split (2.51) into

$$\begin{aligned}
&\int_Y\left\langle \partial_t|\lambda-u^0(t,x,y)|+\sum_{i=1}^N\partial_{x_i}\eta_i(y,\lambda,u^0(t,x,y)),\nu_{t,x,y}\right\rangle\,dy\\
=\;&\int_Y\left\langle \partial_t|\lambda-u^0(t,x,y)|+\sum_{i=1}^N\partial_{x_i}\eta_i^{y,\lambda}(\bar u),\nu_{t,x,y}\right\rangle\,dy &(2.55)\\
&+\sum_{i=1}^N\int_Y\left\langle\partial_{x_i}\left[\eta_i(y,\lambda,u^0(t,x,y))-\eta_i^{y,\lambda}(\bar u)\right],\nu_{t,x,y}\right\rangle\,dy &(2.56)
\end{aligned}$$

Thanks to (2.54), (2.55) is nonpositive. Let us now focus on (2.56) : set

$$\begin{aligned}
f^i(y,\lambda,p)\;&:=\;\frac{\partial}{\partial p}\left[\eta_i(y,\lambda,v(y,p))-\eta_i^{y,\lambda}(p)\right]\\
&=\;\operatorname{sgn}(v(y,p)-\lambda)\frac{\partial v}{\partial p}\left[a_i(y,v(y,p))-\bar a_i(p)\right]
\end{aligned}$$

Using once again lemma 2.3.3 and the definition of $\bar a_i$ yields

$$\int_Y\left\langle f^i(y,\lambda,p),\nu_{t,x,y}\right\rangle\,dy=0\quad\forall p\in\mathbb{R}. \tag{2.57}$$

Set

$$F^i(y,\lambda,q):=\int_0^q f^i(y,\lambda,p)\,dp;$$

Then (2.56) is equal to

$$\begin{aligned}
&\sum_{i=1}^N\int_Y\left\langle\partial_{x_i}F^i(y,\lambda,\bar u(t,x)),\nu_{t,x,y}\right\rangle\,dy\\
=\;&\sum_{i=1}^N\partial_{x_i}\int_Y\left\langle F^i(y,\lambda,\bar u(t,x)),\nu_{t,x,y}\right\rangle\,dy-\int_Y\left\langle F^i(y,\lambda,\bar u(t,x)),\partial_{x_i}\nu_{t,x,y}\right\rangle\,dy.
\end{aligned}$$

(2.57) entails that

$$\int_Y\left\langle F^i(y,\lambda,q),\nu_{t,x,y}\right\rangle\,dy=0\qquad\text{for a.e. }(t,x)\in[0,\infty)\times\mathbb{R}^N\ \forall q\in\mathbb{R},$$

and thus (2.56) is null as well. Hence, we have proved (2.46), and the family $\{\nu_{t,x,y}\}_{t,x,y}$ is reduced to a family of Dirac masses.

□

Remark 2.3.1. *In fact, several regularizations are necessary in order to make the proof rigorous; for instance, we need to regularize the measure ν with respect to t, x, so that the quantities $\partial_t\nu$, $\partial_{x_i}\nu$ are well-defined and the properties of lemma 2.3.3 are preserved, together with inequality* (2.52)*. These calculations are straight-forward and follow the arguments developed by R. DiPerna in [22].*

Let us stress as well that the equality $\nu_{t=0,x,y} = \delta_{u_0(x,y)}$ is not obvious: indeed, uniform bounds in ε on $\left|\left|u^\varepsilon(t) - u_0\left(x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(\mathbb{R}^N)}$, for t close to 0, are not easy to derive; a simple way to prove this fact is to go back to inequality (2.44)*, which yields*

$$\limsup_{t\to 0} \int_{\mathbb{R}^N\times Y} \langle|\lambda - v(y,p)|, \nu_{t,x,y}\rangle\, \varphi(x)\, dx\, dy \leq$$
$$\leq \int_{\mathbb{R}^N\times Y} |v(y,\bar{u}_0(x)) - v(y,p)|\, \varphi(x)\, dx\, dy,$$

for all $p \in \mathbb{R}$, $\varphi \in \mathcal{D}(\mathbb{R}^N)_+$. Hence, for any measure $\mu_{x,y}(\lambda)$ such that there exists a sequence $t_n \to 0$ with $\nu_{t_n,x,y} \rightharpoonup \mu_{x,y}$ w-$M^1(\mathbb{R}_\lambda \times \mathbb{R}^N \times Y)$, we have

$$\int_Y \langle|\lambda - v(y,p)|, \mu_{x,y}\rangle\, dy \leq \int_Y |v(y,\bar{u}_0(x)) - v(y,p)|\, dy$$

for all $p \in \mathbb{R}$ and in the sense of measures for $x \in \mathbb{R}^N$. Taking $p = \bar{u}_0(x)$ gives $\mu_{x,y} = \delta_{u_0(x,y)}$, and thus the whole sequence $\nu_{t,x,y}$ converges in w-M^1 to $\delta_{u_0(x,y)}$ as $t \to 0$.

2.3.5 Proof of lemma 2.3.3

First, observe that $\langle|\lambda - v(y,p)|, \nu_{t,x,y}\rangle$ is a continuous function of p for a.e. $(t,x,y) \in [0,\infty)\times\mathbb{R}^N\times Y$. Moreover, if $\lambda \neq v(y,p_0)$, then the function $f_y(\lambda,p) := |\lambda - v(y,p)|$ has a partial derivative with respect to p at the point (λ, p_0) which is equal to

$$\frac{\partial f_y}{\partial p}(\lambda,p_0) = -\frac{\partial v}{\partial p}(y,p_0)\mathrm{sgn}(\lambda - v(y,p_0)).$$

If $\lambda = v(y,p_0)$, then f_y has a partial derivatives on the right and on the left at the point (λ,p_0) which are equal to

$$\frac{\partial f_y}{\partial p}(v(y,p_0),p_0^+) = \frac{\partial v}{\partial p}(y,p_0),$$
$$\frac{\partial f_y}{\partial p}(v(y,p_0),p_0^-) = -\frac{\partial v}{\partial p}(y,p_0).$$

Additionnally, notice that for all $\lambda \in \mathbb{R}$, $p \neq p_0$

$$\left|\frac{f_y(\lambda,p) - f_y(\lambda,p_0)}{p - p_0}\right| \leq \left|\frac{v(y,p) - v(y,p_0)}{p-p_0}\right| \leq \left|\left|\frac{\partial v}{\partial p}\right|\right|_{L^\infty(Y\times\mathbb{R})}.$$

Hence, using Lebesgue's dominated convergence theorem, we deduce that the function

$$F_{t,x,y}(p) := \langle|\lambda - v(y,p)|, \nu_{t,x,y}\rangle$$

has derivatives on the right and on the left with respect to p for almost every $(t,x,y)\in[0,\infty)\times\mathbb{R}^N\times Y$:

$$F'_{t,x,y}(p_0^+) = -\frac{\partial v}{\partial p}(y,p_0)\left\langle \mathrm{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\right\rangle + \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\frac{\partial v}{\partial p}(y,p_0),$$
$$F'_{t,x,y}(p_0^-) = -\frac{\partial v}{\partial p}(y,p_0)\left\langle \mathrm{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\right\rangle - \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\frac{\partial v}{\partial p}(y,p_0).$$

In a similar fashion, the function

$$G^i_{t,x,y}(p) := \left\langle \mathrm{sgn}(\lambda - v(y,p))\left[A_i(y,\lambda) - A_i(y,v(y,p))\right],\nu_{t,x,y}\right\rangle$$

has derivatives on the right and on the left with respect to p at $p=p_0$ which are equal to

$$G^{i\prime}_{t,x,y}(p_0^+) = -\frac{\partial v}{\partial p}(y,p_0)a_i(y,v(y,p_0))\left[\left\langle \mathrm{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\right\rangle - \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\right],$$
$$G^{i\prime}_{t,x,y}(p_0^-) = -\frac{\partial v}{\partial p}(y,p_0)a_i(y,v(y,p_0))\left[\left\langle \mathrm{sgn}(\lambda - v(y,p_0)),\nu_{t,x,y}\right\rangle + \nu_{t,x,y}\left(\{v(y,p_0)\}\right)\right].$$

Thus, setting

$$r(t,x,y,p) := \frac{\partial v}{\partial p}(y,p)\left[\left\langle \mathrm{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\right\rangle - \nu_{t,x,y}\left(\{v(y,p)\}\right)\right],$$
$$l(t,x,y,p) := \frac{\partial v}{\partial p}(y,p)\left[\left\langle \mathrm{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\right\rangle + \nu_{t,x,y}\left(\{v(y,p)\}\right)\right],$$

we see that l and r both satisfy for all $p\in\mathbb{R}$ the elliptic equation

$$-\Delta_y g + \mathrm{div}_y\left(a(y,v(y,p))g\right) = 0 \tag{2.58}$$

a.e. on $[0,\infty)\times\mathbb{R}^N$ and in the sense of distributions on Y. Thus $l,r\in H^1_{\mathrm{per}}(Y)$ for all $p\in\mathbb{R}$ and for a.e. $(t,x)\in[0,\infty)\times\mathbb{R}^N$, and the equation is satisfied in the variational sense for elliptic equations.

Comparing (2.58) to (2.17), and using the Krein-Rutman theorem (see lemma 2.2.1), we deduce that there exist constants $C_r=C_r(t,x,p)$ and $C_l=C_l(t,x,p)$ such that

$$r(t,x,y,p) = C_r(t,x,p)\frac{\partial v}{\partial p}(y,p),$$
$$l(t,x,y,p) = C_l(t,x,p)\frac{\partial v}{\partial p}(y,p)$$

Since $\frac{\partial v}{\partial p}$ is a positive function which does not vanish on Y (see lemma 2.2.1), this yields

$$\left\langle \mathrm{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\right\rangle - \nu_{t,x,y}\left(\{v(y,p)\}\right) = C_r(t,x,p),$$
$$\left\langle \mathrm{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\right\rangle + \nu_{t,x,y}\left(\{v(y,p)\}\right) = C_l(t,x,p).$$

Thus,

$$\left\langle \mathrm{sgn}(\lambda - v(y,p)),\nu_{t,x,y}\right\rangle = \frac{1}{2}\left(C_l(t,x,p) + C_r(t,x,p)\right) = C(t,x,p).$$

and the proof is complete. Notice that we have also proved that $\nu_{t,x,y}\left(\{v(y,p)\}\right)$ does not depend on y.

□

Chapitre 3

Formulation cinétique d'une loi de conservation scalaire parabolique. Application à un problème d'homogénéisation

On construit dans ce chapitre une formulation cinétique pour la loi de conservation parabolique

$$\partial_t u + \text{div}_y A(y, u) - \Delta_y u = 0.$$

En s'appuyant sur cette formulation, on définit une notion de solution faible dans L^1 de la loi de conservation ci-dessus, et on montre que ces solutions faibles obéissent à un principe de contraction. Par ailleurs, on donne une nouvelle preuve du résultat d'homogénéisation du chapitre précédent. La démonstration présentée ici repose sur la structure de l'équation de formulation cinétique plutôt que sur les mesures d'Young à deux échelles. La relative simplicité de cette preuve permet de la transposer à des situations plus complexes : on traite ici le cas où le flux A possède une dépendance macroscopique.

Ce chapitre a fait l'objet d'une publication dans SIAM Journal on Mathematical Analysis (2007), à l'exception de l'appendice en français.

3.1 Introduction

This paper is devoted to the study of the solution u of the equation

$$\left\{\begin{array}{l}\partial_t u(t,y) + \mathrm{div}_y A(y,u(t,y)) - \Delta_y u(t,y) = 0, \quad t>0,\ y\in Y\\ u(t=0,y) = u_0(y),\end{array}\right. \tag{3.1}$$

where $Y=[0,1]^N$ is the N-dimensional torus; $A=A(y,v)\in\mathbb{R}^N$, $y\in Y$, $v\in\mathbb{R}$ is a given N-dimensional flux, periodic in the space variable y. The function u belongs to $\mathcal{C}([0,\infty),L^1(Y))\cap L^2_{\mathrm{loc}}([0,\infty),H^1_{\mathrm{per}}(Y))\cap L^\infty_{\mathrm{loc}}([0,\infty)\times Y)$.

In [15], a kinetic formulation was derived for such heterogeneous conservation laws (in fact, this work was achieved for hyperbolic laws, but it can be generalized to parabolic laws with no difficulty), based on the previous papers of P.-L. Lions, B. Perthame and E. Tadmor concerning hyperbolic homogeneous conservation laws (see [53], [52], [60], [58], and the general presentation in [59]). However, this formulation is not entirely satisfactory : indeed, it is based on the comparison between the solution $u(t,y)$ of the conservation law and the constants via the function $\mathbf{1}_{v<u(t,y)}$, where v is an additional fluctuation variable. But the constants, which happen to be stationary solutions of homogeneous conservation laws, no longer play a special role in the context of heterogeneous conservation laws. Hence, our goal in this article is to derive a kinetic formulation based on the study of the stationary solutions of (3.1). Let us mention a related work of E. Audusse and B. Perthame [8], which defines a notion of entropy solution which is not based on Kruzkhov's inequalities, but rather on the comparison with special stationary solutions, and which is sufficient to derive the L^1 contraction principle.

Let us precise a few notations which will be used later on : if $\mathcal{C}^\infty_{\mathrm{per}}(Y)$ denotes the space of Y-periodic functions in $\mathcal{C}^\infty(\mathbb{R}^N)$, then

$$\begin{aligned}W^{k,p}_{\mathrm{per}}(Y) &:= \overline{\mathcal{C}^\infty_{\mathrm{per}}(Y)}^{W^{k,p}(Y)},\\ W^{1,\infty}_{\mathrm{per,loc}}(Y\times\mathbb{R}) &:= \{u=u(y,v)\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{N+1}), u \text{ is } Y-\text{periodic in } y\},\\ \mathcal{D}_{\mathrm{per}}([0,\infty)\times Y\times\mathbb{R}) &:= \{u=u(t,y,v)\in\mathcal{C}^\infty([0,\infty)\times\mathbb{R}^{N+1}),\ u \text{ is periodic in } y\\ &\qquad \text{and } \exists R>0,\ u(t,y,v)=0 \text{ if } t+|v|\geq R\},\\ \langle v\rangle &:= \frac{1}{|Y|}\int_Y v(y)\,dy \quad \forall v\in L^1(Y).\end{aligned}$$

First, let us recall a few results on the stationary solutions of (3.1), which were studied in [14]:

Proposition 3.1.1. *Let $A=A(y,v)\in W^{1,\infty}_{per,loc}(Y\times\mathbb{R})^N$. Let $a_i(y,v):=\partial_v A_i(y,v)$, $1\leq i\leq N$, $b(y,v):=\mathrm{div}_y A(y,v)\in L^\infty_{loc}(\mathbb{R}^{N+1})$. Assume that there exist real numbers $C_0>0$, $m\in[0,\infty)$, $n\in[0,\frac{N+2}{N-2})$ when $N\geq 3$, such that for all $(y,p)\in Y\times\mathbb{R}$*

$$|a_i(y,p)|\leq C_0\,(1+|p|^m) \quad \forall\ 1\leq i\leq N, \tag{3.2}$$

$$|b(y,p)|\leq C_0\,(1+|p|^n)\,. \tag{3.3}$$

Assume as well that

$$a(y,\cdot)\in\mathcal{C}(\mathbb{R})^N \quad \textit{for almost every } y\in Y, \tag{3.4}$$

and that the couple (m,n) *satisfies at least one of the following conditions*

$$m=0 \tag{3.5}$$

$$\textit{or} \quad 0\le n<1 \tag{3.6}$$

$$\textit{or} \quad n<\min\left(\frac{N+2}{N},2\right) \quad \textit{and } \exists p_0\in\mathbb{R},\ \forall y\in Y\ b(y,p_0)=0. \tag{3.7}$$

Then for all $p\in\mathbb{R}$*, there exists a unique solution* $v(\cdot,p)\in H^1_{per}(Y)$ *of the equation*

$$-\Delta_y v(y,p)+\mathrm{div}_y A(y,v(y,p))=0, \quad \langle v(\cdot,p)\rangle=p. \tag{3.8}$$

For all $p\in\mathbb{R}$*,* $v(\cdot,p)$ *belongs to* $W^{2,q}_{per}(Y)$ *for all* $1<q<+\infty$ *and satisfies the following a priori estimate : for all* $R>0$*, there exists a constant* $C_R>0$ *depending only on* N*,* Y*,* C_0*,* m*,* n*,* q*,* p_0 *and* R*, such that*

$$||v(\cdot,p)||_{W^{2,q}(Y)}\le C_R \quad \forall p\in\mathbb{R},\ |p|\le R. \tag{3.9}$$

Moreover, for all $p\in\mathbb{R}$*,* $\partial_p v(\cdot,p)\in H^1_{per}(Y)$ *and is a solution of*

$$-\Delta_y\frac{\partial v}{\partial p}+\mathrm{div}_y\left[a(y,v(y,p))\frac{\partial v}{\partial p}\right]=0, \quad \left\langle\frac{\partial v}{\partial p}\right\rangle=1. \tag{3.10}$$

And for all $R>0$*, there exists* $\alpha>0$ *depending only on* N*,* Y*,* C_0*,* m*,* n*,* q*,* p_0 *and* R*, such that for all* $(y,p)\in Y\times(-R,R)$*,*

$$\frac{\partial v}{\partial p}(y,p)\ge\alpha>0.$$

Equation (3.8) is also called “cell problem”, on account of its significance in homogenization problems.

Following the idea of E. Audusse and B. Perthame (see [8]), we now give a notion of entropy solution for equation (3.1) based on the comparison with stationary solutions :

Definition 3.1.1. *Assume the hypotheses of proposition 3.1.1 are satisfied.*

Let $u\in\mathcal{C}([0,\infty),L^1(Y))\cap L^2_{loc}([0,\infty),H^1_{per}(Y))\cap L^\infty_{loc}([0,\infty)\times Y)$ *be a solution of* (3.1)*. We say that* u *is an entropy solution of* (3.1) *if* u *satisfies the inequality*

$$\partial_t(u(t,y)-v(y,p))_+ +\mathrm{div}_y\left[\mathbf{1}_{u>v(y,p)}(A(y,u)-A(y,v(y,p)))\right] \\ -\Delta_y(u(t,y)-v(y,p))_+\le 0 \tag{3.11}$$

for all $p\in\mathbb{R}$ *and in the sense of distributions on* $[0,\infty)\times Y$*.*

Notice that this notion of entropy solution is different (at least in its formulation) from the one of Kruzkhov, since the latter is based on the comparison with constants. However, inequality (3.11) was known by Kruzkhov, since it can be considered as a particular case of the comparison principle (notice that $v(y,p)$ is a stationary solution of (3.1)). It will be proved in the second section, under suitable regularity assumptions on the flux function A, that all solutions of (3.1) are entropy solutions in the sense of definition 3.1.1.

Let us mention here an important application of inequality (3.11) and of the kinetic formulation which follows from (3.11) : we give in this paper another proof for a homogenization result proved in [14], which we recall here for the reader's convenience :

Proposition 3.1.2. *Assume that* $A \in W^{1,\infty}_{per,loc}(\mathbb{R}^{N+1})^N$ *satisfies the assumptions of proposition 3.1.1, and that* $\partial_{y_j} a_i \in L^1_{loc}(\mathbb{R}^{N+1})$, $\partial_v a_i \in L^1_{loc}(\mathbb{R}^{N+1})$ *for* $1 \le i \le N+1$, $1 \le j \le N$.

For $\varepsilon > 0$, *let* $v^\varepsilon \in L^\infty_{loc}([0,\infty)\times\mathbb{R}^N)\cap\mathcal{C}([0,\infty), L^1_{loc}(\mathbb{R}^N))\cap L^2_{loc}([0,\infty), H^1_{loc}(\mathbb{R}^N))$ *be a solution of the parabolic scalar conservation law :*

$$\frac{\partial v^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, v^\varepsilon(t,x)\right) - \varepsilon \Delta_x v^\varepsilon = 0 \quad t \ge 0,\ x \in \mathbb{R}^N, \tag{3.12}$$

$$v^\varepsilon(t=0) = v_0\left(x, \frac{x}{\varepsilon}\right). \tag{3.13}$$

Let $p \in \mathbb{R}$, *and let* $v = v(y,p)$ *be the unique solution in* $H^1_{per}(Y)$ *of the cell problem* (3.8). *Define*

$$\bar{A}_i(p) := \frac{1}{|Y|}\int_Y A_i(y, v(y,p))\, dy. \tag{3.14}$$

Assume also that v_0 *is "well-prepared", i.e. satisfies*

$$v_0(x,y) = v(y, \bar{v}_0(x)) \tag{3.15}$$

for some $\bar{v}_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$.

Then as ε *goes to 0,*

$$v^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{v}(t,x)\right) \to 0 \quad \text{in } L^2_{loc}([0,\infty)\times\mathbb{R}^N),$$

where $\bar{v} = \bar{v}(t,x) \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty)\times\mathbb{R}^N)$ *is the unique entropy solution of the hyperbolic scalar conservation law*

$$\begin{cases} \dfrac{\partial \bar{v}}{\partial t} + \displaystyle\sum_{i=1}^N \frac{\partial \bar{A}_i(\bar{v}(t,x))}{\partial x_i} = 0, \\ \bar{v}(t=0,x) = \bar{v}_0(x) \in L^1 \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{3.16}$$

Actually, the result proved in section 3.3 is more general than proposition 3.1.2, but is much more complicated to state at this stage. In particular, we work in a L^1 rather than L^∞ setting, which appears to us to be entirely new for this kind of equation; this point will be developed a little further in remark 3.3.2. We emphasize that inequality (3.11) was already used in [14], but we believe that the proof given here gives a better insight of the homogenization process.

Let us mention related results of Weinan E (see [24], [25]), and Weinan E and Denis Serre (see [27]), which use two-scale Young measures instead of the kinetic formulation in a hyperbolic context. In fact, the proof of [14] is close to the ones of these articles, although the viscous term in (3.12) is absent from the problems studied by Weinan E in [24], and Weinan E and Denis Serre in [27]. Indeed, the scaling in our problem is chosen so that the viscosity has the same order of magnitude than the size of the oscillations in the flux function, and thus the viscosity has an effect at a microscopic level only. Notice that the (macroscopic) homogenized problem (3.16) is hyperbolic; this justifies the use of hyperbolic tools, such as Young measures or a kinetic fomulation, in the study of equation (3.12).

We also wish to point out that the expression of the homogenized flux in the case studied by Weinan E and Denis Serre in [27] when $N = 1$ is the same as in (3.14). However, the corrector v appearing in the expression is not the same in both cases : indeed, in the hyperbolic case studied by Weinan E and Denis Serre, v is a solution of

$$\partial_y A(y, v(y, p)) = 0.$$

In particular, v is not unique in general, although the homogenized flux is. We refer the interested reader to [27] and [49] for details; the latter uses an equivalent formulation using Hamilton-Jacobi equations.

The organization of this article is as follows : first we derive a kinetic formulation for equation (3.1). As usual, this allows us to define a weaker notion of solutions of the parabolic conservation law (3.1), called *kinetic solutions*. We also derive formally the L^1 contraction principle for kinetic solutions of equation (3.1). Then we use this formulation to give another proof of proposition 3.1.2 in section 3.3. Eventually, in section 3.4 we give a rigorous proof for the derivation of the L^1 contraction principle announced in section 3.2.

3.2 Kinetic formulation

This section is devoted to the derivation of a kinetic formulation for equation (3.1). Throughout the section, we assume that the hypotheses of proposition 3.1.1 are satisfied, that is, $A \in W^{1,\infty}_{\text{per,loc}}(Y \times \mathbb{R})$, and A satisfies either (3.5), or (3.6), or (3.7), together with (3.4).

Under such hypotheses, the following result is easily deduced from proposition 3.1.1 :

Lemma 3.2.1. *For a.e.* $y \in Y$, $p \mapsto v(y, p)$ *is a* $\mathcal{C}^1$ *diffeomorphism from* $\mathbb{R}$ *to* $(\alpha_-(y), \alpha_+(y))$, *where* $\alpha_\pm(y) = \lim_{p \to \pm\infty} v(y, p)$.

Its reciprocal application is denoted by $w(y,\cdot)$

$$w(y) : (\alpha_-(y), \alpha_+(y)) \to \mathbb{R}.$$

Remark 3.2.1. *Notice that* $+\infty$ *(resp.* $-\infty$*) is an admissible value for* α_+ *(resp.* α_-*). In fact, it can be checked that*

$$\langle \alpha_\pm \rangle = \pm\infty,$$

and there are cases when

$$\alpha_\pm(y) = \pm\infty \quad \forall y \in Y.$$

Indeed, for all $y \in Y$*, the family* $(v(y,p) - v(y,0))_{p>0}$ *is increasing in* p *and nonnegative. Moreover,*

$$\langle v(\cdot,p) - v(\cdot,0)\rangle = p \quad \forall p \in \mathbb{R}.$$

Hence according to Lebesgue's monotone convergence theorem, $\langle \alpha_+ - v(\cdot,0)\rangle = +\infty$*, and thus* $\langle \alpha_+ \rangle = +\infty$*. If we assume additionnally that* $m = 0$ *in hypothesis* (3.2) *(i.e. we assume that* (3.5) *is satisfied), then it is proved in [14], lemma 6 (see also* (2.20), (2.21)*), that*

$$\lim_{p\to+\infty} \inf_{y\in Y} v(y,p) = +\infty.$$

In that case, $\alpha_+(y) = +\infty$ *for all* $y \in Y$*.*

We begin our study of equation (3.1) with the following

Lemma 3.2.2. *Let* $u \in \mathcal{C}([0,\infty); L^1(Y)) \cap L^2_{loc}(0,\infty; H^1_{per}(Y)) \cap L^\infty_{loc}([0,\infty)\times Y)$ *be an arbitrary solution of* (3.1)*. Assume that the flux* $A \in W^{1,\infty}_{per,loc}(Y\times\mathbb{R})$ *satisfies* (3.4) *and the hypotheses of proposition 3.1.1.*

Then the function u *satisfies the following equality in the sense of distributions on* $[0,\infty)\times Y \times \mathbb{R}_p$

$$\partial_t (u - v(y,p))_+ + \mathrm{div}_y \left[\mathbf{1}_{u>v(y,p)}(A(y,u) - A(y,v(y,p)))\right] - \Delta_y (u - v(y,p))_+ = -m, \tag{3.17}$$

where

$$m(t,y,p) = \frac{1}{\frac{\partial v}{\partial p}(y,p)} \left|\nabla_y (u(t,y) - v(y,p))\right|^2 \delta(p = w(y,u(t,y)))$$

is a nonnegative measure on $(0,\infty)\times Y\times\mathbb{R}$*.*

Consequently, u *is an entropy solution of* (3.1) *in the sense of definition 3.1.1.*

We postpone the proof of lemma 3.2.2 to the end of section 3.2. Let us stress that equality (3.17) is to be understood in the sense of distributions in $[0,\infty)\times Y\times\mathbb{R}$. Such an equality would indeed be meaningless were it considered for $p \in \mathbb{R}$ fixed.

Let us now write down the kinetic formulation for equation (3.1). Let u be an entropy solution of (3.1); differentiating equality (3.17) with respect to p leads to

$$\frac{\partial}{\partial t}\left(\frac{\partial v(y,p)}{\partial p} f^+\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v(y,p)}{\partial p} a_i(y,v(y,p)) f^+\right) - \Delta_y \left(\frac{\partial v(y,p)}{\partial p} f^+\right) = \frac{\partial m(t,y,p)}{\partial p} \tag{3.18}$$

where $f^+(t,y,p) = \mathbf{1}_{u(t,y)>v(y,p)}$.

The same kind of equation holds for $f^- = \mathbf{1}_{u(t,y)<v(y,p)} = 1 - f^+$ (recall (3.10))

$$\frac{\partial}{\partial t}\left(\frac{\partial v(y,p)}{\partial p} f^-\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v(y,p)}{\partial p} a_i(y,v(y,p)) f^-\right) - \Delta_y \left(\frac{\partial v(y,p)}{\partial p} f^-\right) = -\frac{\partial m}{\partial p} \tag{3.19}$$

This leads to a notion of kinetic solution :

Definition 3.2.1. *Assume that the flux A satisfies the hypotheses of proposition 3.1.1. Let* $u = u(t,y) \in \mathcal{C}([0,\infty); L^1(Y)) \cap L^2_{loc}(0,\infty; H^1_{per}(Y))$ *such that*

$$\alpha_-(y) < u(t,y) < \alpha_+(y) \quad \text{for a.e. } (t,y) \in [0,\infty) \times Y.$$

We say that u is a kinetic solution of (3.1) *if* $f^+ = \mathbf{1}_{u(t,y)>v(y,p)}$ *satisfies* (3.18) *in the sense of distributions with the initial data* $f^+(t=0,y,p) = \mathbf{1}_{u_0(y)>v(y,p)}$*, and if there exists a function* $\mu \in L^\infty(\mathbb{R})$ *such that* $\mu(p) \to 0$ *as* $|p| \to \infty$*, and*

$$\int_0^\infty \int_Y m(t,y,p)\, dy\, dt \le \mu(p) \quad \text{in } \mathcal{D}'(\mathbb{R}). \tag{3.20}$$

Precisely, u is a kinetic solution of (3.1) *if* (3.20) *holds and if for all test function* $\psi = \psi(t,y,p) \in \mathcal{D}_{per}([0,\infty) \times Y \times \mathbb{R})$*, we have*

$$\int_0^\infty \int_{Y\times\mathbb{R}} f^+(t,y,p) \frac{\partial v(y,p)}{\partial p} \left\{\partial_t \psi + a_i(y,v(y,p))\partial_{y_i}\psi + \Delta_y \psi\right\} dt\, dy\, dp =$$
$$= \int_0^\infty \int_{Y\times\mathbb{R}} m(t,y,p)\partial_p \psi(t,y,p)\, dt\, dy\, dp - \int_{Y\times\mathbb{R}} \mathbf{1}_{u_0(y)>v(y,p)} \frac{\partial v(y,p)}{\partial p} \psi(0,y,p)\, dy\, dp. \tag{3.21}$$

Notice that without any loss of generality, we can choose a function μ in (3.20) which is nonincreasing on $(0,\infty)$ and nondecreasing on $(-\infty,0)$.

It is easily checked that the notions of entropy and kinetic solutions are equivalent as long as u is bounded in some kind of L^∞ norm :

Proposition 3.2.1. *Assume that A satisfies* (3.4) *and the hypotheses of proposition 3.1.1. Let* $u = u(t,y) \in \mathcal{C}([0,\infty); L^1(Y)) \cap L^2_{loc}(0,\infty; H^1_{per}(Y))$*. Assume that there exist real numbers* $\beta_1, \beta_2 \in \mathbb{R}$ *such that*

$$v(y,\beta_1) \le u(t,y) \le v(y,\beta_2) \quad \text{for a.e. } (t,y) \in (0,\infty) \times Y. \tag{3.22}$$

Then u is an entropy solution of (3.1) *if and only if u is a kinetic solution.*

We are then able to prove the L^1 contraction principle thanks to the kinetic formulation; we wish to emphasize that when u satisfies (3.22), this result is not new by any means, and has been known since the articles of Kruzkhov [69, 70]. However, we present here a different proof (see section 3.4), using merely regularizations by convolution following [58, 59]. Moreover, we prove that the L^1 contraction principle holds for a larger class of solutions.

Theorem 3. *Assume the hypotheses of proposition 3.1.1 are satisfied, with* $a \in W^{1,1}_{per,loc}(Y \times \mathbb{R})^N$*, and*

$$\partial_v a \in L^\infty_{loc}(Y \times \mathbb{R})^N, \tag{3.23}$$

$$\forall R > 0,\ \exists\, \alpha, C > 0, \forall (y, y') \in Y^2\ \forall v \in (-R, R) \quad |a(y, v) - a(y', v)| \le C|y - y'|^\alpha. \tag{3.24}$$

Let u_1*,* u_2 *be two kinetic solutions of* (3.1)*. Then*

$$||(u_1(t) - u_2(t))_+||_{L^1(Y)} \le ||(u_1(t = 0) - u_2(t = 0))_+||_{L^1(Y)}. \tag{3.25}$$

Moreover, if for all $T > 0$

$$\int_0^T \int_Y \int_{\mathbb{R}} \frac{\partial v(y, p)}{\partial p} |a(y, v(y, p))|\, \mathbf{1}_{u_2(t,y) < v(y,p) < u_1(t,y)}\, dt\, dy\, dp < +\infty, \tag{3.26}$$

then the following inequality holds, in the sense of distributions on $[0, \infty) \times Y$

$$\frac{\partial}{\partial t}(u_1 - u_2)_+ + \frac{\partial}{\partial y_i}\left[\mathbf{1}_{u_1 > u_2}\left(A_i(y, u_1) - A_i(y, u_2)\right)\right] - \Delta_y (u_1 - u_2)_+ \le 0. \tag{3.27}$$

Remark 3.2.2. *Hypothesis* (3.26) *is necessary in order to retrieve inequality* (3.27)*. However, if the sole purpose is to derive the* L^1 *contraction inequality* (3.25)*, hypothesis* (3.26) *is no longer required. Hypothesis* (3.26) *implies that the function*

$$(t, y) \mapsto \mathbf{1}_{u_1 > u_2}\left[A(y, u_1(t, y)) - A(y, u_2(t, y))\right]$$

belongs to $L^1((0, T) \times Y)^N$ *for all* $T > 0$*. Notice that such an integrability property is not obvious in general, since we no longer assume that* $u \in L^\infty_{loc}$*, and thus* $A(\cdot, u)$ *does not belong to* L^∞_{loc} *either.*

Let us explain formally how inequality (3.27) is derived : let u_1, u_2 be two kinetic solutions of (3.1). We set $f_1 = \mathbf{1}_{u_1(t,y) > v(y,p)}$, $f_2 = \mathbf{1}_{u_2(t,y) < v(y,p)}$,

$$m_i = |\nabla_y u_i(t, y) - \nabla_y v(y, p)|^2 \frac{1}{\frac{\partial v(y,p)}{\partial p}} \delta(p = w(y, u_i(t, y))), \quad i = 1, 2.$$

Then

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_1\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v}{\partial p} a_i(y, v(y, p)) f_1\right) - \Delta_y \left(\frac{\partial v}{\partial p} f_1\right) = \frac{\partial m_1}{\partial p}, \tag{3.28}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_2\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v}{\partial p} a_i(y, v(y, p)) f_2\right) - \Delta_y \left(\frac{\partial v}{\partial p} f_2\right) = -\frac{\partial m_2}{\partial p}. \tag{3.29}$$

Multiply (3.28) by f_2, and (3.29) by f_1; recalling equation (3.10), we add the two equations thus obtained and we are led to

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_1 f_2\right) + \frac{\partial}{\partial y_i}\left(\frac{\partial v}{\partial p} a_i(y, v(y, p)) f_1 f_2\right) - \Delta_y \left(\frac{\partial v}{\partial p} f_1 f_2\right) = \\ = \frac{\partial m_1}{\partial p} f_2 - \frac{\partial m_2}{\partial p} f_1 - 2\frac{\partial v}{\partial p} \nabla_y f_1 \cdot \nabla_y f_2. \tag{3.30}$$

Set $\varphi_i(t,y) = w(y, u_i(t,y))$ $(i = 1, 2)$, i.e. $v(y, \varphi_i(t,y)) = u_i(t,y)$. Then

$$\nabla_y \varphi_i(t,y) = \frac{1}{\frac{\partial v}{\partial p}(y, \varphi_i(t,y))} \left[\nabla_y u_i(t,y) - \nabla_y v(y, \varphi_i(t,y))\right].$$

Notice that

$$f_1 = \mathbf{1}_{u_1(t,y)>v(y,p)} = \mathbf{1}_{\varphi_1(t,y)>p},$$
$$f_2 = \mathbf{1}_{u_2(t,y)<v(y,p)} = \mathbf{1}_{\varphi_2(t,y)<p},$$

and thus, setting $\eta_1 = 1$ and $\eta_2 = -1$,

$$\frac{\partial f_i}{\partial p} = -\eta_i\, \delta(p = \varphi_i(t,y)),$$
$$\nabla_y f_i = \eta_i \nabla_y \varphi_i(t,y) \delta(p = \varphi_1(t,y)).$$

We refer to the proof of lemma 3.2.2, at the end of the present section, for a derivation of the above equalities in the sense of distributions on $[0, \infty) \times Y \times \mathbb{R}$.

On the other hand, for any function $G \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} G'(v(y,p)) f_1 f_2 \frac{\partial v(y,p)}{\partial p}\, dp &= \int_{\mathbb{R}} G'(v(y,p)) \mathbf{1}_{u_2(t,y)<v(y,p)<u_1(t,y)} \frac{\partial v(y,p)}{\partial p}\, dp \\ &= \mathbf{1}_{u_2(t,y)<u_1(t,y)} \left[G(u_1(t,y)) - G(u_2(t,y))\right].\end{aligned}$$

Hence, integrating (3.30) with respect to p on $\mathbb{R}$ yields

$$\begin{aligned}&\frac{\partial}{\partial t}(u_1 - u_2)_+ + \frac{\partial}{\partial y_i} \mathbf{1}_{u_2(t,y)<u_1(t,y)} \left[A_i(y, u_1(t,y)) - A_i(y, u_2(t,y))\right] - \Delta_y (u_1 - u_2)_+ \\ = &\int_{\mathbb{R}} -m_1 \partial_p f_2 + m_2 \partial_p f_1 - 2\frac{\partial v}{\partial p} \nabla_y f_1 \cdot \nabla_y f_2\, dp \\ = &-\int_{\mathbb{R}} |\nabla_y u_1(t,y) - \nabla_y v(y, \varphi_1)|^2 \frac{1}{\frac{\partial v(y,p)}{\partial p}} \delta(p = \varphi_1) \delta(p = \varphi_2) dp \\ &-\int_{\mathbb{R}} |\nabla_y u_2(t,y) - \nabla_y v(y, \varphi_2)|^2 \frac{1}{\frac{\partial v(y,p)}{\partial p}} \delta(p = \varphi_2) \delta(p = \varphi_1) dp \\ &+2\int_{\mathbb{R}} \frac{\partial v}{\partial p}(y,p)\ \nabla_y \varphi_1(t,y) \cdot \nabla_y \varphi_2(t,y) \delta(p = \varphi_1) \delta(p = \varphi_2) dp \\ = &-\int_{\mathbb{R}} \frac{1}{\frac{\partial v(y,p)}{\partial p}} \delta(p = \varphi_1) \delta(p = \varphi_2) |\nabla_y (u_1 - u_2) - \nabla_y v(y, \varphi_1) + \nabla_y v(y, \varphi_2)|^2\, dp \\ \leq &\ 0\end{aligned}$$

which is exactly the L^1 contraction principle between u_1 and u_2.

However, the calculations above are entirely formal, since the product of Dirac masses is not a well-defined object, and f_1, f_2 do not have enough regularity to perform nonlinear calculations. Thus, regularizations are necessary in order to justify the contraction principle, which is proved in section 3.4.

Proof of lemma 3.2.2. Notice first that since $u(t,y)$ and $v(y,p)$ are both solutions of (3.1), we always have

$$\partial_t\left[u(t,y)-v(y,p)\right]+\mathrm{div}_y\left[A(y,u)-A(y,v(y,p))\right]-\Delta_y\left[u(t,y)-v(y,p)\right]=0.$$

Thanks to the regularizing parabolic (resp. elliptic) term, the regularity of u (resp. v) is sufficient for us to use the chain rule, and thus

$$\begin{aligned}\mathbf{1}_{u(t,y)>v(y,p)}\partial_t\left[u(t,y)-v(y,p)\right]&=\partial_t\left[u(t,y)-v(y,p)\right]_+,\\ \mathbf{1}_{u(t,y)>v(y,p)}\mathrm{div}_y\left[A(y,u)-A(y,v(y,p))\right]&=\mathrm{div}_y\left[\mathbf{1}_{u(t,y)>v(y,p)}(A(y,u)-A(y,v(y,p)))\right],\\ \mathbf{1}_{u>v(y,p)}\Delta_y\left[u-v(y,p)\right]&=\Delta_y\left[u-v(y,p)\right]_+-\nabla_y\mathbf{1}_{u>v(y,p)}\cdot\nabla_y\left[u-v(y,p)\right]\end{aligned}$$

Similar calculations can be found for instance in [69, 70], and are in fact at the heart of Kruzkhov's method for proving the L^1 contraction principle.

The major difficulty comes from the term $\nabla_y\mathbf{1}_{u(t,y)>v(y,p)}$. Notice that

$$\mathbf{1}_{u(t,y)>v(y,p)}=\mathbf{1}_{w(y,u(t,y))>p}.$$

When $p\in\mathbb{R}$ is considered as a fixed parameter, we have

$$\nabla_y\mathbf{1}_{u>v(y,p)}=\nu\otimes\mathcal{H}^{n-1}_{\partial\omega}$$

where $\omega:=\{y\in Y;\ w(y,u(t,y))>p\}$, $\mathcal{H}^{n-1}_{\partial\omega}$ is the $(n-1)$-dimensional Hausdorff measure along $\{w(y,u(t,y))=p\}$, and ν is the unit normal vector field oriented from $\{w(y,u(t,y))<p\}$ to $\{w(y,u(t,y))>p\}$. In general, no simplification occurs. However, when deriving a kinetic formulation for equation (3.1), we are only interested in the computation of $\nabla_y\mathbf{1}_{u>v(y,p)}$ in the sense of distributions on $[0,\infty)\times Y\times\mathbb{R}_p$ (see for instance [53], [52], and section 3.2 in [59]). In that case, we can give another expression for the gradient of $\mathbf{1}_{u>v(y,p)}$, namely

$$\begin{aligned}\nabla_y\mathbf{1}_{u(t,y)>v(y,p)}&=\nabla_y\mathbf{1}_{w(y,u(t,y))>p}\\ &=\nabla_y\left(w(y,u(t,y))\right)\delta(p=w(y,u(t,y)))\\ &=\frac{1}{\frac{\partial v}{\partial p}(y,p)}\nabla_y(u(t,y)-v(y,p))\delta(p=w(y,u(t,y))).\end{aligned}$$

Notice that the above expression, although meaningless if considered for $p\in\mathbb{R}$ fixed, is well-defined in the sense of distributions on $[0,\infty)\times Y\times\mathbb{R}_p$.

Thus (3.17) is proved. Consequently, all solutions of (3.1) satisfy inequality (3.11) in the sense of distributions on $[0,\infty)\times Y\times\mathbb{R}$. And it is then easily checked that if a solution u of (3.1) satisfies (3.11) in the sense of distributions in t,y,p, then u satisfies (3.11) for all p in the sense of distributions in t, y. □

3.3 An application to homogenization

We provide here a proof for proposition 3.1.2. The kinetic formulation derived above allows a better understanding of the homogenization process, and the proof is much more elegant than the original one in [14], which used two-scale Young measures.

We will work in the context of kinetic solutions of equation (3.12) : let $\varepsilon > 0$, and let $u^\varepsilon \in L^\infty_{\text{loc}}([0,\infty); L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^2_{\text{loc}}(0,+\infty; H^1_{\text{loc}}(\mathbb{R}^N))$. We assume that

$$f^\varepsilon(t,x,p) := \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}$$

is a solution in the sense of distributions of

$$\partial_t \left(v_p \left(\frac{x}{\varepsilon}, p \right) f^\varepsilon \right) + \partial_{x_i} \left[a_i \left(\frac{x}{\varepsilon}, v\left(\frac{x}{\varepsilon}, p \right) \right) v_p \left(\frac{x}{\varepsilon}, p \right) f^\varepsilon \right] - \varepsilon \Delta_x \left(v_p \left(\frac{x}{\varepsilon}, p \right) f^\varepsilon \right) = \partial_p m^\varepsilon,$$
$$f^\varepsilon(t=0) = \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u_0\left(x,\frac{x}{\varepsilon}\right)} \qquad (3.31)$$

where

$$m^\varepsilon(t,x,p) := \varepsilon \left| \nabla_x u^\varepsilon(t,x) - \nabla_y v\left(\frac{x}{\varepsilon}, p \right) \right|^2 \frac{1}{v_p\left(\frac{x}{\varepsilon},p\right)} \delta\left(p = w\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x) \right) \right).$$

We assume that the hypotheses of proposition 3.1.1 are satisfied, together with (3.4), so that $w(y,p)$ is well-defined (see lemma 3.2.1). We have used the notation $v_p(y,p) = \partial_p v(y,p)$.

The hypotheses on f^ε are the following:

(H1) $u_0(x,y) = v(y, \bar{u}_0(x))$, for some $\bar{u}_0 \in L^1(\mathbb{R}^N)$;

(H2) $u_0 - v(y,0) \in L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$; this means that

$$\int_{\mathbb{R}^N} \sup_{y\in Y} |v(y,\bar{u}_0(x)) - v(y,0)|\, dx < +\infty,$$

which is slightly stronger than requiring $\bar{u}_0 \in L^1$.

(H3) $f^\varepsilon(t,x,p) \to 0$ (resp. $1 - f^\varepsilon \to 0$) as $p \to +\infty$ (resp. as $p \to -\infty$) for a.e. $(t,x) \in [0,\infty) \times \mathbb{R}^N$ and for all $\varepsilon > 0$. Equivalently,

$$\alpha_-\left(\frac{x}{\varepsilon}\right) < u^\varepsilon(t,x) < \alpha_+\left(\frac{x}{\varepsilon}\right) \quad \text{for a.e. } (t,x) \in (0,\infty) \times \mathbb{R}^N,$$

where α_- and α_+ were defined in lemma 3.2.1.

(H4) For all $\varepsilon > 0$, there exists a function $\mu_\varepsilon \in L^\infty(\mathbb{R})$ such that $\mu_\varepsilon(p) \to 0$ as $|p| \to \infty$ and

$$\int_0^\infty \int_{\mathbb{R}^N} m^\varepsilon(t,x,p)\, dt\, dx \le \mu_\varepsilon(p) \quad \forall p \in \mathbb{R}.$$

(H5) For all $\varepsilon > 0$, the function

$$(t,x,p) \mapsto \frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon}, p \right) [\mathbf{1}_{p>0} f^\varepsilon(t,x,p) + \mathbf{1}_{p<0}(1 - f^\varepsilon(t,x,p))]$$

belongs to $L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^{N+1}))$. Equivalently, the function

$$(t,x) \mapsto u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, 0 \right)$$

belongs to $L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^N))$.

A function $u^\varepsilon \in L^\infty_{\text{loc}}([0,\infty); L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^2_{\text{loc}}(0,+\infty; H^1_{\text{loc}}(\mathbb{R}^N))$ such that f^ε is a solution of (3.31) and such that **(H3)-(H5)** are satisfied is called a *kinetic solution* of the parabolic scalar conservation law (3.12). Notice that we do not assume that (3.12) is satisfied in the sense of distributions.

Let us now state the result we prove in this section.

Theorem 4. *Assume that A satisfies the hypotheses of proposition 3.1.1 and* (3.4). *Let $u^\varepsilon \in L^\infty_{loc}([0,\infty); L^1_{loc}(\mathbb{R}^N)) \cap L^2_{loc}(0,+\infty; H^1_{loc}(\mathbb{R}^N))$ be a kinetic solution of* (3.12) *such that hypotheses* **(H1) - (H5)** *are satisfied. Then*

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0$$

in $L^1_{loc}([0,\infty)\times\mathbb{R}^N)$, where $\bar{u} \in L^\infty([0,\infty), L^1(\mathbb{R}^N))$ is the kinetic solution of (3.16) *with initial data $\bar{u}_0$.*

Remark 3.3.1. *When hypothesis* **(H1)** *on the microscopic profile of the initial data is not satisfied, it is proved in the L^∞ case in [18] that there is an initial layer of typical size ε, during which the solution adapts itself to the profile dictated by the microscopic structure. The proof of this result relies on the parabolic structure of the equation, which cannot be used here since the kinetic formulation is essentially a hyperbolic tool.*

Remark 3.3.2. *It is can be checked that* **(H2) - (H5)** *are always satisfied when $\bar{u}_0 \in L^\infty \cap L^1(\mathbb{R}^N)$ and $u^\varepsilon \in L^\infty_{loc}$ is an entropy solution. However, we wish to stress that hypothesis* **(H3)** *does not imply that $u^\varepsilon \in L^\infty_{loc}([0,\infty)\times\mathbb{R}^N)$ in general. For instance, in the case when hypothesis* (3.5) *is satisfied, we have $\alpha_\pm = \pm\infty$, as explained in remark 3.2.1. Hence in that case, hypothesis* **(H3)** *is always satisfied, and the only bound required on u^ε is* **(H5)**, *which is an L^1 bound. Consequently, we refer to* **(H2) - (H5)** *as an "L^1 setting", by contrast with the "L^∞ setting" of entropy solutions.*

At last, let us mention that the function μ_ε in hypothesis **(H4)** *can in fact be derived from equation* (3.31) *(see lemma 3.3.1 below), if it is known that* **(H4)** *holds for some function μ_ε; nonetheless,* **(H4)** *cannot be avoided and is necessary for lemma 3.3.1 to hold.*

We will prove the convergence in several steps; first, we introduce the two-scale weak limit $f(t,x,y,p)$ of f^ε. Then, the key point in the analysis is to show that $f(t,x,y,p) = \mathbf{1}_{p<\bar{u}(t,x)}$, where $\bar{u}$ is the solution of the homogenized problem. Hence, we first prove that f does not depend on y. Then we derive the macroscopic equation solved by f and we prove that $f(t=0) = \mathbf{1}_{p<\bar{u}_0(x)}$; this entails that $f = \mathbf{1}_{p<\bar{u}}$, and $\bar{u}$ can be identified thanks to the equation solved by f. Eventually, we prove the strong convergence in L^1_{loc}.

We begin with a few preliminary bounds on m^ε and f^ε, of which we only give a rough idea of the proof (see for instance [59], proposition 4.1.7 and lemma 3.1.7 for the derivation of similar inequalities):

Lemma 3.3.1. *Assume that* **(H1) - (H5)** *are satisfied.*

- *There exists a constant $C > 0$ such that for all $\varepsilon > 0$, for a.e. $t > 0$,*

$$\int_{\mathbb{R}^{N+1}} v_p\left(\frac{x}{\varepsilon}, p\right)\left(\mathbf{1}_{p>0} f^\varepsilon(t,x,p) + \mathbf{1}_{p<0}(1-f^\varepsilon)(t,x,p)\right)\, dx\, dp \le C.$$

- *There exists a constant $C > 0$ such that for all $p_0 > 0$, $\varepsilon > 0$,*

$$\int_0^\infty \int_{\mathbb{R}^N} m^\varepsilon(t,x,p_0)\, dx\, dt \le \int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right)_+ dx \le C.$$

The same kind of bound holds for $p_0 < 0$.

Thus $m^\varepsilon((0,+\infty) \times \mathbb{R}^N \times (-R,R))$ is bounded for all $R > 0$ uniformly in ε.

- *For all $t \ge 0$, for all $p_0 > 0$, for all $\varepsilon > 0$*

$$\int_{\mathbb{R}^N} \left(u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, p_0\right)\right)_+ dx \le \int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right)_+ dx \tag{3.32}$$

We deduce from the second bound in the lemma that we can take in **(H4)**

$$\begin{aligned} \mu_\varepsilon(p) \ :=\ & \mathbf{1}_{p>0} \int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p\right)\right)_+ dx \\ & + \mathbf{1}_{p<0} \int_{\mathbb{R}^N} \left(v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p\right)\right)_- dx. \end{aligned}$$

Then μ_ε is bounded in L^∞, uniformly in ε. Moreover, it will be proved in the very last step of the proof that for all p, $\mu_\varepsilon(p)$ converges as $\varepsilon \to 0$ towards $\mu_0(p)$, for some function $\mu_0 \in L^\infty(\mathbb{R})$ vanishing at infinity.

Proof. Thanks to the integrability assumptions **(H4)-(H5)** on f^ε and m^ε, we prove that for any test function $S' \in \mathcal{D}(\mathbb{R})$, for all $t > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} S'(p) f^\varepsilon(t,x,p) v_p\left(\frac{x}{\varepsilon}, p\right) dx\, dp - \int_{\mathbb{R}^{N+1}} S'(p) f^\varepsilon(t=0,x,p) v_p\left(\frac{x}{\varepsilon}, p\right) dx\, dp = \\ = - \int_0^t \int_{\mathbb{R}^{N+1}} m^\varepsilon(t,x,p) S''(p)\, dt\, dx\, dp. \end{aligned}$$

Then, using the fact that μ_ε vanishes at infinity , we prove that the above equality holds for more general functions S. In particular, the choice $S'(p) = \mathbf{1}_{p>0}$ (and thus $S''(p) = \delta(p = 0)$) yields the first bound on f^ε, and the choice $S'(p) = \mathbf{1}_{p>p_0}$ for some $p_0 > 0$ gives the one on m^ε. Moreover

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} \mathbf{1}_{p>p_0} f^\varepsilon(t,x,p) v_p\left(\frac{x}{\varepsilon}, p\right) dx\, dp \ &=\ \int_{\mathbb{R}^{N+1}} \mathbf{1}_{v\left(\frac{x}{\varepsilon}, p_0\right) < v\left(\frac{x}{\varepsilon}, p\right) < u^\varepsilon(t,x)} v_p\left(\frac{x}{\varepsilon}, p\right) dx\, dp \\ &=\ \int_{\mathbb{R}^N} \left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx, \end{aligned}$$

and thus the choice $S'(p) = \mathbf{1}_{p>p_0}$ also yields the bound on u^ε. □

We now use the concept of two scale convergence, defined by Grégoire Allaire in [3] following an idea of Gabriel N'Guetseng (see [56]), in order to find a two-scale limit for f^ε :

Proposition 3.3.1. *Let $\{v^\varepsilon\}_{\varepsilon>0}$ be a bounded sequence of $L^2(\Omega)$, where Ω is an open set of $\mathbb{R}^N$. Then as $\varepsilon \longrightarrow 0$, there exists a subsequence, still denoted by ε, and $v^0 \in L^2(\Omega \times Y)$, such that*

$$\int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) v^\varepsilon(x)\, dx \longrightarrow \int_{\Omega\times Y} \psi(x,y) v^0(x,y)\, dx\, dy$$

for all $\psi \in \mathcal{C}_{per}(Y, L^2(\Omega))$.

It is then said that the sequence $\{v^\varepsilon\}_{\varepsilon>0}$ "two-scale" converges to v_0.

This concept is easily generalized to functions in L^∞ (the proof goes along the same lines as the one given in [3]), which allows us to prove the following :

Lemma 3.3.2. *There exists a function $f(t,x,y,p) \in L^\infty((0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ and a subsequence, still denoted by ε, such that f^ε two-scale converges to f.*

It is easily checked that $0 \le f \le 1$ a.e. Since v_p, f and $1-f$ are nonnegative, lemma 3.3.1 entails that there exists a constant C such that

$$\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \{\mathbf{1}_{p>0} f(t,x,y,p) + \mathbf{1}_{p<0}(1-f(t,x,y,p))\}\, v_p(y,p)\, dx\, dy\, dp \le C \quad \text{a.e. t>0.}$$

The goal is now to identify the equations solved by f in order to prove that f is an indicator function. Hence, we now focus on the microscopic (i.e. in y) and macroscopic (i.e. in t,x) equations solved by f.

First step. Microscopic profile. Multiplying (3.31) by a test function of the form $\varepsilon\varphi(t,x,x/\varepsilon,p)$, with $\varphi \in \mathcal{D}_{\rm per}((0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ and passing to the limit as $\varepsilon \longrightarrow 0$ leads to the equation

$$-\Delta_y\left(\frac{\partial v}{\partial p} f\right) + \mathrm{div}_y\left(a(y,v(y,p))\frac{\partial v}{\partial p} f\right) = 0 \tag{3.33}$$

in the sense of distributions on $(0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R}$. Let us point out that $a(y,v(y,p))$ is an "admissible" test function in the sense of G. Allaire (see [3]) thanks to the continuity assumption (3.4).

Then, we regularize the equation (3.33) in the variables t,x,y,p thanks to a convolution kernel, and pass to the limit as the parameter of the regularization vanishes. We easily deduce that equation (3.33) in fact holds almost everywhere in t,x,p, in the variational sense in y.

Notice that the constant function equal to 1 on Y, denoted by $\bar{1}$, is a positive solution of the dual problem

$$-\Delta_y \bar{1} - a(y,v(y,p))\cdot\nabla_y\bar{1} = 0.$$

Consequently, by the Krein-Rutman theorem, we infer that any solution g of the equation

$$-\Delta_y g + \mathrm{div}_y\left(a(y,v(y,p))g\right) = 0$$

can be written $g(y) = c\frac{\partial v(y,p)}{\partial p}$, where c is a constant in y.

Thus $f(t,x,y,p)$ does not depend on y, and $f = f(t,x,p)$.

Second step. Evolution equation. Now, we multiply (3.31) by a test function of the form $\varphi(t,x,p)$, with $\varphi(t,x,p) = 0$ when $|p| \geq R$, $R > 0$ arbitrary; thanks to lemma 3.3.1, $m^\varepsilon((0,\infty) \times \mathbb{R}^N \times (-R,R))$ is bounded uniformly in ε, and thus up to the extraction of a subsequence, there exists a measure $\bar{m}_R$ such that

$$m^\varepsilon \rightharpoonup \bar{m}_R \quad \text{in } w - M^1((0,\infty) \times \mathbb{R}^N \times (-R,R)).$$

We define, for any $p \in \mathbb{R}$,

$$\bar{a}(p) = \frac{1}{|Y|}\int_Y a(y, v(y,p))\frac{\partial v}{\partial p}\,dy;$$

recall also that

$$\frac{1}{|Y|}\int_Y \frac{\partial v}{\partial p}\,dy = 1.$$

Then f satisfies, in the sense of distributions on $(0,\infty) \times \mathbb{R}^N \times (-R,R)$

$$\partial_t f + \operatorname{div}_x(\bar{a}(p)f) = \frac{\partial \bar{m}_R}{\partial p}. \tag{3.34}$$

We deduce that for any $0 < R < R'$, $\bar{m}_R = \bar{m}_{R'}$ on $(0,\infty) \times \mathbb{R}^N \times (-R,R)$. Consequently, the measure $\bar{m}$, defined by $\bar{m} = \bar{m}_R$ on $(0,\infty) \times \mathbb{R}^N \times (-R,R)$ is well-defined. Hence equation (3.34) holds in $(0,\infty) \times \mathbb{R}^{N+1}$ with $\bar{m}_R$ replaced by $\bar{m}$, and $\bar{m} \in \mathcal{C}(\mathbb{R}_p, w - M^1([0,\infty) \times \mathbb{R}^N_x)$. Moreover the measure $\bar{m}$ inherits the following property from the bounds on m^ε: for almost every $p \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}^N \times Y} \bar{m}(t,x,y,p)\,dt\,dx\,dy \leq \mu_0(p), \tag{3.35}$$

and μ_0 belongs to L^∞ and vanishes at infinity, as we shall prove in the fourth step.

Equation (3.34) looks very much like the kinetic formulation for a homogeneous and hyperbolic scalar conservation law (see for instance [53], [52], and [59], chapter 3). However we have to work out a few points before jumping to a conclusion.

Third step. Identification of f as an indicator function. First, the function which occurs in the kinetic formulation is the function $\chi : \mathbb{R}^2 \to \{1, -1, 0\}$ defined by

$$\chi(v,u) := \begin{cases} 1 & \text{if } 0 < v < u, \\ -1 & \text{if } u < v < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here, if $u^\varepsilon(t,x) - v(x/\varepsilon, \bar{u}(t,x))$ converges strongly to 0, as we intend to prove, then $f = \mathbf{1}_{v(y,p)<v(y,\bar{u}(t,x))} = \mathbf{1}_{p<\bar{u}(t,x)}$; hence, a good candidate for a function $\chi(v,\bar{u}(t,x))$ seems to be

$$g(t,x,p) = \mathbf{1}_{p>0}f - \mathbf{1}_{p<0}(1-f) = f - \mathbf{1}_{p<0}.$$

The function g satisfies the same equation as f, and

$$\operatorname{sgn}(p)g = \mathbf{1}_{p>0}f + \mathbf{1}_{p<0}(1-f) = |g| \in [0,1].$$

Moreover,

$$\frac{\partial g}{\partial p} = \delta(p=0) + \partial_p f. \tag{3.36}$$

Recall that

$$\partial_p f^\varepsilon(t,x,p) = -\delta\left(p - w\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right)\right);$$

Hence $-\partial_p f^\varepsilon(t,x,p)$ is a nonnegative measure, uniformly bounded in ε on compact sets of $(0,\infty)\times\mathbb{R}^{N+1}$. Since $\partial_p f^\varepsilon$ weakly converges to $\partial_p f$, we deduce that $\partial_p f$ is a nonpositive locally finite measure.

There remains to check that

$$g(t=0,x,p) = \chi(p, \bar{u}_0(x)); \tag{3.37}$$

this equality is in fact not obvious : if $f^\varepsilon(t=0,x,p) = f_0\left(x, x/\varepsilon, p\right)$, then it is false in general that $f(t=0,x,y,p) = f_0(x,y,p)$. Indeed, there might be initial layers of typical size ε. These are not taken into account when passing to the two-scale limit because the test functions do not select the microscopic information in time. In order to see the possible initial layers, we should have taken test functions of the kind

$$\psi\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}, p\right).$$

Here, it is unnecessary to consider test functions which have microscopic oscillations in time because the initial data is well-prepared. Hence, there is no initial layer in this case. In other words, the u^ε are uniformly continuous in time at time $t=0$ (with values in L^1_{loc}). In terms of the kinetic formulation, this result follows directly from the fact that

$$f^\varepsilon(t=0,x,p) = \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right) < v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)} = \mathbf{1}_{p<\bar{u}_0(x)}.$$

Hence $f^\varepsilon(t=0)$ does not depend on ε. Consequently, multiplying (3.31) by a test function $\varphi(t,x,p)\in\mathcal{D}([0,\infty)\times\mathbb{R}^{N+1})$ yields

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}^{N+1}} f^\varepsilon(t,x,p)\frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon},p\right)\left\{\partial_t\varphi + a_i\left(\frac{x}{\varepsilon}, v\left(\frac{x}{\varepsilon},p\right)\right)\partial_{x_i}\varphi + \varepsilon\Delta_x\varphi\right\}\,dt\,dx\,dp \\ = &\int_0^\infty\int_{\mathbb{R}^{N+1}} m^\varepsilon(t,x,p)\partial_p\varphi(t,x,p)\,dt\,dx\,dp \\ &-\int_{\mathbb{R}^{N+1}}\frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon},p\right)\mathbf{1}_{p<\bar{u}_0(x)}\varphi(t=0,x,p)\,dx\,dp \end{aligned}$$

Passing to the limit as $\varepsilon\to 0$ entails that

$$f(t=0,x,p) = \mathbf{1}_{p<\bar{u}_0(x)},$$

and thus

$$g(t=0,x,p) = \chi(p,\bar{u}_0(x)).$$

Gathering (3.34), (3.35), (3.36), (3.37), we infer that g is a generalized kinetic solution (see definition 4.1.2 in [59]) of the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial \bar{A}_i(u)}{\partial x_i} = 0,$$

where

$$\bar{A}_i'(p) = \bar{a}_i(p).$$

Now, we can apply theorem 4.3.1 in [59] : there exists $\bar{u}(t,x) \in L^\infty([0,\infty); L^1(\mathbb{R}^N))$ such that $g(t,x,p) = \chi(p, \bar{u}(t,x))$ a.e., and $\bar{u}$ is a kinetic solution of the above scalar conservation law.

And since

$$\mathbf{1}_{p>0} f - \mathbf{1}_{p<0}(1-f) = \chi(p, \bar{u}(t,x)),$$

we deduce that

$$f(t,x,p) = \mathbf{1}_{p<\bar{u}(t,x)}$$

almost everywhere.

Fourth step. Strong convergence. Let us now prove that this result entails that

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0$$

in L^1_{loc}.

(i) *Convergence of* $u^\varepsilon \wedge v(x/\varepsilon, p_0)$ *for all* $p_0 > 0$*:* take an arbitrary cut-off function $\varphi = \varphi(t,x)$ with compact support in $[0,\infty) \times \mathbb{R}^N$, $p_0 > 0$ and set

$$\psi(t,x,y,p) := \mathbf{1}_{\bar{u}(t,x)<p<p_0} \frac{\partial v}{\partial p}(y,p)\, \varphi(t,x).$$

Since $f^\varepsilon(t,x,p) = \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}$ two-scale converges to $f = \mathbf{1}_{p<\bar{u}(t,x)}$, we deduce that as $\varepsilon \to 0$

$$\int_0^\infty \int_{\mathbb{R}^{N+1}} \psi\left(t,x,\frac{x}{\varepsilon},p\right) \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}\, dp\, dx\, dt \to 0.$$

And the left-hand side can be transformed as follows

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^{N+1}} \psi\left(t,x,\frac{x}{\varepsilon},p\right) \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}\, dp\, dx\, dt \\
=\; &\int_0^\infty \int_{\mathbb{R}^{N+1}} \mathbf{1}_{v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)<v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)\wedge v\left(\frac{x}{\varepsilon},p_0\right)}\, \varphi(t,x) \frac{\partial v}{\partial p}\left(\frac{x}{\varepsilon},p\right) dp\, dx\, dt \\
=\; &\int_0^\infty \int_{\mathbb{R}^{N+1}} \mathbf{1}_{v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)<v<u^\varepsilon(t,x)\wedge v\left(\frac{x}{\varepsilon},p_0\right)}\, \varphi(t,x) dv\, dx\, dt \\
=\; &\int_0^\infty \int_{\mathbb{R}^N} \varphi(t,x) \left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon},p_0\right) - v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)\right]_+ dx\, dt.
\end{aligned}$$

Take any compact set $K \subset [0,\infty) \times \mathbb{R}^N$, and choose a test function $\varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^N)$ such that $0 \le \varphi \le 1$, and $\varphi \equiv 1$ on K. Then for all $\varepsilon > 0$,

$$\begin{aligned}
&\left\| \left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon},p_0\right) - v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)\right]_+ \right\|_{L^1(K)} \\
\le\; &\int_0^\infty \int_{\mathbb{R}^N} \varphi(t,x) \left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon},p_0\right) - v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)\right]_+ dx\, dt
\end{aligned}$$

In the inequality above, we have used the fact that $u_+ = \max(u, 0)$ is always non-negative. Thus we deduce that for all $p_0 > 0$

$$\left|\left|\left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^N)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The same kind of result holds for $p_0 < 0$.

(ii) *Convergence of u^ε:* let $T > 0$, $R > 0$, and set $Q := (0,T) \times B_R$. For $p_0 > 0$ arbitrary, we have

$$\begin{aligned}
&\left|\left|\left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} \\
\leq\ & \left|\left|\left[u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} + \left|\left|\left[u^\varepsilon - u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+\right|\right|_{L^1(Q)} \\
\leq\ & \left|\left|\left[u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} + \left|\left|\left[u^\varepsilon - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+\right|\right|_{L^1(Q)} \\
\leq\ & \left|\left|\left[u^\varepsilon \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+\right|\right|_{L^1(Q)} \\
& + T\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx
\end{aligned}$$

thanks to inequality (3.32).

According to **(H2)**, we have $[v(y,\bar{u}_0) - v(y,p_0)]_+ \in L^1(\mathbb{R}^N; \mathcal{C}_{\mathrm{per}}(Y))$; thus, using to a result of Grégoire Allaire (see [3]), we deduce

$$\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx \to \int_{\mathbb{R}^N\times Y}\left[v\left(y, \bar{u}_0(x)\right) - v\left(y, p_0\right)\right]_+ dx\, dy$$

as $\varepsilon \to 0$, for all $p_0 > 0$. Since $||\,(v(y,p) - v(y,p'))_+\,||_{L^1(Y)} = (p - p')_+$ for all $p, p' \in \mathbb{R}$, we derive the following bound

$$\begin{aligned}
&\int_0^T\int_{B_R}\left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+ dx\, dt \\
\leq\ & \int_0^T\int_{B_R}\left[u^\varepsilon(t,x) \wedge v\left(\frac{x}{\varepsilon}, p_0\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+ dx\, dt \\
& + T\left|\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, p_0\right)\right]_+ dx - ||(\bar{u}_0 - p_0)_+||_{L^1(\mathbb{R}^N)}\right| \\
& + T||(\bar{u}_0 - p_0)_+||_{L^1(\mathbb{R}^N)}
\end{aligned}$$

In the above inequality, take p_0 large enough so that $||(\bar{u}_0 - p_0)_+||_{L^1(\mathbb{R}^N)}$ is small enough, and then for this p_0, take $\varepsilon > 0$ small enough so that the two other terms are small (notice that the first one vanishes thanks to the first step). We deduce that

$$\left[u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)\right]_+ \to 0$$

in $L^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N)$, and theorem 4 is proved.

Moreover, we have proved that for all $p > 0$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\mu_\varepsilon(p) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left[v\left(\frac{x}{\varepsilon},\bar{u}_0(x)\right)-v\left(\frac{x}{\varepsilon},p\right)\right]_+ dx\\ &= \int_{\mathbb{R}^N\times Y}\left[v\left(y,\bar{u}_0(x)\right)-v\left(y,p\right)\right]_+ dx\,dy\\ &= ||(\bar{u}_0-p)_+||_{L^1(\mathbb{R}^N)} =: \mu_0(p).\end{aligned}$$

Thus μ_0 vanishes at infinity, and the result stated after lemma 3.3.1 holds.

3.4 Rigorous proof of the L^1 contraction principle

This section is devoted to the proof of inequality (3.27) under assumption (3.26) and the hypotheses of theorem 3. The main ideas behind the proof were exposed in the formal calculations in section 3.2; however, regularizations are necessary in order to justify nonlinear manipulations of the type

$$f_1\partial_t f_2 + f_2\partial_t f_1 = \partial_t(f_1 f_2),$$

as well as the reduction of the right hand-side.

As in [58], [59] (Chapter 4), we will merely regularize the equation by convolution; let $\varepsilon > 0$ be a small parameter, $\zeta_1\in\mathcal{D}(\mathbb{R})$, $\zeta_2\in\mathcal{D}(\mathbb{R}^N)$, $\zeta_3\in\mathcal{D}(\mathbb{R})$ such that

$$\begin{gathered}\zeta_i\geq 0\quad (i=1,2,3),\\ \operatorname{Supp}\zeta_1\subset[-1,0],\quad \operatorname{Supp}\zeta_2\subset B_1,\quad \operatorname{Supp}\zeta_3\subset[-1,1],\quad \zeta_1(0)=0,\\ \int_{\mathbb{R}}\zeta_1=\int_{\mathbb{R}^N}\zeta_2=\int_{\mathbb{R}}\zeta_3=1.\end{gathered}$$

We set, for $\varepsilon > 0$, $(t,x,p)\in\mathbb{R}^{N+2}$

$$\phi_\varepsilon(t,x,p) := \frac{1}{\varepsilon^{N+2}}\zeta_1\left(\frac{t}{\varepsilon}\right)\zeta_2\left(\frac{x}{\varepsilon}\right)\zeta_3\left(\frac{p}{\varepsilon}\right),$$

and for $(t,x,p)\in[0,\infty)\times\mathbb{R}^N\times\mathbb{R}$

$$\begin{aligned}f_i^\varepsilon(t,x,p) &= \int_{\mathbb{R}^{N+2}} f_i(s,z,q)\phi_\varepsilon(t-s,x-z,p-q)\,ds\,dz\,dq,\\ m_i^\varepsilon(t,x,p) &= \int_{\mathbb{R}^{N+2}} m_i(s,z,q)\phi_\varepsilon(t-s,x-z,p-q)\,ds\,dz\,dq.\end{aligned}$$

(Notice that the convolution in the space variable x is meant in the whole of $\mathbb{R}^N$: f_i is thus considered as a function defined on $[0,\infty)\times\mathbb{R}^N\times\mathbb{R}$, periodic with period Y in its second variable. The function f_i^ε is of course Y-periodic as well.)

We begin with the derivation of the equation solved by f^ε :

Lemma 3.4.1. *Set $\tilde{a}_i(y,p) = a_i(y,v(y,p))\frac{\partial v(y,p)}{\partial p}$ for $1\leq i\leq N$, $y\in Y$, $p\in\mathbb{R}$.*

Then for $\varepsilon < 1/2$, f_j^ε ($j = 1, 2$) *is a classical solution of*

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_j^\varepsilon\right) + \frac{\partial}{\partial y_i}\left(\tilde{a}_i(y,p) f_j^\varepsilon\right) - \Delta_y \left(\frac{\partial v}{\partial p} f_j^\varepsilon\right) = \eta_j \frac{\partial m_j^\varepsilon}{\partial p} + r_j^\varepsilon \tag{3.38}$$

where $\eta_1 = 1$, $\eta_2 = -1$, *and the error term* r_j^ε *is equal to*

$$\begin{aligned} r_j^\varepsilon(t,y,p) \;=\; & \frac{\partial}{\partial t}\left[\frac{\partial v}{\partial p}(y,p) f_j^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p} f_j\right) * \phi_\varepsilon(t,y,p)\right] \\ & + \frac{\partial}{\partial y_i}\left[\tilde{a}_i(y,p) f_j^\varepsilon(t,y,p) - (\tilde{a}_i f_j) * \phi_\varepsilon(t,y,p)\right] \\ & - \Delta_y \left[\frac{\partial v}{\partial p}(y,p) f_j^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p} f_j\right) * \phi_\varepsilon(t,y,p)\right]. \end{aligned}$$

Moreover, for all $0 < \varepsilon < 1/2$, *for all* $p \in \mathbb{R}$,

$$\int_0^\infty \int_Y m_i^\varepsilon(t,y,p)\, dt\, dy \le \max(\mu_i(p+1), \mu_i(p-1)),$$

where the functions μ_i *were introduced in hypothesis* (3.20) *in definition 3.2.1.*

We postpone the proof of lemma 3.4.1 to the end of the section.

Now, since f_j^ε is smooth we can multiply (3.38) written for f_1^ε (resp. f_2^ε) by f_2^ε (resp. f_1^ε), and add the two equations thus obtained. Following the calculations in section 3.2 leads to

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p} f_1^\varepsilon f_2^\varepsilon\right) + \frac{\partial}{\partial y_i}\left(\tilde{a}_i(y,p) f_1^\varepsilon f_2^\varepsilon\right) - \Delta_y \left(\frac{\partial v}{\partial p} f_1^\varepsilon f_2^\varepsilon\right) \\ = \; & \frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p} \nabla_y f_1^\varepsilon \cdot \nabla_y f_2^\varepsilon \\ & + r_1^\varepsilon f_2^\varepsilon + r_2^\varepsilon f_1^\varepsilon. \end{aligned}$$

Let $R > 0$ arbitrary, and let $K_R \in \mathcal{D}(\mathbb{R})$ be a cut-off function such that

$$\begin{gathered} 0 \le K_R(p) \le 1, \quad |K_R'(p)| \le 2 \quad \forall p \in \mathbb{R}, \\ K_R(p) = 1 \quad \forall p \in [-R, R], \\ K_R(p) = 0 \quad \forall p \in (-\infty, -R-1] \cup [R+1, +\infty). \end{gathered}$$

Classically, the following convergence results hold for any test function $\theta = \theta(t,y) \in \mathcal{D}_{\text{per}}([0,\infty) \times Y)$ (recall (3.26))

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_0^\infty \int_{Y\times\mathbb{R}} \frac{\partial v}{\partial p}(y,p) f_1^\varepsilon f_2^\varepsilon \theta(t,y) K_R(p)\, dt\, dy\, dp = \int_Y (u_1 - u_2)_+ \theta(t,y)\, dt\, dy,$$

$$\begin{aligned} & \lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_0^\infty \int_{Y\times\mathbb{R}} \tilde{a}_i(y,p) f_1^\varepsilon f_2^\varepsilon \partial_{y_i}\theta(t,y) K_R(p)\, dt\, dy\, dp = \\ & \qquad = \int_0^\infty \int_Y \mathbf{1}_{u_1 > u_2} \left[A_i(y,u_1) - A_i(y,u_2)\right] \partial_{y_i}\theta(t,y)\, dt\, dy. \end{aligned}$$

(If one is interested in deriving (3.25), without assumption (3.26), instead of (3.27), one should merely take $\theta \in \mathcal{D}([0,\infty))$, independent of y, at this stage; the left-hand side in the second equality above is zero in that case. The rest of the proof remains unchanged.)

On the other hand, it is easily proved that the first order terms in r_j^ε go to 0 in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^{N+1})$ as $\varepsilon \to 0$ thanks to the assumption $a \in W^{1,1}_{\text{loc}}$. Hence, we now focus on the convergence of

$$\frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon \cdot \nabla_y f_2^\varepsilon$$

and the second order terms in r_j^ε, that is

$$\begin{aligned}&-\Delta_y\left[\frac{\partial v}{\partial p}(y,p) f_1^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p} f_1\right) * \phi_\varepsilon(t,y,p)\right] f_2^\varepsilon\\ &-\Delta_y\left[\frac{\partial v}{\partial p}(y,p) f_2^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p} f_2\right) * \phi_\varepsilon(t,y,p)\right] f_1^\varepsilon.\end{aligned}$$

In the following, we set

$$\varphi_i(t,y) = w(y,u_i(t,y)) \quad (\text{i.e. } v(y,\varphi_i(t,y)) = u_i(t,y)\ ), \tag{3.39}$$

$$\gamma_i(t,y) = \frac{1}{\frac{\partial v}{\partial p}(y,\varphi_i(t,y))}\left[\nabla_y u_i(t,y) - (\nabla_y v)(y,\varphi_i(t,y))\right] = \nabla_y \varphi_i(t,y). \tag{3.40}$$

We recall that

$$\begin{aligned}
m_i(t,y,p) &= |\nabla_y\varphi_i(t,y)|^2 \frac{\partial v}{\partial p}(y,\varphi_i(t,y))\, \delta(p=\varphi_i(t,y)) &(3.41)\\
&= |\gamma_i|^2(t,y)\frac{\partial v}{\partial p}(y,\varphi_i(t,y))\, \delta(p=\varphi_i(t,y)), &(3.42)\\
\nabla_y f_i(t,y,p) &= \eta_i \nabla_y \varphi_i(t,y)\delta(p=\varphi_i(t,y)) &(3.43)\\
&= \eta_i\gamma_i(t,y)\delta(p=\varphi_i(t,y)) &(3.44)\\
\partial_p f_i &= -\eta_i \delta(p=\varphi_i(t,y)), &(3.45)
\end{aligned}$$

for $i=1,2$, where $\eta_1 = 1$ and $\eta_2=-1$.

First, for any test function $\theta = \theta(t,y) \in \mathcal{D}_{\text{per}}([0,+\infty)\times Y)$ such that $\theta \geq 0$, for $\varepsilon < 1$, $R>1$, we claim that

$$\begin{aligned}
&\int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon\cdot\nabla_y f_2^\varepsilon\right]\theta(t,y)K_R(p)\,dt\,dy\,dp\\
\leq\ & \int_0^\infty \int_{\mathbb{R}^{2N+3}\times Y} \phi^\varepsilon(t-s_1,y-y_1,p-\varphi_1)\phi^\varepsilon(t-s_2,y-y_2,p-\varphi_2)\theta(t,y)K_R(p)\times\\
&\times 2\left[\gamma_1\cdot\gamma_2\left(\frac{\partial v}{\partial p}(y,p) - \sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1)\frac{\partial v}{\partial p}(y_2,\varphi_2)}\right)\right]dy\,dp\,dy_1\,dy_2\,ds_1\,ds_2\,dt\\
&+2||\theta||_\infty\left[\mu_1(R-1)+\mu_1(-R+1)+\mu_2(R-1)+\mu_2(-R+1)\right]
\end{aligned}\tag{3.46}$$

In the integral of the right-hand side above, γ_i, φ_i are evaluated at (s_i,y_i) $(i=1,2)$.

The derivation of this inequality is rather technical, but straightforward if one follows the formal calculations of section 3.2. Let us focus on the first term of the left-hand side, namely

$$\begin{aligned} I_\varepsilon &:= \int_0^\infty \int_{Y\times\mathbb{R}} \frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon \theta(t,y) K_R(p)\, dt\, dy\, dp \\ &= -\int_0^\infty \int_{Y\times\mathbb{R}} m_1^\varepsilon\, \partial_p f_2^\varepsilon\, \theta(t,y) K_R(p)\, dt\, dy\, dp \\ &\quad -\int_0^\infty \int_{Y\times\mathbb{R}} m_1^\varepsilon f_2^\varepsilon\, \theta(t,y)\, K_R'(p)\, dt\, dy\, dp \\ &=: -(I_{\varepsilon,1}+I_{\varepsilon,2}). \end{aligned}$$

Remembering (3.42) and (3.45), we have

$$m_1^\varepsilon(t,y,p) = \int |\gamma_1(s_1,y_1)|^2 \frac{\partial v}{\partial p}(y_1,\varphi_1(s_1,y_1))\phi^\varepsilon(t-s_1,y-y_1,p-\varphi_1(s_1,y_1))\, ds_1\, dy_1,$$

$$\partial_p f_2^\varepsilon = \int_{\mathbb{R}^{N+1}} \phi^\varepsilon(t-s_2,y-y_2,p-\varphi_2(s_2,y_2))\, ds_2\, dy_2,$$

and thus

$$\begin{aligned} I_{\varepsilon,1} &= \int_0^\infty \int_{\mathbb{R}^{2N+2}} \int_{Y\times\mathbb{R}} \phi^\varepsilon(t-s_1,y-y_1,p-\varphi_1)\phi^\varepsilon(t-s_2,y-y_2,p-\varphi_2)\times \\ &\quad \times\theta(t,y)K_R(p)\left[\sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1(s_1,y_1))}\gamma_1(s_1,y_1)\right]^2 dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt. \end{aligned}$$

On the other hand, according to lemma 3.4.1 and the assumptions on K_R,

$$\begin{aligned} |I_{\varepsilon,2}| &\le \int_0^\infty \int_{Y\times\mathbb{R}} m_1^\varepsilon \theta(t,y)|K_R'(p)|\, dt\, dy\, dp \\ &\le 2||\theta||_\infty \left[\mu_1(R-1)+\mu_1(-R+1)\right]. \end{aligned}$$

The two other terms are treated in a similar way; eventually, we use the inequality $-(|a|^2+|b|^2) \le -2a\cdot b$ for all $a,b\in\mathbb{R}^N$ with $a=\sqrt{v_p(y_1,\varphi_1)}\,\gamma_1$, $b=\sqrt{v_p(y_2,\varphi_2)}\,\gamma_2$, and γ_i, φ_i are evaluated at $(s_i,y_i)\in[0,\infty)\times\mathbb{R}^N$. This concludes the derivation of (3.46).

Next, we investigate the second order terms in r_j^ε, i.e.,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^{N+1}} -\Bigg\{ &\Delta_y\left[\frac{\partial v}{\partial p}(y,p)f_1^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p}f_1\right) * \phi_\varepsilon(t,y,p)\right] f_2^\varepsilon \\ &+\Delta_y\left[\frac{\partial v}{\partial p}(y,p)f_2^\varepsilon(t,y,p) - \left(\frac{\partial v}{\partial p}f_2\right) * \phi_\varepsilon(t,y,p)\right] f_1^\varepsilon \Bigg\}\theta(t,y)K_R(p)\, dt\, dy\, dp. \end{aligned}$$

Integrating by parts, we obtain for instance

$$\begin{aligned}
&\int_0^\infty \int_{Y\times\mathbb{R}} -\Delta_y \left[\frac{\partial v}{\partial p} f_1^\varepsilon - \left(\frac{\partial v}{\partial p} f_1\right) * \phi_\varepsilon\right] f_2^\varepsilon\, \theta(t,y) K_R(p)\, dt\, dy\, dp \\
= &\int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon - \left(\frac{\partial v}{\partial p}(\nabla_y f_1)\right) * \phi_\varepsilon\right] \cdot \nabla_y f_2^\varepsilon\, \theta(t,y) K_R(p)\, dt\, dy\, dp \\
&+ \int_0^\infty \int_{Y\times\mathbb{R}} \left[\nabla_y \frac{\partial v}{\partial p} f_1^\varepsilon - \left(\left(\nabla_y \frac{\partial v}{\partial p}\right) f_1\right) * \phi_\varepsilon\right] \cdot \nabla_y f_2^\varepsilon\, \theta(t,y) K_R(p)\, dt\, dy\, dp \\
&- \int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial v}{\partial p} f_1^\varepsilon - \left(\frac{\partial v}{\partial p} f_1\right) * \phi_\varepsilon\right] \nabla_y f_2^\varepsilon \cdot \nabla\theta(t,y) K_R(p)\, dt\, dy\, dp \\
&- \int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial v}{\partial p} f_1^\varepsilon - \left(\frac{\partial v}{\partial p} f_1\right) * \phi_\varepsilon\right] f_2^\varepsilon \Delta\theta(t,y) K_R(p)\, dt\, dy\, dp \\
=: &\ J_{\varepsilon,1} + J_{\varepsilon,2} + J_{\varepsilon,3} + J_{\varepsilon,4}.
\end{aligned}$$

Notice that hypothesis (3.23) ensures that $\frac{\partial^2 v}{\partial p^2}$ exists and is Hölder continuous in y, with locally uniform bounds in p (see theorem 8.24 in [34]), and hypothesis (3.24) entails that $\nabla_y \frac{\partial v}{\partial p}$ is Hölder continuous in y, with locally uniform bounds in p (see theorem 8.32 in [34]).

Hence, $\frac{\partial v}{\partial p}$ belongs to $\mathcal{C}(\mathbb{R}, \mathcal{C}^\alpha_{\text{per}}(Y))$ for some $\alpha \in (0,1)$. Moreover, thanks to classical H^1 bounds for elliptic equations, we deduce that $\nabla_y \frac{\partial v}{\partial p}$ belongs to $\mathcal{C}(\mathbb{R}, L^2(Y))$. Together with identity (3.44), these regularity results easily entail that for all $R > 0$,

$$\lim_{\varepsilon\to 0} J_{\varepsilon,i} = 0 \quad \text{for } i = 2,3,4.$$

In the following, we denote by $\omega_R : [0,\infty) \to [0,\infty)$ a function such that $\lim_{0^+} \omega_R = 0$ and

$$|J_{\varepsilon,2}| + |J_{\varepsilon,3}| + |J_{\varepsilon,4}| \le \omega_R(\varepsilon) \quad \forall R > 0,\ \forall \varepsilon > 0.$$

Without loss of generality, we can also assume that the first order terms in r_j^ε are bounded by $\omega_R(\varepsilon)$.

We now focus on the term $J_{\varepsilon,1}$; thanks to identity (3.44), we have

$$\begin{aligned}
J_{\varepsilon,1} \ = \ &- \int_0^\infty \int_{\mathbb{R}^{2N+2}} \int_{Y\times\mathbb{R}} \phi_\varepsilon(t-s_1, y-y_1, p-\varphi_1)\phi_\varepsilon(t-s_2, y-y_2, p-\varphi_2) \\
&\times \left[\frac{\partial v}{\partial p}(y,p) - \frac{\partial v}{\partial p}(y_1,\varphi_1)\right] \gamma_1 \cdot \gamma_2\, \theta(t,y)\, K_R(p)\, dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt,
\end{aligned}$$

and γ_i, φ_i are evaluated at t_i, s_i. Gathering all the terms, we infer

$$\begin{aligned}
&\int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon \cdot \nabla_y f_2^\varepsilon + r_1^\varepsilon f_2^\varepsilon + r_2^\varepsilon f_1^\varepsilon\right]\theta K_R \\
\leq\ & \int \phi_\varepsilon(t-s_1, y-y_1, p-\varphi_1)\phi_\varepsilon(t-s_2, y-y_2, p-\varphi_2)\theta(t,y)K_R(p)\gamma_1\cdot\gamma_2 \\
\times\ & \left(\frac{\partial v}{\partial p}(y_1,\varphi_1) + \frac{\partial v}{\partial p}(y_2,\varphi_2) - 2\sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1)\frac{\partial v}{\partial p}(y_2,\varphi_2)}\right) dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt \\
+\ & 2||\theta||_\infty \left[\mu_1(R-1) + \mu_1(-R+1) + \mu_2(R-1) + \mu_2(-R+1)\right] + C\omega_R(\varepsilon) \\
\leq\ & \int_0^\infty \int_{\mathbb{R}^{2N+2}} \int_{Y\times\mathbb{R}} \phi_\varepsilon(t-s_1, y-y_1, p-\varphi_1)\phi_\varepsilon(t-s_2, y-y_2, p-\varphi_2)\theta(t,y)K_R(p) \\
& \times\gamma_1\cdot\gamma_2 \left[\sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1)} - \sqrt{\frac{\partial v}{\partial p}(y_2,\varphi_2)}\right]^2 dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt \\
& +2||\theta||_\infty \left[\mu_1(R-1) + \mu_1(-R+1) + \mu_2(R-1) + \mu_2(-R+1)\right] + C\omega_R(\varepsilon).
\end{aligned}$$

The function $\frac{\partial v}{\partial p}$ belongs to $W^{1,\infty}_{\text{per,loc}}(Y\times\mathbb{R})$ and is bounded away from zero on bounded subsets of $Y\times\mathbb{R}$. As a consequence, there exists a constant $C_R > 0$, depending on R, such that for all $\varepsilon > 0$,

$$\left|\sqrt{\frac{\partial v}{\partial p}(y_1,\varphi_1)} - \sqrt{\frac{\partial v}{\partial p}(y_2,\varphi_2)}\right| \leq C_R\, \varepsilon$$

whenever $|y_1 - y_2| \leq 2\varepsilon$, $|\varphi_1 - \varphi_2| \leq 2\varepsilon$, and $|\varphi_1|, |\varphi_2| \leq R$. We set

$$\phi(t,x,p) := \zeta_1(t)\,\zeta_2(x)\,\zeta_3(p).$$

Performing changes of variables in the integral in the right-hand side, we obtain

$$\begin{aligned}
&\int_0^\infty \int_{Y\times\mathbb{R}} \left[\frac{\partial m_1^\varepsilon}{\partial p} f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p} f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon \cdot \nabla_y f_2^\varepsilon + r_1^\varepsilon f_2^\varepsilon + r_2^\varepsilon f_1^\varepsilon\right]\theta K_R \\
\leq\ & C_R\,\varepsilon^2 \int_0^\infty \int_{\mathbb{R}^{2N+2}} \int_{Y\times\mathbb{R}} \phi_\varepsilon(t-s_1, y-y_1, p-\varphi_1)\phi_\varepsilon(t-s_2, y-y_2, p-\varphi_2) \\
& \times\theta(t,y)K_R(p)\gamma_1\cdot\gamma_2\, dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt \\
& +2||\theta||_\infty \left[\mu_1(R-1) + \mu_1(-R+1) + \mu_2(R-1) + \mu_2(-R+1)\right] + C\omega_R(\varepsilon) \\
\leq\ & C_R\,\varepsilon \int_0^\infty \int_{\mathbb{R}^{2N+2}} \int_{Y\times\mathbb{R}} \theta(t,y)K_R(p + \varepsilon\varphi_1(t-\varepsilon s_1, y-\varepsilon y_1)) \\
& \times\phi(s_1, y_1, p)\phi\left(s_2, y_2, p + \frac{\varphi_1(t-\varepsilon s_1, y-\varepsilon y_1) - \varphi_2(t-\varepsilon s_2, y-\varepsilon y_2)}{\varepsilon}\right) \\
& \times\gamma_1(t-\varepsilon s_1, y-\varepsilon y_1)\cdot\gamma_2(t-\varepsilon s_2, y-\varepsilon y_2)\, dy\, dp\, dy_1\, dy_2\, ds_1\, ds_2\, dt \\
& +2||\theta||_\infty \left[\mu_1(R-1) + \mu_1(-R+1) + \mu_2(R-1) + \mu_2(-R+1)\right] + C\omega_R(\varepsilon) \\
\leq\ & C_R\,\varepsilon + C\omega_R(\varepsilon) \\
& +2||\theta||_\infty \left[\mu_1(R-1) + \mu_1(-R+1) + \mu_2(R-1) + \mu_2(-R+1)\right],
\end{aligned}$$

so that eventually, for all test functions $\theta(t,y) \in \mathcal{D}_{\text{per}}([0,\infty)\times\mathbb{R}^N)$ such that $\theta \geq 0$,

$$\limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\int_0^\infty\int_{Y\times\mathbb{R}}\left[\frac{\partial m_1^\varepsilon}{\partial p}f_2^\varepsilon - \frac{\partial m_2^\varepsilon}{\partial p}f_1^\varepsilon - 2\frac{\partial v}{\partial p}\nabla_y f_1^\varepsilon\cdot\nabla_y f_2^\varepsilon + r_1^\varepsilon f_2^\varepsilon + r_2^\varepsilon f_1^\varepsilon\right]\times$$
$$\times\,\theta(t,y)K_R(p)dt\,dy\,dp \leq 0 \quad (3.47)$$

Consequently, in the limit, we obtain for any test function $\theta(t,y)\in\mathcal{D}_{\text{per}}([0,\infty)\times\mathbb{R}^N)$ such that $\theta\geq 0$

$$\int_0^\infty\int_Y (u_1-u_2)_+\partial_t\theta(t,y) + \mathbf{1}_{u_1>u_2}\left[A(y,u_1 - A(y,u_2)\right]\cdot\nabla_y\theta(t,y)\,dtdy$$
$$\geq \int_Y (u_1(t=0,y)-u_2(t=0,y))_+\,\theta(t=0,y)\,dy,$$

which means exactly that

$$\partial_t(u_1-u_2)_+ + \operatorname{div}_y\left[1_{u_1>u_2}\left(A(y,u_1)-A(y,u_2)\right)\right]\leq 0 \quad (3.48)$$

in the sense of distributions.

Integrating this last inequality on $(0,T)\times Y$ for any $T>0$ yields

$$||(u_1(t=T)-u_2(t=T))_+||_{L^1(Y)} \leq ||(u_1(t=0)-u_2(t=0))_+||_{L^1(Y)}. \quad (3.49)$$

Hence the derivation of (3.25) and (3.27) is complete; there only remains to prove lemma 3.4.1. The argument goes along the same lines as lemma 4.2.1 in [59].

Proof of Lemma 3.4.1. Notice that equation (3.38) is equivalent to

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial p}f_j\right)*\phi^\varepsilon + \frac{\partial}{\partial y_i}\left(\tilde{a}_i f_j\right)*\phi^\varepsilon - \Delta_y\left(\frac{\partial v}{\partial p}f_j\right)*\phi^\varepsilon = \eta_j\frac{\partial m_j^\varepsilon}{\partial p}. \quad (3.50)$$

Thus we focus on the derivation of (3.50) for f_1; let $(t,y,p)\in[0,\infty)\times Y\times\mathbb{R}$ be arbitrary. Following [59], one is tempted to consider the test function

$$(s,z,q)\mapsto\phi^\varepsilon(t-s,y-z,p-q) = \frac{1}{\varepsilon^{N+2}}\zeta_1\left(\frac{t-s}{\varepsilon}\right)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\zeta_3\left(\frac{p-q}{\varepsilon}\right)$$

in the definition 3.2.1 of kinetic solutions. However, such a test function is not periodic in z, as required in definition 3.2.1; but the support of $z\mapsto\zeta_2\left((y-z)/\varepsilon\right)$ is a subset of $\bar{B}(y,\varepsilon)$, the closed ball centered on y and of radius ε. Thus, for $0<\varepsilon<1/2$,

$$\operatorname{Supp}\zeta_2\left((y-\cdot)/\varepsilon\right)\subset\bar{B}(y,\varepsilon)\subset\Pi_{i=1}^N\left(y_i-\frac{1}{2},y_i+\frac{1}{2}\right).$$

Hence for $\varepsilon<1/2$, we can extend $\zeta_2\left((y-\cdot)/\varepsilon\right)$ by periodicity on $\mathbb{R}^N$; the function thus obtained is denoted by $\tilde{\zeta}_{y,\varepsilon}$, and belongs to $\mathcal{C}^\infty_{\text{per}}(\mathbb{R}^N)$.

Now, for fixed $(t,y,p)\in[0,\infty)\times Y\times\mathbb{R}$, we define the test function

$$\psi:(s,z,q)\mapsto\frac{1}{\varepsilon^{N+2}}\zeta_1\left(\frac{t-s}{\varepsilon}\right)\tilde{\zeta}_{y,\varepsilon}(z)\zeta_3\left(\frac{p-q}{\varepsilon}\right).$$

By construction, ψ belongs to $\mathcal{D}_{\text{per}}([0,\infty)\times Y\times\mathbb{R})$. Thus ψ is an admissible test function, and according to definition 3.2.1,

$$\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)\frac{\partial v(z,q)}{\partial q}\left\{\partial_s\psi + a_i(z,v(z,q))\partial_{z_i}\psi + \Delta_z\psi\right\} ds\,dz\,dq =$$
$$= \int_0^\infty\int_{Y\times\mathbb{R}} m(s,z,q)\partial_q\psi(s,z,q)\;ds\,dz\,dq - \int_{Y\times\mathbb{R}} \mathbf{1}_{u_0(z)>v(z,q)}\psi(0,z,q)\frac{\partial v(z,q)}{\partial q}\,dz\,dq.$$

First, notice that since $\text{Supp}\zeta_1\subset[-1,0]$, we have $\psi(0,z,q)=0$ for all z,q. Moreover, since f_1 and ψ are Y-periodic in their second variable, we have for instance, setting $Y_y=\Pi_{i=1}^N\left(y_i-1/2,y_i+1/2\right)=y-e+Y$, where $e:=(1/2,\cdots,1/2)\in\mathbb{R}^N$,

$$\int_0^\infty\int_{Y\times\mathbb{R}} f_1\frac{\partial v}{\partial q}\partial_s\psi = \int_0^\infty\int_{Y_y\times\mathbb{R}} f_1\frac{\partial v}{\partial q}\partial_s\psi.$$

And when $z\in Y_y$, $\psi(s,z,q)=\phi^\varepsilon(t-s,y-z,p-q)$ by definition. Thus, using once again the assumption on the support of ζ_2,

$$\begin{aligned}
&\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_s\psi(s,z,q)ds\,dz\,dq\\
=\;&\int_0^\infty\int_{Y_y\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_s\phi^\varepsilon(t-s,y-z,p-q)ds\,dz\,dq\\
=\;&\int_0^\infty\int_{\mathbb{R}^N\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_s\phi^\varepsilon(t-s,y-z,p-q)ds\,dz\,dq\\
=\;&-\int_0^\infty\int_{\mathbb{R}^N\times\mathbb{R}} f_1(s,z,q)\frac{\partial v}{\partial q}(z,q)\partial_t\phi^\varepsilon(t-s,y-z,p-q)ds\,dz\,dq\\
=\;&-\partial_t\left[(f_1v_p)*\phi^\varepsilon\right](t,y,p).
\end{aligned}$$

The other terms are treated in a similar way; we obtain

$$\begin{aligned}
&\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)\tilde{a}_i(z,q)\partial_{z_i}\psi(s,z,q)\;ds\,dz\,dq\\
=\;&\int_0^\infty\int_{\mathbb{R}^N\times\mathbb{R}} f_1(s,z,q)\tilde{a}_i(z,q)\partial_{z_i}\phi^\varepsilon(t-s,y-z,p-q)\;ds\,dz\,dq\\
=\;&-\partial_{y_i}\left[f_1\tilde{a}_i\right]*\phi^\varepsilon(t,y,p),\\
&\int_0^\infty\int_{Y\times\mathbb{R}} f_1(s,z,q)v_q(z,q)\Delta_z\psi(s,z,q)ds\,dz\,dq = \Delta_y\left[(f_1v_p)*\phi^\varepsilon\right](t,y,p),\\
&\int_0^\infty\int_{Y\times\mathbb{R}} m(s,z,q)\partial_q\psi(s,z,q)\;ds\,dz\,dq = -\partial_p m_1^\varepsilon(t,y,p).
\end{aligned}$$

There remains to derive the bound on m_1^ε : by definition,

$$\begin{aligned}
&\int_0^\infty \int_Y m_1^\varepsilon(t,y,p)\,dt\,dy \\
= &\int_0^\infty \int_Y \int_{\mathbb{R}^{N+2}} m_1(s,z,q)\phi^\varepsilon(t-s,y-z,p-q)ds\,dz\,dq\,dt\,dy \\
= &\int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty \int_0^\infty m_1(s,z,q)\phi^\varepsilon(t-s,y-z,p-q)\,dt\,ds\,dz\,dq\,dy \\
= &\int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty \int_{-s}^\infty m_1(s,z,q)\phi^\varepsilon(u,y-z,p-q)\,du\,ds\;\,dz\,dq\,dy \\
\le &\int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty \int_{\mathbb{R}} m_1(s,z,q)\phi^\varepsilon(u,y-z,p-q)\,du\,ds\;\,dz\,dq\,dy \\
\le &\frac{1}{\varepsilon^{N+1}} \int_Y \int_{\mathbb{R}^{N+1}} \int_0^\infty m_1(s,z,q)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\zeta_3\left(\frac{p-q}{\varepsilon}\right)\,ds\;\,dz\,dq\,dy.
\end{aligned}$$

Then, with the same notations as earlier,

$$\begin{aligned}
&\int_Y \int_{\mathbb{R}^N} m_1(s,z,q)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\,dz\,dy \\
= &\int_Y dy \left(\int_{Y_y} dz\; m_1(s,z,q)\zeta_2\left(\frac{y-z}{\varepsilon}\right)\right) \\
= &\int_{Y\times Y} m_1(s,y+y'-e,q)\zeta_2\left(\frac{-y'+e}{\varepsilon}\right)dy\,dy' \\
= &\int_{Y\times Y} m_1(s,y,q)\zeta_2\left(\frac{-y'+e}{\varepsilon}\right)dy\,dy' \\
= &\left(\int_Y m_1(s,y,q)\,dy\right)\times\left(\int_{\mathbb{R}^N}\zeta_2\left(\frac{-y'}{\varepsilon}\right)\,dy'\right).
\end{aligned}$$

In the one before last step, we have used the periodicity of m_1.

Thus

$$\begin{aligned}
\int_0^\infty \int_Y m_1^\varepsilon(t,y,p)\,dt\,dy &\le \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{z\in Y}\int_0^\infty m_1(s,z,q)\zeta_3\left(\frac{p-q}{\varepsilon}\right)\,ds\;\,dz\,dq \\
&\le \frac{1}{\varepsilon}\int_{\mathbb{R}} \mu_1(q)\zeta_3\left(\frac{p-q}{\varepsilon}\right)\,dq \\
&\le \int_{-1}^1 \mu_1(p-\varepsilon q)\zeta_3(q)\,dq.
\end{aligned}$$

The monotonicity of μ_1 yields the desired result.

3.5 Appendice : cas d'un flux avec une dépendance macroscopique

On se concentre à présent sur le cas où le flux possède une dépendance explicite en la variable d'espace $x \in \mathbb{R}^N$. Autrement dit, on considère un flux $A \in W^{2,\infty}_{\text{per,loc}}(\mathbb{R}^N \times$

$Y \times \mathbb{R})^N$, et on s'intéresse au comportement de la solution entropique u^ε de la loi de conservation scalaire

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(x, \frac{x}{\varepsilon}, v^\varepsilon(t,x)\right) - \varepsilon \Delta_x u^\varepsilon = 0 \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{3.51}$$

$$u^\varepsilon(t=0) = u_0\left(x, \frac{x}{\varepsilon}\right), \quad u_0 \in L^\infty(\mathbb{R}^N \times Y). \tag{3.52}$$

Pour simplifier l'étude, on se place dans un cadre L^∞ (voir l'hypothèse (3.54)) ; en effet, il est un peu plus compliqué de trouver des bornes *a priori* sur la suite u^ε que dans le cas où le flux ne possède pas de dépendance macroscopique, et l'étude du comportement des solutions cinétiques de l'équation apparaît, pour l'instant, hors de portée.

Le lemme suivant donne un résultat de régularité pour les solutions du problème de la cellule :

Lemme 3.5.1. *Soit $A \in L^\infty_{loc}(\mathbb{R}^N_x, W^{2,\infty}_{per}(Y \times \mathbb{R}))^N$. Alors pour presque tout $x \in \mathbb{R}^N$, pour tout $p \in \mathbb{R}$, il existe une unique solution $v(x,\cdot,p)$ du problème de la cellule*

$$-\Delta_y v(x,y,p) + \mathrm{div}_y A(x,y,v(x,y,p)) = 0, \quad \langle v(x,\cdot,p)\rangle = p. \tag{3.53}$$

De plus, si le flux A est tel que

$$\nabla_x(\mathrm{div}_y A),\ \nabla_x \partial_p A,\ \partial_p^2 A,\ \Delta_x A \in L^\infty_{per}(\mathbb{R}^N \times Y \times \mathbb{R}), \tag{3.54}$$

alors $v \in L^\infty(\mathbb{R}^N_x \times \mathbb{R}^N_p, W^{2,q}_{per}(Y))$ pour tout $q \in (1,\infty)$, et il existe une constante $C > 0$ telle que pour presque tout x, y, p,

$$|v(x,y,p)|,\ |\Delta_x v(x,y,p)|,\ |\Delta_{xy} v(x,y,p)|,\ \left|\frac{\partial v}{\partial p}(x,y,p)\right| \leq C.$$

Le résultat principal de cette partie est énoncé dans la proposition suivante :

Proposition 3.5.1. *Soit $A \in L^\infty(\mathbb{R}^N_x, W^{2,\infty}_{per}(Y \times \mathbb{R}))^N$; on suppose que la condition* (3.54) *est vérifiée. Soit $\bar{u}_0 \in L^\infty(\mathbb{R}^N)$, et soit*

$$u_0(x,y) := v(x,y,\bar{u}_0(x)) \in L^\infty(\mathbb{R}^N \times Y).$$

On définit le flux homogénéisé $\bar{A} \in W^{1,\infty}(\mathbb{R}^{N+1})$ par

$$\bar{A}(x,p) := \frac{1}{Y}\int_Y A(x,y,v(x,y,p))\,dy, \quad (x,p) \in \mathbb{R}^{N+1},$$

et on considère la solution entropique $\bar{u} \in \mathcal{C}([0,\infty), L^1_{loc}(\mathbb{R}^N)) \cap L^\infty_{loc}([0,\infty), L^\infty(\mathbb{R}^N))$ de la loi de conservation scalaire

$$\frac{\partial \bar{u}(t,x)}{\partial t} + \mathrm{div}_x \bar{A}(x,\bar{u}(t,x)) = 0, \quad t > 0,\ x \in \mathbb{R}^N, \tag{3.55}$$

$$\bar{u}(t=0,x) = \bar{u}_0(x). \tag{3.56}$$

Soit u^ε la solution entropique de l'équation (3.51)*. Alors lorsque $\varepsilon \to 0$, on a*

$$u^\varepsilon(t,x) - v\left(x, \frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0$$

dans $L^1_{loc}([0,\infty) \times \mathbb{R}^N)$.

La preuve de la proposition 3.5.1 est divisée en trois parties : dans le paragraphe 3.5.1, on montre le lemme 3.5.1. Le paragraphe 3.5.2 est consacré à l'obtention de bornes uniformes en ε dans $L^\infty_{\text{loc}}([0,\infty), L^\infty(\mathbb{R}^N))$ sur la famille u^ε, et le paragraphe 3.5.3 est dédié au passage à la limite proprement dit.

3.5.1 Régularité des solutions du problème de la cellule

Le lemme 3.5.1 est une conséquence facile de la proposition 3.1.1 et des résultats démontrés dans l'article [14] (voir le chapitre 2). Tout d'abord, à $x \in \mathbb{R}^N$ fixé, l'existence et l'unicité des solutions du problème de la cellule est assurée par l' inégalité

$$|\text{div}_y A(x,y,p)| + |\partial_p A(x,y,p)| \leq ||\text{div}_y A||_{L^\infty(\mathbb{R}^N\times Y\times\mathbb{R})} + ||\partial_p A||_{L^\infty(\mathbb{R}^N\times Y\times\mathbb{R})}.$$

Autrement dit, les hypothèses (3.2) et (3.3) sont satifaites avec $m = n = 0$, et donc (3.5) est vérifiée également. On en déduit que $v \in L^\infty(\mathbb{R}^N_x \times \mathbb{R}^N_p, W^{2,q}_{\text{per}}(Y))$ pour tout $q < \infty$. Et d'après [14], il existe une constante $\alpha > 0$ telle que

$$\frac{\partial v}{\partial p}(x,y,p) \geq \alpha \quad \forall (x,y,p) \in \mathbb{R}^N \times Y \times \mathbb{R}.$$

Par ailleurs, sous l'hypothèse (3.54), on montre que la fonction v est régulière en x. Précisément, $\partial v/\partial x_i$ est solution de l'équation

$$\begin{aligned}
&-\Delta_y \frac{\partial v}{\partial x_i} + \text{div}_y \left(a(x,y,v(x,y,p)) \frac{\partial v}{\partial x_i} \right) \\
= \ & -\text{div}_y \left(\frac{\partial A}{\partial x_i}(x,y,v(x,y,p)) \right) \\
= \ & -\frac{\partial \text{div}_y A}{\partial x_i}(x,y,v(x,y,p)) - \frac{\partial a}{\partial x_i}(x,y,v(x,y,p)) \cdot \nabla_y v(x,y,p).
\end{aligned}$$

Le membre de droite appartient à $L^\infty(\mathbb{R}^N \times Y \times \mathbb{R})$; par un argument de *"boot-strap"*, on en déduit que $\nabla_x v \in L^\infty(\mathbb{R}^N_x \times \mathbb{R}_p, W^{2,q}(Y))$ pour tout $q < \infty$, et donc $\nabla_x v \in L^\infty(\mathbb{R}^N_x \times \mathbb{R}_p, W^{1,\infty}(Y))$.

En dérivant une nouvelle fois l'équation précédente par rapport à x_i, on obtient

$$\begin{aligned}
&-\Delta_y \Delta_x v + \text{div}_y \left(a(x,y,v(x,y,p)) \Delta_x v \right) \\
= \ & -\left[\text{div}_y \left(\nabla_x \left(a(x,y,v(x,y,p)) \cdot \nabla_x v \right) + \text{div}_y \left((\Delta_x A)(x,y,v(x,y,p)) \right) \right) \right] \\
& -\text{div}_y \left((\nabla_x a)(x,y,v(x,y,p)) \cdot \nabla_x v(x,y,p) \right).
\end{aligned}$$

Le membre de droite de l'équation précédente appartient à $L^\infty(\mathbb{R}^N_x \times \mathbb{R}_p, W^{-1,\infty}(Y))$; en utilisant une fois encore un argument de *boot-strap*, on conclut que $\Delta_x v \in L^\infty(\mathbb{R}^N_x \times \mathbb{R}_p, W^{1,q}(Y))$ pour tout $q < \infty$, puis que $\Delta_x v \in L^\infty(\mathbb{R}^N \times Y \times \mathbb{R})$.

3.5.2 Bornes *a priori* dans L^∞ sur la famille u^ε

Dans le cas où le flux ne dépend que de la variable microscopique en espace, l'obtention de bornes *a priori* dans L^∞ est réalisée en comparant la fonction u^ε aux solutions stationnaires de (3.12), c'est à dire aux fonctions $v(x/\varepsilon, p)$, où v est solution du problème de la cellule. Ici, en raison de la dépendance macroscopique du flux, la fonction $v(x, x/\varepsilon, p)$ n'est pas une solution stationnaire de (3.51) ; néanmoins, il est possible de construire des sur-solutions (et des sous-solutions) de (3.51) à partir des solutions de (3.53). Précisément, pour tout $p \in \mathbb{R}$, on a

$$\frac{\partial}{\partial x_i} A_i \left(x, \frac{x}{\varepsilon}, v\left(x, \frac{x}{\varepsilon}, p\right)\right) - \varepsilon \Delta_x v\left(x, \frac{x}{\varepsilon}, p\right) = b_\varepsilon\left(x, \frac{x}{\varepsilon}, p\right), \tag{3.57}$$

où

$$b_\varepsilon(x, y, p) = \mathrm{div}_x\left(A(x, y, v(x, y, p))\right) - 2\Delta_{xy} v(x, y, p) - \varepsilon \Delta_x v.$$

D'après le lemme 3.5.1, il existe une constante $C_0 > 0$ telle que

$$|b_\varepsilon(x, y, p)| \le C_0 \quad \forall (x, y, p) \in \mathbb{R}^{2N+1},\ \forall \varepsilon \in (0, 1).$$

Posons

$$v^\varepsilon(t, x) := v\left(x, \frac{x}{\varepsilon}, p_+(t)\right),$$

où la fonction $p_+ : \mathbb{R} \to \mathbb{R}$ est donnée par

$$p_+(t) = p_0 + \frac{C_0}{\alpha} t, \tag{3.58}$$

où

$$\begin{aligned} p_0 &= \sup \bar{u}_0 \\ \alpha &= \inf_{p \in \mathbb{R}} \inf_{y \in Y} \frac{\partial v}{\partial p}(y, p). \end{aligned}$$

Si $\varepsilon \in (0, 1)$, on a

$$\begin{aligned} &\frac{\partial}{\partial t} v^\varepsilon + \frac{\partial}{\partial x_i} A_i\left(x, \frac{x}{\varepsilon}, v^\varepsilon\right) - \varepsilon \Delta_x v^\varepsilon = \\ &= b_\varepsilon\left(x, \frac{x}{\varepsilon}, p_+(t)\right) + \frac{\partial v}{\partial p}\left(x, \frac{x}{\varepsilon}, p_+(t)\right) \dot{p_+}(t) \\ &= b_\varepsilon\left(x, \frac{x}{\varepsilon}, p_+(t)\right) + C_0 \\ &\ge 0. \end{aligned}$$

De plus, presque partout en x, on a

$$v^\varepsilon(t = 0, x) \ge u^\varepsilon(t = 0, x).$$

On en déduit que pour tout $t \ge 0$, pour presque tout $x \in \mathbb{R}^N$,

$$u^\varepsilon(t, x) \le v^\varepsilon(t, x) = v\left(x, \frac{x}{\varepsilon}, \sup \bar{u}_0 + t\frac{C_0}{\alpha}\right). \tag{3.59}$$

De même, on montre que

$$u^\varepsilon(t, x) \ge v\left(x, \frac{x}{\varepsilon}, \inf \bar{u}_0 - t\frac{C_0}{\alpha}\right). \tag{3.60}$$

Par conséquent, la famille u^ε est bornée dans $L^\infty((0, T) \times \mathbb{R}^N)$ pour tout $T > 0$.

3.5.3 Passage à la limite et homogénéisation

Le but de ce paragraphe est de montrer le résultat de convergence énoncé dans la proposition 3.5.1 ; pour cela, la méthode est exactement la même que lorsque le flux n'a pas de dépendance macroscopique ; par conséquent, nous passerons rapidement sur les points qui sont similaires à ceux développés dans la partie 3.3.

Commençons par écrire une formulation cinétique pour l'équation (3.51). D'après (3.57), on a

$$\partial_t\left(u^\varepsilon - v\left(x,\frac{x}{\varepsilon},p\right)\right)_+ + \partial_{x_i}\left[\mathbf{1}_{v\left(x,\frac{x}{\varepsilon},p\right)<u^\varepsilon}\left(A_i\left(x,\frac{x}{\varepsilon},u^\varepsilon\right) - A_i\left(x,\frac{x}{\varepsilon},v\left(x,\frac{x}{\varepsilon},p\right)\right)\right)\right]$$
$$+\mathbf{1}_{v\left(x,\frac{x}{\varepsilon},p\right)<u^\varepsilon} b_\varepsilon\left(x,\frac{x}{\varepsilon},p\right) - \varepsilon\Delta_x\left(u^\varepsilon - v\left(x,\frac{x}{\varepsilon},p\right)\right)_+ =: -m^\varepsilon(t,x,p) \le 0.$$

L'égalité précédente est entendue au sens des distributions, et m^ε est une mesure positive. En dérivant cette égalité par rapport à p, on obtient une équation cinétique vérifiée par la fonction $f^\varepsilon(t,x,p) := \mathbf{1}_{v\left(x,\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)}$:

$$\begin{aligned}\frac{\partial m^\varepsilon}{\partial p} &= \frac{\partial}{\partial t}\left(f^\varepsilon \frac{\partial v}{\partial p}\left(x,\frac{x}{\varepsilon},p\right)\right) + \frac{\partial}{\partial x_i}\left[a_i\left(x,\frac{x}{\varepsilon},v\left(x,\frac{x}{\varepsilon},p\right)\right)\frac{\partial v}{\partial p}\left(x,\frac{x}{\varepsilon},p\right) f^\varepsilon(t,x,p)\right]\\ &\quad -\frac{\partial}{\partial p}\left[f^\varepsilon(t,x,p) b_\varepsilon\left(x,\frac{x}{\varepsilon},p\right)\right] - \varepsilon\Delta_x\left(f^\varepsilon\frac{\partial v}{\partial p}\left(x,\frac{x}{\varepsilon},p\right)\right).\end{aligned} \tag{3.61}$$

On fixe un temps $T>0$. Compte tenu des bornes sur f^ε et m^ε, il existe une fonction $f \in L^\infty([0,T]\cap\mathbb{R}^N \times Y \times \mathbb{R})$ et une mesure $m \in M^1((0,T)\times\mathbb{R}^{N+1})$ telles que, à extraction d'une sous-suite près,

$$\begin{aligned} f^\varepsilon(t,x,p) &\overset{2\text{ éch.}}{\rightharpoonup} f(t,x,y,p),\\ m^\varepsilon(t,x,p) &\rightharpoonup m(t,x,p) \quad w-M^1.\end{aligned}$$

De plus, f et m vérifient les propriétés suivantes :

$$\begin{gathered}\partial_\xi f \le 0, \quad m \ge 0,\\ 0 \le f \le 1,\\ f(t,x,y,p) = 1 \quad \text{si } p < \inf \bar u_0 - t\frac{C_0}{\alpha},\\ f(t,x,y,p) = 0 \quad \text{si } p > \sup \bar u_0 + t\frac{C_0}{\alpha},\\ m(t,x,y,p) = 0 \quad \text{si } p < \inf \bar u_0 - t\frac{C_0}{\alpha} \text{ ou si } p > \sup \bar u_0 + t\frac{C_0}{\alpha}.\end{gathered}$$

Comme dans la partie 3.3, on commence par choisir des fonctions tests du type

$$\varepsilon\varphi\left(t,x,\frac{x}{\varepsilon},p\right),$$

avec $\varphi \in \mathcal{D}_{\text{per}}([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$. En passant à la limite double-échelle lorsque $\varepsilon \to 0$, on obtient (voir (3.33))

$$-\Delta_y\left(\frac{\partial v}{\partial p} f\right) + \text{div}_y\left(a(y,v(y,p))\frac{\partial v}{\partial p} f\right) = 0.$$

On en déduit alors que la fonction f est indépendante de la variable microscopique y :

$$f = f(t, x, p).$$

Dans un deuxième temps, on passe à la limite faible dans l'équation (3.61). On trouve alors que la fonction f est solution d'une équation de transport linéaire :

$$\partial_t f + \partial_{x_i} \left(\bar{a}_i(x,p) f\right) + \partial_p \left(\bar{a}_{N+1}(x,p) f\right) = \partial_p m \tag{3.62}$$

avec

$$\begin{aligned}\bar{a}_i(x,p) &:= \left\langle a_i(x,\cdot,v(x,\cdot,p)) \frac{\partial v}{\partial p}(x,\cdot,p) \right\rangle = \frac{\partial \bar{A}_i(x,p)}{\partial p},\\ \bar{a}_{N+1}(x,p) &:= -\left\langle \mathrm{div}_x A(x,y,v(x,y,p)) \right\rangle = -\mathrm{div}_x \bar{A}(x,p),\end{aligned}$$

où

$$\bar{A}(x,p) := \left\langle A(x,\cdot,v(x,\cdot,p)) \right\rangle.$$

En utilisant un théorème de rigidité pour la formulation cinétique d'une loi de conservation scalaire hétérogène (voir [15]), on en déduit qu'il existe une fonction $\bar{u} \in \mathcal{C}([0,\infty), L^1_{\mathrm{loc}}(\mathbb{R}^N)) \cap L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}^N))$ telle que

$$f(t,x,p) = \mathbf{1}_{p<\bar{u}(t,x)} \quad \text{p.p.}$$

et $\bar{u}$ est l'unique solution entropique de la loi de conservation scalaire

$$\begin{aligned}&\frac{\partial \bar{u}(t,x)}{\partial t} + \mathrm{div}_x \bar{A}(x, \bar{u}(t,x)) = 0, \quad t>0,\ x \in \mathbb{R}^N,\\ &\bar{u}(t=0,x) = \bar{u}_0(x).\end{aligned}$$

Le résultat de convergence forte énoncé dans la proposition 3.5.1 est ensuite une conséquence facile de la propriété

$$\mathbf{1}_{v\left(x,\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)} \overset{2\text{ éch.}}{\rightharpoonup} \mathbf{1}_{p<\bar{u}(t,x)}.$$

La preuve de ce résultat est identique à celle de la partie 3.3. Par conséquent, nous laissons au lecteur le soin de l'écrire dans ce contexte.

Chapitre 4

Homogénéisation d'une loi de conservation scalaire avec viscosité évanescente. Partie II : données mal-préparées

On étudie ici le comportement lorsque $\varepsilon \to 0$ des solutions de l'équation

$$\partial_t u^\varepsilon + \mathrm{div}_x \left[A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\right] - \varepsilon \Delta_x u^\varepsilon = 0.$$

Dans les chapitres précédents, on a démontré un résultat d'homogénisation pour cette équation, mais la validité de celui-ci est limitée au cas de données bien préparées, c'est-à-dire au cas où le profil microscopique de la condition initiale est adapté au milieu oscillant. Dans ce chapitre, on s'intéresse au cas de données mal préparées ; on montre qu'il se forme alors une couche limite en temps, de taille typique ε, pendant laquelle la solution u^ε s'adapte au profil microscopique dicté par l'environnement. La preuve repose sur la nature parabolique de l'équation.

Ce chapitre a fait l'objet d'une publication dans Archive for Rational Mechanics and Analysis (2007).

4.1 Introduction

We study the homogenization of equation

$$\frac{\partial u^\varepsilon}{\partial t}(t,x)+\sum_{i=1}^N \frac{\partial}{\partial x_i}A_i\left(\frac{x}{\varepsilon},u^\varepsilon(t,x)\right)-\varepsilon\Delta u^\varepsilon=0 \quad t\geq 0,\ x\in\mathbb{R}^N, \tag{4.1}$$

$$u^\varepsilon(t=0)=u_0\left(x,\frac{x}{\varepsilon}\right). \tag{4.2}$$

It is well-known (see for instance [69, 70]) that under suitable regularity assumptions on A and u_0, there exists a unique solution $u^\varepsilon(t,x)$ of (4.1) belonging to $\mathcal{C}([0,\infty),L^1_{\text{loc}}(\mathbb{R}^N))\cap L^\infty_{\text{loc}}([0,\infty)\times\mathbb{R}^N)\cap L^2_{\text{loc}}([0,\infty),H^1_{\text{loc}}(\mathbb{R}^N))$. The first part of the homogenization process was already performed in [14], the main results of which we recall below. Precisely, it is proved in [14] that if the initial data is already adapted to the microstructure, then $u^\varepsilon(t,x)$ behaves in $L^2_{\text{loc}}([0,\infty)\times\mathbb{R}^N)$ like some function $v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)$, where $v(y,p)$ is the solution of a microscopic cell problem depending on the parameter $p\in\mathbb{R}$, and $\bar{u}(t,x)$ is the entropy solution of a nonlinear scalar conservation law. This result was proved with the help of two-scale Young measures, a tool introduced by Weinan E in [24]. Although equation (4.1) is parabolic, the proof bears a lot of resemblance with the ones of Weinan E in [24] and Weinan E and Denis Serre in [27], both of which tackle hyperbolic problems. However, this is not surprising if we take into account the scaling of the viscosity: indeed, since the viscosity is of order ε, it has an effect on the microscopic asymptotic profile of the solution u^ε, but it disappears from the macroscopic homogenized problem, which is hyperbolic.

This article is the next step in the analysis of equation (4.1); indeed, we are able to prove that homogenization holds even when the initial data is not well-prepared, i.e. when it cannot be written as $u_0\left(x,\frac{x}{\varepsilon}\right)=v\left(\frac{x}{\varepsilon},\bar{u}_0(x)\right)$ for some function $\bar{u}_0\in L^1\cap L^\infty(\mathbb{R}^N)$. In that case, there is an initial layer of order ε during which the solution adapts itself to the microstructure. The proof relies strongly on the parabolic form of the equation, which compels the solutions of (4.1) to match the microscopic profile $v\left(\frac{x}{\varepsilon},\bar{u}(t,x)\right)$ exponentially fast. In the case when the viscosity is zero, nonlinearity assumptions on the flux A are most probably necessary in order to obtain the same kind of result, but few cases are known: in [28], Bjorn Engquist and Weinan E prove that homogenization holds in dimensions one and two for a nonlinear homogeneous flux in the case where the initial data is oscillating. In [25], Weinan E studies a particular kind of heterogeneous conservation law in dimension one, for which he proves a result similar to our theorem 5 under a strict convexity assumption (see also [24] for further results in the linear case). Let us point out that in the hyperbolic case, these issues are certainly linked to the compactness of solutions of conservation laws and to the cancellation of oscillations in the case when the flux is nonlinear (see [53], [52]). However, even if this connection seems natural, it is still an open problem how to handle initial layers (or for that matter, homogenization in general) in the hyperbolic case when the dimension is greater than or equal to two (for N=1, an equivalence with Hamilton-Jacobi equations allows us to use the results of P.-L. Lions, G. Papanicolaou and S.R.S. Varadhan in [49], and thus to identify the weak limit of u^ε as $\varepsilon\to 0$).

Throughout this article, we use the notation $Y := \Pi_{i=1}^{N}(0,T_i)$, $T_i > 0$ for $1 \le i \le N$ (Y is the unit cell), and

$$\langle v \rangle := \frac{1}{|Y|}\int_Y v(y)\,dy;$$

we will work in the following functional spaces: if $\mathcal{C}^\infty_{\mathrm{per}}(Y)$ denotes the space of Y-periodic functions in $\mathcal{C}^\infty(\mathbb{R}^N)$, then:

$$\begin{gathered}
H^1_{\mathrm{per}}(Y) := \overline{\mathcal{C}^\infty_{\mathrm{per}}(Y)}^{H^1(Y)},\ \ ||\cdot||_{H^1_{\mathrm{per}}(Y)} = ||\cdot||_{H^1(Y)},\\
V := \{v \in H^1_{\mathrm{per}}(Y), \langle v\rangle_Y = 0\},\ \ ||v||_V = ||\nabla v||_{L^2(Y)}\\
\mathcal{C}^\infty_{\mathrm{per}}(Y\times\mathbb{R}) := \{f = f(y,v) \in \mathcal{C}^\infty(\mathbb{R}^N\times\mathbb{R}); f \text{ is } Y-\text{periodic in } y\},\\
W^{k,\infty}_{\mathrm{per}}(Y\times\mathbb{R}) := \overline{\mathcal{C}^\infty_{\mathrm{per}}(Y\times\mathbb{R})}^{W^{k,\infty}(Y\times\mathbb{R})}, \quad k\in\mathbb{N},\\
W^{1,\infty}_{\mathrm{per,loc}}(Y\times\mathbb{R}) := \{u = u(y,v) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{N+1}), u \text{ is } Y-\text{periodic in } y\}.
\end{gathered}$$

Thanks to the Poincaré-Wirtinger inequality, the norm on V is equivalent to the H^1 norm.

We will often use the following notations:

$$a_i(y,v) := \frac{\partial A_i(y,v)}{\partial v} \quad (1\le i\le N), \quad a_{N+1}(y,v) := -\sum_{i=1}^{N}\frac{\partial A_i(y,v)}{\partial y_i}.$$

Let us now recall the main results of [14]; the first one is about the cell problem:

Proposition 4.1.1. *Let $A \in W^{1,\infty}_{per,loc}(Y\times\mathbb{R})^N$. Assume that there exist $C_0 > 0$, $m \in [0,\infty)$, $n \in [0, \frac{N+2}{N-2})$ when $N > 2$, such that for all $(y,p) \in Y\times\mathbb{R}$*

$$|a_i(y,p)| \le C_0\,(1+|p|^m) \quad \forall\ 1\le i\le N, \tag{4.3}$$

$$|a_{N+1}(y,p)| \le C_0\,(1+|p|^n)\,. \tag{4.4}$$

Assume as well that at least one of the following conditions holds:

$$m = 0 \tag{4.5}$$

$$\text{or} \quad 0 \le n < 1 \tag{4.6}$$

$$\text{or} \quad n < \min\left(\frac{N+2}{N}, 2\right) \text{ and } \exists p_0 \in \mathbb{R} \text{ s.t. } a_{N+1}(y,p_0) = 0\ \forall y \in Y. \tag{4.7}$$

Then for all $p \in \mathbb{R}$, there exists a unique solution $\tilde{u} \in V$ of the cell problem

$$-\Delta_y \tilde{u} + \mathrm{div}_y A(y, p+\tilde{u}) = 0; \tag{4.8}$$

For all $p \in \mathbb{R}$, $\tilde{u}(\cdot,p)$ belongs to $W^{2,q}_{per}(Y)$ for all $1 < q < +\infty$ and satisfies the following a priori estimate for all $R > 0$

$$||\tilde{u}(\cdot,p)||_{W^{2,q}(Y)} \le C \quad \forall p\in\mathbb{R}, |p|\le R, \tag{4.9}$$

for some constant C depending only on N, Y, C_0, m, n, q and R.

Moreover, setting $v(y,p) := p + \tilde{u}(y,p)$, the sequence v is increasing in p: for every $p > p'$,

$$v(y,p) > v(y,p') \quad \forall y \in Y.$$

The homogenization result proved in [14] is stated in the following

Proposition 4.1.2. *Assume that $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$ satisfies the hypotheses of proposition 4.1.1, and that $\frac{\partial a_i}{\partial y_j} \in L^\infty_{loc}(Y \times \mathbb{R})$, $\frac{\partial a_i}{\partial v} \in L^\infty_{loc}(Y \times \mathbb{R})$ for $1 \leq i \leq N+1$, $1 \leq j \leq N$.*

Let $p \in \mathbb{R}$, and let $\tilde{u}(\cdot, p)$ be the unique solution in V of the cell problem (4.8).

Assume that u_0 is "well-prepared", i.e. satisfies

$$u_0(x,y) = v(y, \bar{u}_0(x)) \tag{4.10}$$

for some $\bar{u}_0 \in L^1 \cap L^\infty(\mathbb{R}^N)$.

Let

$$\bar{A}_i(p) := \frac{1}{|Y|} \int_Y A(y, p + \tilde{u}(y,p))\, dy, \tag{4.11}$$

and let $\bar{u} = \bar{u}(t,x) \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty) \times \mathbb{R}^N)$ be the unique entropy solution of the hyperbolic scalar conservation law

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial t} + \displaystyle\sum_{i=1}^N \frac{\partial \bar{A}_i(\bar{u}(t,x))}{\partial x_i} = 0, \\ \bar{u}(t=0,x) = \bar{u}_0(x) \in L^1 \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{4.12}$$

Then as ε goes to 0,

$$u^\varepsilon(t,x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right) \to 0 \quad \text{in } L^2_{loc}([0,\infty) \times \mathbb{R}^N).$$

We now state the main results of this paper; the first one addresses the long-time behavior of the solutions of a parabolic problem, which is derived by inserting in (4.1) a two-scale Ansatz in both space and time, namely

$$u^\varepsilon(t,x) \approx u^0\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \cdots$$

Theorem 5. *Let $u_0 \in L^\infty(Y)$. Assume that $A \in W^{1,\infty}_{per,loc}(Y \times \mathbb{R})^N$ satisfies the hypotheses of proposition 4.1.1, and that $\frac{\partial a_i}{\partial y_j} \in L^\infty_{loc}(Y \times \mathbb{R})$. Assume also that there exist constants $\beta_1, \beta_2 \in \mathbb{R}$ such that*

$$v(y, \beta_1) \leq u_0(y) \leq v(y, \beta_2). \tag{4.13}$$

Let $u \in \mathcal{C}([0,\infty), L^1(Y)) \cap L^\infty([0,\infty) \times Y) \cap L^2_{loc}([0,\infty), H^1_{per}(Y))$ be the unique solution of the parabolic equation

$$\begin{cases} \dfrac{\partial u(\tau,y)}{\partial \tau} + \mathrm{div}_y\left[A(y, u(\tau,y))\right] - \Delta_y u(\tau,y) = 0, \quad \tau \geq 0,\ y \in Y, \\ u(\tau = 0, y) = u_0(y). \end{cases} \tag{4.14}$$

Let $p = \langle u_0 \rangle$, and let $v(y,p) \in H^1_{per}(Y)$ be the solution of the associated cell problem (4.8).

Then as $\tau \to \infty$,

$$||u(\tau, y) - v(y, p)||_{L^\infty(Y)} \to 0. \tag{4.15}$$

Consequently, there exist constants $c, \mu > 0$, *with* μ *depending only on* Y, N, *and* $\max_{1\le i\le N} ||a_i||_{L^\infty(Y\times(-K,K))}$, *where* $K = \sup_{\beta_1\le p\le\beta_2} ||v(\cdot, p)||_{L^\infty(Y)}$ *such that*

$$||u(\tau, y) - v(y, p)||_{L^\infty(Y)} \le c||u_0(y) - v(y, p)||_{L^2(Y)}e^{-\mu\tau} \quad \forall \tau \ge 1. \tag{4.16}$$

The proof of the first part of theorem 5, i.e. of the convergence result (4.15), is given is section 4.2 and relies strongly on the parabolic form of equation (4.14). Thus, no condition of nonlinearity on A is required. The same kind of result has been proved for hyperbolic scalar conservation laws under strict nonlinearity conditions (see [28], [48], [53] and the references therein). The second part of the theorem, i.e. the exponential decay result stated in (4.16), will be a straightforward consequence of one of the lemmas in section 4.3.

Combining the results of [14] and of theorem 5, we obtain the following homogenization result:

Theorem 6. *Let* $u_0 \in L^1_{loc}(\mathbb{R}^N; \mathcal{C}_{per}(Y))$ *such that there exist constants* $\beta_1, \beta_2 \in \mathbb{R}$ *such that*

$$v(y, \beta_1) \le u_0(x, y) \le v(y, \beta_2) \quad \text{for a.e. } x \in \mathbb{R}^N, y \in Y. \tag{4.17}$$

Assume that A *satisfies the hypotheses of theorem 5 and that* $\partial_v a_i(y, \cdot) \in \mathcal{C}(\mathbb{R})$ *for a.e.* $y \in Y$ *and for* $1 \le i \le N + 1$. *Then for all* $T > 0$, *for all* $R > 0$

$$\left|\left|u^\varepsilon(t, x) - v\left(\frac{x}{\varepsilon}, \bar{u}(t, x)\right)\right|\right|_{L^1((0,T)\times B_R)} \to 0 \quad \text{as } \varepsilon \to 0,$$

where $\bar{u}$ *is the solution of the homogenized problem* (4.12) *with initial data*

$$\bar{u}(t = 0, x) = \bar{u}_0(x) = \langle u_0(x, \cdot)\rangle .$$

Remark 4.1.1. *Notice that hypothesis* (4.13) *(or hypothesis* (4.17)*) is somehow a generalization of the well-preparedness hypothesis on the initial data in proposition 4.1.2. It implies in particular that* $u^0 \in L^\infty(\mathbb{R}^N \times Y)$. *Conversely, if* u^0 *is any function in* $L^\infty(\mathbb{R}^N \times Y)$, *and if*

$$\begin{aligned} &\lim_{p\to+\infty} \inf_Y v(y, p) = +\infty, \\ &\lim_{p\to-\infty} \sup_Y v(y, p) = -\infty, \end{aligned} \tag{4.18}$$

then we can always find constants $\beta_1, \beta_2 \in \mathbb{R}$ *such that* (4.17) *is satisfied. And* (4.18) *is always satisfied when* $a_i \in L^\infty(Y \times \mathbb{R})$ *for* $1 \le i \le N$ *(see [14]).*

However, when (4.18) *is not satisfied (see [14] for examples in which this condition is violated), we could not find more general hypothesis on the initial data. Basically, hypothesis* (4.17) *provides a subsolution and a supersolution of* (4.1) *which are bounded uniformly in* ε *in* $L^\infty((0, T) \times \mathbb{R}^N)$. *In particular, this implies uniform* L^∞ *bounds on* u^ε, *which are not easy to derive otherwise.*

4.2 Long time behavior of solutions of (4.14) - proof of (4.15)

This section is devoted to the first part of the proof of theorem 5; the proof uses Harnack's inequality and therefore relies strongly on the parabolic form of equation (4.14). Equation (4.14) is derived by means of a formal double-scale expansion in time and space variables :

$$u^\varepsilon(t,x) = u^0\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon u^1\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \cdots$$

Inserting this expansion in equation (4.1) and identifying the coefficient of ε^{-1} yields equation (4.14).

Before tackling the proof of theorem 5, let us recall a few facts about the solutions of equations (4.14) and (4.8):

1. $v(y,p) \geq v(y,p')$ for all $y \in Y$ for $p \geq p'$;
2. $v \in \mathcal{C}(\mathbb{R}_p, \mathcal{C}^{0,\gamma}(\overline{Y}))$ for some $\gamma \in (0,1)$;
3. If u_1, u_2 are solutions of (4.14), then for $0 \leq s \leq t$

$$||u_1(t) - u_2(t)||_{L^1(Y)} \leq ||u_1(s) - u_2(s)||_{L^1(Y)}, \tag{4.19}$$
$$||\left(u_1(t) - u_2(t)\right)_+||_{L^1(Y)} \leq ||\left(u_1(s) - u_2(s)\right)_+||_{L^1(Y)}; \tag{4.20}$$

4. If u is a solution of (4.14) with initial data satisfying (4.13), then u belongs to $L^\infty([0,\infty) \times Y)$ and $v(y,\beta_1) \leq u(\tau,y) \leq v(y,\beta_2)$ for all $\tau \geq 0,\ y \in Y$.

We skip the proof of these properties; the first two are proved in [14]. In particular, the second one follows from the fact that $v \in L^\infty_{\text{loc}}(\mathbb{R}_p; W^{2,q}(Y)) \cap \mathcal{C}(\mathbb{R}_p; H^1_{\text{per}}(Y))$ for all $q \in [1,\infty)$. (4.19) and (4.20) can be shown by Kruzkhov's method (see [64] for instance in the case of a hyperbolic homogeneous conservation law). The last property is a consequence of (4.20).

We are now ready to prove theorem 5 : define, for $y \in Y$, $t \in [0,\infty)$

$$U(t,y) := \sup_{\tau \geq t} u(\tau, y),$$
$$\text{and } p^*(t) := \inf\{p; v(y,p) \geq U(t,y) \text{ for a.e. } y \in Y\} \leq \beta_2 < \infty.$$

Then it is easily proved that

1. $U \in L^\infty([0,\infty) \times Y)$;
2. $U(t,y) \leq U(t',y)$ for $t \geq t'$ and for a.e. $y \in Y$;
3. $v(y, p^*(t)) \geq U(t,y)$ for all $t \in [0,\infty)$, for a.e. $y \in Y$;
4. $p^*(t)$ is a bounded non-increasing function of t; let $p^* := \lim_{t\to\infty} \downarrow p^*(t)$;

5. for all $t \in [0, \infty)$, if $p < p^*(t)$, then the set

$$E(p, t) := \{y \in Y; v(y, p) < U(t, y)\}$$

has positive measure.

Set $\delta > 0$; since v is a continous function of p, there exists $\eta > 0$ such that

$$|p - p^*| \leq \eta \Rightarrow ||v(\cdot, p) - v(\cdot, p^*)||_{L^\infty(Y)} \leq \delta;$$

take $t_0 \in \mathbb{R}$ such that if $t \geq t_0$, then $|p^* - p^*(t)| \leq \eta$, and take $p_\delta < p^*$ such that $|p_\delta - p^*| \leq \eta$. Then

$$||v(\cdot, p^*(t)) - v(\cdot, p_\delta)||_{L^\infty(Y)} \leq 2\delta \quad \forall t \geq t_0$$

and the set $E(p_\delta, t)$ has positive measure for all $t \geq 0$. Hence, for $t \geq t_0$, for $y \in E(p_\delta, t)$, we have

$$v(y, p^*(t)) - 2\delta \leq v(y, p_\delta) \leq U(t, y) \leq v(y, p^*(t)). \tag{4.21}$$

Now, take any sequence $t_n \to \infty$, and for all $n \in \mathbb{N}$ choose $y_n^\delta \in E(p_\delta, t_n + 1)$; there exists $\tau_n^\delta \geq t_n + 1$ such that

$$\left|u(\tau_n^\delta, y_n^\delta) - U(t_n + 1, y_n^\delta)\right| \leq \delta; \tag{4.22}$$

then for n large enough, $t_n \geq t_0$ and gathering (4.21) and (4.22) leads us to

$$v(y_n^\delta, p^*(t_n)) - 2\delta \leq v(y_n^\delta, p_\delta) \leq U(t_n + 1, y_n^\delta) \leq v(y_n^\delta, p^*(t_n + 1)) \leq v(y_n^\delta, p^*(t_n)),$$
$$\left|u(\tau_n^\delta, y_n^\delta) - v(y_n^\delta, p^*(t_n))\right| \leq 3\delta.$$

Set, for $s \in [-1, 1]$ and $y \in Y$,

$$w_n^\delta(s, y) := v(y, p^*(t_n)) - u(\tau_n^\delta + s, y);$$

since $\tau_n^\delta \geq t_n + 1$, according to the definition of $p^*(t_n)$ and to that of $U(t_n)$, w_n^δ is a nonnegative function for all $n \in \mathbb{N}, \delta > 0$. Moreover, thanks to our preliminary analysis, for n large enough,

$$w_n^\delta(s = 0, y_n^\delta) \leq 3\delta.$$

w_n^δ is therefore a nonnegative solution of the parabolic equation

$$\frac{\partial w_n^\delta}{\partial s} + \sum_{i=1}^N \frac{\partial}{\partial y_i}\left(b_i^{n,\delta}(s, y) w_n^\delta(s, y)\right) - \Delta_y w_n^\delta = 0,$$

where

$$b_i^{n,\delta}(s, y) := \int_0^1 a_i(y, \tau v(y, p^*(t_n)) + (1 - \tau) u(\tau_n^\delta + s, y))\, d\tau.$$

Let $K > 0$ such that

$$-K \leq v(y, \beta_1) \leq v(y, \beta_2) \leq K \quad \forall y \in Y.$$

Then for $1 \leq i \leq N$,

$$||b_i^{n,\delta}||_{L^\infty((-1,1)\times Y)} \leq ||a_i||_{L^\infty(Y\times(-K,K))} \quad \forall n \in \mathbb{N}\ \forall \delta > 0.$$

According to Harnack's inequality, there exists a constant C depending only on Y and $||a||_{L^\infty(Y\times(-K,K))^N}$ such that for n large enough

$$\sup_{y\in Y} w_n^\delta(-\frac{1}{2}, y) \leq C \inf_{y\in Y} w_n^\delta(0, y). \tag{4.23}$$

Hence, we have proved that for n large enough,

$$0 \leq v(y, p^*(t_n)) - u(\tau_n^\delta - \frac{1}{2}, y) \leq C\delta \quad \forall n \in \mathbb{N}\ \forall y \in Y,$$

But as $n \to \infty$

$$v(y, p^*(t_n)) \to v(y, p^*) \quad \text{in } L^\infty(Y).$$

Thus there exists a sequence $T_n \to \infty$ such that

$$||u(T_n, y) - v(y, p^*)||_{L^\infty(Y)} \to 0. \tag{4.24}$$

There remains to prove that $p^* = \langle u_0 \rangle$ and that

$$u(t, \cdot) \to v(\cdot, p^*) \quad \text{as } t \to \infty$$

in $L^\infty(Y)$; since $\int_Y u(t)$ is conserved by the equation, as $n \to \infty$

$$\langle u(T_n) \rangle = \langle u_0 \rangle \to \langle v(\cdot, p^*) \rangle = p^*,$$

and $p^* = \langle u_0 \rangle$. Using the L^1 contraction property (4.19) for equation (4.14) with $u_1 = u$, $u_2 = v(y, p^*)$ (u_2 is a stationary solution of (4.14)), $s = T_n$ and $t \geq T_n$ shows that

$$u(t, \cdot) \to v(\cdot, p^*) \quad \text{in } L^1(Y).$$

The strong convergence in L^∞ norm follows from parabolic regularity. □

4.3 Homogenization of equation (4.1) in case of ill-prepared initial data

The proof of theorem 6 is rather technical, but the ideas involved are quite simple. Our guess is that u^ε behaves in L^1_{loc} as

$$z^\varepsilon(t, x) := w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) + v^\varepsilon(t, x)$$

where

- $w(\cdot, X, \cdot)$ is the solution of (4.14) with periodic initial data $u_0(X, \cdot)$ ($X \in \mathbb{R}^N$ being treated as a parameter),

- $\bar{u}_0(X) := \langle u_0(X,\cdot)\rangle$ for $X \in \mathbb{R}^N$,
- $v^\varepsilon(t,x)$ is the solution of (4.1) with initial data $v^\varepsilon(t=0,x) = v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$.

Let us point out the roles of the different terms in z^ε : for small times, v^ε is close to $v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)$, and thus $z^\varepsilon \approx w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) \approx w\left(\tau = 0, x, \frac{x}{\varepsilon}\right)$. On the contrary, on time scales which are large with respect to ε, $w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) = \mathcal{O}(e^{-\frac{\mu t}{\varepsilon}})$, and thus $z^\varepsilon \approx v^\varepsilon$. But according to proposition 4.1.2, v^ε behaves in L^2_{loc} as $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$, where $\bar{u} = \bar{u}(t,x)$ is the solution of the homogenized problem (4.12) with initial data $\bar{u}_0$ (in proposition 4.1.2, this result is stated for $\bar{u}_0 \in L^1 \cap L^\infty$, but it can be generalized easily to $\bar{u}_0 \in L^\infty$ as explained in paragraph 3.2.1). In other words, there is an initial layer of size ε which is described by $w\left(\frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right)$, and after which z^ε behaves as $v\left(\frac{x}{\varepsilon}, \bar{u}(t,x)\right)$.

In order to prove that $u^\varepsilon - z^\varepsilon$ goes to 0 in $L^1_{\text{loc}}((0,+\infty)\times\mathbb{R}^N)$, we could compute

$$f^\varepsilon := \frac{\partial z^\varepsilon}{\partial t}(t,x) + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, z^\varepsilon(t,x)\right) - \varepsilon\Delta z^\varepsilon$$

and prove that f^ε goes to 0 in $L^1_{\text{loc}}((0,+\infty)\times\mathbb{R}^N)$ (notice that $u^\varepsilon(t=0) = z^\varepsilon(t=0)$). However, this involves rather heavy and unnecessary calculations. Instead, we notice that since u^ε and v^ε are both solutions of (4.1), we can use the L^1 contraction principle:

$$\partial_t|u^\varepsilon - v^\varepsilon| + \sum_{i=1}^N \frac{\partial}{\partial x_i}\eta_i\left(\frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon\right) - \varepsilon\Delta_x|u^\varepsilon - v^\varepsilon| \le 0, \tag{4.25}$$

where

$$\eta_i(y,v,w) := \operatorname{sgn}(v-w)\left[A_i(y,v) - A_i(y,w)\right].$$

First, let us prove that u^ε and v^ε are uniformly bounded in L^∞: since equation (4.1) is order preserving, for all $t,\tau \ge 0$, for a.e. $x \in \mathbb{R}^N$, $y \in Y$ we have

$$v\left(\frac{x}{\varepsilon}, \beta_1\right) \le u^\varepsilon(t,x) \le v\left(\frac{x}{\varepsilon}, \beta_2\right) \quad \forall t \ge 0 \text{ for a.e. } x \in \mathbb{R}^N,$$
$$\beta_1 \le \bar{u}_0(x) \le \beta_2 \quad \text{for a.e. } x \in \mathbb{R}^N,$$
$$v\left(\frac{x}{\varepsilon}, \beta_1\right) \le v^\varepsilon(t,x) \le v\left(\frac{x}{\varepsilon}, \beta_2\right) \quad \forall t \ge 0 \text{ for a.e. } x \in \mathbb{R}^N.$$

Consequently, there exists a constant $K > 0$ depending only on $\beta_1, \beta_2, N, Y, n, m$ such that

$$|u^\varepsilon(t,x)| + |v^\varepsilon(t,x)| \le K \quad \forall t \ge 0 \text{ for a.e. } x \in \mathbb{R}^N.$$

Now, let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $\varphi(x) = e^{-|x|}$ when $|x| \ge 1$, and $\frac{1}{e} \le \varphi(x) \le 1$ for $|x| \le 1$. Notice that thanks to the L^∞ bound, it is enough to prove that

$$\int_\alpha^T \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - v^\varepsilon(t,x)|\ \varphi(x)\, dx\, dt \to 0$$

as $\varepsilon \to 0$ for all $0 < \alpha < T$ (this fact will be explained more thoroughly in paragraph 3.2.1). Moreover, there exists a constant C such that

$$|\nabla_x\varphi(x)|, |\Delta_x\varphi(x)| \le C\varphi(x) \quad \forall x \in \mathbb{R}^N.$$

Hence, multiplying (4.25) by $\varphi(x)$ and integrating on $(\alpha, T) \times \mathbb{R}^N$ with $0 < \alpha < T$ arbitrary yields

$$\begin{aligned}\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\ dx \ \leq\ & \int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx \\ & + C\int_\alpha^T \int_{\mathbb{R}^N} \left|\eta\left(\frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon\right)\right| \varphi(x)\ dx\, dt \\ & + \varepsilon C \int_\alpha^T \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - v^\varepsilon(t,x)|\varphi(x)\ dx\, dt\end{aligned}$$

Thanks to the L^∞ bound, we deduce that

$$\left|\eta_i\left(\frac{x}{\varepsilon}, u^\varepsilon, v^\varepsilon\right)\right| \leq ||a_i||_{L^\infty(Y\times(-K,K))}|u^\varepsilon - v^\varepsilon|$$

and thus for all $\varepsilon \in (0,1)$

$$\begin{aligned}\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\ dx \ \leq\ & \int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx \\ & + C\int_\alpha^T \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - v^\varepsilon(t,x)|\varphi(x)\ dx\, dt\end{aligned}$$

where the constant C depends only on $||a_i||_{L^\infty(Y\times(-K,K))}$. Using Gronwall's lemma, we deduce that

$$\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\ dx \leq e^{C(T-\alpha)} \int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx.$$

On the other hand, for all $R > 1$, we have

$$\int_{\mathbb{R}^N} |u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)|\varphi(x)\ dx \leq ||u^\varepsilon(\alpha) - v^\varepsilon(\alpha)||_{L^1(B_R)} + Ce^{-R},$$

where C depends only on N and K.

Thus, for all $R > 1$

$$\int_{\mathbb{R}^N} |u^\varepsilon(T,x) - v^\varepsilon(T,x)|\varphi(x)\, dx \leq e^{C(T-\alpha)}\left[||u^\varepsilon(\alpha) - v^\varepsilon(\alpha)||_{L^1(B_R)} + Ce^{-R}\right]. \quad (4.26)$$

It remains to prove that for all $R > 0$, we can choose ε and α small enough so that $||u^\varepsilon(\alpha) - v^\varepsilon(\alpha)||_{L^1(B_R)}$ is arbitrarily small. But for small α, our guess is that

$u^\varepsilon(\alpha, x)$ behaves as $w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right)$. If we follow this intuition, we are led to

$$\begin{aligned}
||u^\varepsilon(\alpha,x) - v^\varepsilon(\alpha,x)||_{L^1(B_R)} \leq\ & \left|\left|u^\varepsilon(\alpha,x) - w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_R)} \\
& + \left|\left|w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right)\right|\right|_{L^1(B_R)} \\
& + \left|\left|v\left(\frac{x}{\varepsilon}, \bar{u}_0(x)\right) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)} \\
& + \left|\left|v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right) - v^\varepsilon(\alpha,x)\right|\right|_{L^1(B_R)} \\
\leq\ & \left|\left|u^\varepsilon(\alpha,x) - w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_R)} \\
& + \left|\left|w\left(\frac{\alpha}{\varepsilon}, x, y\right) - v\left(y, \bar{u}_0(x)\right)\right|\right|_{L^1(B_R;\mathcal{C}_{\rm per}(Y))} \\
& + ||v\left(y, \bar{u}_0(x)\right) - v\left(y, \bar{u}(\alpha,x)\right)||_{L^1(B_R;\mathcal{C}_{\rm per}(Y))} \\
& + \left|\left|v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha,x)\right) - v^\varepsilon(\alpha,x)\right|\right|_{L^1(B_R)}.
\end{aligned}$$

In the next subsections we prove that each of the terms of the right-hand side of the above inequality goes to 0 as $\alpha \to 0$ and $\varepsilon \to 0$.

We have used above the following proposition, due to Allaire ([3]):

Proposition 4.3.1. *Let Ω be an open set of $\mathbb{R}^N$.*

Let $\psi(x,y) \in L^1(\Omega; \mathcal{C}_{per}(Y))$. Then, for any positive value of ε, $\psi\left(x, \frac{x}{\varepsilon}\right)$ is a measurable function on Ω such that

$$\left|\left|\psi\left(x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(\Omega)} \leq ||\psi(x,y)||_{L^1(\Omega;\mathcal{C}_{per}(Y))} := \int_\Omega \sup_{y\in Y} |\psi(x,y)|\, dx.$$

4.3.1 Preliminary lemmas

First of all, let us recall the following result, which is the basis of all our analysis:

Lemma 4.3.1. *Let $b \in L^\infty(Y)^N$, $v_0 \in L^2(Y)$ with $\langle v_0 \rangle = 0$ and let $v = v(\tau, y) \in L^2_{loc}(0,\infty; H^1_{per}(Y))$ be the solution of*

$$\begin{cases} \partial_\tau v + \mathrm{div}_y(bv) - \Delta_y v = 0, \\ v(\tau = 0) = v_0. \end{cases} \tag{4.27}$$

Then there exist $\mu, C > 0$, depending only on N, Y and $||b||_{L^\infty}$, such that

$$\begin{aligned} ||v(\tau)||_{L^\infty} \leq C||v_0||_{L^2} e^{-\mu\tau} \quad \forall \tau \geq 1, \\ ||v(\tau)||_{L^2} \leq C||v_0||_{L^2} e^{-\mu\tau} \quad \forall \tau \geq 0. \end{aligned}$$

Proof. This result is linked to the existence of gaps in the spectrum of the operator

$$Lw := -\Delta w + \mathrm{div}(bw).$$

We use the result stated in lemma 1.1 of [55] (see also the references therein). Let π be the invariant probability measure associated to the equation, i.e. π is the solution of

$$-\Delta_y \pi + \operatorname{div}_y(b\pi) = 0, \quad \langle \pi \rangle = 1.$$

$\pi \in H^1_{\text{per}}(Y)$ exists and is unique, and there exists $\alpha > 0$ such that $\pi \geq \alpha$ by the Krein-Rutman theorem, and α depends only on Y, N and $||b||_\infty$.

Now, according to [55] we have for any $\mathcal{C}^2$ function $H : \mathbb{R} \to \mathbb{R}$

$$\partial_t \left[\pi H\left(\frac{v}{\pi}\right)\right] + \operatorname{div}_y \left[b\pi H\left(\frac{v}{\pi}\right)\right] - \frac{\partial}{\partial x_i}\left\{\pi^2 \frac{\partial}{\partial x_i}\left[\frac{1}{\pi} H\left(\frac{v}{\pi}\right)\right]\right\} = -\pi H''\left(\frac{v}{\pi}\right)\left|\nabla\left(\frac{v}{\pi}\right)\right|^2.$$

Take now $H(p) = \frac{1}{2}|p|^2$. Integrating this equality on Y yields

$$\frac{1}{2}\frac{d}{dt}\int_Y \pi \left|\frac{v}{\pi}\right|^2 = -\int_Y \pi \left|\nabla\left(\frac{v}{\pi}\right)\right|^2.$$

But according to the Poincaré inequality for the measure π, there exists a constant $\nu > 0$ such that for any $w \in H^1_{\text{per}}(Y)$,

$$\nu \int_Y \pi |w - \langle w \rangle_\pi|^2 \leq \int_Y \pi |\nabla w|^2,$$

where we have used the notation

$$\langle w \rangle_\pi := \frac{\int_Y w(y)\pi(y)\,dy}{\int_Y \pi(y)\,dy}.$$

(this inequality can be proved exactly along the same lines as the usual Poincaré inequality, for which $\pi \equiv 1$.) Hence

$$\frac{1}{2}\frac{d}{dt}\int_Y \pi \left|\frac{v}{\pi}\right|^2 \leq -\nu \int_Y \pi \left|\frac{v}{\pi} - \left\langle \frac{v}{\pi}\right\rangle_\pi\right|^2.$$

And since

$$\left\langle \frac{v}{\pi}\right\rangle_\pi = \frac{\int_Y v}{\int_Y \pi} = 0$$

we deduce that

$$\int_Y \pi \left|\frac{v}{\pi}\right|^2 \leq e^{-2\nu t}\int_Y \pi \left|\frac{v_0}{\pi}\right|^2$$

and we obtain the result announced in lemma 4.3.1. The exponential convergence in L^∞ norm follows from parabolic regularity.

□

Remark 4.3.1. *For the reader's convenience, we provide here a short proof of the parabolic regularity result used here, for which we have failed to find an explicit reference. Assume that $u = u(\tau, y)$ is a solution of*

$$\partial_\tau u + \operatorname{div}_y(bu) - \Delta_y u = 0, \tag{4.28}$$

where $b = b(\tau, y) \in L^\infty((0,\infty) \times Y)^N$ (we allow b to depend on time). Then there exists a constant $C > 0$, depending only on $||b||_{L^\infty}$, N and Y, such that for all $\tau \geq 0$,

$$||u(\tau+1)||_{L^\infty(Y)} \leq C||u(\tau)||_{L^1(Y)}. \tag{4.29}$$

Indeed, if u is a nonnegative solution of (4.28), then (4.29) follows from Harnack's inequality: there exists a constant C depending only on $||b||_{L^\infty}$, N and Y, such that for all $\tau \geq 0$

$$\max_{y\in Y} u(\tau+1, y) \leq C \min_{y \in Y} u(\tau+2, y).$$

Thus

$$||u(\tau+1)||_{L^\infty(Y)} \leq C \langle u(\tau+2)\rangle = C\langle u(\tau)\rangle = \frac{C}{|Y|}||u(\tau)||_{L^1(Y)}.$$

Hence (4.29) is proved for nonnegative solutions of (4.28). Now, if u is an arbitrary solution of (4.28) and $\tau_0 \geq 0$, we write $u(\tau = \tau_0) =: u_0 = (u_0)_+ - (u_0)_-$, with $a_+ := \max(a, 0)$ for any $a \in \mathbb{R}$, and $a_- = (-a)_+$. We denote by u^+, u^- the solutions of (4.28) for $\tau \geq \tau_0$ corresponding to initial data $u^+(\tau = \tau_0) = u_0^+$, $u^-(\tau = \tau_0) = u_0^-$. Then u^+ and u^- are nonnegative solutions of (4.28) and $u = u^+ - u^-$ for $\tau \geq \tau_0$. Thus

$$\max_Y u(\tau_0+1) \leq \max_Y u^+(\tau_0+1) \leq C\int_Y u^+(\tau_0),$$
$$\inf_Y u(\tau_0+1) \geq -\max_Y u^-(\tau_0+1) \geq -C\int_Y u^-(\tau_0)$$

and consequently

$$||u(\tau_0+1)||_{L^\infty} \leq C\int_Y (u_0^+ + u_0^-) \leq C\int_Y |u(\tau_0)|$$

which is the desired inequality for an arbitrary solution of (4.28).

We now generalize lemma 4.3.1 to the case when the coefficients might depend on the time variable t and on a parameter $x \in \mathbb{R}^N$. We wish to emphasize that the results stated in the two next lemmas are not optimal, but merely adapted to the problem addressed here.

Lemma 4.3.2. *Let $R > 0$.*

Let $b = b(\tau, x, y) \in L^\infty((0,\infty) \times B_R \times Y)^N$ such that $\mathrm{div}_y b \in L^\infty((0,\infty) \times B_R; L^2(Y))$, $v_0 = v_0(x,y) \in L^\infty(B_R; L^2(Y))$ with $\langle v_0(x,\cdot)\rangle = 0$ for almost every $x \in B_R$.

Let $v = v(\tau, x, y) \in L^\infty(B_R; L^2_{loc}(0,\infty; H^1_{per}(Y)))$ be the solution of

$$\begin{cases} \partial_\tau v + \mathrm{div}_y(bv) - \Delta_y v = 0, \\ v(\tau = 0) = v_0. \end{cases} \tag{4.30}$$

(x is a parameter of the above equation.)

Assume that there exists $b_\infty \in L^\infty(B_R; L^\infty(Y))$ with $\mathrm{div}_y b_\infty \in L^\infty(B_R; L^2(Y))$ such that as $\tau \to \infty$

$$||b(\tau) - b_\infty||_{L^\infty(B_R;L^\infty(Y))} \to 0,$$
$$||\mathrm{div}_y b(\tau) - \mathrm{div}_y b_\infty||_{L^\infty(B_R;L^2(Y))} \to 0.$$

Then there exists a constant C depending only on N, Y and the bounds on b and b_∞ such that for all $\tau \geq 1$,

$$||v||_{L^\infty(B_R;L^2(\tau,\tau+1;H^2(Y)))} + ||v(\tau)||_{L^\infty(B_R;H^1(Y))} \leq C||v(\tau-1)||_{L^\infty(B_R;L^2(Y))}. \quad (4.31)$$

Moreover, there exists a constant C depending on N, Y, b and b_∞, and a constant $\mu > 0$ depending only on N, Y, and $||b_\infty||_{L^\infty(B_R\times Y)}$ such that

$$||v(\tau)||_{L^\infty(B_R;L^\infty(Y))} \leq C||v_0||_{L^\infty(B_R;L^2(Y))}e^{-\mu\tau} \quad \forall\tau \geq 1,$$
$$||v(\tau)||_{L^\infty(B_R;L^2(Y))} \leq C||v_0||_{L^\infty(B_R;L^2(Y))}e^{-\mu\tau} \quad \forall\tau \geq 0.$$

Proof. Let us treat x as a parameter. Let $L_\infty(x)$ be the differential operator

$$L_\infty(x)w(y) := -\Delta_y w(y) + \mathrm{div}_y(b_\infty(x,y)w(y));$$

then v satisfies the equation

$$\partial_\tau v + L_\infty(x)v = f(\tau,x,y),$$

where $f(\tau,x,y) = \mathrm{div}_y((b_\infty(x,y) - b(\tau,x,y))v(\tau,x,y))$.

According to standard regularity results on parabolic equations (see for instance [32]), we have on the one hand, for all $T > 0$ and for a.e. $x \in B_R$

$$\sup_{T\leq t\leq T+\frac{3}{2}} ||v(t,x)||_{H^1} + ||v(x)||_{L^2(T,T+\frac{3}{2};H^2(Y))}$$
$$\leq C\left(||f(x)||_{L^2((T,T+\frac{3}{2})\times Y)} + ||v(T,x))||_{H^1}\right),$$

and on the other hand, for all $\tau \geq \frac{1}{2}$,

$$\sup_{\tau-\frac{1}{2}\leq t\leq\tau+\frac{1}{2}} ||v(t,x)||_{L^2(Y)} + ||v(x)||_{L^2(\tau-\frac{1}{2},\tau+\frac{1}{2};H^1(Y))} \leq C\left|\left|v\left(\tau-\frac{1}{2},x\right)\right|\right|_{L^2},$$

where C is a constant which depends only on N, Y and the bounds on b and b_∞.

For a.e. $x \in B_R$, we choose $T \in [\tau-\frac{1}{2},\tau]$ such that

$$||v(T)||_{H^1} \leq \sqrt{2}||v(x)||_{L^2(\tau-\frac{1}{2},\tau;H^1(Y))}$$

and we evaluate

$$||f(x)||_{L^2((T,T+\frac{3}{2})\times Y)} \leq C\left[||v(x)||_{L^2(\tau-\frac{1}{2},\tau+\frac{3}{2};H^1)} + ||v(x)||_{L^\infty((\tau-\frac{1}{2},\tau+\frac{3}{2})\times Y)}\right].$$

Always thanks to parabolic regularity, there exists a constant C (depending only on N, Y and $||b||_{L^\infty}$) such that for all $\tau \geq 1$

$$||v(x)||_{L^\infty((\tau-\frac{1}{2},\tau+\frac{3}{2})\times Y)} \leq C||v(\tau-1,x)||_{L^2(Y)}.$$

Gathering all the terms, we obtain inequality (4.31) (notice that $[\tau,\tau+1] \subset [T,T+\frac{3}{2}]$).

Let us now prove the exponential convergence: let $U_\infty(\tau; x)$ be the evolution operator associated to the equation $\partial_\tau w + L_\infty(x)w = 0$, that is, $U_\infty(\tau; x)w_0 = w(\tau, y; x)$, where $w(\cdot; x) \in L^2_{\text{loc}}(0, \infty; H^1_{\text{per}}(Y))$ is the solution of the system

$$\begin{cases} \partial_\tau w + \text{div}_y(b_\infty w) - \Delta_y w = 0, \\ w(\tau = 0) = w_0. \end{cases}$$

According to lemma 4.3.1, for all $w \in L^2(Y)$ such that $\langle w \rangle = 0$ and for almost every $x \in B_R$

$$||U_\infty(\tau; x)w||_{L^2} \leq C||w||_{L^2} e^{-\mu\tau} \quad \forall \tau \geq 0,$$

where C and μ are constants which depend only on N,Y, and the bounds on b_∞.

We use Duhamel's formula: for all $\tau > 0$, $x \in B_R$

$$v(\tau, x) = U_\infty(\tau; x)v_0 + \int_0^\tau U_\infty(\tau - \sigma; x) f(\sigma, x)\, d\sigma.$$

Notice that f has mean value zero; consequently,

$$||v(\tau, x)||_{L^2} \leq Ce^{-\mu\tau}||v_0(x)||_{L^2} + \int_0^\tau e^{-\mu(\tau-\sigma)}||f(\sigma, x)||_{L^2}\, d\sigma.$$

We bound $||f(\sigma, x)||_{L^2}$ by

$$\begin{aligned} &||b(\sigma) - b_\infty||_{L^\infty(Y)}||v(\sigma)||_{H^1} + ||\text{div}_y b(\sigma) - \text{div}_y b_\infty||_{L^2(Y)}||v(\sigma)||_{L^\infty} \\ \leq\ & C\left(||b(\sigma) - b_\infty||_{L^\infty(Y)} + ||\text{div}_y b(\sigma) - \text{div}_y b_\infty||_{L^2(Y)}\right)||v(\sigma - 1)||_{L^2}. \end{aligned}$$

Let $\delta > 0$ arbitrary. There exist $\sigma_\delta > 0$ such that if $\sigma \geq \sigma_\delta$, then for a.e. $x \in B_R$

$$||b(\sigma, x) - b_\infty(x)||_{L^\infty(Y)} + ||\text{div}_y b(\sigma, x) - \text{div}_y b_\infty(x)||_{L^2(Y)} \leq \delta.$$

Hence,

$$\begin{aligned} ||v(\tau, x)||_{L^2} \ \leq\ & Ce^{-\mu\tau}||v_0(x)||_{L^2} + C\int_0^{\sigma_\delta} e^{-\mu(\tau-\sigma)}||v(\sigma, x)||_{H^1} d\sigma \\ & + C\delta \int_{\sigma_\delta}^\tau e^{-\mu(\tau-\sigma)}||v(\sigma - 1, x)||_{L^2} d\sigma. \end{aligned}$$

Returning to equation (4.30), it is easily proved that

$$\int_0^{\sigma_\delta} ||\nabla_y v(\sigma, x)||^2_{L^2}\, d\sigma \leq e^{||b||^2_\infty \sigma_\delta}||v_0(x)||^2_{L^2}.$$

(Other bounds are possible).

We are eventually led to

$$||v(\tau, x)||_{L^2} \leq C_\delta e^{-\mu\tau}||v_0(x)||_{L^2} + C\delta \int_{\sigma_\delta - 1}^\tau e^{-\mu(\tau-\sigma)}||v(\sigma, x)||_{L^2}\, d\sigma,$$

where the constants C and C_δ depend on N, Y, b, b_∞, and the constant C_δ also depends on δ. We use Gronwall's lemma and we find

$$||v(\tau, x)||_{L^2} \leq C_\delta||v_0(x)||_{L^2} e^{-(\mu - C\delta)\tau}.$$

We take $\delta = \frac{\mu}{2C}$ and use once more parabolic regularity; since the inequality is true for almost every $x \in B_R$, the lemma is proved.

□

We now prove a third lemma which will be needed in the course of the proof.

Lemma 4.3.3. *Let* $b = b(\tau, x, y) \in L^\infty_{loc}(\mathbb{R}^N; L^\infty((0,\infty) \times Y))^N$, $b_\infty = b_\infty(x, y) \in L^\infty_{loc}(\mathbb{R}^N; L^\infty(Y))^N$ *two vector fields satisfying the same hypotheses as in lemma 4.3.2 for any* $R > 0$.

Let $f = f(\tau, x, y) \in L^\infty_{loc}((0,\infty) \times \mathbb{R}^N, L^2(Y))$, *and* $v_0 \in L^\infty_{loc}(\mathbb{R}^N, L^\infty(Y))$. *Assume that* $\langle f(\tau, x, \cdot)\rangle = \langle v_0(x, \cdot)\rangle = 0$ *for almost every* $(\tau, x) \in (0,\infty) \times \mathbb{R}^N$, *and that for all* $R > 0$

$$||f||_{L^\infty((0,\infty)\times B_R; L^2(Y))} < \infty \tag{4.32}$$

Let $v = v(\tau, x, y)$ *be the solution of*

$$\begin{cases} \partial_\tau v(\tau, x, y) + \mathrm{div}_y(b(\tau, x, y)v(\tau, x, y)) - \Delta_y v(\tau, x, y) = f(\tau, x, y), \\ v(\tau = 0, x, y) = v_0(x, y). \end{cases} \tag{4.33}$$

Then for all $R > 0$, *there exists a constant* C_R *depending on* N, Y, b, b_∞, *and* $||f||_{L^\infty((0,\infty)\times B_R; L^2(Y))}$, *such that*

$$||v(\tau)||_{L^\infty(B_R; L^2(Y))} \le C_R \left(1 + ||v_0||_{L^\infty(B_R, L^2(Y))}\right) \quad \forall \tau \ge 0.$$

Moreover, if f *can be written*

$$f = \sum_{i=1}^N \partial_{y_i} f_i,$$

with $f_i \in L^\infty((0, +\infty) \times B_R \times Y)$ *for all* $R > 0$ *and for* $1 \le i \le N$, *then for all* $R > 0$, *there exists a constant* C_R *depending only on* N, Y, $\max_{1\le i\le N} ||f_i||_{L^\infty((0,+\infty)\times B_R\times Y)}$, b *and* b_∞ *such that*

$$||v(\tau)||_{L^\infty(B_R; L^\infty(Y))} \le C_R \left(1 + ||v_0||_{L^\infty(B_R, L^2(Y))}\right) \quad \forall \tau \ge 1, \tag{4.34}$$

$$||v(\tau)||_{L^\infty(B_R; L^\infty(Y))} \le C_R \left(1 + ||v_0||_{L^\infty(B_R, L^\infty(Y))}\right) \quad \forall \tau \le 1. \tag{4.35}$$

Proof. Let $U(\tau, \sigma; x)$ be the evolution operator associated to the equation

$$\partial_\tau w + \mathrm{div}_y(bw) - \Delta_y w = 0$$

(x is still treated as a parameter). In other words, for any $\tau \ge \sigma \ge 0$, $\varphi \in L^2(Y)$, $w(\tau, x, y) := U(\tau, \sigma; x)\varphi$ satisfies

$$\begin{cases} \partial_\tau w + \mathrm{div}_y(b(\tau)w) - \Delta_y w = 0 \quad \text{for } \tau > \sigma, \\ w(\tau = \sigma, x, y) = \varphi(y). \end{cases}$$

In lemma 4.3.2, we have proved that for every $R > 0$, there exists $c_R, \mu_R > 0$ such that if $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; L^2(Y))$ satisfies $\langle \varphi(x, \cdot)\rangle = 0$ for almost every x, then

$$||U(\tau, \sigma; x)\varphi||_{L^\infty(B_R; L^\infty(Y))} \le c_R ||\varphi||_{L^\infty(B_R; L^2(Y))} e^{-\mu_R(\tau-\sigma)} \quad \forall\, \tau \ge \sigma + 1 \ge 1,$$

$$||U(\tau, \sigma; x)\varphi||_{L^\infty(B_R; L^2(Y))} \le c_R ||\varphi||_{L^\infty(B_R; L^2(Y))} e^{-\mu_R(\tau-\sigma)} \quad \forall\, \tau \ge \sigma \ge 0.$$

And by Duhamel's formula, we also have

$$v(\tau, x, y) = U(\tau, 0; x)v_0(x, y) + \int_0^\tau U(\tau, \sigma; x) f(\sigma, x, y)\, d\sigma.$$

Thus, for any $R > 0$, for all $\tau \geq 0$,

$$\begin{aligned} ||v(\tau)||_{L^\infty(B_R;L^2(Y))} &\leq c_R e^{-\mu_R \tau} ||v_0||_{L^\infty(B_R;L^2(Y))} + c_R \int_0^\tau e^{-\mu_R(\tau-\sigma)}\, d\sigma \\ &\leq c_R \left(||v_0||_{L^\infty(B_R;L^2(Y))} + 1 \right). \end{aligned}$$

Inequalities (4.34) and (4.35) are direct consequences of theorems 8.1 and 7.1 in chapter III of [46]. □

4.3.2 Homogenization

We now apply the lemmas of the preceding subsection to the function $r(\tau, x, y) = w(\tau, x, y) - v(y, \bar{u}_0(x))$ and its derivatives. The proof is divided into six steps: in the first step, we restrict ourselves to smooth initial datas. In the second step, we prove that r converges towards 0 in $L^\infty_{\text{loc}}(\mathbb{R}^N; L^\infty(Y))$. In the third step, we prove that r converges to 0 exponentially fast, thanks to the second step and lemma 4.3.2. In the fourth step, we focus on what happens for small times : precisely, we derive a bound on $w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - u^\varepsilon(\alpha, x)$. This step is quite long because bounds on the derivatives of r are required. The fifth step is also concerned with the behavior of the solutions of (4.1) and (4.12) for small times, but is much shorter and less involved. Eventually, in the sixth and last step, we gather all the bounds derived in the previous steps and we prove the convergence result announced in theorem 6.

A Restriction to smooth initial datas

As often in the study of conservation laws, or more generally, of evolution problems which admit a contraction principle, it is enough to prove the result for smooth initial datas: indeed, choose, by density, a family of functions u_0^δ such that $u_0^\delta \in \mathcal{C}^\infty_{\text{per}}(\mathbb{R}^N \times Y) \cap L^1_{\text{loc}}(\mathbb{R}^N; \mathcal{C}_{\text{per}}(Y))$, with the following properties:

$$\begin{gathered} \left|\left|u_0 - u_0^\delta\right|\right|_{L^1(B_R;\mathcal{C}_{\text{per}(Y)})} \to 0 \quad \text{as } \delta \to 0 \quad \forall R > 0, \\ v(y, \beta_1 - 1) \leq u_0^\delta(x, y) \leq v(y, \beta_2 + 1) \quad \forall \delta > 0\ \forall (x, y) \in \mathbb{R}^N \times Y, \\ \bar{u}_0^\delta(x) := \left\langle u_0^\delta(x, \cdot) \right\rangle \in L^1(\mathbb{R}^N). \end{gathered}$$

We denote by $u_\delta^\varepsilon(t, x)$, $v_\delta^\varepsilon(t, x)$ the solutions of (4.1) corresponding to initial datas $u_0^\delta\left(x, \frac{x}{\varepsilon}\right)$, $v\left(\frac{x}{\varepsilon}, \bar{u}_0^\delta(x)\right)$ respectively, and by $\bar{u}_\delta$ the entropy solution of (4.12) with initial data $\bar{u}_0^\delta$. Then there exists a constant $K' \geq K$ depending only on N, Y, β_1, β_2, m, n and C_0 such that

$$|u_\delta^\varepsilon(t, x)|, |v_\delta^\varepsilon(t, x)| \leq K' \quad \forall (\tau, x, y) \in [0, \infty) \times \mathbb{R}^N \times Y,\ \forall \varepsilon, \delta > 0$$

and

$$-K' \leq v(y, p) \leq K' \quad \forall y \in Y\ \forall p \in [\beta_1 - 1, \beta_2 + 1].$$

Hence the following inequalities hold, according to the L^1 contraction principle for equation (4.1)

$$||u^\varepsilon(T) - u^\varepsilon_\delta(T)||_{L^1(B_R)} \leq e^{\frac{CT}{R}}\left(e^{-CR} + ||u_0 - u_0^\delta||_{L^1(B_R;\mathcal{C}_{\rm per}(Y))}\right) \tag{4.36}$$

$$||\bar u(T) - \bar u_\delta(T)||_{L^1(B_R)} \leq e^{\frac{CT}{R}}\left(e^{-CR} + ||\bar u_0 - \bar u_0^\delta||_{L^1(B_R)}\right) \tag{4.37}$$

for all $R>0$, $T>0$, and for some constant C depending only on $||a||_{L^\infty(Y\times(-K',K'))}$.

Assume that we can prove that

$$u^\varepsilon_\delta - v^\varepsilon_\delta$$

goes to 0 in $L^1_{\rm loc}((0,\infty)\times\mathbb{R}^N)$ as $\varepsilon\to 0$, for all $\delta>0$. Then for any $0<T_1<T$, $R>1$, setting $Q := (0,T)\times B_R$, and $Q_1 := (T_1,T)\times B_R$,

$$\begin{aligned}
\left|\left|u^\varepsilon - v\left(\frac{x}{\varepsilon},\bar u(t,x)\right)\right|\right|_{L^1(Q)} \leq\ & ||u^\varepsilon - u^\varepsilon_\delta||_{L^1(Q)} + K'\,|B_R|T_1 + ||u^\varepsilon_\delta - v^\varepsilon_\delta||_{L^1(Q_1)}\\
&+\left|\left|v^\varepsilon_\delta - v\left(\frac{x}{\varepsilon},\bar u_\delta(t,x)\right)\right|\right|_{L^1(Q)}\\
&+\left|\left|v\left(\frac{x}{\varepsilon},\bar u_\delta(t,x)\right) - v\left(\frac{x}{\varepsilon},\bar u(t,x)\right)\right|\right|_{L^1(Q)}\\
\leq\ & C\left(e^{-CR} + ||u_0 - u_0^\delta||_{L^1(B_R;\,\mathcal{C}_{\rm per}(Y))} + ||u^\varepsilon_\delta - v^\varepsilon_\delta||_{L^1(Q_1)}\right)\\
&+C\left|\left|v^\varepsilon_\delta - v\left(\frac{x}{\varepsilon},\bar u_\delta(t,x)\right)\right|\right|_{L^1(Q)} + C\,K'\,|B_R|T_1.
\end{aligned}$$

In the above inequality, we have used the bound

$$\begin{aligned}
\left|\left|v\left(\frac{x}{\varepsilon},\bar u_\delta(t,x)\right) - v\left(\frac{x}{\varepsilon},\bar u(t,x)\right)\right|\right|_{L^1(Q)} \leq\ & \left|\left|\frac{\partial v}{\partial p}\right|\right|_{L^\infty(Y\times[\beta_1-1,\beta_2+1])} ||\bar u_\delta - \bar u||_{L^1(Q)}\\
\leq\ & C(e^{-CR} + ||\bar u_0 - \bar u_0^\delta||_{L^1(B_R)})\\
\leq\ & C(e^{-CR} + ||u_0 - u_0^\delta||_{L^1(B_R;\mathcal{C}_{\rm per}(Y))}).
\end{aligned}$$

Let $\eta>0$ arbitrary, $T>0$ fixed. Take $R>0$ large enough so that $Ce^{-CR}\leq\eta$. For this $R>0$, we now choose $\delta>0$ and $T_1>0$ such that

$$C\left(||u_0 - u_0^\delta||_{L^1(B_R;\mathcal{C}_{\rm per}(Y))} + K'\,|B_R|T_1\right)\leq\eta.$$

Now, for this choice of R, T_1 and δ, we take $\varepsilon_0>0$ small enough so that for all $\varepsilon\leq\varepsilon_0$

$$C\left(||u^\varepsilon_\delta - v^\varepsilon_\delta||_{L^1((T_1,T)\times B_R)} + \left|\left|v^\varepsilon_\delta - v\left(\frac{x}{\varepsilon},\bar u_\delta(t,x)\right)\right|\right|_{L^1(Q)}\right)\leq\eta$$

thanks to proposition 4.1.2 (notice that $\bar u_0^\delta$ belongs to $L^1\cap L^\infty$). Hence, for all $\eta>0$, $T>0$, $R>0$, we have found $\varepsilon_0>0$ so that for all $\varepsilon\leq\varepsilon_0$

$$\left|\left|u^\varepsilon - v\left(\frac{x}{\varepsilon},\bar u\right)\right|\right|_{L^1((0,T)\times B_R)}\leq 2\eta.$$

This is exactly saying that $u^\varepsilon - v\left(\frac{x}{\varepsilon},\bar u\right)$ goes to 0 in $L^1_{\rm loc}([0,\infty)\times\mathbb{R}^N)$.

Thus, we now restrict ourselves to initial datas which have as much regularity as desired (this hypothesis will be made precise in the course of the proof). We work with $\delta > 0$ fixed, and thus we drop all δ's, and we write K instead of K'. The goal of the next steps is to prove that for sufficiently smooth initial data, we have $u^\varepsilon - v^\varepsilon \to 0$ in $L^1_{\text{loc}}((0,\infty)\times\mathbb{R}^N)$ as $\varepsilon \to 0$.

We wish to stress here that we have proved that the hypothesis $\bar{u}_0 \in L^1$ in proposition 4.1.2 is irrelevant : indeed, if theorem 6 is true for smooth initial data, then it is true for initial data which merely satisfy (4.17) according to the above calculations. And thus proposition 4.1.2 is true for $\bar{u}_0 \in L^\infty$ only.

B Convergence of r in L^∞

Let us prove that

$$||r(\tau)||_{L^\infty_{\text{loc}}(\mathbb{R}^N;L^\infty(Y))} \to 0 \quad \text{as } \tau \to \infty. \tag{4.38}$$

Thanks to section 4.2, we already know that for almost every $x \in \mathbb{R}^N$,

$$||r(\tau,x)||_{L^\infty(Y)} \to 0 \quad \text{as } \tau \to \infty.$$

We deduce easily that

$$||r(\tau)||_{L^p_{\text{loc}}(\mathbb{R}^N;L^\infty(Y))} \to 0 \quad \text{as } \tau \to \infty$$

for all $p \in [1,\infty)$. However, we need to prove that the same convergence holds with $p = \infty$: indeed, if we try and prove lemma 4.3.2 with the hypotheses on b replaced by

$$||b(\tau) - b_\infty||_{L^p_{\text{loc}}(\mathbb{R}^N,L^2(Y))} \to 0$$

for some $p < \infty$ (idem with $\text{div}b - \text{div}b_\infty$), then in the course of the proof, we are led to an inequality of the type

$$\begin{aligned}
&||v(\tau)||_{L^q(B_R;L^1(Y))}\\
\leq\ & Ce^{-\mu\tau}||v_0||_{L^q(B_R;L^1(Y))}\\
&+C\int_0^\tau e^{-\mu(\tau-\sigma)}||b(\sigma)-b_\infty||_{L^p(B_R,L^2(Y))}||v(\sigma-1)||_{L^r(B_R;L^1(Y))}\,d\sigma\\
&+C\int_0^\tau e^{-\mu(\tau-\sigma)}||\text{div}b(\sigma)-\text{div}b_\infty||_{L^p(B_R,L^2(Y))}||v(\sigma-1)||_{L^r(B_R;L^1(Y))}\,d\sigma
\end{aligned}$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. If we cannot take $p = \infty$, then $r > q$ and we can no longer apply Gronwall's lemma.

In order to prove (4.38), we go back to the proof of theorem (5) and we define the quantities $U(t,x,y)$, $p^*(t,x)$, $p^*(x)$ (x is a parameter of the equation).

If we look closely at the proof, we see that it is enough to prove that $p^*(t,x)$ converges to $p^*(x) = \bar{u}_0(x)$ locally uniformly in x as $t \to \infty$. Since the functions $p^*(t,x)$ are decreasing w.r.t. $t > 0$, and $\bar{u}_0(x)$ is continuous in x if $u_0(x,y)$ is continuous in x uniformly in $y \in Y$, according to Dini's theorem we only have to prove that $p^*(t,x)$ is continuous w.r.t. x for every $t > 0$.

Let $\delta > 0$, $R > 0$. We assume that $u_0 \in \mathcal{C}_{\text{per}}(\mathbb{R}^N \times Y)$. There exists $\eta > 0$ such that

$$\forall x, x' \in B_R \quad |x - x'| \leq \eta \Rightarrow ||u_0(x, \cdot) - u_0(x', \cdot)||_{L^1(Y)} \leq \delta.$$

Let $x \in B_R$ such that $|x - x'| \leq \eta$. By the L^1 contraction property,

$$||u(\tau, x) - u(\tau, x')||_{L^1(Y)} \leq \delta \quad \forall \tau \geq 0,$$

which entails, thanks to parabolic regularity results (see remark 4.3.1)

$$||u(\tau, x) - u(\tau, x')||_{L^\infty(Y)} \leq C\delta \quad \forall \tau \geq 1$$

for some constant C depending only on N,Y, R and $||a||_{L^\infty(Y \times (-K,K))}$.

From this we deduce easily that

$$||U(\tau, x) - U(\tau, x')||_{L^\infty(Y)} \leq C\delta \quad \forall \tau \geq 1.$$

Let $p < p^*(\tau, x')$ arbitrary, $\tau \geq 1$. There exists $y \in Y$ such that

$$v(y, p) < U(\tau, x', y) \leq U(\tau, x, y) + C\delta \leq v(y, p^*(\tau, x)) + C\delta.$$

Hence, either $p \leq p^*(\tau, x)$ or

$$V(y) := v(y, p) - v(y, p^*(\tau, x)) \geq 0 \quad \forall y \in Y,$$
$$\inf_{y \in Y} V(y) \leq C\delta$$

and V satisfies an elliptic equation of the type

$$-\Delta_y V + \text{div}_y(bV) = 0,$$

with $b \in L^\infty(Y)$, $||b||_{L^\infty(Y)} \leq ||a||_{L^\infty(Y \times (-K,K))}$.

In the second case, according to Harnack's inequality (see [34]) there exists a constant C depending only on N,Y, and $||a||_{L^\infty(Y \times (-K,K))}$, such that $\sup_Y V \leq C \inf_Y V \leq C\delta$, and in that case

$$|p - p^*(\tau, x)| = \frac{1}{|Y|} ||v(\cdot, p) - v(\cdot, p^*(\tau, x))||_{L^1(Y)} \leq C\delta.$$

In all cases, for all $p \in \mathbb{R}$,

$$p < p^*(\tau, x') \Rightarrow p \leq p^*(\tau, x) + C\delta$$

and thus $p^*(\tau, x') \leq p^*(\tau, x) + C\delta$. Of course, the inequality obtained by exchanging the roles of x and x' holds and $p^*(\tau)$ is thus continuous in x for all $\tau \geq 1$.

C r converges exponentially fast to 0

$r(\tau, x, y)$ satisfies an equation of the type (4.30), with

$$b(\tau, x, y) := \int_0^1 a(y, v(y, \bar{u}_0(x)) + sr(\tau, x, y))\, ds.$$

Consequently, setting

$$b_\infty(x, y) := a(y, v(y, \bar{u}_0(x))),$$

we have

$$||b(\tau) - b_\infty||_{L^\infty(B_R;L^\infty(Y))} \le c_0 ||r(\tau)||_{L^\infty(B_R;L^\infty(Y))},$$
$$||\mathrm{div}_y b(\tau) - \mathrm{div}_y b_\infty||_{L^\infty(B_R;L^2(Y))} \le c_0 \left(1 + ||r(\tau)||_{L^\infty(B_R;H^1_{\mathrm{per}}(Y))}\right),$$

where

$$c_0 := ||\partial_v a||_{L^\infty(Y\times(-K,K))^N} + ||\mathrm{div}_y a||_{L^\infty(Y\times(-K,K))}.$$

Notice that in lemma 4.3.2, inequality (4.31) is established without using the assumption that $||\mathrm{div}_y b(\tau) - \mathrm{div}_y b_\infty|| \to 0$; in fact, we only need to prove that

$$||\mathrm{div}_y b - \mathrm{div}_y b_\infty||_{L^\infty(B_R;L^2((\tau-\frac{1}{2},\tau+\frac{3}{2})\times Y))}$$

is bounded (uniformly in $\tau \ge \frac{1}{2}$), and thus that

$$||r||_{L^\infty(B_R;L^2(\tau-\frac{1}{2},\tau+\frac{3}{2};H^1_{\mathrm{per}}(Y)))}$$

is bounded uniformly in τ. But we have

$$||r||_{L^\infty(B_R;L^2(\tau-\frac{1}{2},\tau+\frac{3}{2};H^1_{\mathrm{per}}(Y)))} \le C||r(\tau - \frac{1}{2})||_{L^\infty(B_R;L^2(Y))},$$

where C is a constant which depends only on N, Y, and $||b||_{L^\infty(0,\infty)\times B_R\times Y} \le ||a||_{L^\infty(Y\times(-K,K))^N}$. Hence inequality (4.31) is satisfied for r. And the continuity of $\mathrm{div}_y a$ and $\partial_v a$ entails that $\mathrm{div} b(\tau) - \mathrm{div} b_\infty$ converges to 0 in $L^\infty(B_R; L^2(Y))$ as $\tau \to \infty$.

Consequently the hypotheses of lemma 4.3.2 are satisfied and for all $R > 0$, there exists a constant $c_R > 0$ such that

$$||r(\tau)||_{L^\infty(B_R;L^\infty(Y))} \le c_R e^{-\mu\tau} ||r_0||_{L^\infty(B_R;L^2(Y))} \quad \forall \tau \ge 1 \tag{4.39}$$

where $\mu > 0$ is a constant which depends only on N, Y, and $||a||_{L^\infty(Y\times(-K,K))}$.

D Behavior of $u^\varepsilon(t, x)$ for small times

In this paragraph, we derive a bound on $w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right) - u^\varepsilon(\alpha, x)$.

Assume that the initial data $u_0(x, y)$ is smooth in x and y, namely

$$\nabla_x u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathcal{C}^1_{\mathrm{per}}(Y))^N \quad \text{and } \Delta_x u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathcal{C}_{\mathrm{per}}(Y)).$$

Then $\nabla_x w(\tau,x,y)\in L^\infty_{\text{loc}}((0,\infty)\times\mathbb{R}^N;\mathcal{C}^1_{\text{per}}(Y))^N$, $\Delta_x w\in L^\infty_{\text{loc}}((0,\infty)\times\mathbb{R}^N;\mathcal{C}_{\text{per}}(Y))$.

Let us compute

$$\begin{aligned}
&\frac{\partial}{\partial t}w\left(\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)+\sum_{i=1}^N\frac{\partial}{\partial x_i}A_i\left(\frac{x}{\varepsilon},w\left(\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)\right)-\varepsilon\Delta_x w\left(\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)\\
=\;&\sum_{i=1}^N\frac{\partial w}{\partial x_i}\left(\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)a_i\left(\frac{x}{\varepsilon},w\left(\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)\right)-\Delta_{xy}w\left(\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)-\varepsilon\Delta_x\left(\frac{t}{\varepsilon},x,\frac{x}{\varepsilon}\right)\\
=:\;&f_\varepsilon(t,x)
\end{aligned}$$

and

$$||f_\varepsilon(t)||_{L^1(B_R)}\leq C\left(\left|\left|\nabla_x w\left(\frac{t}{\varepsilon}\right)\right|\right|_{L^1(B_R;W^{1,\infty}(Y))}+||\Delta_x w||_{L^1(B_R;L^\infty(Y))}\right)$$

where the constant C depends only on $||a||_{L^\infty(Y\times[-K,K])}$. Thus, we have to prove that $\nabla_x w(\tau,x,y)$ is bounded in $L^1(B_R;W^{1,\infty}(Y))$ uniformly in τ and that $\Delta_x w(\tau,x,y)$ is bounded in $L^1(B_R;L^\infty(Y))$ uniformly in τ.

With this aim in view, we define

$$R_i(\tau,x,y):=\frac{\partial r}{\partial x_i}(\tau,x,y)=\frac{\partial w}{\partial x_i}(\tau,x,y)-\frac{\partial\bar u_0(x)}{\partial x_i}m(y,\bar u_0(x)),$$

where $m(\cdot,p)\in H^1_{\text{per}}(Y)$ is the solution of

$$-\Delta_y m(y,p)+\text{div}_y(a(y,v(y,p))m(y,p))=0.$$

Thanks to elliptic regularity results and the regularity hypotheses on a, $m(\cdot,p)\in W^{2,q}_{\text{per}}(Y)$ for all $p\in\mathbb{R}$ and for all $1\leq q<\infty$, and for all $R>0$ there exists a constant $c_R>0$ depending only on N, Y, a and R such that

$$||m(\cdot,p)||_{L^\infty(Y)}+||m(\cdot,p)||_{H^1(Y)}\leq c_R\quad\forall|p|\leq R.$$

R_i satisfies

$$\begin{aligned}
&\partial_\tau R_i+\text{div}_y(c(\tau,x,y)R_i)-\Delta_y R_i\\
=\;&\text{div}_y\left((b_\infty(x,y)-c(\tau,x,y))\frac{\partial\bar u_0(x)}{\partial x_i}m(y,\bar u_0(x))\right)=:f_i(\tau,x,y)
\end{aligned}$$

where

$$\begin{aligned}
c(\tau,x,y)&:=a(y,w(\tau,x,y)),\\
b_\infty(x,y)&=a(y,v(y,\bar u_0(x))).
\end{aligned}$$

As a direct application of lemma 4.3.3, we deduce that R_i is bounded in $L^\infty([0,\infty)\times B_R;\mathcal{C}_{\text{per}}(Y))$ for all $R>0$. Indeed we only have to check that f_i is bounded in $L^\infty((0,\infty)\times B_R;L^2(Y))$; since

$$\begin{aligned}
&||\text{div}_y c(\tau)-\text{div}_y b_\infty||_{L^\infty(B_R;L^2(Y))}\\
\leq\;&C\left(1+||\nabla_y r(\tau)||_{L^\infty(B_R;L^2(Y))}+\sup_{p\in[-K,K]}||\nabla_y v(y,p)||_{L^2(Y)}\right),
\end{aligned}$$

we thus obtain a bound on f_i thanks to the first step, inequality (4.31) and proposition 4.1.1. Hence $\nabla_x w(\tau, x, y)$ is bounded in $L^\infty(B_R, \mathcal{C}_{\text{per}}(Y))$ uniformly in τ.

In order to prove that the y derivatives are bounded as well, we intend to use theorem 11.1 in chapter III of [46]; hence we have to prove that

$$\frac{\partial c_j}{\partial y_i} \in L^{2r}(T, T+1; L^{2q}(Y)), \quad 1 \le i, j \le N$$

with uniform bounds in $T > 0$, $x \in B_R$, for some $q \ge 1, r \ge 1$ such that

$$\frac{1}{r} + \frac{N}{2q} = 1 - \kappa, \tag{4.40}$$

with $\kappa \in (0,1)$ if $N \ge 2$ and $\kappa \in (0, \frac{1}{2})$ if $N = 1$. Consequently we must prove that

$$\nabla_y w(\tau, x, y) \in L^{2r}(T, T+1; L^{2q}(Y))$$

with uniform bounds in $T > 0$ and $x \in B_R$. However at the moment we only know that

$$\nabla_y w(\tau, x, y) \in L^\infty((0,\infty) \times B_R; L^2(Y)) \cap L^\infty(B_R; L^2_{\text{loc}}((0,\infty); H^1_{\text{per}}(Y)))^N$$

which is not sufficient to ensure that w has the desired regularity when N is large. Hence we first need to prove the

Lemma 4.3.4. *There exist q, r satisfying* (4.40)*, and a constant C_R depending only on N, Y, R, K and $||a||_{W^{1,\infty}(Y\times(-K,K))}$ such that for all $T > 0$*

$$||\nabla_y w||_{L^\infty(B_R; L^{2r}(T,T+1; L^{2q}(Y)))} \le C_R.$$

Proof. Let

$$S_i(\tau, x, y) := \frac{\partial r(\tau, x, y)}{\partial y_i}.$$

Then S_i satisfies

$$\partial_\tau S_i + \operatorname{div}(c S_i) - \Delta_y S_i = \operatorname{div}(F_i),$$

where

$$F_i(\tau, x, y) := -\frac{\partial c(\tau, x, y)}{\partial y_i} r(\tau, x, y).$$

Then

$$|F_i(\tau, x, y)| \le C(1 + |S_i(\tau, x, y)|),$$

where the constant C depends only on $||\partial_v a||_{L^\infty(Y\times(-K,K))}$, $||\partial_{y_i} a||_{L^\infty(Y\times(-K,K))}$, and K. Moreover,

$$||c||_{L^\infty([0,\infty)\times\mathbb{R}^N\times Y)} \le ||a||_{L^\infty(Y\times(-K,K))}.$$

Thus, according to theorem 9.1 and corollary 9.2 in chapter III of [46], we have

$$\left\{\begin{array}{l} S_i \in L^\infty(B_R; L^{2r}(T,T+1; L^{2q}(Y))) \\ \text{with } \dfrac{1}{r}+\dfrac{N}{2q}=1+\dfrac{N\theta}{2},\ \theta\in(0,1)\end{array}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} S_i \in L^\infty(B_R; L^{2r'}(T,T+1; L^{2q'}(Y))) \\ \text{with } \dfrac{1}{r'}+\dfrac{N}{2q'}=\dfrac{N\theta}{2}\end{array}\right. \tag{4.41}$$

and $||S_i||_{L^\infty(B_R;L^{2r'}(T,T+1;L^{2q'}(Y)))}$ is bounded by a constant which depends only on N, Y, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, θ and $||S_i||_{L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))}$.

Moreover, by interpolation we know that

$$\begin{aligned} ||S_i||_{L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))} &\le ||S_i||^{1-\theta}_{L^\infty(B_R;L^\infty(T,T+1;L^2(Y)))}||S_i||^{\theta}_{L^\infty(B_R;L^2(T,T+1;L^{q_0}(Y)))} \\ &\le C||S_i||^{1-\theta}_{L^\infty(B_R;L^\infty(T,T+1;L^2(Y)))}||S_i||^{\theta}_{L^\infty(B_R;L^2(T,T+1;H^1(Y)))} \end{aligned}$$

where the constant C depends only on N and Y and where

$$\frac{1}{q_0}=\frac{1}{2}-\frac{1}{N},$$
$$\theta=\frac{1}{r}\in[0,1],$$
$$\frac{1}{2q}=\frac{\theta}{q_0}+\frac{1-\theta}{2}.$$

Hence we have $S_i \in L^\infty(B_R; L^{2r_1}(T,T+1; L^{2q_1}(Y)))$ for all $(q_1,r_1)\in[1,\infty)$ such that

$$\frac{N}{2q_1}+\frac{1}{r_1}=\frac{N}{2}=1+\frac{N\theta_1}{2},$$

where $\theta_1 = 1-\frac{2}{N}$ (notice that in the case when $N=1$, we need not go further). Define the sequence $(\theta_k)_{k\ge 0}$ in $\mathbb{R}$ by

$$1+\frac{N\theta_{k+1}}{2}=\frac{N\theta_k}{2}.$$

Then it is easily proved by induction that as long as $\theta_k\in(0,1)$,

$$\left\{\begin{array}{l} S_i \in L^\infty(B_R; L^{2r}(T,T+1; L^{2q}(Y))) \\ \forall (q,r)\in[1,\infty)^2 \text{ s.t. } \exists\, \theta\in(\theta_k,1),\ \dfrac{1}{r}+\dfrac{N}{2q}=\dfrac{N\theta}{2}.\end{array}\right.$$

And the bound on $||S_i||_{L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))}$ depends only on N, Y, K, θ and $||a||_{W^{1,\infty}(Y\times(-K,K))}$. In particular, it is uniform in T, R.

Since $\theta_k = 1-k\frac{2}{N}$, we can choose k_0 such that $\theta_k\in(0,1)$ for $k\le k_0$ and $0\le \frac{N\theta_k}{2}<1$. Then we choose $\theta\in(\theta_k,1)$ such that $0<\frac{N\theta}{2}<1$, and $q,r\ge 1$ such that $\frac{1}{r}+\frac{N}{2q}=\frac{N\theta}{2}$. According to the above remarks, $S_i\in L^\infty(B_R;L^{2r}(T,T+1;L^{2q}(Y)))$ with uniform bounds in $T>0$. Since

$$\nabla_y w = (S_1,\cdots,S_N)+\nabla_y v(y,\bar{u}_0(x)),$$

this concludes the proof of the lemma.

□

Consequently, according to theorem 11.1 in chapter III of [46], there exists a constant C_R depending on $R > 0$, N, Y, K, $||a||_{W^{1,\infty}(Y\times(-K,K))}$ and $||\nabla_x u_0||_{L^\infty(B_R,\mathcal{C}_{\text{per}}(Y))}$ such that

$$||\partial^2_{y_i x_j} w||_{L^\infty([1,\infty)\times B_R\times Y)} \leq C_R \quad \forall\, 1 \leq i,j \leq N.$$

And for every $\tau \in [0,1]$,

$$||\partial^2_{y_i x_j} w(\tau)||_{L^\infty(B_R\times Y)} \leq C_R \left(1 + ||u_0||_{W^{1,\infty}(B_R;\mathcal{C}^1_{\text{per}}(Y))}\right).$$

It remains to prove that $\Delta_x w$ is bounded in $L^\infty([0,\infty)\times B_R \times Y)$. The equation satisfied by $R(\tau,x,y) := \Delta_x w(\tau,x,y)$ is

$$\partial_\tau R + \text{div}_y(cR) - \Delta_y R = \frac{\partial f_i}{\partial x_i} - \text{div}_y((\partial_{x_i} c)R);$$

the right-hand side of the above equation belongs to $W^{-1,\infty}(Y, L^\infty([0,\infty)\times B_R))$ for all $R > 0$ according to the preceding steps. Thus $\Delta_x w$ is bounded in $L^\infty([0,\infty)\times B_R \times Y)$ by lemma 4.3.3.

Then, we multiply the L^1 contraction principle inequality between the functions u^ε and $w\left(t/\varepsilon, x, x/\varepsilon\right)$ by a test function $\varphi\left(\frac{x}{R}\right)$, with $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ when $|x| \leq 1$, $\varphi(x) = 0$ when $|x| \geq 2$, and we integrate over $(0,\alpha)\times\mathbb{R}^N$.

We deduce that there exist constants C, C_R such that C depends only on N and $||a||_{L^\infty(Y\times[-K,K])}$ and C_R depends on N, Y, K, R, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, and the norms $||u_0||_{W^{1,\infty}(B_{2R};\mathcal{C}^1_{\text{per}}(Y))}$ and $||\Delta_x u_0||_{L^\infty(B_{2R};\mathcal{C}_{\text{per}}(Y))}$ such that

$$\begin{aligned}\left|\left|u^\varepsilon(\alpha) - w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_R)} &\leq \underbrace{||u^\varepsilon(t=0) - w\left(\tau = 0, x, \frac{x}{\varepsilon}\right)||_{L^1(B_{2R})}}_{=0} \\ &\quad + \frac{C}{R}\int_0^\alpha \left|\left|u^\varepsilon(s) - w\left(\frac{s}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_{2R})} ds \\ &\quad + C_R\alpha.\end{aligned}$$

For $s > 0$,

$$\begin{aligned}\left|\left|u^\varepsilon(s) - w\left(\frac{s}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_{2R})} &\leq ||u^\varepsilon(s)||_{L^1(B_{2R})} + \left|\left|w\left(\frac{s}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_{2R})} \\ &\leq 2\,|B_{2R}|\,K.\end{aligned}$$

Thus

$$\left|\left|u^\varepsilon(\alpha) - w\left(\frac{\alpha}{\varepsilon}, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(B_R)} \leq C_R\alpha$$

where the constant C_R depends only on R, N, Y, K, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, and on the bounds $||u_0||_{W^{1,\infty}(B_{2R};\mathcal{C}^1_{\text{per}}(Y))}$ and $||\Delta_x u_0||_{L^\infty(B_{2R};\mathcal{C}_{\text{per}}(Y))}$.

E Bound on $\left|\left|v\left(\frac{x}{\varepsilon},\bar{u}(\alpha,x)\right)-v\left(\frac{x}{\varepsilon},\bar{u}_0(x)\right)\right|\right|_{L^1(B_R)}+\left|\left|v^\varepsilon(\alpha)-v\left(\frac{x}{\varepsilon},\bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)}$

- First, $\beta_1 \le \bar{u}_0(x) \le \beta_2$ a.e. in $\mathbb{R}^N$. Hence, $\beta_1 \le \bar{u}(t,x) \le \beta_2$ for a.e. $(t,x) \in (0,\infty)\times\mathbb{R}^N$. Thus

$$\begin{aligned}&\left|\left|v\left(\frac{x}{\varepsilon},\bar{u}(\alpha,x)\right)-v\left(\frac{x}{\varepsilon},\bar{u}_0(x)\right)\right|\right|_{L^1(B_R)} &(4.42)\\ \le\ &\left|\left|\frac{\partial v(y,p)}{\partial p}\right|\right|_{L^\infty(Y\times[\beta_1,\beta_2])}||\bar{u}(\alpha)-\bar{u}_0(x)||_{L^1(B_R)}\\ \le\ &\left|\left|\frac{\partial v(y,p)}{\partial p}\right|\right|_{L^\infty(Y\times[\beta_1,\beta_2])}\omega(\alpha) &(4.43)\end{aligned}$$

  where ω is the modulus of continuity in time for $\bar{u}$:

$$\omega(k) := \sup_{0\le\tau\le k}||\bar{u}(\tau)-\bar{u}_0||_{L^1(\mathbb{R}^N)} \underset{k\to 0}{\longrightarrow} 0$$

- For all $R>0$ and for almost every $\alpha>0$,

$$\left|\left|v^\varepsilon(\alpha)-v\left(\frac{x}{\varepsilon},\bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)} \to 0 \tag{4.44}$$

  as $\varepsilon\to 0$ according to the homogenization result in case of well-prepared initial data.

F Conclusion

Gathering (4.43), (4.44), and the exponential result proved in the second step, we obtain, for all $R>0$

$$||u^\varepsilon(\alpha)-v^\varepsilon(\alpha)||_{L^1(B_R)} \le C_R\alpha + C\omega(\alpha) + \left|\left|v^\varepsilon(\alpha,x)-v\left(\frac{x}{\varepsilon},\bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)} + C'_R e^{-\frac{\mu\alpha}{\varepsilon}},$$

as long as $\alpha\ge\varepsilon>0$, where :

- the constant C_R depends on R, N, Y, K, $||a||_{W^{1,\infty}(Y\times(-K,K))}$, and on the norms $||u_0||_{W^{1,\infty}(B_{2R};\mathcal{C}^1_{\mathrm{per}}(Y))}$ and $||\Delta_x u_0||_{L^\infty(B_{2R};\mathcal{C}_{\mathrm{per}}(Y))}$;
- the constants C, μ depend only on N, Y, and $||a||_{W^{1,\infty}(Y\times[-K',K'])}$;
- the constant C'_R depends on R, N, Y, K, $||a||_{W^{1,\infty}(Y\times[-K,K])}$, $||u_0(x,y)-v(y,\bar{u}_0(x))||_{L^\infty(B_R;L^2(Y))}$, and the speed of convergence of $w(\tau,x,y)$ towards $v(y,\bar{u}_0(x))$ in $L^\infty(B_R\times Y)$.

Going back to inequality (4.26), and using all the bounds derived in the preceding steps leads us to

$$\begin{aligned}||u^\varepsilon(T)-v^\varepsilon(T)||_{L^1(B_R)} \le\ & Ce^{CT}\Big[e^{-R}+C_R\alpha+\omega(\alpha)+\\ &+\left|\left|v^\varepsilon(\alpha,x)-v\left(\frac{x}{\varepsilon},\bar{u}(\alpha,x)\right)\right|\right|_{L^1(B_R)}+C'_R e^{-\frac{\mu\alpha}{\varepsilon}}\Big].\end{aligned}$$

The inequality holds for all $R > 0$ and for all $T > \alpha \geq \varepsilon > 0$.

Let $\eta > 0$ arbitrary, $T > 0$ fixed. First, we choose $R > 0$ large enough so that

$$Ce^{CT}e^{-R} \leq \eta.$$

For this $R > 0$, we choose $\alpha_0 > 0$ small enough so that

$$Ce^{CT}\left(C_R\alpha + C\omega(\alpha)\right) \leq \eta \quad \text{for } 0 < \alpha \leq \alpha_0.$$

We pick $\alpha \in (0, \alpha_0)$ such that

$$\left|\left|v^\varepsilon(\alpha, x) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha, x)\right)\right|\right|_{L^1(B_R)} \to 0 \quad \text{as } \varepsilon \to 0.$$

At last, for this R and $\alpha > 0$, we take $0 < \varepsilon_0 \leq \alpha$ small enough so that

$$Ce^{CT}\left(\left|\left|v^\varepsilon(\alpha, x) - v\left(\frac{x}{\varepsilon}, \bar{u}(\alpha, x)\right)\right|\right|_{L^1(B_R)} + C'_R e^{-\frac{\mu\alpha}{\varepsilon}}\right) \leq \eta \quad \forall \varepsilon \in (0, \varepsilon_0].$$

Hence, we have proved that

$$||u^\varepsilon - v^\varepsilon||_{L^\infty_{\mathrm{loc}}((0,\infty);L^1_{\mathrm{loc}}(\mathbb{R}^N))} \to 0 \quad \text{as } \varepsilon \to 0$$

as long as the initial data has the smoothness required in the fourth and the first steps. Thus according to the first step, theorem 6 is true for any initial data satisfying (4.17).

Chapitre 5

Homogénéisation de lois de conservation scalaires hyperboliques

On étudie ici le comportement lorsque $\varepsilon \to 0$ des solutions de l'équation

$$\partial_t u^\varepsilon + \mathrm{div}_x \left[A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\right] = 0.$$

On montre que la famille u^ε converge à deux échelles vers une fonction $u(t,x,y)$, et on peut identifier u de façon univoque comme solution d'un problème d'évolution limite. L'originalité du résultat réside dans le fait que le problème limite n'est pas une loi de conservation scalaire, mais plutôt une équation cinétique, dans laquelle les variables macroscopiques et microscopiques sont couplées.

Ce chapitre a été accepté pour publication dans Archive for Rational Mechanics and Analysis.

5.1 Introduction

This article is concerned with the asymptotic behavior of the sequence u^ε as the parameter ε vanishes, where $u^\varepsilon \in \mathcal{C}([0,\infty), L^1_{\text{loc}}(\mathbb{R}^N))$ is the entropy solution of the scalar conservation law

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i\left(\frac{x}{\varepsilon}, u^\varepsilon(t,x)\right) = 0 \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{5.1}$$

$$u^\varepsilon(t=0) = u_0\left(x, \frac{x}{\varepsilon}\right). \tag{5.2}$$

The functions $A_i = A_i(y,v)$ $(y \in \mathbb{R}^N,\ v \in \mathbb{R})$ are assumed to be Y-periodic, where $Y = \Pi_{i=1}^N (0,T_i)$ is the unit cell, and u_0 is also assumed to be periodic in its second variable.

Under regularity hypotheses on the flux, namely $A \in W^{2,\infty}_{\text{per,loc}}(\mathbb{R}^{N+1})$, and when the initial data $u^\varepsilon(t=0)$ belongs to L^∞, it is known that there exists a unique entropy solution u^ε of the above system for all $\varepsilon > 0$ given (see [13, 69, 70, 62, 63]). The study of the homogenization of such hyperbolic scalar conservation laws has been investigated by several authors, see for instance [24, 25, 27], and in the linear case [42, 43]. In dimension one, there is also an equivalence with Hamilton-Jacobi equations which allows to use the results of [49]. In general, the results obtained by these authors can be summarized as follows: there exists a function $u^0 = u^0(t,x,y)$ such that

$$u^\varepsilon - u^0\left(t, x, \frac{x}{\varepsilon}\right) \to 0 \quad \text{in } L^1_{\text{loc}}((0,\infty) \times \mathbb{R}^N). \tag{5.3}$$

The function $u^0(t,x,y)$ satisfies a microscopic equation, called cell problem, and an evolution equation, which is a scalar conservation law in which the coefficients depend on the microscopic variable y. In general, there is no "decoupling" of the macroscopic variables t,x, and the microscopic variable y: the average of u^0 with respect to the variable y is not the solution of an "average" conservation law.

To our knowledge, there are no results as soon as the dimension is strictly greater than one when the flux does not satisfy a structural condition of the type $A(y,\xi) = a(y)g(\xi)$. Here, we investigate the behavior of the family u^ε for arbitrary fluxes. We prove that (5.3) still holds, in a sense which will be made clear later on, and the function u^0 is a solution of a microscopic cell problem. Precisely, we prove that even though there is no simple evolution equation satisfied by the function u^0 itself, the function

$$f(t,x,y,\xi) = \mathbf{1}_{\xi < u^0}$$

is the unique solution of a linear transport equation, with a source term which is a Lagrange multiplier accounting for the constraints on f. This statement is reminiscent of the kinetic formulation for scalar conservation laws (see [52, 53, 58], the general presentation in [59], and [15] for the heterogeneous case); this is not surprising since our method of proof relies on the kinetic formulation for equation (5.1). However, in general, it is unclear whether u^0 is the solution of a scalar conservation law. Thus the kinetic formulation appears as the "correct" vision of the entropy solutions of (5.1), at least as far as homogenization is concerned.

The rest of this introduction is devoted to the presentation of the main results. We begin with the description of the asymptotic problem, and then we state the convergence results.

5.1.1 Description of the asymptotic evolution problem

We first introduce the asymptotic evolution problem, for which we state an existence and uniqueness result; then we explain how this asymptotic problem can be understood formally.

In the following, we set, for $(y,\xi)\in\mathbb{R}^{N+1}$,

$$a_i(y,\xi)=\frac{\partial A_i}{\partial\xi}(y,\xi),\quad 1\le i\le N,$$
$$a_{N+1}(y,\xi)=-\mathrm{div}_y A(y,\xi).$$

We set $a(y,\xi)=(a_1(y,\xi),\cdots,a_{N+1}(y,\xi))\in\mathbb{R}^{N+1}$. Notice that $\mathrm{div}_{y,\xi}a(y,\xi)=0$. These notations were introduced in [15].

Before giving the Definition of the limit system, let us recall the kinetic formulation for equation (5.1), which may shed some light on the structure of the asymptotic evolution problem. Let u^ε be an entropy solution of (5.1). Then there exists a non-negative measure $m^\varepsilon\in M^1((0,\infty)\times\mathbb{R}^{N+1})$ such that $f^\varepsilon=\mathbf{1}_{\xi<u^\varepsilon(t,x)}$ is a solution of the transport equation

$$\partial_t f^\varepsilon+a_i\left(\frac{x}{\varepsilon},\xi\right)\partial_{x_i}f^\varepsilon+\frac{1}{\varepsilon}a_{N+1}\left(\frac{x}{\varepsilon},\xi\right)\partial_\xi f^\varepsilon=\partial_\xi m^\varepsilon, \tag{5.4}$$
$$f^\varepsilon(t=0,x,\xi)=\mathbf{1}_{\xi<u_0\left(x,\frac{x}{\varepsilon}\right)}. \tag{5.5}$$

In fact, this equation was derived in [15] for the function $g^\varepsilon(t,x,\xi)=\chi(\xi,u^\varepsilon(t,x))$, where $\chi(\xi,u)=\mathbf{1}_{0<\xi<u}-\mathbf{1}_{u<\xi<0}$, for $u,\xi\in\mathbb{R}$, and under the additional assumption $a_{N+1}(y,0)=0$ for all $y\in\mathbb{R}^N$. However, it is easily proved, using the identity $f^\varepsilon=g^\varepsilon+\mathbf{1}_{\xi<0}$, that f^ε satisfies (5.4), even when $a_{N+1}(y,0)$ does not vanish.

We now define the limit system, which is reminiscent of equation (5.4) :

Definition 5.1.1. *Let $f\in L^\infty([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$, $u_0\in L^\infty(\mathbb{R}^N\times Y)$. We say that f is a generalized kinetic solution of the limit problem, with initial data $\mathbf{1}_{\xi<u_0}$, if there exists a distribution $\mathcal{M}\in\mathcal{D}'_{per}([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ such that f and $\mathcal{M}$ satisfy the following properties:*

1. *Compact support in ξ: there exists a constant $M>0$ such that*

$$\mathrm{Supp}\,\mathcal{M}\subset[0,\infty)\times\mathbb{R}^N\times Y\times[-M,M], \tag{5.6}$$
$$f(t,x,y,\xi)=1\quad \textit{if }\xi<-M, \tag{5.7}$$
$$f(t,x,y,\xi)=0\quad \textit{if }\xi>M. \tag{5.8}$$

2. *Microscopic equation for f: there exists a measure $m\in M^1((0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ such that $m\ge 0$ and f is a solution in the sense of distributions of*

$$\mathrm{div}_{y,\xi}(a(y,\xi)f(t,x,y,\xi))=\partial_\xi m, \tag{5.9}$$

and $\operatorname{Supp} m \subset [0,\infty) \times \mathbb{R}^N \times Y \times [-M, M]$.

and $\operatorname{Supp} m \subset [0,\infty) \times \mathbb{R}^N \times Y \times [-M, M]$.

3. *Evolution equation: the couple* $(f, \mathcal{M})$ *is a solution in the sense of distributions of*

$$\begin{cases} \partial_t f + \sum_{i=1}^N a_i(y,\xi)\partial_{x_i} f = \mathcal{M}, \\ f(t=0,x,y,\xi) = \mathbf{1}_{\xi < u_0(x,y)} =: f_0(x,y,\xi); \end{cases} \tag{5.10}$$

In other words, for any test function $\phi \in \mathcal{D}_{per}([0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R})$,

$$\int f(t,x,y,\xi) \left\{ \partial_t \phi(t,x,y,\xi) + \sum_{i=1}^N a_i(y,\xi)\partial_{x_i}\phi(t,x,y,\xi) \right\} dt\, dx\, dy\, d\xi =$$
$$= -\langle \phi, \mathcal{M} \rangle_{\mathcal{D},\mathcal{D}'} - \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \mathbf{1}_{\xi < u_0(x,y)} \phi(t=0,x,y,\xi)\, dx\, dy\, d\xi.$$

4. *Conditions on* f*: there exists a non-negative measure* $\nu \in M^1_{per}([0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R})$ *such that*

$$\partial_\xi f \leq 0 \quad \text{in } \mathcal{D}', \tag{5.11}$$

$$0 \leq f(t,x,y,\xi) \leq 1 \quad \text{almost everywhere.} \tag{5.12}$$

And for all compact set $K \subset \mathbb{R}^N$,

$$\frac{1}{\tau}\int_0^\tau ||f(s) - f_0||_{L^2(K\times Y \times \mathbb{R})}\, ds \underset{\tau \to 0}{\longrightarrow} 0. \tag{5.13}$$

5. *Condition on* $\mathcal{M}$*: define the set*

$$\begin{aligned} \mathcal{G} := \{\psi \in L^\infty_{loc}(Y \times \mathbb{R}),\ \partial_\xi \psi \geq 0, \text{ and } \exists \mu \in M^1_{per}(Y\times\mathbb{R}),\ \exists C > 0, \exists \alpha_- \in \mathbb{R}, \\ \operatorname{div}_{y,\xi}(a\psi) = -\partial_\xi \mu,\ \operatorname{Supp} \mu \subset Y \times [-C, C],\ \mu \geq 0, \\ \psi(y,\xi) = \alpha_- \text{ if } \xi < -C\}. \end{aligned}$$

Then for all $\varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^N)$ *such that* $\varphi \geq 0$*, the function* $\mathcal{M} *_{t,x} \varphi$ *belongs to* $\mathcal{C}([0,\infty)\times\mathbb{R}^N, L^2(Y\times\mathbb{R}))$*, and*

$$\forall (t,x) \in [0,\infty) \times \mathbb{R}^N, \quad \forall \psi \in \mathcal{G}, \quad \int_{Y\times\mathbb{R}} (\mathcal{M} *_{t,x} \varphi)(t,x,\cdot)\, \psi \leq 0. \tag{5.14}$$

We now state an existence and uniqueness result for solutions of the limit problem:

Theorem 7. *Let* $A \in W^{2,\infty}_{per,loc}(Y \times \mathbb{R})$.

1. *Existence: let* $u_0 \in L^1_{loc}(\mathbb{R}^N; \mathcal{C}_{per}(Y)) \cap L^\infty(\mathbb{R}^N)$*, and let* $f_0(x,y,\xi) = \mathbf{1}_{\xi < u_0(x,y)}$*, for* $(x,y,\xi) \in \mathbb{R}^N \times Y \times \mathbb{R}$*. Assume that there exists non-negative measure* $m_0 = m_0(x,y,\xi)$ *such that* f_0 *is a solution of*

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(a_i(y,\xi)f_0\right) + \frac{\partial}{\partial \xi}\left(a_{N+1}(y,\xi)f_0\right) = \frac{\partial m_0}{\partial \xi} \tag{5.15}$$

   and $\operatorname{Supp} m_0 \subset \mathbb{R}^N \times Y \times [-M,M]$*, where* $M = ||u_0||_\infty$*.*

   Assume that there exist functions u_1*,* $u_2 \in L^\infty(Y)$ *such that* $\mathbf{1}_{\xi<u_i}$ *is a solution of* (5.15) *for* $i = 1,2$*, for some non-negative measures* m_1*,* m_2*, and*

$$u_1(y) \le u_0(x,y) \le u_2(y) \quad \textit{for a.e. } x \in \mathbb{R}^N, y \in Y. \tag{5.16}$$

   Then there exists a generalized kinetic solution f *of the limit problem (in the sense of Definition 5.1.1), with initial data* f_0*.*

2. *"Rigidity": let* $u_0 \in L^\infty(\mathbb{R}^N \times Y)$*, and let* $f \in L^\infty([0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R})$ *be a generalized kinetic solution of the limit problem, with initial data* $f_0 = \mathbf{1}_{\xi<u_0}$*. Then there exists a function* $u \in L^\infty([0,\infty) \times \mathbb{R}^N \times Y)$ *such that*

$$f(t,x,y,\xi) = \mathbf{1}_{\xi<u(t,x,y)} \quad \textit{almost everywhere.}$$

3. *Uniqueness and contraction principle: let* $u_0, v_0 \in L^\infty(\mathbb{R}^N \times Y)$*, and let* f, g *be two generalized kinetic solutions of the limit problem with initial data* $\mathbf{1}_{\xi<u_0}$ *and* $\mathbf{1}_{\xi<v_0}$ *respectively. Then there exists a constant* $C > 0$ *such that for all* $t > 0$*, for all* $R, R' > 0$*,*

$$||f(t) - g(t)||_{L^1(B_R \times Y \times \mathbb{R})} \le e^{Ct+R}\left(||u_0 - v_0||_{L^1(B_{R'} \times Y)} + e^{-R'}\right). \tag{5.17}$$

As a consequence, for all $u_0 \in L^\infty(\mathbb{R}^N \times Y) \cap L^1_{loc}(\mathbb{R}^N, \mathcal{C}_{per}(Y))$ *satisfying* (5.15) *and* (5.16)*, there exists a unique generalized kinetic solution* $f \in L^\infty([0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R})$ *of the limit problem.*

Remark 5.1.1. *Notice that for any function* $v \in L^\infty(Y)$*,* v *is an entropy solution of the cell problem*

$$\operatorname{div}_y A(y, v(y)) = 0 \tag{5.18}$$

if and only if there exists a non-negative measure $m \in M^1_{per}(Y \times \mathbb{R})$ *such that equation*

$$\operatorname{div}_{y,\xi}(a(y,\xi)\mathbf{1}_{\xi<v(y)}) = \partial_\xi m$$

is satisfied in the sense of distributions on $Y \times \mathbb{R}$*. Hence equation* (5.15) *entails that* u_0 *is an entropy solution of the cell equation* (5.18)*. In that case, it is said that the initial data* u_0 *is "well-prepared".*

In the case where A *is divergence-free, condition* (5.15) *becomes*

$$\sum_{i=1}^N \partial_{y_i}(a_i(y,\xi)\mathbf{1}_{\xi<v(y)}) = 0.$$

Indeed, in that case, v satisfies

$$\sum_{i=1}^N \partial_{y_i}(a_i(y,\xi)\mathbf{1}_{\xi<v(y)}) = \partial_\xi m$$

for some non-negative measure m such that $\operatorname{Supp} m \subset Y \times [-M, M]$. *Consequently, for $\xi \geq M$, we have*

$$\sum_{i=1}^N \partial_{y_i}\left(\int_{-M-1}^{\xi} a_i(y,w)\mathbf{1}_{w<v(y)}\,dw\right) = m(y,\xi) \geq 0.$$

Since the left-hand side has zero mean-value on Y for all $\xi \in [-M, M]$, we deduce that $m = 0$. Thus, in the case where the flux A is divergence free, the limit system takes a slightly simpler form: conditions (5.9), (5.14) *become*

$$\begin{aligned}
&\operatorname{div}_y(a(y,\xi)f(t,x,y,\xi)) = 0,\\
&\partial_t f + \sum_{i=1}^N a_i(y,\xi)\partial_{x_i} f = \mathcal{M},\\
&\left\{\begin{array}{l}\displaystyle\int_{Y\times\mathbb{R}} (\mathcal{M} *_{t,x} \varphi)(t,x,\cdot)\,\psi \leq 0,\\ \forall \psi \in L^\infty_{loc}(Y\times\mathbb{R}),\ \operatorname{div}_y(a\psi) = 0,\ \text{and } \partial_\xi\psi \geq 0.\end{array}\right.
\end{aligned}\tag{5.19}$$

All the other properties remain the same.

Remark 5.1.2. *Assume that the flux A is divergence-free, and set*

$$\begin{aligned}
C_1 &:= \{\psi \in L^2_{loc}(Y\times\mathbb{R}),\ \sum_{i=1}^N \frac{\partial}{\partial y_i}(a_i(y,\xi)\psi(y,\xi)) = 0\},\\
C_2 &:= \{\psi \in L^\infty_{loc}(Y\times\mathbb{R}),\ \partial_\xi\psi \geq 0\}.
\end{aligned}$$

Then C_1, C_2 are convex sets of the vector space $L^2_{loc}(Y\times\mathbb{R})$. Thus condition (5.14) *can be written as follows: for all $\varphi \in \mathcal{D}((-\infty,0)\times\mathbb{R}^N)$ such that $\varphi \geq 0$, for all $(t,x) \in (0,\infty)\times\mathbb{R}^N$, we have*

$$\mathcal{M} * \varphi(t,x) \in (C_1 \cap C_2)^\circ,$$

where C° denotes the normal cone of C. Let us recall that when the space dimension is finite (that is, if C_1, C_2 are convex cones in $\mathbb{R}^d$ for some $d \in \mathbb{N}$), then

$$(\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2))^\circ = \mathrm{cl}\left(C_1^\circ + C_2^\circ\right),$$

where $\mathrm{cl}(A)$ *denotes the closure of the set A.*

If we forget about the closure and the fact that we are considering convex sets in an infinite dimensional space, then we are tempted to write

$$\mathcal{M} * \varphi(t,x) \in (C_1 \cap C_2)^\circ = \mu_1 + \mu_2,$$

with $\mu_i \in C_i^\circ$, $i = 1, 2$. *Moreover, very formally, we have*

$$C_2^\circ \text{``=''} \{\partial_\xi m,\ m \text{ non-negative measure}\}.$$

Thus, we may think of $\mathcal{M}$ *as some distribution of the form*

$$\mathcal{M} = \partial_\xi m + \mu_1,$$

with m *a non-negative measure on* $[0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R}$, *and* $\mu_1 \in C_1^\circ$.

Of course, these computations are not rigorous, but we believe they may help the reader to understand the action of the distribution $\mathcal{M}$ *(at least in the divergence-free case), even though the precise structure of* $\mathcal{M}$ *shall not be needed in the proof. Inequality* (5.14) *is sufficient for all the applications in this paper.*

Let us stress that uniqueness for the limit problem holds, even though the cell problem does not have a unique solution in general; indeed, in the linear divergence free case, that is, if $A(y, \xi) = b(y)\xi$, with $\text{div}_y b = 0$, then a function u is a solution of the cell problem if

$$\text{div}_y(b(y)u(y)) = 0, \quad \langle u \rangle_Y = 0.$$

The constant function equal to zero is a solution of this equation, but in general there are other entropy solutions: for instance, let us consider the case where $N = 2$, and

$$b(y_1, y_2) = (-\partial_2\phi(y_1, y_2), \partial_1\phi(y_1, y_2)),$$

for some function $\phi \in \mathcal{C}^2_{\text{per}}(Y)$. Then any function u of the form $g(\phi) - \langle g(\phi) \rangle$, with g a continuous function, is an entropy solution. Let us emphasize that nonlinearity assumptions on the flux are not enough to ensure uniqueness of solutions either, see for instance [49].

In Theorem 7, uniqueness of solutions of the limit system follows from a contraction principle associated with the macroscopic evolution equation, rather than the microscopic cell equation. The well-preparedness of the initial data, that is, the fact that $u_0(x, \cdot)$ is an entropy solution of the cell problem, is fundamental.

On the other hand, the lack of uniqueness of solutions of the cell problem entails that in general, there is no notion of homogenized problem. Indeed, if u is a solution of

$$\text{div}_y A(y, p + u(y)) = 0, \quad \langle u \rangle_Y = 0,$$

then in general, the quantity

$$\langle A(\cdot, p + u(\cdot)) \rangle$$

depends on u (except when $N = 1$, and in some special cases, when $N = 2$; see [31, 49]). Hence the macroscopic and microscopic scales cannot be decoupled: if $\mathbf{1}_{\xi < u(t,x,y)}$ is a solution of the limit evolution problem, then $\bar{u}(t, x) = \langle u(t, x, \cdot) \rangle$ does not satisfy any remarkable equation. This is the main consequence of the absence of uniqueness for the cell problem.

Let us mention an important particular case of Theorem 7, which we call the "separate case". We now assume that the flux A can be written $A(y, \xi) = a_0(y)g(\xi)$, with $\text{div}_y a_0 = 0$. This case has already been thoroughly investigated by Weinan E in [24] in the case where $g'(\xi) \neq 0$ for all ξ, that is, when the function g is strictly monotonous. Here, we prove that his results hold with no restriction on g.

Let us introduce the so-called "constraint space"

$$\mathbb{K}_0 := \{f \in L^1(Y);\ \mathrm{div}_y(a_0 f) = 0 \quad \text{in } \mathcal{D}'\},$$

and the orthogonal projection P_0 on $\mathbb{K}_0 \cap L^2(Y)$ for the scalar product in $L^2(Y)$.

Then the following properties hold: for all $f, g \in L^2(Y)$, if $f \in \mathbb{K}_0$, then

$$P_0(fg) = f P_0(g).$$

And if $f, g \in \mathbb{K}_0 \cap L^2(Y)$, then the product fg belongs to $\mathbb{K}_0$. Notice also that all functions which do not depend on y belong to $\mathbb{K}_0$, and that $L^\infty(Y)$ is stable by P_0.

Proposition 5.1.1. *Let $u_0 \in L^1(\mathbb{R}^N, \mathcal{C}_{per}(Y)) \cap L^\infty(\mathbb{R}^N \times Y)$ such that $u_0(x, \cdot) \in \mathbb{K}_0$ for a.e. $x \in \mathbb{R}^N$.*

For $1 \leq i \leq N$, define the vector valued function $\tilde{a} \in L^\infty(Y)^N$ by $\tilde{a}_{0,i} = P_0(a_{0,i})$. Let $u = u(t, x; y)$ be the entropy solution of the scalar conservation law

$$\begin{cases} \partial_t u(t, x; y) + \mathrm{div}_x\left(\tilde{a}(y) g(u(t, x; y))\right) = 0, \quad t > 0,\ x \in \mathbb{R}^N,\ y \in Y, \\ u(t = 0, x; y) = u_0(x, y). \end{cases} \tag{5.20}$$

Then the function $f(t, x, y, \xi) = \mathbf{1}_{\xi < u(t,x,y)}$ is the unique generalized kinetic solution of the limit problem (5.10) *with initial data $\mathbf{1}_{\xi < u_0(x,y)}$. In that case the distribution $\mathcal{M}$ is given by*

$$\mathcal{M} = \frac{\partial m}{\partial \xi} + g'(\xi)(\tilde{a}(y) - a_0(y)) \cdot \nabla_x f,$$

where m is the kinetic entropy defect measure associated with the function u, that is, f is a solution of

$$\partial_t f + g'(\xi)\tilde{a}(y) \cdot \nabla_x f = \partial_\xi m.$$

As a consequence, the solution $u(t, x; y)$ of (5.20) *is an entropy solution of*

$$\mathrm{div}_y A(y, u) = 0$$

for almost every $(t, x) \in (0, \infty) \times \mathbb{R}^N$.

5.1.2 Convergence results

Our first result is concerned with entropy solutions of (5.1).

Theorem 8. *Let $A \in W^{2,\infty}_{per,loc}(\mathbb{R}^{N+1})$.*

Assume that the initial data $u_0 \in L^1_{loc}(\mathbb{R}^N, \mathcal{C}_{per}(Y))$ satisfies (5.15), (5.16)*. Let $f = \mathbf{1}_{\xi < u}$ be the unique generalized kinetic solution of the limit problem, with initial data $\mathbf{1}_{\xi < u_0}$; the existence of f follows from Theorem 7. Then as ε vanishes,*

$$\mathbf{1}_{\xi < u^\varepsilon(t,x)} \stackrel{2\ sc.}{\rightharpoonup} \mathbf{1}_{\xi < u(t,x,y)}. \tag{5.21}$$

As a consequence, for all regularization kernels φ^δ of the form

$$\varphi^\delta(x) = \frac{1}{\delta^N}\varphi\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}^N,$$

with $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\int \varphi = 1$, $0 \le \varphi \le 1$, we have, for all compact $K \subset [0,\infty) \times \mathbb{R}^N$,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left|\left|u^\varepsilon(t,x) - u *_x \varphi^\delta\left(t,x,\frac{x}{\varepsilon}\right)\right|\right|_{L^1(K)} = 0. \tag{5.22}$$

Remark 5.1.3. *Assumption* (5.15) *means that u_0 is "well-prepared", that is, $u_0(x,\cdot)$ is an entropy solution of*

$$\operatorname{div}_y\left(A(y,u_0(x,y))\right) = 0$$

for a.e. $x \in \mathbb{R}^N$. If this hypothesis is not satisfied, then it is expected that the behavior of the sequence u^ε will depend on the nature of the flux. If the flux is linear, then oscillations will propagate, and the cell equation (5.9) *shall not be satisfied in general. If the flux satisfies some strong nonlinearity assumption, on the contrary, the conjecture is that the solution u^ε re-prepares itself in order to match the microscopic profile dictated by the equation. Few results in this direction are known in the hyperbolic case; the reader may consult for instance [5, 25, 28, 65]. In [18], the author studies the same equation as* (5.1) *in which a viscosity term of order ε is added, and proves such a result, but the method relies strongly on the parabolicity of the equation.*

Remark 5.1.4. *The way in which Theorem 8 is stated might seem slightly peculiar; indeed, convergence results of the type*

$$u^\varepsilon(t,x) - u\left(t,x,\frac{x}{\varepsilon}\right) \to 0 \quad \text{in } L^1_{loc}$$

are expected to hold. In order to establish such a result, it seems necessary to prove that

$$\lim_{\delta\to 0}\int_K \sup_{y\in Y}\left|u(t,x,y) - u *_x \varphi^\delta\,(t,x,y)\right|\,dt\,dx = 0.$$

But the evolution equation for u (or rather, for $\mathbf{1}_{\xi<u}$) is given by Definition 5.1.1; since the distribution $\mathcal{M}$ hinders most computations, it seems difficult to derive such estimates.

The next result generalizes Theorem 5.1.1 to weaker solutions of equation (5.1), called kinetic solutions. In order to simplify the presentation, let us restrict the statement to the divergence-free case; it is explained in the remark following the Theorem how to derive an analogous result when the flux A is arbitrary.

Thus, for the reader's convenience, we first recall the Definition of kinetic solutions in the divergence-free case (see [15] for the heterogeneous case, and the presentation in [59] for the homogeneous case) :

Definition 5.1.2 (Kinetic solutions of (5.1))**.** *Let $u^\varepsilon \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N))$. Assume that there exists a non-negative measure $m^\varepsilon \in \mathcal{C}(\mathbb{R}_\xi, M^1_w([0,\infty)\times\mathbb{R}^N))$ such that for all $T>0$, the function*

$$\xi \mapsto \int_0^T\int_{\mathbb{R}^N} m^\varepsilon(t,x,\xi)\,dt\,dx$$

is bounded on $\mathbb{R}$, and vanishes as $|\xi| \to \infty$.
Assume also that $f^\varepsilon(t,x,\xi) := \chi(\xi, u^\varepsilon(t,x))$ is a solution in the sense of distributions of the linear transport equation

$$\frac{\partial f^\varepsilon}{\partial t} + \sum_{i=1}^N a_i\left(\frac{x}{\varepsilon}, \xi\right) \partial_{x_i} f^\varepsilon = \frac{\partial m^\varepsilon}{\partial \xi} \quad t \geq 0,\ x \in \mathbb{R}^N, \tag{5.23}$$

$$f^\varepsilon(t=0) = \chi\left(\xi, u_0\left(x, \frac{x}{\varepsilon}\right)\right), \tag{5.24}$$

Then it is said that u^ε is a kinetic solution of equation (5.1).

Remark 5.1.5. *Let us recall the Definition of the space $\mathcal{C}(\mathbb{R}_\xi, M^1_w([0,\infty) \times \mathbb{R}^N))$. Let $m \in M^1([0,\infty) \times \mathbb{R}^N)$; for $\theta \in \mathcal{C}_c([0,\infty) \times \mathbb{R}^N))$, define $\mu_\theta \in M^1(\mathbb{R})$ by*

$$\mu_\theta := \int_0^\infty \int_{\mathbb{R}^N} m(t,x,\cdot)\theta(t,x)\, dt\, dx.$$

Then

$$\mathcal{C}(\mathbb{R}_\xi, M^1_w([0,\infty) \times \mathbb{R}^N)) := \\ \{m \in M^1([0,\infty) \times \mathbb{R}^N);\ \forall \theta \in \mathcal{C}_c([0,\infty) \times \mathbb{R}^N)),\ \mu_\theta \in \mathcal{C}(\mathbb{R})\}.$$

The existence of kinetic solutions of (5.1) is only known when the flux satisfies additionnal regularity assumptions. Assume that $a_i \in \mathcal{C}^1_{\text{per}}(Y \times \mathbb{R})$ for $1 \leq i \leq N$, and that there exists a constant C such that

$$|a(y,\xi)| \leq C\,(1 + |\xi|) \quad \forall y \in Y\ \forall \xi \in \mathbb{R}. \tag{5.25}$$

Under such hypotheses, it is proved in [15] that for all $u_0 \in L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$, there exists a unique function $u^\varepsilon \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N))$ such that $\chi(\xi, u^\varepsilon)$ is a solution of (5.1); u^ε is called the kinetic solution of (5.1)-(5.2). And if u^ε is bounded in $L^\infty((0,T) \times \mathbb{R}^N)$ for all $T > 0$, then u^ε is the entropy solution of (5.1). Moreover, a contraction principle holds between kinetic solutions.

Let us now state the convergence result for kinetic solutions :

Theorem 9. *Let $A \in W^{2,\infty}_{per,loc}(Y \times \mathbb{R})$ such that $\operatorname{div}_y A(y,\xi) = 0$ for all y, ξ. Assume that $a_i \in \mathcal{C}^1_{per}(Y \times \mathbb{R})$ for $1 \leq i \leq N$, and that* (5.25) *is satisfied. Assume that the initial data u_0 belongs to $L^1(\mathbb{R}^N, \mathcal{C}^1_{per}(Y))$ and satisfies*

$$\sum_{i=1}^N \frac{\partial}{\partial y_i}\left(a_i(y,\xi)\chi(\xi,u_0)\right) = 0.$$

Let $u^\varepsilon \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N))$ be the kinetic solution of (5.1) *with initial data $u_0(x, x/\varepsilon)$. Then there exists a function $u \in L^\infty([0,\infty), L^1(\mathbb{R}^N \times Y))$ such that the convergence results* (5.21) *and* (5.22) *hold, and*

$$\frac{\partial}{\partial y_i}(a(y,\xi)\chi(\xi, u(t,x,y))) = 0 \quad \text{in } \mathcal{D}'.$$

Moreover, if we set

$$\mathcal{M} := \frac{\partial}{\partial t}\chi(\xi, u) + \sum_{i=1}^N a_i(y,\xi)\frac{\partial}{\partial x_i}\chi(\xi, u) \in \mathcal{D}',$$

then $\mathcal{M}$ satisfies (5.19).

Remark 5.1.6. *Let us explain how this result can be generalized to the general case. First, the L^1 setting is not adapted to this case, because the L^1 norm is not conserved by the equation in general. Hence another notion of kinetic solutions is needed; the correct functional space should be of the type $V + L^1(\mathbb{R}^N)$, where V is a fixed solution of the cell problem.*

Then, the crucial point in Theorem 9 is to find a sequence u_0^n such that u_0^n converges towards u_0 in $L^1(\mathbb{R}^N, \mathcal{C}_{per}(Y))$, and for all $n \in \mathbb{N}$, u_0^n satisfies (5.15), (5.16). *Finding such a sequence is easy in the divergence-free case, but seems more difficult in the general case, since solutions of the cell problem are not known. This seems to be the main obstacle to the generalization of Theorem 9 to arbitrary fluxes. If this step is admitted, it is likely that the proof of Theorem 9 can be adapted to general settings.*

The plan of the paper is the following: in section 5.2 we prove, under the hypotheses of Theorem 8, that the two-scale limit of the sequence $\mathbf{1}_{\xi < u^\varepsilon(t,x)}$ is a generalized kinetic solution of the limit system. In section 5.3, we study the limit problem introduced in Definition 5.1.1 and we prove the rigidity and uniqueness results in Theorem 7; hence Theorem 7 and 8 will be proved by the end of section 5.3. In section 5.4, we study a relaxation model of BGK type, approaching the limit system in the divergence free case. In section 5.5, we prove Proposition 5.1.1. Eventually, in section 5.6, we have gathered further remarks on the notion of limit evolution problem.

5.2 Asymptotic behavior of the sequence u^ε

In this section, we prove that the two-scale limit of the sequence $f^\varepsilon = \mathbf{1}_{\xi < u^\varepsilon(t,x)}$, say $f^0(t,x,y,\xi)$, is a generalized kinetic solution of the limit system; thus the existence result of Theorem 7 follows from this section. The organization is the following: we first derive some basic (microscopic) properties for the function f^0. Then we explain how regularization by convolution can be used in two-scale problems. The two other subsections are devoted to the other properties of the limit system, namely condition (5.14) and the strong continuity at time $t = 0$.

5.2.1 Basic properties of f^0

We use the concept of two-scale convergence, formalized by G. Allaire after an idea of G. N'Guetseng (see [3, 56]). The fundamental result in [3] can be generalized to the present setting as follows:

Corollary 5.2.1. *Let $(g^\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^\infty((0,\infty)\times\mathbb{R}^{N+1})$. Then there exists a function $g^0\in L^\infty((0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$, and a subsequence (ε_n) such that $\varepsilon_n\to 0$ as $n\to\infty$, such that*

$$\int_0^\infty\int_{\mathbb{R}^{N+1}} g^{\varepsilon_n}(t,x,\xi)\psi\left(t,x,\frac{x}{\varepsilon},\xi\right)\,dt\,dx\,d\xi \to$$
$$\to \int_0^\infty\int_{\mathbb{R}^N\times Y\times\mathbb{R}} g^0(t,x,y,\xi)\psi(t,x,y,\xi)\,dt\,dx\,dy\,d\xi$$

for all functions $\psi\in L^1((0,\infty)\times\mathbb{R}^{N+1};\mathcal{C}_{per}(Y))$.

Here, the sequence f^ε is bounded by 1 in L^∞; hence we can extract a subsequence, still denoted by ε, and find a function $f^0\in L^\infty((0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ such that (f^ε) two-scale converges towards f^0. It is easily checked that f^0 inherits the following properties from the sequence f^ε

$$0\le f^0(t,x,y,\xi)\le 1, \tag{5.26}$$
$$\partial_\xi f^0\le 0. \tag{5.27}$$

Now, let us prove (5.7)-(5.8): let

$$M := \max\left(||u_1||_\infty, ||u_2||_\infty\right),$$

where u_1,u_2 are the functions appearing in assumption (5.16). Since $u_i\left(x/\varepsilon\right)$ is a stationary solution of (5.1), by a comparison principle for equation (5.1), we deduce that

$$u_1\left(\frac{x}{\varepsilon}\right)\le u^\varepsilon(t,x)\le u_2\left(\frac{x}{\varepsilon}\right)\quad\text{for almost every } t>0,\ x\in\mathbb{R}^N.$$

Thus $||u^\varepsilon||_{L^\infty([0,\infty)\times\mathbb{R}^N}\le M$, and for almost every t,x,ξ, for all $\varepsilon>0$,

$$f^\varepsilon(t,x,\xi)=1\quad\text{if } \xi<-M,$$
$$f^\varepsilon(t,x,\xi)=0\quad\text{if } \xi>M.$$

Passing to the two-scale limit, we infer (5.7) and (5.8).

Now, we derive a microscopic equation for f^0. First, multiplying (5.4) by $S'(\xi)$, with $S'\in\mathcal{D}(\mathbb{R})$, and integrating on $(0,T)\times B_R\times\mathbb{R}$, with $T>0$, $R>0$, yields

$$\begin{aligned}
&\int_{B_R}\left(S(u^\varepsilon(T,x)) - S\left(u_0\left(x,\frac{x}{\varepsilon}\right)\right)\right)\,dx\\
+&\int_0^T\int_{\mathbb{R}}\int_{\partial B_R} a\left(\frac{x}{\varepsilon},\xi\right)\cdot n_R(x)f^\varepsilon S'(\xi)\,d\sigma_R(x)\,d\xi\,dt\\
-&\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}}\int_{B_R} a_{N+1}\left(\frac{x}{\varepsilon},\xi\right)f^\varepsilon S''(\xi)\,dx\,d\xi\,dt\\
=&-\int_0^T\int_{\mathbb{R}}\int_{B_R} m^\varepsilon(t,x,\xi)S''(\xi)\,dx\,d\xi\,dt,
\end{aligned}$$

where $n_R(x)$ is the outward-pointing normal to B_R at a given point $x\in\partial B_R$, and $d\sigma_R(x)$ is the Lebesgue measure on ∂B_R.

Hence we obtain the following bound on m^ε

$$\varepsilon m^\varepsilon((0,T)\times B_R\times\mathbb{R})\le C_{T,R}$$

for all $\varepsilon>0$, $R>0$, $T>0$, and $\operatorname{Supp} m^\varepsilon\subset(0,\infty)\times\mathbb{R}\times[-M,M]$.

Consequently, there exists a further subsequence, still denoted by ε, and a non-negative measure $m^0=m^0(t,x,y,\xi)$ such that $\varepsilon m^\varepsilon$ two-scale converges to m^0 (the concept of two-scale convergence can easily be generalized to measures; the arguments are the same as in [3], the only difference lies in the functional spaces). Moreover, $\operatorname{Supp} m^0\subset(0,\infty)\times\mathbb{R}\times Y\times[-M,M]$.

We now multiply (5.4) by test functions of the type $\varepsilon\varphi(t,x,x/\varepsilon,\xi)$, with $\varphi\in\mathcal{D}_{\text{per}}([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$, and we pass to the two-scale limit. We obtain, in the sense of distributions on $(0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R}$

$$\frac{\partial}{\partial y_i}\left(a_i(y,\xi)f^0\right)+\frac{\partial}{\partial\xi}\left(a_{N+1}(y,\xi)f^0\right)=\frac{\partial m^0}{\partial\xi}. \tag{5.28}$$

Thus (5.9) is satisfied, which completes the derivation of the basic properties of f^0.

Now, we define the distribution

$$\mathcal{M}:=\frac{\partial f^0}{\partial t}+\sum_{i=1}^N a_i(y,\xi)\frac{\partial f^0}{\partial x_i}.$$

The distribution $\mathcal{M}$ obviously satisfies (5.6). The next step is to prove that $\mathcal{M}$ satisfies (5.14); since regularizations by convolution are involved in condition (5.14), we now describe the links between convolution and two-scale convergence.

5.2.2 Convolution and two-scale convergence

In this subsection, we wish to make a few remarks concerning the links between convolution and two-scale convergence. Indeed, it is a well-known fact that if a sequence (f_n) weakly converges in $L^2(\mathbb{R}^N)$ towards a function f, then for all convolution kernels $\varphi=\varphi(x)$, the sequence $(f_n*\varphi)$ two-scale converges in L^2 towards $f*\varphi$. It would be convenient to have an analogue property for two-scale limits. However, in general, if a sequence $f^\varepsilon=f^\varepsilon(x)$ is bounded in $L^2(\mathbb{R}^N)$ and two-scale converges towards a function $f=f(x,y)\in L^2(\mathbb{R}^N\times Y)$, then $f^\varepsilon*\varphi$ does not two-scale converge towards $f*_x\varphi$. Indeed, if $\psi=\psi(x,y)\in L^2(\mathbb{R}^N,\mathcal{C}_{\text{per}}(Y))$, then

$$\begin{aligned}&\int_{\mathbb{R}^N} f^\varepsilon*\varphi(x)\psi\left(x,\frac{x}{\varepsilon}\right)\,dx\\ =\;&\int_{\mathbb{R}^{2N}} f^\varepsilon(x')\varphi(x-x')\psi\left(x,\frac{x}{\varepsilon}\right)\,dx\,dx'\\ =\;&\int_{\mathbb{R}^N} dx'\,f^\varepsilon(x')\left[\int_{\mathbb{R}^N}\varphi(x-x')\psi\left(x,\frac{x}{\varepsilon}\right)\,dx\right].\end{aligned}$$

In general, the quantity between brackets in the last integral cannot be written as a function of x' and x'/ε, and it seems difficult to pass to the limit as $\varepsilon\to0$.

In order to get rid of this difficulty, let us suggest the following construction, which is reminiscent of the doubling of variables in the papers of Kružkov, see [69, 70]. With the same notations as above, consider the test function $(\psi *_x \check{\varphi})\left(x, \frac{x}{\varepsilon}\right)$, where $\check{\varphi}(x) := \varphi(-x)\ \forall x \in \mathbb{R}^N$. Then by definition of the two-scale convergence,

$$\int_{\mathbb{R}^N} f^\varepsilon(x)\,[\psi *_x \check{\varphi}]\left(x, \frac{x}{\varepsilon}\right)\,dx \to \int_{\mathbb{R}^N\times Y} f(x,y)\,[\psi *_x \check{\varphi}]\,(x,y)\,dx\,dy$$

And

$$\int_{\mathbb{R}^N} f^\varepsilon(x)\,[\psi *_x \check{\varphi}]\left(x, \frac{x}{\varepsilon}\right)\,dx = \int_{\mathbb{R}^{2N}} f^\varepsilon(x')\varphi(x-x')\psi\left(x, \frac{x'}{\varepsilon}\right)\,dx\,dx',$$
$$\int_{\mathbb{R}^N\times Y} f(x,y)\,[\psi *_x \check{\varphi}]\,(x,y)\,dx\,dy = \int_{\mathbb{R}^N\times Y} [f *_x \varphi]\,(x,y)\psi(x,y)\,dx\,dy.$$

Consequently, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^{2N}} f^\varepsilon(x')\varphi(x-x')\psi\left(x, \frac{x'}{\varepsilon}\right)\,dx\,dx' \to \int_{\mathbb{R}^N\times Y} [f *_x \varphi]\,(x,y)\psi(x,y)\,dx\,dy \quad (5.29)$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$, for all $\psi \in L^2(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$.

In fact, different assumptions on the function ψ can be chosen; the key point is that ψ should be an admissible test function in the sense of Allaire (see [3]). In particular, if there exist functions $\psi_1 \in \mathcal{D}(\mathbb{R}^N)$, $\psi_2 \in L^\infty(Y)$ such that

$$\psi(x,y) = \psi_1(x)\psi_2(y),$$

then ψ is an admissible test function, and (5.29) holds.

5.2.3 Proof of the condition on $\mathcal{M}$

The goal of this subsection is to prove that with

$$\mathcal{M} = \partial_t f^0 + \sum_{i=1}^N a_i(y,\xi)\partial_i f^0,$$

condition (5.14) holds; hence, let $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^N)$, $\theta \in \mathcal{D}([0,\infty)\times\mathbb{R}^N)$, such that

$$\varphi \geq 0,\ \ \theta \geq 0,$$
$$\varphi(t,x) = 0\ \forall t \geq 0\ \forall x \in \mathbb{R}^N.$$

the function φ shall be used as a convolution kernel, which explains the hypothesis on its support. We do not assume that $\theta(t=0) = 0$.
Let $\psi \in \mathcal{G}$ arbitrary (the definition of the set $\mathcal{G}$ is given in definition 5.1.1). We have to prove that the quantity

$$A := \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{2N}\times Y\times\mathbb{R}} d\xi\,dy\,dx\,dz\,ds\,dt\; f^0(s,z,y,\xi)\psi(y,\xi)\theta(t,x) \times$$
$$\times \left\{ \partial_t\varphi(t-s,x-z) + \sum_{i=1}^N a_i(y,\xi)\partial_i\varphi(t-s,x-z) \right\}$$

in non-positive.

Before going into the technicalities, let us explain formally why the property is true; let us forget about the convolution and the regularity issues, and take the test function

$$\theta(t,x)\psi\left(\frac{x}{\varepsilon},\xi\right)$$

in equation (5.4).

Let $R > \max(M, C+1)$; recall that M and C are such that $\mathrm{Supp}\, f^0 \subset [0,\infty)\times\mathbb{R}^N\times Y\times[-M,M]$, and $\psi(y,\xi)=\alpha_-$ if $\xi < -C$. Integrating on $[0,\infty)\times\mathbb{R}^N\times[-R,R]$, we obtain

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^N}\int_{-R}^R f^\varepsilon(t,x,\xi)\left[\partial_t\theta(t,x)+a_i\left(\frac{x}{\varepsilon},\xi\right)\partial_{x_i}\theta(t,x)\right]\ \psi\left(\frac{x}{\varepsilon},\xi\right)\ dx\, d\xi\, dt\\
&-\frac{1}{\varepsilon}\int_0^\infty\int_{\mathbb{R}^N}\int_{-R}^R f^\varepsilon(t,x,\xi)\frac{\partial\mu}{\partial\xi}\left(\frac{x}{\varepsilon},\xi\right)\theta(t,x)\, dx\, d\xi\, dt\\
&+\alpha_-\int_0^\infty\int_{\mathbb{R}^N}\frac{1}{\varepsilon}a_{N+1}\left(\frac{x}{\varepsilon},-R\right)\theta(t,x)\, dt\, dx\\
=\ &\int_0^\infty\int_{\mathbb{R}^N}\int_{-R}^R m^\varepsilon(s,z,\xi)\partial_\xi\psi\left(\frac{x}{\varepsilon},\xi\right)\ dz\, d\xi\, ds\\
&-\int_{\mathbb{R}^N}\int_{-R}^R \mathbf{1}_{\xi<u_0\left(x,\frac{x}{\varepsilon}\right)}\ \theta(t=0,x)\psi\left(\frac{x}{\varepsilon},\xi\right)\ dx\, d\xi.
\end{aligned}$$

Notice that

$$\frac{1}{\varepsilon}a_{N+1}\left(\frac{x}{\varepsilon},-R\right) = -\mathrm{div}_x A\left(\frac{x}{\varepsilon},-R\right),$$

and thus

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^N}\int_{-R}^R f^\varepsilon(t,x,\xi)\left[\partial_t\theta(t,x)+a_i\left(\frac{x}{\varepsilon},\xi\right)\partial_{x_i}\theta(t,x)\right]\ \psi\left(\frac{x}{\varepsilon},\xi\right)\ dx\, d\xi\, dt\\
=\ &\int_0^\infty\int_{\mathbb{R}^N}\int_{-R}^R \left[m^\varepsilon(s,z,\xi)\partial_\xi\psi\left(\frac{x}{\varepsilon},\xi\right)-\frac{1}{\varepsilon}\mu\left(\frac{x}{\varepsilon},\xi\right)\partial_\xi f^\varepsilon(t,x,\xi)\right]\theta(t,x)\, dz\, d\xi\, ds\\
&-\alpha_-\int_0^\infty\int_{\mathbb{R}^N}A_i\left(\frac{x}{\varepsilon},-R\right)\partial_i\theta(t,x)\, dt\, dx\\
&-\int_{\mathbb{R}^N}\int_{-R}^R \mathbf{1}_{\xi<u_0\left(x,\frac{x}{\varepsilon}\right)}\ \theta(t=0,x)\psi\left(\frac{x}{\varepsilon},\xi\right)\ dx\, d\xi\\
\geq\ &-\alpha_-\int_0^\infty\int_{\mathbb{R}^N}A_i\left(\frac{x}{\varepsilon},-R\right)\partial_i\theta(t,x)\, dt\, dx\\
&-\int_{\mathbb{R}^N}\int_{-R}^R \mathbf{1}_{\xi<u_0\left(x,\frac{x}{\varepsilon}\right)}\ \theta(t=0,x)\psi\left(\frac{x}{\varepsilon},\xi\right)\ dx\, d\xi.
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, we retrieve

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} \int_{-R}^{R} f^0(t,x,y,\xi)\left[\partial_t\theta(t,x) + a_i(y,\xi)\,\partial_{x_i}\theta(t,x)\right]\,\psi(y,\xi)\;dx\,dy\,d\xi\,dt\\
\geq\;& -\alpha_- \int_0^\infty \int_{\mathbb{R}^N\times Y} A_i(y,-R)\partial_i\theta(t,x)\,dt\,dx\\
&-\int_{\mathbb{R}^N}\int_{-R}^{R} \mathbf{1}_{\xi<u_0(x,y)}\,\theta(t=0,x)\psi(y,\xi)\;dx\,d\xi\\
=\;& -\int_{\mathbb{R}^N}\int_{-R}^{R} \mathbf{1}_{\xi<u_0(x,y)}\,\theta(t=0,x)\psi(y,\xi)\;dx\,d\xi.
\end{aligned}$$

This means exactly that

$$\frac{\partial}{\partial t}\int_{Y\times\mathbb{R}} f^0\psi + \frac{\partial}{\partial x_i}\int_{Y\times\mathbb{R}} a_i f^0\psi \leq 0,$$

or in other words, that $\int_{Y\times\mathbb{R}} \mathcal{M}\psi \leq 0$ in the sense of distributions on $[0,\infty)\times\mathbb{R}^N$.

Now, we go back to the regularizations by convolution. According to the preceding subsection,

$$\begin{aligned}
A = \lim_{\varepsilon\to 0}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{2N}\times Y\times\mathbb{R}} & d\xi\,dx\,dz\,ds\,dt f^\varepsilon(s,z,\xi)\psi\left(\frac{z}{\varepsilon},\xi\right)\theta(t,x)\times\\
&\times\left\{\partial_t\varphi(t-s,x-z) + \sum_{i=1}^N a_i\left(\frac{z}{\varepsilon},\xi\right)\partial_i\varphi(t-s,x-z)\right\}
\end{aligned}$$

Hence, in (5.4), we consider the test function

$$\phi(s,z,\xi) = \left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]\,\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)K(\xi),$$

where

- K is a cut-off function such that $0\leq K\leq 1$, $K\in\mathcal{D}(\mathbb{R})$, $K(\xi)=1$ if $|\xi|\leq R$ (R is the same parameter as before, and satisfies $R\geq\max(M,C+1)$);

- $\psi_\delta := \psi *_y \varphi_1^\delta *_\xi \varphi_2^\delta$, with $\varphi_1\in\mathcal{D}(\mathbb{R}^N)$, $\varphi_2\in\mathcal{D}(\mathbb{R})$, $0\leq\varphi_i\leq 1$, $\int\varphi_i = 1$ for $i=1,2$, and

$$\varphi_1^\delta(y) = \frac{1}{\delta^N}\varphi_1\left(\frac{y}{\delta}\right),\quad \varphi_2^\delta(\xi) = \frac{1}{\delta}\varphi_2\left(\frac{\xi}{\delta}\right)\quad (0<\delta<1).$$

According to (5.4), we have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^{N+1}} f^\varepsilon(s,z,\xi)\left[\partial_s\phi(s,z,\xi)+\sum_{i=1}^N a_i\left(\frac{z}{\varepsilon},\xi\right)\partial_{z_i}\phi(s,z,\xi)\right]\,dz\,d\xi\,ds\\
+&\ \frac{1}{\varepsilon}\int_0^\infty\int_{\mathbb{R}^{N+1}} f^\varepsilon(s,z,\xi)a_{N+1}\left(\frac{z}{\varepsilon},\xi\right)\partial_\xi\phi(s,z,\xi)\,dz\,d\xi\,ds \qquad (5.30)\\
-&\int_0^\infty\int_{\mathbb{R}^{N+1}} m^\varepsilon(s,z,\xi)\partial_\xi\phi(s,z,\xi)\,dz\,d\xi\,ds \qquad (5.31)\\
+&\int_{\mathbb{R}^{N+1}}\chi\left(\xi,u_0\left(z,\frac{z}{\varepsilon}\right)\right)\,\phi(s=0,z,\xi)\,dz\,d\xi\\
=&\ 0.
\end{aligned}$$

And

$$\begin{aligned}
\partial_s\phi(s,z,\xi) &= -\left[\int_0^\infty\int_{\mathbb{R}^N}\partial_t\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]\,\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)K(\xi),\\
\nabla_z\phi(s,z,\xi) &= -\left[\int_0^\infty\int_{\mathbb{R}^N}\nabla_x\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]\,\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)K(\xi)\\
&\quad+\frac{1}{\varepsilon}\left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]\,(\nabla_y\psi_\delta)\left(\frac{z}{\varepsilon},\xi\right)K(\xi),\\
\partial_\xi\phi(s,z,\xi) &= \left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]K(\xi)\,\partial_\xi\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)\\
&\quad+\left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)\partial_\xi K(\xi)\\
\phi(s=0,z,\xi) &= \left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t,x-z)\,\theta(t,x)\,dt\,dx\right]\,\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)K(\xi)=0.
\end{aligned}$$

Using the assumption on the sign of θ,φ, and the fact that

$$\partial_\xi\psi_\delta=(\partial_\xi\psi)*_y\varphi_1^\delta*_\xi\varphi_2^\delta\geq 0,$$

we derive

$$\left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]K(\xi)\,\partial_\xi\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)\geq 0.$$

Moreover, due to (5.7), (5.8), and the assumptions on ψ and K, we have $\partial_\xi K=0$ on $\operatorname{Supp} m^\varepsilon$, and

$$\begin{aligned}
&\left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)\partial_\xi K(\xi)f^\varepsilon(s,z,\xi)\\
=&\ \alpha_-\left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\,dt\,dx\right]\partial_\xi K(\xi)\mathbf{1}_{\xi<M}.
\end{aligned}$$

Hence, we obtain, for all $\varepsilon,\delta>0$,

$$
\begin{aligned}
&-\int d\xi\, dx\, dz\, ds\, dt\; f^\varepsilon(s,z,\xi)\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)\theta(t,x)\times\\
&\qquad\times\left\{\partial_t\varphi(t-s,x-z)+\sum_{i=1}^N a_i\left(\frac{z}{\varepsilon},\xi\right)\partial_i\varphi(t-s,x-z)\right\}\\
&+\frac{1}{\varepsilon}\int f^\varepsilon(s,z,\xi)\, a\left(\frac{z}{\varepsilon},\xi\right)\cdot\nabla_{y,\xi}\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)\varphi(t-s,x-z)\theta(t,x)K(\xi)\, dt\, dx\, ds\, dz\, d\xi\\
&+\frac{\alpha_-}{\varepsilon}\int\varphi(t-s,x-z)\,\theta(t,x)\,\partial_\xi K(\xi)a_{N+1}\left(\frac{z}{\varepsilon},\xi\right)\, dt\, dx\, ds\, dz\, d\xi\\
&\geq 0.
\end{aligned}
$$

Following the formal calculations above, we have to investigate the sign of the term

$$\int f^\varepsilon(s,z,\xi)\, a\left(\frac{z}{\varepsilon},\xi\right)\cdot\nabla_{y,\xi}\psi_\delta\left(\frac{z}{\varepsilon},\xi\right)\varphi(t-s,x-z)\,\theta(t,x)\,K(\xi)\, dt\, dx\, ds\, dz\, d\xi.$$

Since $\operatorname{div}_{y,\xi}(a\psi)=-\partial_\xi\mu$, we have

$$\operatorname{div}_{y,\xi}(a\psi_\delta)=-\frac{\partial\mu_\delta}{\partial\xi}+r_\delta$$

where $\mu_\delta=\mu *_y \varphi_1^\delta *_\xi \varphi_2^\delta$, and r_δ is a remainder term. Then

$$
\begin{aligned}
&-\int_0^\infty\int_{\mathbb{R}^{N+1}} f^\varepsilon(s,z,\xi)\frac{\partial\mu_\delta}{\partial\xi}\left(\frac{x}{\varepsilon},\xi\right)\left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\, dt\, dx\right] ds\, dz\, d\xi\\
&=-\int_0^\infty\int_{\mathbb{R}^{N+1}}\delta(\xi=u^\varepsilon(t,x))\mu_\delta\left(\frac{x}{\varepsilon},\xi\right)\times\\
&\qquad\times\left[\int_0^\infty\int_{\mathbb{R}^N}\varphi(t-s,x-z)\,\theta(t,x)\, dt\, dx\right]\leq 0.
\end{aligned}
$$

Hence, we have to prove that as $\delta\to 0$,

$$r_\delta\to 0\quad\text{in } L^1_{\text{loc}}(Y\times\mathbb{R}).$$

The proof is quite classical. We have

$$
\begin{aligned}
r_\delta(y,\xi) &= a(y,\xi)\psi *\left(\nabla_{y,\xi}\varphi_1^\delta\varphi_2^\delta\right)-[a(y,\xi)\psi]*\left(\nabla_{y,\xi}\varphi_1^\delta\varphi_2^\delta\right)\\
&= \sum_{i=1}^N\int[a_i(y,\xi)-a_i(y_1,\xi_1)]\,\psi(y_1,\xi_1)\partial_{y_i}\varphi_1^\delta(y-y_1)\varphi_2^\delta(\xi-\xi_1)\, dy_1\, d\xi_1\\
&\quad+\int[a_{N+1}(y,\xi)-a_{N+1}(y_1,\xi_1)]\,\psi(y_1,\xi_1)\varphi_1^\delta(y-y_1)\partial_\xi\varphi_2^\delta(\xi-\xi_1)\, dy_1\, d\xi_1
\end{aligned}
$$

Thus, we compute, for $(y,y_1,\xi,\xi_1)\in\mathbb{R}^{2N+2}$, $1\leq i\leq N+1$,

$$
\begin{aligned}
a_i(y,\xi)-a_i(y_1,\xi_1) &= (y-y_1)\cdot\int_0^1\nabla_y a_i(\tau y+(1-\tau)y_1,\tau\xi+(1-\tau)\xi_1)\, d\tau\\
&\quad+(\xi-\xi_1)\cdot\int_0^1\partial_\xi a_i(\tau y+(1-\tau)y_1,\tau\xi+(1-\tau)\xi_1)\, d\tau.
\end{aligned}
$$

Set, for $1 \leq k, i \leq N$, $y \in \mathbb{R}^N$, $\xi \in \mathbb{R}$,

$$\phi_{k,i}(y,\xi) = y_k \frac{\partial \varphi_1}{\partial y_i}(y)\varphi_2(\xi), \qquad \phi_{k,N+1}(y,\xi) = y_k \frac{\partial \varphi_2}{\partial \xi}(\xi)\varphi_1(y),$$
$$\zeta_i(y,\xi) = \xi \frac{\partial \varphi_1}{\partial y_i}(y)\varphi_2(\xi), \qquad \zeta_{N+1}(y,\xi) = \xi \frac{\partial \varphi_2}{\partial \xi}(\xi)\varphi_1(y).$$

Notice that

$$\int_{\mathbb{R}^{N+1}} \phi_{k,i} = -\delta_{k,i}, \qquad \int_{\mathbb{R}^{N+1}} \zeta_i = -\delta_{N+1,i}.$$

Then, setting $X = (y,\xi)$, $X_1 = (y_1, \xi_1)$,

$$\begin{aligned} r_\delta(X) \;=\; & \sum_{i=1}^{N+1} \int \frac{\partial a_i}{\partial y_k}(\tau X + (1-\tau)X_1)\psi(X_1)\phi^\delta_{k,i}(X - X_1)\, dy_1\, d\xi_1\, d\tau \\ & + \sum_{i=1}^{N+1} \int \frac{\partial a_i}{\partial \xi}(\tau X + (1-\tau)X_1)\psi(X_1)\zeta^\delta_i(X - X_1)\, dy_1\, d\xi_1\, d\tau. \end{aligned}$$

Hence as $\delta \to 0$, r_δ converges to

$$-\mathrm{div}_{y,\xi}(a(y,\xi))\, \psi(y,\xi) = 0$$

in $L^p_{\mathrm{loc}}(\mathbb{R}^{N+1})$ for any $p < \infty$ and for all $(t,x) \in [0,\infty) \times \mathbb{R}^N$. We now pass to the limit as $\delta \to 0$, with ε fixed, and we obtain

$$\begin{aligned} & - \int d\xi\, dx\, dz\, ds\, dt\; f^\varepsilon(s,z,\xi)\psi\left(\frac{z}{\varepsilon},\xi\right)\theta(t,x) \times \\ & \qquad \times \left\{ \partial_t\varphi(t-s,x-z) + \sum_{i=1}^N a_i\left(\frac{z}{\varepsilon},\xi\right)\partial_i\varphi(t-s,x-z) \right\} \\ & -\alpha_- \int \theta(t,x)\, \partial_\xi K(\xi) A\left(\frac{z}{\varepsilon},\xi\right) \cdot \nabla_x\varphi(t-s,x-z)\, dt\, dx\, ds\, dz\, d\xi \\ \geq \;& 0. \end{aligned}$$

Passing to the limit as ε vanishes, we are led to

$$\begin{aligned} & - \int d\xi\, dx\, dz\, ds\, dy\, dt\; f^0(s,z,y,\xi)\psi\,(y,\xi)\,\theta(t,x) \times \\ & \qquad \times \left\{ \partial_t\varphi(t-s,x-z) + \sum_{i=1}^N a_i\,(y,\xi)\,\partial_i\varphi(t-s,x-z) \right\} \\ & -\alpha_- \int \theta(t,x)\, \partial_\xi K(\xi) A\,(y,\xi) \cdot \nabla_x\varphi(t-s,x-z)\, dt\, dx\, ds\, dy\, dz\, d\xi \\ \geq \;& 0. \end{aligned}$$

Since

$$\int \theta(t,x)\nabla_x\varphi(t-s,x-z)dtdxdsdz = -\left(\int \theta(t,x)\, dt\, dx\right)\left(\int \nabla_z\varphi(s,z)\, ds\, dz\right) = 0,$$

we deduce that

$$\int d\xi\, dx\, dz\, ds\, dy\, dt\; f^0(s,z,y,\xi)\psi\,(y,\xi)\,\theta(t,x)\times$$
$$\times\left\{\partial_t\varphi(t-s,x-z)+\sum_{i=1}^N a_i\,(y,\xi)\,\partial_i\varphi(t-s,x-z)\right\}\leq 0,$$

which means that f^0 satisfies condition (5.14). There only remains to check the strong continuity of f at time $t=0$.

5.2.4 Strong continuity at time $t = 0$

The continuity property forf^0 is inherited from uniform continuity properties at time $t=0$ for the sequence f^ε. This is strongly linked to the well-preparedness of the initial data (condition (5.9)), that is, the fact that for all $x\in\mathbb{R}^N$, $u_0(x,\cdot)$ is an entropy solution of the cell problem

$$\mathrm{div}_y A(y,u_0(x,y))=0.$$

First, let us consider a regularization of the initial data

$$g_n^\delta=f_0 *_x \rho_n *_y \varphi_1^\delta *_\xi \varphi_2^\delta.$$

with $\rho_n\in\mathcal{D}(\mathbb{R}^N_x)$ a convolution kernel ($n\in\mathbb{N}$), $\delta>0$, and φ_i^δ defined as in the previous subsection. Then we can write

$$\begin{aligned}&\sum_{i=1}^N a_i\left(\frac{x}{\varepsilon},\xi\right)\cdot\frac{\partial}{\partial x_i}\left[g_n^\delta\left(x,\frac{x}{\varepsilon},\xi\right)\right]+\frac{1}{\varepsilon}a_{N+1}\left(\frac{x}{\varepsilon},\xi\right)\frac{\partial}{\partial\xi}g_n^\delta\left(x,\frac{x}{\varepsilon},\xi\right)\\ =\;&\frac{1}{\varepsilon}a\left(\frac{x}{\varepsilon},\xi\right)\cdot\left(\nabla_{y,\xi}g_n^\delta\right)\left(x,\frac{x}{\varepsilon},\xi\right)+\sum_{i=1}^N a_i\left(\frac{x}{\varepsilon},\xi\right)\left(\frac{\partial}{\partial x_i}g_n^\delta\right)\left(x,\frac{x}{\varepsilon},\xi\right) \quad (5.32)\\ :=\;& r^\varepsilon_{n,\delta}.\end{aligned}$$

Notice that

$$||\nabla_x g_n^\delta||_{L^\infty(\mathbb{R}^N\times Y\times\mathbb{R})}\leq||\nabla_x\rho_n||_{L^1(\mathbb{R}^N)},$$

and

$$a\,(y,\xi)\,\nabla_{y,\xi}g_n^\delta\,(x,y,\xi)=\partial_\xi m_n^\delta+r_n^\delta,$$

where

$$\begin{aligned}m_n^\delta&=m_0 *_x \rho_n *_y \varphi_1^\delta *_\xi \varphi_2^\delta,\\ r_n^\delta(x,y,\xi)&=a\,(y,\xi)\,\nabla_{y,\xi}g_n^\delta\,(x,y,\xi)-\left[af_0 *_x \rho_n\right] *_{y,\xi}\nabla_{y,\xi}\varphi_1^\delta(y)\varphi_2^\delta(\xi).\end{aligned}$$

Then for all $n\in\mathbb{N}$, for all $x\in\mathbb{R}^N$, r_n^δ vanishes as $\delta\to 0$ in $L^1_{\mathrm{loc}}(Y\times\mathbb{R})$ and almost everywhere. The proof of this fact is exactly the same as in the preceding subsection, and thus, we leave the details to the reader. As a consequence, we can write

$$r^\varepsilon_{n,\delta}(x,\xi)=\frac{1}{\varepsilon}\partial_\xi m_n^\delta\left(x,\frac{x}{\varepsilon},\xi\right)+R^\varepsilon_{n,\delta}(x,\xi),$$

and there exists a constant C_n, independent of ε, such that for all $n \in \mathbb{N}$, for all $\varepsilon > 0$, and for almost every x, ξ

$$\limsup_{\delta \to 0} |R^\varepsilon_{n,\delta}(x,\xi)| \leq C_n.$$

Moreover, $R^\varepsilon_{n,\delta}(x,\xi) = 0$ if $\xi > M + \delta$. In the following, we take $\delta < 1$.

Now, we multiply (5.4) by $1-2g^\delta_n(x, x/\varepsilon, \xi)$, and (5.32) by $1-2f^\varepsilon(t,x,\xi)$. Setting

$$\begin{aligned} h^\varepsilon_{n,\delta}(t,x,\xi) &:= f^\varepsilon(t,x,\xi)\left[1 - 2g^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right] + g^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right)\left[1 - 2f^\varepsilon(t,x,\xi)\right] \\ &= \left|f^\varepsilon(t,x,\xi) - g^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right|^2 + g^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right) - \left|g^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right|^2, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} h^\varepsilon_{n,\delta}(t,x,\xi) + \sum_{i=1}^N a_i\left(\frac{x}{\varepsilon}, \xi\right)\partial_{x_i} h^\varepsilon_{n,\delta}(t,x,\xi) + \frac{1}{\varepsilon} a_{N+1}\left(\frac{x}{\varepsilon}, \xi\right)\partial_\xi h^\varepsilon_{n,\delta}(t,x,\xi) \\ = \; &\frac{\partial m^\varepsilon}{\partial \xi}\left[1 - 2g^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right] + \frac{1}{\varepsilon}\partial_\xi m^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right)\left[1 - 2f^\varepsilon(t,x,\xi)\right] \\ &+ R^\varepsilon_{n,\delta}(x,\xi)\left[1 - 2f^\varepsilon(t,x,\xi)\right]. \end{aligned} \tag{5.33}$$

Notice that

$$\begin{aligned} \partial_\xi\left[1 - 2f^\varepsilon(t,x,\xi)\right] &= 2\delta(\xi = u^\varepsilon(t,x)), \\ \frac{\partial}{\partial \xi}\left(1 - 2g^\delta_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right) &= \geq 0. \end{aligned}$$

Notice also that $f^\varepsilon(t,x,\xi) - g^\delta_n(x, x/\varepsilon, \xi) = 0$ if $|\xi|$ is large enough ($|\xi| > M + 1$), whence $h^\varepsilon_{n,\delta}$ has compact support in ξ.

Take a cut-off function $\zeta \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\zeta(x) = e^{-|x|}$ when $|x| \geq 1$, and $\frac{1}{e} \leq \zeta(x) \leq 1$ for $|x| \leq 1$. Then there exists a constant C such that

$$|\nabla_x \zeta(x)| \leq C\zeta(x) \quad \forall x \in \mathbb{R}^N.$$

Hence, multiplying (5.33) by $\zeta(x)$ and integrating on $\mathbb{R}^{N+1}$, we obtain a bound of the type

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^{N+1}} h^\varepsilon_{n,\delta}(t,x,\xi)\zeta(x)\,dx\,d\xi &\leq C\int_{\mathbb{R}^{N+1}} h^\varepsilon_{n,\delta}(t,x,\xi)\zeta(x)\,dx\,d\xi \\ &+ \int_{\mathbb{R}^{N+1}} \left|R^\varepsilon_{n,\delta}(x,\xi)\right| \left|1 - 2f^\varepsilon(t,x,\xi)\right|\zeta(x)\,dx\,d\xi. \end{aligned}$$

Using Gronwall's lemma and passing to the limit as $\delta \to 0$ with ε and $n \in \mathbb{N}$ fixed, we retrieve, for all $t \geq 0$,

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}} \left|f^\varepsilon(t,x,\xi) - g_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right|^2 \zeta(x)\,dx\,d\xi \\ \leq \; &e^{Ct}\int_{\mathbb{R}^{N+1}} \left|f_0\left(x, \frac{x}{\varepsilon}, \xi\right) - g_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right|^2 \zeta(x)\,dx\,d\xi \\ &+ e^{Ct}\int_{\mathbb{R}^{N+1}} \left[g_n\left(x, \frac{x}{\varepsilon}, \xi\right) - \left|g_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right|^2\right]\zeta(x)\,dx\,d\xi \\ &+ C_n(e^{Ct} - 1), \end{aligned}$$

where the constant C_n does not depend on ε, and $g_n = f_0 *_x \rho_n$. And for all $n \in \mathbb{N}$, $\varepsilon > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}} \left| f_0\left(x, \frac{x}{\varepsilon}, \xi\right) - g_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right|^2 \zeta(x)\, dx\, d\xi \\
\leq\ &\int_{\mathbb{R}^{N+1}} \int_{\mathbb{R}^N} \left| f_0\left(x, \frac{x}{\varepsilon}, \xi\right) - f_0\left(x', \frac{x}{\varepsilon}, \xi\right)\right|^2 \rho_n(x-x')\zeta(x)\, dx\, dx'\, d\xi \\
\leq\ &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| u_0\left(x, \frac{x}{\varepsilon}, \xi\right) - u_0\left(x', \frac{x}{\varepsilon}, \xi\right)\right| \rho_n(x-x')\zeta(x)\, dx\, dx' \\
\leq\ &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \sup_{y\in Y} \left| u_0\left(x, y, \xi\right) - u_0\left(x', y, \xi\right)\right| \rho_n(x-x')\zeta(x)\, dx\, dx'.
\end{aligned}$$

The right-hand side of the above inequality vanishes as $n \to \infty$ because $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$. Similarly,

$$\begin{aligned}
&\int_{\mathbb{R}^{N+1}} \left[g_n\left(x, \frac{x}{\varepsilon}, \xi\right) - \left| g_n\left(x, \frac{x}{\varepsilon}, \xi\right)\right|^2\right] \zeta(x)\, dx\, d\xi \\
\leq\ &\int_{\mathbb{R}^{N+1}} \left[g_n\left(x, \frac{x}{\varepsilon}, \xi\right) - f_0\left(x, \frac{x}{\varepsilon}, \xi\right)\right] \zeta(x)\, dx\, d\xi \\
&+ \int_{\mathbb{R}^{N+1}} \left[f_0\left(x, \frac{x}{\varepsilon}, \xi\right)^2 - g_n\left(x, \frac{x}{\varepsilon}, \xi\right)^2\right] \zeta(x)\, dx\, d\xi \\
\leq\ &3\int_{\mathbb{R}^{N+1}} \left| g_n\left(x, \frac{x}{\varepsilon}, \xi\right) - f_0\left(x, \frac{x}{\varepsilon}, \xi\right)\right| \zeta(x)\, dx\, d\xi \\
\leq\ &3\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \sup_{y\in Y} \left| u_0\left(x, y, \xi\right) - u_0\left(x', y, \xi\right)\right| \rho_n(x-x')\zeta(x)\, dx\, dx'.
\end{aligned}$$

Hence, we deduce that there exists a function $\omega : [0,\infty) \to [0,\infty)$, independent of ε and satisfying $\lim_{t\to 0} \omega(t) = 0$, such that

$$\int_{\mathbb{R}^{N+1}} \left| f^\varepsilon(t,x,\xi) - f_0\left(x, \frac{x}{\varepsilon}, \xi\right)\right| \zeta(x)\, dx\, d\xi \leq \omega(t)$$

for all $t > 0$.

Then, we prove that the same property holds for the function f^0, that is, the two-scale limit of the sequence f^ε. Indeed, we write

$$\left| f^\varepsilon(t,x,\xi) - \mathbf{1}_{\xi < u_0\left(x, \frac{x}{\varepsilon}\right)} \right| = f^\varepsilon - 2f^\varepsilon \mathbf{1}_{\xi < u_0\left(x, \frac{x}{\varepsilon}\right)} + \mathbf{1}_{\xi < u_0\left(x, \frac{x}{\varepsilon}\right)};$$

let $\theta \in L^\infty([0,\infty))$ with compact support and such that $\theta \geq 0$. Then for all $\varepsilon > 0$,

$$\int_0^\infty \int_{\mathbb{R}^{N+1}} \left[f^\varepsilon - 2f^\varepsilon \mathbf{1}_{\xi < u_0\left(x, \frac{x}{\varepsilon}\right)} + \mathbf{1}_{\xi < u_0\left(x, \frac{x}{\varepsilon}\right)} \right] \zeta(x)\theta(t)\, dx\, d\xi\, dt \leq \int_0^\infty \omega(t)\theta(t)\, dt.$$

Since $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$, it is an admissible test function in the sense of G. Allaire (see [3]); we deduce that $\mathbf{1}_{\xi < u_0}$ is also an admissible test function. This is not entirely obvious because it is a discontinuous function of u_0. However, this difficulty can be overcome thanks to an argument similar to the one developed below in subsection 5.3.3, and which we do not reproduce here. Thus, we can pass to the two-scale limit in the above inequality. We obtain

$$\int_0^\infty \int_{\mathbb{R}^{N+1}\times Y} \left(f^0(t,x,y,\xi) - |f^0(t,x,y\xi)|^2 + |f^0(t,x,y,\xi) - \mathbf{1}_{\xi<u_0(x,y)}|^2\right) \times \\ \times \theta(t)\zeta(x)\,dt\,dx\,dy\,d\xi \le \int_0^\infty \theta(t)\omega(t)\,dt$$

Notice that $f^0 - |f^0|^2 \ge 0$ almost everywhere. As a consequence, taking $\theta(t) = \mathbf{1}_{0<t<\tau}$, with $\tau > 0$ arbitrary, we deduce that

$$\frac{1}{\tau}\int_0^\tau \left|f^0(t) - \chi(\xi, u_0(x,y))\right|^2 \zeta(x)\,dt\,dx\,dy \le \frac{1}{\tau}\int_0^\tau \omega(t)\,dt,$$

and the left-hand side vanishes as $\tau \to 0$. Thus the continuity property is satisfied at time $t = 0$.

Hence, we have proved that any two-scale limit of the sequence f^ε is a solution of the limit system. Thus the existence result in Theorem 7 is proved, as well as the convergence result of Theorem 8. We now tackle the proof of the uniqueness and rigidity results of Theorem 7. The strong convergence result of Theorem 5.1.1 will follow from the rigidity.

5.3 Uniqueness of solutions of the limit evolution problem

In this section, we prove the second and the third point in Theorem 7, that is, if f is any solution of the limit evolution problem, then there exists a function $u \in L^\infty([0,\infty)\times\mathbb{R}^N\times Y)$ such that $f(t,x,y,\xi) = \mathbf{1}_{\xi<u(t,x,y)}$ almost everywhere, and if $f_1 = \mathbf{1}_{\xi<u_1}$, $f_2 = \mathbf{1}_{\xi<u_2}$ are two generalized kinetic solutions, then the contraction principle (5.17) holds.

5.3.1 The rigidity result

Let f be a generalized kinetic solution of the limit problem, with initial data $\mathbf{1}_{\xi<u_0}$. The rigidity result relies on the comparison between f and f^2. Precisely, we prove that $f = f^2$ almost everywhere, and since $\partial_\xi f \le 0$, there exists a function u such that $f = \mathbf{1}_{\xi<u}$. Thus, we now turn to the derivation of the equality $f = f^2$.

Let $\delta > 0$ arbitrary, and let $\theta_1 \in \mathcal{D}(\mathbb{R}), \theta_2 \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\theta_1 \ge 0,\ \ \theta_2 \ge 0,$$
$$\int_{\mathbb{R}} \theta_1 = \int_{\mathbb{R}^N} \theta_2 = 1,$$
$$\operatorname{Supp}\theta_1 \subset [-1,0] \text{ and } \theta_1(0) = 0.$$

We set, for $(t,x) \in \mathbb{R}^{N+1}$

$$\theta^\delta(t,x) = \frac{1}{\delta^{N+1}}\theta_1\left(\frac{t}{\delta}\right)\theta_2\left(\frac{x}{\delta}\right).$$

Set $f^\delta := f *_{t,x} \theta^\delta$, $\mathcal{M}^\delta := \mathcal{M} *_{t,x} \theta^\delta$. Then f^δ is a solution of

$$\frac{\partial f^\delta}{\partial t} + \sum_{i=1}^N a_i(y,\xi) \frac{\partial f^\delta}{\partial x_i} = \mathcal{M}^\delta.$$

Moreover, f^δ satisfies the following properties

$$0 \le f^\delta \le 1, \tag{5.34}$$
$$\operatorname{div}_{y,\xi}(a(y,\xi) f^\delta) = \partial_\xi m *_{t,x} \theta^\delta, \tag{5.35}$$
$$\partial_\xi f^\delta \le 0, \tag{5.36}$$
$$f^\delta(\cdot,\xi) = 0 \quad \text{if } \xi > M, \quad f^\delta(\cdot,\xi) = 1 \quad \text{if } \xi < -M, \tag{5.37}$$

whereas $\mathcal{M}^\delta$ satisfies

$$\mathcal{M}^\delta \in \mathcal{C}((0,T)\times\mathbb{R}^N, L^2(Y\times\mathbb{R})) \cap L^\infty([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R}), \tag{5.38}$$
$$\mathcal{M}^\delta(\cdot,\xi) = 0 \quad \text{if } |\xi| > M, \tag{5.39}$$
$$\int_{Y\times\mathbb{R}} \mathcal{M}^\delta(t,x)\psi \le 0 \quad \forall \psi \in \mathcal{G}\ \forall (t,x) \in [0,\infty)\times\mathbb{R}^N. \tag{5.40}$$

In particular, notice that $(1 - 2f^\delta(t,x)) \in \mathcal{G}$ for all t, x, and

$$f^\delta(t,x,y,\xi) - f^\delta(t,x,y,\xi)^2 = 0 \text{ if } |\xi| > M.$$

Let $\zeta \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a cut-off function as in the previous subsection. We multiply by $(1-2f^\delta)\zeta(x)$ the equation satisfied by f^δ, and we integrate over $\mathbb{R}^N \times Y \times \mathbb{R}$. We obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta - |f^\delta|^2\right)\zeta - \int_{\mathbb{R}^N\times Y\times\mathbb{R}} a_i(y,\xi)\partial_i\zeta(x)\left(f^\delta - |f^\delta|^2\right) \\ = &\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}^\delta\left(1-2f^\delta\right)\zeta \le 0. \end{aligned}$$

We then deduce successively, using Gronwall's lemma,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta - |f^\delta|^2\right)\zeta &\le C\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta - |f^\delta|^2\right)\zeta, \\ \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta(t) - |f^\delta(t)|^2\right)\zeta &\le e^{Ct}\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta(t=0) - |f^\delta(t=0)|^2\right)\zeta \quad \forall t > 0, \\ \int_0^T\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta - |f^\delta|^2\right)\zeta &\le \frac{e^{CT}-1}{C}\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta(t=0) - |f^\delta(t=0)|^2\right)\zeta, \end{aligned} \tag{5.41}$$

and the constant C depends only on $||a||_{L^\infty(Y\times[-R,R])}$.

Now, let us check that $f^\delta(t=0)$ strongly converges towards $\mathbf{1}_{\xi<u_0} = f_0$ at time $t=0$. In fact, the main difference between the rigidity result of Theorem 7 and the one for generalized kinetic solutions of scalar conservation laws (see chapter 4 in [59]) lies in this particular point. Indeed, in the case of scalar conservation laws, the continuity property can be inferred from the equation itself; in the present case, the

lack of structure of the right-hand side $\mathcal{M}$ prevents us from deriving such a result, and hence the continuity of solutions at time $t = 0$ is a necessary assumption in Definition 5.1.1.

Using hypothesis (5.13), we write, for almost every x, y, ξ,

$$\begin{aligned} f^\delta(t=0,x,y,\xi) &= \int_{\mathbb{R}^{N+1}} f(s,z,y,\xi)\theta^\delta(-s,x-z)\,ds\,dz \\ \left[f^\delta(0) - f_0 *_x \theta_2^\delta\right](x,y,\xi) &= \int_{\mathbb{R}^{N+1}} \left(f(s,z,y,\xi) - f_0(z,y,\xi)\right)\theta^\delta(-s,x-z)\,ds\,dz. \end{aligned}$$

As a consequence, for all $\delta > 0$

$$\begin{aligned} &\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left|f^\delta(t=0) - f_0 *_x \theta_2^\delta\right|^2 \zeta(x)\,dx\,dy\,d\xi \\ \le\; &\int_{\mathbb{R}^N\times Y\times\mathbb{R}}\int_{\mathbb{R}^{N+1}} \left|f(s,z,y,\xi) - f_0(z,y,\xi)\right|^2 \zeta(x)\theta^\delta(-s,x-z)\,dx\,dy\,d\xi\,ds\,dz \\ \le\; &\int_{\mathbb{R}} ||f(s) - f_0||^2_{L^2(\mathbb{R}^N\times Y\times\mathbb{R},\zeta(x)\,dx\,dy\,d\xi)} \frac{1}{\delta}\theta_1\left(\frac{-s}{\delta}\right)\,ds\,dx\,dy\,d\xi \\ &+||\zeta - \zeta * \check{\theta}_2^\delta||_{L^1(\mathbb{R}^N)} \\ \le\; &\frac{C}{\delta}\int_0^\delta ||f(s) - f_0||^2_{L^2(\mathbb{R}^N\times Y\times\mathbb{R},\zeta(x)\,dx\,dy\,d\xi)}\,ds + ||\zeta - \zeta * \check{\theta}_2^\delta||_{L^1(\mathbb{R}^N)}. \end{aligned}$$

The right-hand side of the last inequality vanishes as $\delta \to 0$, and thus $f^\delta(t = 0)$ converges towards f_0 as $\delta \to 0$ in $L^2(\mathbb{R}^N \times Y \times \mathbb{R}, \zeta(x)\,dx\,dy\,d\xi)$, and hence also in $L^1(\mathbb{R}^N \times Y \times \mathbb{R}, \zeta(x)\,dx\,dy\,d\xi)$. Consequently,

$$\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f^\delta(t=0) - f^\delta(t=0)^2\right)\zeta \to 0 \quad \text{as } \delta \to 0.$$

Above, we have used the fact that $f_0 = \mathbf{1}_{\xi<u_0}$, and thus $f_0 = f_0^2$.

Now, we pass to the limit as $\delta \to 0$ in (5.41); we obtain, for all $T > 0$,

$$\int_0^T\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left(f - f^2\right)\varphi \le 0.$$

Since the integrand in the left-hand side is non-negative, we deduce that $f = f^2$ almost everywhere. The rigidity property follows.

5.3.2 Contraction principle

Let f_1, f_2 be two generalized kinetic solutions of the limit problem; we denote by M_1, M_2, and $\mathcal{M}_1, \mathcal{M}_2$, the constants and distributions associated to f_1, f_2, respectively. Without loss of generality, we assume that $M_1 \le M_2$. According to the rigidity result, there exist functions $u_1, u_2 \in L^\infty([0,\infty) \times \mathbb{R}^N \times Y) \cap L^\infty([0,\infty), L^1(\mathbb{R}^N \times Y))$ such that $f_i = \mathbf{1}_{\xi<u_i}$.

As in the previous subsection, we regularize $f_i, \mathcal{M}_i$ by convolution in the variables t, x, and we denote by $f_i^\delta, \mathcal{M}_i^\delta$ the functions thus obtained. The strategy of the proof is the same as in [59], Theorem 4.3.1. The idea is to derive an inequality of the type

$$\begin{aligned}&\frac{d}{dt}\int |f_1(t,x,y,\xi)-f_2(t,x,y,\xi)|\zeta(x)\,dx\,dy\,d\xi\\ \leq\ & C\int |f_1(t,x,y,\xi)-f_2(t,x,y,\xi)|\zeta(x)\,dx\,dy\,d\xi,\end{aligned}$$

where ζ is a cut-off function as in the previous section.

Since $|f_1(t)-f_2(t)|=|f_1(t)-f_2(t)|^2=f_1+f_2-2f_1f_2$, let us first write the equation satisfied by $g^\delta:=f_1^\delta+f_2^\delta-2f_1^\delta f_2^\delta$. We compute

$$\left\{\partial_t f_1^\delta+\sum_{i=1}^N a_i(y,\xi)\frac{\partial}{\partial x_i}f_1^\delta=\mathcal{M}_1^\delta\right\}\quad\times 1-2f_2^\delta,$$

$$\left\{\partial_t f_2^\delta+\sum_{i=1}^N a_i(y,\xi)\frac{\partial}{\partial x_i}f_2^\delta=\mathcal{M}_2^\delta\right\}\quad\times 1-2f_1^\delta.$$

Adding the two equations thus obtained leads to

$$\partial_t g^\delta+\sum_{i=1}^N a_i(y,\xi)\frac{\partial}{\partial x_i}g^\delta=\mathcal{M}_1^\delta\left[1-2f_2^\delta\right]+\mathcal{M}_2^\delta\left[1-2f_1^\delta\right].$$

Notice that thanks to (5.7), (5.8) and the microscopic constraints (5.9), (5.11), $1-2f_i^\delta(t,x)\in\mathcal{G}$ for all (t,x). Hence

$$\int_{Y\times\mathbb{R}}\mathcal{M}_2^\delta(t,x)\left[1-2f_1^\delta(t,x)\right]\leq 0\quad\forall (t,x)\in[0,\infty)\times\mathbb{R}^N,$$

and the same inequality holds if the roles of f_1 and f_2 are exchanged.

Now, take a cut-off function $\zeta\in\mathcal{C}^\infty(\mathbb{R}^N)$ satisfying the same assumptions as in the previous subsection; multiply the equation on g^δ by $\zeta(x)$, and integrate over $\mathbb{R}^N\times Y\times\mathbb{R}$; this yields

$$\frac{d}{dt}\int_{\mathbb{R}^N\times Y\times\mathbb{R}}g^\delta(t,x,y,\xi)\zeta(x)\,dx\,dy\,d\xi\leq C\int_{\mathbb{R}^N\times Y\times\mathbb{R}}g^\delta(t,x,y,\xi)\zeta(x)\,dx\,dy\,d\xi\quad\forall t>0,$$

and thus

$$\int_{\mathbb{R}^N\times Y\times\mathbb{R}}g^\delta(t,x,y,\xi)\zeta(x)\,dx\,dy\,d\xi\leq e^{Ct}\int_{\mathbb{R}^N\times Y\times\mathbb{R}}g^\delta(t=0,x,y,\xi)\zeta(x)\,dx\,dy\,d\xi.$$

According to the strong convergence results of $f_i^\delta(t=0)$ derived in the previous section, we can pass to the limit as $\delta\to 0$. We infer that for almost every $t>0$,

$$\begin{aligned}&\int_{\mathbb{R}^N\times Y\times\mathbb{R}}|f_1(t,x,y,\xi)-f_2(t,x,y,\xi)|\zeta(x)\,dx\,dy\,d\xi\\ \leq\ & e^{Ct}\int_{\mathbb{R}^N\times Y\times\mathbb{R}}|f_1(t=0,x,y,\xi)-f_2(t=0,x,y,\xi)|\zeta(x)\,dx\,dy\,d\xi.\end{aligned}\tag{5.42}$$

This completes the derivation of the contraction principle for the limit system. Uniqueness of solutions of the limit system follows. In particular, we deduce that the whole sequence f^ε of solutions of (5.4) two-scale converges towards f^0.

5.3.3 Strong convergence result

Here, we explain why the strong convergence result stated in Theorem 8 holds, that is, we prove (5.22). This fact is rather classical, and is a direct consequence of the fact that

$$\mathbf{1}_{\xi<u^\varepsilon(t,x)} \overset{2\text{ sc.}}{\rightharpoonup} \mathbf{1}_{\xi<u(t,x,y)}.$$

Let us express this result in terms of Young measures: the above two-scale convergence is strictly equivalent to the fact that the two-scale Young measure $\nu_{t,x,y}$ associated with the sequence u^ε is the Dirac mass $\delta(\xi = u(t,x,y))$ (see [59], Chapter 2). And it is well-known (see [24]) that if u is a smooth function, then

$$d\nu_{t,x,y}(\xi) = \delta(\xi = u(t,x,y)) \quad \Longleftrightarrow \quad u^\varepsilon - u\left(t,x,\frac{x}{\varepsilon}\right) \to 0 \quad \text{in } L^1_{\text{loc}}.$$

However, here, the function u is not smooth, but this issue is bypassed by using convolution kernels. For the reader's convenience, we now prove the result without using two-scale Young measures. We define $u_\delta = u *_x \varphi_\delta$, with φ_δ a standard mollifier. Let $K \in \mathcal{D}(\mathbb{R})$ such that $0 \le K \le 1$, and $K(\xi) = 1$ if $|\xi| \le M$. Consider also a sequence $\theta_n \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \le \theta_n \le 1$, and

$$\theta_n(\xi) = 1 \text{ if } \xi < -\frac{1}{n}, \quad \theta_n(\xi) = 0 \text{ if } \xi > \frac{1}{n}.$$

Then we have

$$\begin{aligned}
\left|\mathbf{1}_{\xi<u^\varepsilon(t,x)} - \mathbf{1}_{\xi<u_\delta\left(t,x,\frac{x}{\varepsilon}\right)}\right|^2 &= \mathbf{1}_{\xi<u^\varepsilon(t,x)} - 2\mathbf{1}_{\xi<u_\delta\left(t,x,\frac{x}{\varepsilon}\right)}\mathbf{1}_{\xi<u^\varepsilon(t,x)} + \mathbf{1}_{\xi<u_\delta\left(t,x,\frac{x}{\varepsilon}\right)}\\
&= \mathbf{1}_{\min\left(u^\varepsilon(t,x),u_\delta\left(t,x,\frac{x}{\varepsilon}\right)\right)<\xi<\max\left(u^\varepsilon(t,x),u_\delta\left(t,x,\frac{x}{\varepsilon}\right)\right)}.
\end{aligned}$$

The function $\mathbf{1}_{\xi<u_\delta\left(t,x,\frac{x}{\varepsilon}\right)}$ is not smooth enough to be used as an oscillating test function. Thus we replace it by

$$\theta_n\left(\xi - u_\delta\left(t,x,\frac{x}{\varepsilon}\right)\right),$$

and we evaluate the difference: for all compact set $C \subset [0,\infty) \times \mathbb{R}^N$, for all $\delta, \varepsilon > 0$,

$$\int_C \int_{\mathbb{R}} \left|\mathbf{1}_{\xi<u_\delta\left(t,x,\frac{x}{\varepsilon}\right)} - \theta_n\left(\xi - u_\delta\left(t,x,\frac{x}{\varepsilon}\right)\right)\right| K(\xi)\,dt\,dx\,d\xi \le \frac{2}{n}|C|.$$

According to the two-scale convergence result, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_C \int_{\mathbb{R}} \theta_n\left(\xi - u_\delta\left(t,x,\frac{x}{\varepsilon}\right)\right)\mathbf{1}_{\xi<u^\varepsilon(t,x)}K(\xi)\,dt\,dx\,d\xi \to\\
&\qquad\qquad \to \int_C \int_{\mathbb{R}} \theta_n\left(\xi - u_\delta(t,x,y)\right)\mathbf{1}_{\xi<u(t,x,y)}K(\xi)\,dt\,dx\,dy\,d\xi.
\end{aligned}$$

Since the sequence $\theta_n(\xi - u_\delta)$ uniformly converges towards $\mathbf{1}_{\xi<u_\delta}$ as $n \to \infty$, we can pass to the limit as $n \to \infty$, and we deduce

$$\int_C \int_{\mathbb{R}} \mathbf{1}_{\xi<u_\delta\left(t,x,\frac{x}{\varepsilon}\right)}\mathbf{1}_{\xi<u^\varepsilon(t,x)}K(\xi)dtdxd\xi \to \int_C \int_{\mathbb{R}\times Y} \mathbf{1}_{\xi<u_\delta(t,x,y)}\mathbf{1}_{\xi<u(t,x,y)}K(\xi)dtdxdyd\xi.$$

Simlarly, as $\varepsilon \to 0$, for all $\delta > 0$,

$$\int_C \int_{\mathbb{R}} \mathbf{1}_{\xi < u_\delta\left(t,x,\frac{x}{\varepsilon}\right)} K(\xi)\, dt\, dx\, d\xi \to \int_C \int_{\mathbb{R}\times Y} \mathbf{1}_{\xi < u_\delta(t,x,y)} K(\xi)\, dt\, dx\, dy\, d\xi,$$
$$\int_C \int_{\mathbb{R}\times Y} \mathbf{1}_{\xi < u^\varepsilon(t,x)} K(\xi)\, dt\, dx\, d\xi \to \int_C \int_{\mathbb{R}} \mathbf{1}_{\xi < u(t,x,y)} K(\xi)\, dt\, dx\, dy\, d\xi.$$

Thus, as $\varepsilon \to 0$, for all $\delta > 0$,

$$\int_C \int_{\mathbb{R}} \left| \mathbf{1}_{\xi < u^\varepsilon(t,x)} - \mathbf{1}_{\xi < u_\delta\left(t,x,\frac{x}{\varepsilon}\right)} \right|^2 K(\xi)\, dt\, dx\, d\xi \to$$
$$\to \int_C \int_{\mathbb{R}\times Y} \left| \mathbf{1}_{\xi < u(t,x,y)} - \mathbf{1}_{\xi < u_\delta(t,x,y)} \right| K(\xi)\, dt\, dx\, dy\, d\xi.$$

On the other hand,

$$\int_C \int_{\mathbb{R}} \left| \mathbf{1}_{\xi < u^\varepsilon(t,x)} - \mathbf{1}_{\xi < u_\delta\left(t,x,\frac{x}{\varepsilon}\right)} \right| K(\xi)\, dt\, dx\, d\xi = \left|\left| u^\varepsilon(t,x) - u_\delta\left(t,x,\frac{x}{\varepsilon}\right) \right|\right|_{L^1(C)},$$
$$\int_C \int_{\mathbb{R}\times Y} \left| \mathbf{1}_{\xi < u(t,x,y)} - \mathbf{1}_{\xi < u_\delta(t,x,y)} \right| K(\xi)\, dt\, dx\, dy\, d\xi = ||u - u_\delta||_{L^1(C\times Y)}.$$

Hence we have proved that for all $\delta > 0$, for all compact set $C \subset [0,\infty) \times \mathbb{R}^N$,

$$\lim_{\varepsilon\to 0} \left|\left| u^\varepsilon(t,x) - u_\delta\left(t,x,\frac{x}{\varepsilon}\right) \right|\right|_{L^1(C)} = ||u - u_\delta||_{L^1(C\times Y)}.$$

Statement (5.22) then follows from standard convolution results.

5.3.4 Application: proof of the convergence result for kinetic solutions

In this subsection, we prove Theorem 9; this result is in fact an easy consequence of the convergence result stated in Theorem 8 for entropy solutions, and of the contraction principle for the limit system. Assume that $a_{N+1} \equiv 0$, and let u^ε be a kinetic solution of equation (5.1), with an initial data $u_0(x, x/\varepsilon)$ such that $u_0 \in L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$ and

$$\sum_{i=1}^N \frac{\partial}{\partial y_i} \left(a_i(y,\xi)\chi(\xi, u_0(x,y))\right) = 0 \tag{5.43}$$

in the sense of distributions.

For $n \in \mathbb{N}$, let $u_0^n := \text{sgn}(u_0)\inf(|u_0|, n)$. Then for all $n \in \mathbb{N}$, u_0^n belongs to $L^\infty(\mathbb{R}^N \times Y)$ and

$$u_0^n \to u_0 \quad \text{as } n \to \infty \quad \text{in } L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y)).$$

Moreover, $\chi(\xi, u_0^n) = \chi(\xi, u_0)\mathbf{1}_{|\xi|<n}$, and thus for all $n \in \mathbb{N}$, u_0^n satisfies (5.43).

For all $n, \varepsilon > 0$, let $u_n^\varepsilon \in \mathcal{C}([0,\infty), L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty)\times\mathbb{R}^N)$ be the unique entropy solution of equation (5.1) with initial data $u_0^n(x, x/\varepsilon)$. Then by the contraction principle for kinetic solutions of scalar conservation laws, we have, for all $n \in \mathbb{N}$,

$$||u^\varepsilon - u_n^\varepsilon||_{L^\infty([0,\infty),L^1(\mathbb{R}^N))} \leq \left|\left|u_0\left(x, \frac{x}{\varepsilon}\right) - u_0^n\left(x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(\mathbb{R}^N)} \leq ||u_0 - u_0^n||_{L^1(\mathbb{R}^N,\mathcal{C}_{\text{per}}(Y))}.$$

On the other hand, for all $n \in \mathbb{N}$, let $\mathbf{1}_{\xi<u_n}$ be the unique solution of the limit system with initial data $\mathbf{1}_{\xi<u_0^n}$. By the contraction principle for solutions of the limit system (see inequality (5.42)), we have, for all integers $n, m \in \mathbb{N}$, for all $t \geq 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N\times Y} |u_n(t,x,y) - u_m(t,x,y)|\, \zeta(x)\, dx\, dy \\
\leq\;& e^{Ct}\int_{\mathbb{R}^N\times Y} |u_0^m(t,x,y) - u_0^n(t,x,y)|\, \zeta(x)\, dx\, dy \\
\leq\;& e^{Ct}||u_0^m - u_0^n||_{L^1(\mathbb{R}^N,\mathcal{C}_{\text{per}}(Y))}. \qquad (5.44)
\end{aligned}$$

where $\zeta \in \mathcal{C}^\infty(\mathbb{R}^N)$ is a cut-off function satisfying the same hypotheses as in the previous subsections.

Consequently, the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^N \times Y, \zeta(x)dx\, dy))$; thus there exists a function $u \in L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^N \times Y, \zeta(x)dx\, dy))$ such that u_n converges towards u as $n \to \infty$ in $L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^N \times Y, \zeta(x)dx\, dy))$. Moreover, the limit u is independent of the chosen sequence u_0^n thanks to (5.44): indeed, let v_0^n, w_0^n be two sequences converging towards u_0 in $L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$, and giving rise to functions v and w respectively. We construct the sequence

$$u_0^n = \begin{cases} v_0^n & \text{if } n \text{ is even,} \\ w_0^n & \text{if } n \text{ is odd.} \end{cases}$$

Then the sequence u_0^n converges towards u_0 in $L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))$, and thus the corresponding sequence u_n converges towards u, while u_{2n} converges towards v and u_{2n+1} towards w. By uniqueness of the limit, $u = v = w$.

On the other hand, since the sequence $f^\varepsilon = \chi(\xi, u^\varepsilon)$ is bounded in L^∞, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive numbers, $\varepsilon_k \to 0$, and a function $f \in L^\infty([0,\infty)\times\mathbb{R}^N \times Y \times \mathbb{R})$, such that

$$\chi(\xi, u^{\varepsilon_k}(t,x)) \stackrel{2\text{ sc.}}{\rightharpoonup} f(t,x,y,\xi).$$

Now, for all $k, n \in \mathbb{N}$,

$$||\chi(\xi, u^{\varepsilon_k}) - \chi(\xi, u_n^{\varepsilon_k})||_{L^\infty([0,\infty),L^1(\mathbb{R}^{N+1}))} \leq ||u_0 - u_0^n||_{L^1(\mathbb{R}^N,\mathcal{C}_{\text{per}}(Y))},$$

and for all $n \in \mathbb{N}$, since $\chi(\xi, u) = \mathbf{1}_{\xi<u} - \mathbf{1}_{\xi<0}$, we have, as $k \to \infty$,

$$\chi(\xi, u_n^{\varepsilon_k}) \stackrel{2\text{ sc.}}{\rightharpoonup} \chi(\xi, u_n).$$

Let $\varphi \in \mathcal{D}_{\text{per}}([0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R})$. According to the definition of two-scale convergence,

$$\int_0^\infty \int_{\mathbb{R}^{N+1}} \left[\chi(\xi, u^{\varepsilon_k}(t,x)) - \chi(\xi, u_n^{\varepsilon_k}(t,x))\right] \varphi\left(t, x, \frac{x}{\varepsilon_k}, \xi\right) \, dt\, dx\, d\xi \rightarrow$$
$$\rightarrow \int_0^\infty \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \left[f(t,x,y,\xi) - \chi(\xi, u_n(t,x,y))\right] \varphi\,(t,x,y,\xi)\, dt\, dx\, dy\, d\xi.$$

And for all $k \in \mathbb{N}$, the following inequality holds:

$$\left| \int_0^\infty \int_{\mathbb{R}^{N+1}} \left[\chi(\xi, u^{\varepsilon_k}(t,x)) - \chi(\xi, u_n^{\varepsilon_k}(t,x))\right] \varphi\left(t, x, \frac{x}{\varepsilon_k}, \xi\right) \, dt\, dx\, d\xi \right| \leq$$
$$\leq ||\varphi||_{L^1([0,\infty), L^\infty(\mathbb{R}^N \times Y \times \mathbb{R}))} ||u_0 - u_0^n||_{L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))}.$$

Passing to the limit as $k \to \infty$, we deduce that for all $n \in \mathbb{N}$, $\varphi \in \mathcal{D}_{\text{per}}([0,\infty) \times \mathbb{R}^N \times Y \times \mathbb{R})$,

$$\left| \int_0^\infty \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \left[f(t,x,y,\xi) - \chi(\xi, u_n(t,x,y))\right] \varphi\,(t,x,y,\xi)\, dt\, dx\, d\xi \right| \leq$$
$$\leq ||u_0 - u_0^n||_{L^1(\mathbb{R}^N, \mathcal{C}_{\text{per}}(Y))} ||\varphi||_{L^1([0,\infty), L^\infty(\mathbb{R}^N \times Y \times \mathbb{R}))}.$$

Thus, we pass to the limit as $n \to \infty$ and we infer that $f = \chi(\xi, u(t,x,y))$ almost everywhere. Hence the limit is unique, and the whole sequence $\chi(\xi, u^\varepsilon)$ converges (in the sense of two-scale convergence).

Eventually, let us pass to the limit as $n \to \infty$ in the limit evolution problem for $\chi(\xi, u_n)$. We set $f = \chi(\xi, u)$, and define the distribution

$$\mathcal{M} := \partial_t f + a(y,\xi) \cdot \nabla_x f.$$

Then $\mathcal{M}_n \rightharpoonup \mathcal{M}$ in the sense of distributions, and it is easily checked that inequality (5.19) is preserved when passing to the (weak) limit. Thus $\mathcal{M}$ satisfies (5.19).

In the divergence-free case, the main difference between the L^∞ and the L^1 setting, that is, Theorem 8 and Theorem 9, lies in the fact that uniqueness for the limit system in the L^1 setting seems difficult to derive; indeed, the proof of uniqueness in the L^∞ case uses several times the fact that the distribution $\mathcal{M}$ has compact support. In a L^1 setting, this assumption ought to be replaced by a hypothesis expressing that $\mathcal{M}$ vanishes as $|\xi| \to \infty$, in some sense. But it is unclear how to retrieve such a property from the hydrodynamic limit (see section 5.4), for instance. The above argument only proves that uniqueness holds among L^1 solutions which are obtained as the limit of a sequence of L^∞ solutions. Thus we have left open the correct notion of limit system in a weak L^1 setting, and the derivation of uniqueness therein.

Nonetheless, we wish to stress that the contraction principle in the L^∞ setting is sufficient to ensure that the whole sequence $\chi(\xi, u^\varepsilon)$ converges, even if uniqueness for the limit system fails.

5.4 A relaxation model for the limit evolution problem

In this section, we exhibit another way of finding solutions of the limit system in the divergence-free case. Indeed, the existence result in Theorem 7 was proved by passing to the two-scale limit in (5.4), and it may be interesting to have another way of constructing solutions, which does not involve a homogenization process.

Hence, we introduce a relaxation model of BGK type, in which we pass to the limit as the relaxation parameter goes to infinity. The drawback of this method lies in the fact that the existence of solutions of the limit system is not a consequence of the construction. Indeed, we shall prove that if a solution of the limit system exists, then the family of solutions of the relaxation model strongly converge towards it in the hydrodynamic limit. Hence the proof is not self-contained, because the existence of a solution of the limit system is required in order to pass to the limit. Nevertheless, the final result may be useful in other applications.

In the whole section, the words "limit system" refer to the modified equations introduced in Remark 5.1.1. In the divergence-free case, it is also slightly more convenient to work with the function $\chi(\xi, u)$, rather than $\mathbf{1}_{\xi<u}$. Hence a solution of the limit problem is a function g satisfying

$$\sum_{i=1}^{N} \frac{\partial}{\partial y_i} \left(a_i(y,\xi) g\right) = 0, \tag{5.45}$$

$$\frac{\partial g}{\partial t} + \sum_{i=1}^{N} a_i(y,\xi) \frac{\partial g}{\partial x_i} = \mathcal{M}, \tag{5.46}$$

$$\partial_\xi g = \delta(\xi) - \nu(t,x,y,\xi), \quad \nu \geq 0, \tag{5.47}$$

and $\mathcal{M}$ is such that for all $\varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^N)$ such that $\varphi \geq 0$, the function $\mathcal{M} *_{t,x} \varphi$ belongs to $\mathcal{C}([0,\infty) \times \mathbb{R}^N, L^2(Y \times \mathbb{R}))$, and

$$\left\{ \begin{array}{l} \displaystyle\int_{Y\times\mathbb{R}} \left(\mathcal{M} *_{t,x} \varphi\right)(t,x,\cdot)\, \psi \leq 0, \\ \forall \psi \in L^\infty_{\text{loc}}(Y \times \mathbb{R}),\ \operatorname{div}_y(a\psi) = 0, \text{ and } \partial_\xi \psi \geq 0. \end{array} \right. \tag{5.48}$$

5.4.1 A relaxation model

The goal of this subsection is to introduce a system approaching (5.45)-(5.48). With this aim in view, a relaxation model of BGK type is introduced, which takes into account the constraints of the limit system, that is, equations (5.45)-(5.48). Let

$$\begin{array}{rcl} M &:=& ||u_0||_{L^\infty(Y\times\mathbb{R})}, \\ E &:=& \{f \in L^2(Y \times \mathbb{R}),\ \text{Supp} f \subset Y \times [-M, M]\}, \\ \mathbb{K} &:=& \{\varphi \in E,\ \operatorname{div}_y(a(y,\xi)\varphi(y,\xi)) = 0 \quad \text{in } \mathcal{D}'\}, \\ \mathcal{K} &:=& \mathbb{K} \cap \left\{\varphi \in E,\ \exists \nu \in M^1_{\text{per}}(Y \times \mathbb{R}),\ \nu \geq 0,\ \partial_\xi \varphi = \delta(\xi) - \nu\right\}. \end{array}$$

Then E endowed with the usual scalar product on L^2 is a Hilbert space, and $\mathcal{K}$ is a nonempty closed convex set in E. Thus the projection $\mathcal{P}$ on $\mathcal{K}$ is well-defined.

The main result of this subsection is the following :

Proposition 5.4.1. *Let $\lambda, T > 0$ be arbitrary. Set*

$$X_T := \mathcal{C}([0,T], L^2(\mathbb{R}_x^N \times Y \times \mathbb{R}_\xi))$$

and

$$f_0(x,y,\xi) = \chi(\xi, u_0(x,y)), \quad (x,y,\xi) \in \mathbb{R}^N \times Y \times \mathbb{R}.$$

Then there exists a unique solution $f_\lambda \in X_T$ of the equation

$$\begin{cases} \partial_t f_\lambda + a(y,\xi)\cdot\nabla_x f_\lambda + \lambda f_\lambda = \lambda \mathcal{P}(f_\lambda), & t>0,\ (x,y,\xi) \in \mathbb{R}^N \times Y \times \mathbb{R}, \\ f_\lambda(t=0,x,y,\xi) = f_0(x,y,\xi). \end{cases} \tag{5.49}$$

The function f_λ has the following properties :

1. *For almost every t, x, y, ξ,*
$$f_\lambda(t,x,y,\xi) = 0 \text{ if } \xi \geq M,$$
$$\operatorname{sgn}(\xi) f_\lambda(t,x,y,\xi) = |f_\lambda(t,x,y,\xi)| \leq 1.$$
2. *L^2 estimate: for all $\lambda > 0$,*
$$||f_\lambda||_{X_T} \leq ||u_0||_{L^1(\mathbb{R}^N\times Y)}. \tag{5.50}$$
3. *Strong continuity at time $t = 0$: there exists a function $\omega : [0,\infty) \to [0,\infty)$, such that $\lim_{0^+} \omega = 0$, and such that for all $\lambda > 0$, $t \geq 0$,*
$$||f_\lambda(t) - f_0||_{L^1(\mathbb{R}^N\times Y\times\mathbb{R})} \leq \omega(t). \tag{5.51}$$
4. *Fundamental inequality for $\mathcal{M}_\lambda := \lambda(\mathcal{P}(f_\lambda) - f_\lambda)$: for all $g \in \mathcal{K}$, for almost every (t,x),*
$$\int_{Y\times\mathbb{R}} \mathcal{M}_\lambda\,(\mathcal{P}(f_\lambda) - g) \leq 0. \tag{5.52}$$

In equation (5.49), the projection $\mathcal{P}$ acts on the variables y, ξ only; since f is a function of t, x, y, ξ, $\mathcal{P}(f)$ should be understood as

$$\mathcal{P}(f)(t,x,\cdot) = \mathcal{P}(f(t,x,\cdot)),$$

and the above equality holds between functions in $L^2(Y\times\mathbb{R})$, almost everywhere in t, x.

Proof. First step. Construction of f_λ. The existence and uniqueness of f_λ follows from a fixed point theorem in X_T. We define the application $\phi_T : X_Y \to X_T$ by $\phi_T(f) = g$, where g is the solution of the linear equation

$$\begin{cases} \partial_t g + a(y,\xi)\cdot\nabla_x g + \lambda g = \lambda\mathcal{P}(f), \\ g(t=0,x,y,\xi) = \chi(\xi, u_0(x,y)) \end{cases}$$

The existence and uniqueness of g follows from well-known results on the theory of linear transport equations (recall that $a \in W^{1,\infty}_{\text{per,loc}}(Y \times \mathbb{R})$). Moreover, if $f_1, f_2 \in X_T$ and $g_i = \phi_T(f_i)$, $i = 1, 2$, then $g = g_1 - g_2$ is a solution of

$$\begin{cases} \partial_t g + a(y,\xi) \cdot \nabla_x g + \lambda g = \lambda \left[\mathcal{P}(f_1) - \mathcal{P}(f_2)\right], \\ g(t = 0, x, y, \xi) = 0. \end{cases}$$

Multiplying the above equation by g, and integrating on $\mathbb{R}^N_x \times Y \times \mathbb{R}_\xi$, we obtain the estimate

$$\frac{1}{2}\frac{d}{dt}||g(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} + \lambda ||g(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} \leq \lambda \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \left[\mathcal{P}(f_1) - \mathcal{P}(f_2)\right] g.$$

Recall that the projection $\mathcal{P}$ is Lipschitz continuous with Lipschitz constant 1. Thus

$$\begin{aligned} \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \left[P(f_1) - P(f_2)\right] g \ &\leq \ \frac{1}{2}||P(f_1) - P(f_2)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} + \frac{1}{2}||g(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} \\ &\leq \ \frac{1}{2}||(f_1 - f_2)(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} + \frac{1}{2}||g(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})}. \end{aligned}$$

Eventually, we obtain

$$\frac{d}{dt}||g(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} + \lambda ||g(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} \leq \lambda ||f_1 - f_2||^2_{X_T}.$$

A straightforward application of Gronwall's lemma yields

$$||g_1 - g_2||_{X_T} \leq \sqrt{1 - e^{-\lambda T}}\, ||f_1 - f_2||_{X_T}.$$

Thus ϕ_T is a contractant application and has a unique fixed point in X_T, which we call f_λ.

Second step. L^2 estimate. Multiplying (5.49) by f_λ and integrating on $\mathbb{R}^N \times Y \times \mathbb{R}$, we infer

$$\frac{1}{2}\frac{d}{dt}||f_\lambda(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} + \lambda ||f_\lambda(t)||^2_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} \leq \lambda \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \mathcal{P}(f_\lambda) f_\lambda.$$

Notice that $0 \in \mathcal{K}$; thus the Lipschitz continuity of $\mathcal{P}$ entails that for almost every t, x

$$||\mathcal{P}(f_\lambda)(t,x)||_E \leq ||f_\lambda(t,x)||_E\,.$$

Hence, using the Cauchy-Schwartz inequality, we deduce that $t \mapsto ||f_\lambda(t)||_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})}$ is nonincreasing on $[0, T]$. The equality

$$\begin{aligned} \int_{\mathbb{R}^N \times Y \times \mathbb{R}} |\chi(\xi, u_0(x,y))|^2 \, dx \, dy \, d\xi \ &= \ \int_{\mathbb{R}^N \times Y \times \mathbb{R}} |\chi(\xi, u_0(x,y))| \, dx \, dy \, d\xi \\ &= \ \int_{\mathbb{R}^N \times Y} |u_0(x,y)| \, dx \, dy \end{aligned}$$

then yields the desired result.

Third step. Compact support in ξ. Let us prove now that $f_\lambda(\cdot,\xi)=0$ if $|\xi|>M$: let $\varphi\in\mathcal{D}(\mathbb{R})$ be an arbitrary test function such that $\varphi(\xi)=0$ when $|\xi|\le M$. Then $\mathcal{P}(f_\lambda)\varphi=0$ since $\mathcal{P}(f_\lambda)\in\mathcal{K}$, and thus $f_\lambda\varphi$ is a solution of

$$\begin{array}{l}\dfrac{\partial}{\partial t}\left(f_\lambda\varphi\right)+a\cdot\nabla_x\left(f_\lambda\varphi\right)+\lambda\left(f_\lambda\varphi\right)=0,\\ \left(f_\lambda\varphi\right)(t=0,x,y,\xi)=0.\end{array}$$

Hence $(f_\lambda\varphi)(t,x,y,\xi)=0$ for almost every t,x,y,ξ, and $f_\lambda(\cdot,\xi)=0$ if $|\xi|>M$.

Fourth step. Sign property. We now prove the sign property, namely

$$\operatorname{sgn}(\xi)f_\lambda=|f_\lambda|\le 1\quad\text{a.e.}$$

This relies on the following fact: if $g\in\mathcal{K}$, then $\operatorname{sgn}(\xi)g(y,\xi)\in[0,1]$ for almost every y,ξ. Indeed, $g(\cdot,\xi)=0$ if $\xi<-M$, and thus if $-M<\xi<0$,

$$g(y,\xi)=-\int_{-M}^{\xi}\nu(y,\xi')\,d\xi'\le 0.$$

Hence $g(y,\cdot)$ is non-positive and nonincreasing on $(-\infty,0)$. Similarly, $g(y,\cdot)$ is non-negative and nondecreasing on $(0,\infty)$. And if $\xi<0<\xi'$, then

$$g(y,\xi')-g(y,\xi)=1-\int_{\xi}^{\xi'}\nu(y,w)\,dw\le 1.$$

Hence the sign property is true for all functions in $\mathcal{K}$.

Multiplying (5.49) by $\operatorname{sgn}(\xi)$, we are led to

$$\frac{\partial}{\partial t}\left(\operatorname{sgn}(\xi)f_\lambda\right)+a(y,\xi)\cdot\nabla_x\left(\operatorname{sgn}(\xi)f_\lambda\right)+\lambda\left(\operatorname{sgn}(\xi)f_\lambda\right)=\lambda\mathcal{P}(f_\lambda)\in[0,\lambda].$$

At time $t=0$, $\operatorname{sgn}(\xi)f_\lambda(t=0)=|\chi(\xi,u_0)|\in[0,1]$. Thus, using a maximum principle for the linear transport equation above, we deduce that the sign property is satisfied for f_λ.

Fifth step. Uniform continuity at time $t=0$. Let $\delta>0$ be arbitrary, and let $f_0^\delta:=f_0*_x\theta^\delta$, with θ^δ a standard mollifier. Then $f_0^\delta(x)\in\mathcal{K}$ for all $x\in\mathbb{R}^N$, and thus $f_\lambda-f_0^\delta$ is a solution of the equation

$$\begin{aligned}\frac{\partial}{\partial t}\left(f_\lambda-f_0^\delta\right)+a(y,\xi)\cdot\nabla_x\left(f_\lambda-f_0^\delta\right)+\lambda\left(f_\lambda-f_0^\delta\right)&=\lambda\left(\mathcal{P}(f_\lambda)-\mathcal{P}(f_0^\delta)\right)\\&\quad-a(y,\xi)\cdot\left(f_0*_x\nabla\theta^\delta\right).\end{aligned}$$

Multiply the above equation by $\left(f_\lambda-f_0^\delta\right)$ and integrate on $\mathbb{R}^N\times Y\times\mathbb{R}$. Using once more the Lipschitz continuity of the projection $\mathcal{P}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||f_\lambda-f_0^\delta||^2_{L^2(\mathbb{R}^N\times Y\times\mathbb{R})}&\le||a||_{L^\infty}||f_\lambda-f_0^\delta||_{L^2}||f_0||_{L^2}||\nabla\theta^\delta||_{L^1(\mathbb{R}^N)}\\ \frac{d}{dt}||f_\lambda-f_0^\delta||_{L^2(\mathbb{R}^N\times Y\times\mathbb{R})}&\le\frac{C}{\delta},\end{aligned}$$

where the L^∞ norm of a is estimated on $Y \times (-M, M)$, and the L^2 norms of f_λ and f_0^δ on $\mathbb{R}^N \times Y \times \mathbb{R}^N$.

As a consequence, we obtain the following estimate, which holds for all $t > 0$, $\lambda > 0$ and $\delta > 0$

$$||f_\lambda(t) - f_0^\delta||_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} \leq \frac{Ct}{\delta} + ||f_0 - f_0^\delta||_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})}.$$

Hence the uniform continuity property is true, with

$$\omega(t) := \inf_{\delta > 0} \left(\frac{Ct}{\delta} + 2\, ||f_0 - f_0^\delta||_{L^2(\mathbb{R}^N \times Y \times \mathbb{R})} \right).$$

Sixth step. Inequality for $\mathcal{M}_\lambda$. Inequality (5.52) is merely a particular case of the inequality

$$\langle \mathcal{P}(f) - f, \mathcal{P}(f) - g \rangle_E \leq 0$$

which holds for all $f \in E$, for all $g \in \mathcal{K}$.

□

5.4.2 The hydrodynamic limit

In this subsection , we prove the following result :

Proposition 5.4.2. *Let $(f_\lambda)_{\lambda > 0}$ be the family of solutions of the relaxation model* (5.49)*, and let $f(t) = \chi(\xi, u)$ be the unique solution of the limit system* (5.45)-(5.48) *with initial data $\chi(\xi, u_0(x, y))$. Then as $\lambda \to \infty$,*

$$f_\lambda \to f \quad \text{in } L^2((0,T) \times \mathbb{R}^N \times Y \times \mathbb{R}).$$

The above Proposition relies on an inequality of the type

$$\frac{d}{dt} \int_{\mathbb{R}^N \times Y \times \mathbb{R}} |f_\lambda - f|^2 \leq r_\lambda(t),$$

with $r_\lambda(t) \to 0$ as $\lambda \to \infty$. The calculations are very similar to those of the contraction principle in the previous section; the only difference lies in the fact that f_λ and f are not solutions of the same equation.

Before tackling the proof itself, let us derive a few properties on the weak limit of the sequence f_λ. Since the sequence f_λ is bounded in $X_T \subset L^2((0,T) \times \mathbb{R}^N \times Y \times \mathbb{R})$, there exists a subsequence, which we relabel f_λ, and a function $g \in L^2((0,T) \times \mathbb{R}^N \times Y \times \mathbb{R})$ such that f_λ weakly converges to g in L^2. Moreover, the sequence $\mathcal{P}(f_\lambda)$ is bounded in $L^2((0,T) \times \mathbb{R}^N \times Y \times \mathbb{R})$, for all $T > 0$. Hence, extracting a further subsequence if necessary, there exists a function $h \in L^2((0,T) \times \mathbb{R}^N \times Y \times \mathbb{R})$ such that $\mathcal{P}(f_\lambda)$ weakly converges towards h as $\lambda \to \infty$. Notice that the convex set $\mathcal{K}$ is closed for the weak topology in L^2. Consequently, $h(t,x) \in \mathcal{K}$ for almost every t, x. At last,

$$\mathcal{P}(f_\lambda) - f_\lambda = \mathcal{O}\left(\frac{1}{\lambda}\right),$$

where the $\mathcal{O}$ is meant in the sense of distributions. Hence, $g = h$, and in particular, we deduce that $g(t, x) \in \mathcal{K}$ for almost every (t, x).

We are now ready to prove the contraction inequality; consider a mollifying sequence θ^δ as in the previous section, and set $f^\delta = f *_{t,x} \theta^\delta$, $f_\lambda^{\delta'} = f_\lambda *_{t,x} \theta^{\delta'}$. Then

$$\partial_t f^\delta + a(y,\xi)\cdot\nabla_x f^\delta = \mathcal{M}^\delta,$$
$$\partial_t f_\lambda^{\delta'} + a(y,\xi)\cdot\nabla_x f_\lambda^{\delta'} = \mathcal{M}_\lambda^{\delta'}.$$

Let us multiply the first equation by $\text{sgn}(\xi) - 2f_\lambda^{\delta'}$, the second by $2(f_\lambda^{\delta'} - f^\delta)$, and add the two identities thus obtained; setting $F_\lambda^{\delta,\delta'} = \text{sgn}(\xi) f^\delta + |f_\lambda^{\delta'}|^2 - 2f^\delta f_\lambda^{\delta'}$, we have

$$\partial_t F_\lambda^{\delta,\delta'} + a(y,\xi)\cdot\nabla_x F_\lambda^{\delta,\delta'} = \mathcal{M}^\delta\left(\text{sgn}(\xi) - 2f_\lambda^{\delta'}\right) + 2\mathcal{M}_\lambda^{\delta'}(f_\lambda^{\delta'} - f^\delta).$$

We integrate over $(0,t)\times\mathbb{R}^N\times Y\times\mathbb{R}$ and obtain

$$\begin{aligned}\int_{\mathbb{R}^N\times Y\times\mathbb{R}} F_\lambda^{\delta,\delta'}(t,x,y,\xi)\,dx\,dy\,d\xi \;=\;& \int_0^t\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}^\delta\left(\text{sgn}(\xi) - 2f_\lambda^{\delta'}\right)\\ &+2\int_0^t\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}_\lambda^{\delta'}(f_\lambda^{\delta'} - f^\delta)\\ &+\int_{\mathbb{R}^N\times Y\times\mathbb{R}} F_\lambda^{\delta,\delta'}(t=0,x,y,\xi)\,dx\,dy\,d\xi.\end{aligned}$$

We now pass to the limit as $\delta' \to 0$, with all the other parameters fixed. Notice that

$$\begin{aligned}\lim_{\delta'\to 0}\int_0^t\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}_\lambda^{\delta'}(f_\lambda^{\delta'} - f^\delta) \;=\;& \int_0^t\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}_\lambda(f_\lambda - f^\delta)\\ =\;& -\lambda\int_0^t\int_{\mathbb{R}^N\times Y\times\mathbb{R}} (f_\lambda - \mathcal{P}(f_\lambda))^2\\ &+\int_0^t\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}_\lambda(\mathcal{P}(f_\lambda) - f^\delta)\\ \le\;& 0,\end{aligned}$$

since $f^\delta(t,x) \in \mathcal{K}$ for all t, x. The passage to the limit in $F_\lambda^{\delta,\delta'}(t=0)$ does not rise any difficulty because of the strong continuity of the functions f_λ at time $t = 0$. Hence, we retrieve

$$\begin{aligned}&\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left\{\left(\left|f^\delta(t)\right| - \left|f^\delta(t)\right|^2\right) + \left|f^\delta(t) - f_\lambda(t)\right|^2\right\}\\ \le\;& \int_0^t\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}^\delta\left(\text{sgn}(\xi) - 2f_\lambda\right)\\ &+\int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left\{\left(\left|f^\delta(t=0)\right| - \left|f^\delta(t=0)\right|^2\right) + \left|f^\delta(t=0) - \chi(\xi,u_0)\right|^2\right\},\end{aligned}$$

and thus, integrating once again this inegality for $t \in [0,T]$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left\{ \left(|f^\delta| - |f^\delta|^2 \right) + \left| f^\delta(t) - f_\lambda \right|^2 \right\} \\
\leq\ & \int_0^T dt \left[\int_0^t \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}^\delta(s) \left(\mathrm{sgn}(\xi) - 2 f_\lambda(s) \right)\, ds \right] \\
&+T \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left\{ \left(\left| f^\delta(t=0) \right| - \left| f^\delta(t=0) \right|^2 \right) + \left| f^\delta(t=0) - \chi(\xi, u_0) \right|^2 \right\}.
\end{aligned}$$

We now pass to the limit as $\lambda \to \infty$, with $\delta > 0$ fixed. Then

$$\liminf_{\lambda\to\infty} \left| \left| f_\lambda - f^\delta \right| \right|^2_{L^2((0,T)\times\mathbb{R}^N\times Y\times\mathbb{R}} \geq \left| \left| g - f^\delta \right| \right|^2_{L^2((0,T)\times\mathbb{R}^N\times Y\times\mathbb{R}},$$

and

$$\begin{aligned}
&\lim_{\lambda\to\infty} \int_0^T dt \left[\int_0^t \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}^\delta(s) \left(\mathrm{sgn}(\xi) - 2 f_\lambda(s) \right)\, ds \right] \\
=\ & \int_0^T dt \left[\int_0^t \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \mathcal{M}^\delta(s) \left(\mathrm{sgn}(\xi) - 2 g(s) \right)\, ds \right] \leq 0.
\end{aligned}$$

Thus, we obtain, for all $\delta > 0$

$$\begin{aligned}
&||g - f^\delta||^2_{L^2((0,T)\times\mathbb{R}^N\times Y\times\mathbb{R}} \\
\leq\ & T \int_{\mathbb{R}^N\times Y\times\mathbb{R}} \left\{ \left(|f^\delta(t=0)| - f^\delta(t=0)^2 \right) + |f^\delta(t=0) - \chi(\xi,u_0)|^2 \right\}.
\end{aligned}$$

We have already proved in the previous section that the family $f^\delta(t=0)$ strongly converges towards $\chi(\xi, u_0)$ as δ vanishes, due to the continuity assumption at time $t = 0$. Hence, we obtain in the limit

$$||g - f||^2_{L^2((0,T)\times\mathbb{R}^N\times Y\times\mathbb{R}} \leq 0,$$

and consequently, $g = f$. Hence the result is proved.

5.5 The separate case : identification of the limit problem

This section is devoted to the proof of Proposition 5.1.1. Thus we focus on the limit system in the case where the flux A can be written as

$$A(y,\xi) = a_0(y) g(\xi), \quad \text{with } \mathrm{div}_y a_0 = 0.$$

The interest of this case lies in the special structure of the limit system; indeed, we shall prove that the function u, which is the two-scale limit of the sequence u^ε, is the solution of the scalar conservation law (5.20). In view of Theorem 7, we wish to emphasize that Proposition 5.1.1 implies in particular that the entropy solution of (5.20) satisfies the constraint equation

$$\mathrm{div}_y\left(a_0(y)g(u(t,x;y))\right)=0$$

for almost every $t>0, x\in\mathbb{R}^N$; this fact is not completely obvious when $g\neq \mathrm{Id}$. We will prove in the sequel that $u(t,x)$ actually belongs to the constraint space $\mathbb{K}_0$ for almost every t,x.

Before addressing the proof of the Theorem, let us mention that the limit problem (5.20) is not the one which is expected from a vanishing viscosity approach. Precisely, for any given $\delta>0$, let u_δ^ε be the solution of

$$\partial_t u_\delta^\varepsilon+\mathrm{div}_x A\left(\frac{x}{\varepsilon},u_\delta^\varepsilon\right)-\varepsilon\delta\Delta_x u_\delta^\varepsilon=0,$$

with the initial data $u_\delta^\varepsilon(t=0,x)=u_0\left(x,x/\varepsilon\right)$. Then for all $\varepsilon>0$, $u_\delta^\varepsilon\to u^\varepsilon$ as $\delta\to 0$; moreover, the behavior of u_δ^ε as $\varepsilon\to 0$ is known for each $\delta>0$ (see [14, 18]). In the divergence-free case, for all $\delta>0$,

$$\lim_{\varepsilon\to 0}u_\delta^\varepsilon=\bar u(t,x)\quad\text{in } L^1_{\mathrm{loc}},$$

where $\bar u$ is the entropy solution of

$$\begin{cases}\partial_t\bar u+\mathrm{div}_x(\langle a\rangle\, g(\bar u))=0, & t>0,\ x\in\mathbb{R}^N,\\ \bar u(t=0,x)=\langle u_0(x,\cdot)\rangle, & x\in\mathbb{R}^N.\end{cases}$$

Hence, it could be expected that the limits $\varepsilon\to 0$ and $\delta\to 0$ can be commuted, that is

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}u_\delta^\varepsilon=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}u_\delta^\varepsilon,$$

which would entail

$$\lim_{\varepsilon\to 0}u^\varepsilon=\bar u.$$

In general, this equality is false, even in a weak sense: a generic counter-example is the one of shear flows (see for instance the calculations in [24]). Indeed, if $N=2$ and $A(y,\xi)=(a_1(y_2)\xi,0)$, then equation (5.20) becomes

$$\begin{cases}\partial_t u+a_1(y_2)\partial_{x_1}u=0, & t>0,\ x\in\mathbb{R}^2,\ y\in[0,1]^2\\ u(t=0,x,y)=u_0(x_1,x_2,y_2), & x\in\mathbb{R}^2,\ y\in[0,1]^2.\end{cases}$$

It is then easily checked that in general, the average of u over $Y=[0,1]^2$ is not the solution of the transport equation

$$\partial_t\bar u+\langle a_1\rangle\,\partial_{x_1}\bar u=0.$$

We now turn to the proof of Proposition 5.1.1. In view of Theorem 7, it is sufficient to prove that the entropy solution of (5.20) belongs to $\mathbb{K}_0$ for a.e. t,x, or in other words, that $\mathbb{K}_0$ is invariant by the semi-group associated to equation (5.20). We prove this result in the slightly more general context of kinetic solutions. The core of the proof lies in the following Proposition:

Proposition 5.5.1. *Let* $u_0 \in L^1(\mathbb{R}^N, L^\infty(Y))$ *such that* $u_0(x,\cdot) \in \mathbb{K}_0$ *for almost every* $x \in \mathbb{R}^N$.

Let $v = v(t,x;y) \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^N \times Y))$ *be the kinetic solution of*

$$\begin{cases} \partial_t v(t,x;y) + \mathrm{div}_x\left(\tilde{a}(y) g(v(t,x;y))\right) = 0, \quad t>0,\ x \in \mathbb{R}^N,\ y \in Y, \\ v(t=0,x;y) = u_0(x,y). \end{cases} \tag{5.53}$$

Then for almost every $t > 0, x \in \mathbb{R}^N,\ v(t,x) \in \mathbb{K}_0$.

Remark 5.5.1. *Let us recall that the function* v *is the kinetic solution of* (5.53) *if there exists a nonnegative measure* $m \in \mathcal{C}(\mathbb{R}_\xi, M^1_w([0,\infty) \times \mathbb{R}^N \times Y)$ *such that the following properties hold:*

- *The function* $f^1(t,x,y,\xi) := \chi(\xi, v(t,x;y))$ *is a solution in the sense of distributions of*

$$\begin{cases} \partial_t f^1 + \tilde{a}(y) \cdot \nabla_x f^1 g'(\xi) = \partial_\xi m, \quad t>0,\ x \in \mathbb{R}^N,\ y \in Y,\ \xi \in \mathbb{R}, \\ f^1(t=0,x,y,\xi) = \chi(\xi, u_0(x,y)); \end{cases} \tag{5.54}$$

- *There exists a function* $\mu \in L^\infty(\mathbb{R})$ *such that*

$$\int_0^\infty \int_{\mathbb{R}\times Y} m(t,x,y,\xi)\,dx\,dy\,dt \le \mu(\xi) \quad \forall \xi \in \mathbb{R},$$
$$\lim_{|\xi|\to\infty} \mu(\xi) = 0.$$

Proof. First, let us recall (see [59, 60]) that for all $T > 0$,

$$f^1 = \lim_{\lambda\to\infty} f_\lambda \quad \text{in } \mathcal{C}([0,T]; L^1(\mathbb{R}^N \times Y \times \mathbb{R})),$$

where $f_\lambda = f_\lambda(t,x,y,\xi)$ $(\lambda > 0)$ is the unique solution of the system

$$\begin{cases} \partial_t f_\lambda + \tilde{a}(y) \cdot \nabla_x f_\lambda\, g'(\xi) + \lambda f_\lambda = \lambda \chi(\xi, u_\lambda), \\ u_\lambda(t,x,y) = \displaystyle\int_{\mathbb{R}} f_\lambda(t,x,y,\xi)\,d\xi, \\ f_\lambda(t=0) = \chi(\xi, u_0). \end{cases} \tag{5.55}$$

Moreover, for every $\lambda > 0$, u_λ is the unique fixed point of the contractant application

$$\phi_\lambda : \begin{array}{ccc} \mathcal{C}((0,T); L^1(\mathbb{R}^N \times Y)) & \to & \mathcal{C}((0,T); L^1(\mathbb{R}^N \times Y)) \\ u_1 & \mapsto & u_2 \end{array}$$

where $u_2 = \int_\xi f$ and f is the solution of

$$\begin{array}{l} \partial_t f + \tilde{a}(y) \cdot \nabla_x f\, g'(\xi) + \lambda f = \lambda \chi(\xi, u_1), \\ f(t=0) = \chi(\xi, u_0). \end{array} \tag{5.56}$$

Thus, it is sufficient to prove that the space

$$\mathcal{I}_0 := \{u \in \mathcal{C}([0,T]; L^1(\mathbb{R}^N \times Y)); u(t,x) \in \mathbb{K}_0 \text{ a.e}\}$$

is invariant by the application ϕ_λ.

First, let us stress that for all $u \in L^1(Y)$,

$$u \in \mathbb{K}_0 \iff \operatorname{div}_y(a(y)\chi(\xi, u)) = 0 \text{ in } \mathcal{D}'(Y \times \mathbb{R}). \tag{5.57}$$

Indeed, if $u \in \mathbb{K}_0$, then for all $\delta > 0$, set $u_\delta = u * \theta^\delta$, with θ^δ a standard mollifier. The function u_δ is a solution of

$$\operatorname{div}_y(a_0 u_\delta) = r_\delta,$$

and the remainder r_δ vanishes strongly in $L^1(Y)$ (see the calculations in the previous sections). Since the function u_δ is smooth, if $G \in \mathcal{C}^1(\mathbb{R}^N)$, we have

$$\operatorname{div}_y(a_0 G(u_\delta)) = G'(u_\delta) r_\delta.$$

Passing to the limit as δ vanishes, we infer $\operatorname{div}_y(a_0 G(u)) = 0$ for all $G \in \mathcal{C}^1(\mathbb{R}^N)$. At last, taking a sequence of smooth functions approaching $\chi(\xi, u)$, we deduce that $\operatorname{div}_y(a_0\chi(\xi, u)) = 0$ in $\mathcal{D}'_{\text{per}}(Y \times \mathbb{R})$. Conversely, assume that $\operatorname{div}_y(a_0\chi(\xi, u)) = 0$; then integrating this equation with respect to ξ yields $u \in \mathbb{K}_0$. Hence (5.57) is proved.

Now, let us prove that the space $\mathcal{I}_0$ is invariant by the application ϕ_λ. Let $u_1 \in C([0, T]; L^1(\mathbb{R}^N \times Y))$ such that $u_1(t, x) \in \mathbb{K}_0$ almost everywhere. Then

$$\operatorname{div}_y(a_0(y)\chi(\xi, u_1(t, x, y))) = 0.$$

Let f be the solution of (5.56); since $\tilde{a} \in \mathbb{K}_0^N$, the distribution $\operatorname{div}_y(a_0 f)$ satisfies the transport equation

$$\partial_t (\operatorname{div}(a_0 f)) + g'(\xi)\tilde{a}(y) \cdot \nabla_x (\operatorname{div}(a_0 f)) + \lambda \operatorname{div}(a_0 f) = 0,$$

and $\operatorname{div}(a_0 f)(t = 0) = 0$ because $u_0(x) \in \mathbb{K}_0$ almost everywhere. Hence

$$\operatorname{div}_y(a_0(y) f(t, x, y, \xi)) = 0 \quad \text{in } \mathcal{D}'.$$

Integrating this equation with respect to ξ yields $u_2 \in \mathbb{K}_0$ almost everywhere.

Consequently, for all $\lambda > 0$, $u_\lambda(t, x; \cdot) \in \mathbb{K}_0$ for almost every t, x. Passing to the limit, we deduce that $v(t, x; \cdot) \in \mathbb{K}_0$ almost everywhere. Hence Proposition 5.5.1 is proved.

□

We now turn to the proof of Proposition 5.1.1: setting $b(y) = a_0(y) - \tilde{a}(y)$, we have

$$\partial_t f^1 + a_0(y)\nabla_x f^1 g'(\xi) = \partial_\xi m - b(y)\nabla_x f^1 g'(\xi) =: \mathcal{M}_1.$$

If $u_0 \in L^\infty(\mathbb{R}^N)$, then $v \in L^\infty([0, \infty) \times \mathbb{R}^N \times Y)$, and it is easily checked that f^1 and $\mathcal{M}_1$ satisfy the compact support assumptions. According to Proposition 5.5.1, f^1 also satisfies (5.45), and thanks to the structure of the right-hand side, the distribution $\mathcal{M}_1$ satisfies (5.48). Thus f^1 is the unique solution of the limit system, and Proposition 5.1.1 is proved.

5.6 Further remarks on the notion of limit system

Here, we have gathered, by way of conclusion, a few remarks on the limit evolution system introduced in Definition 5.1.1. The main idea behind this section is that the limit system is not unique (although its solution always is), and thus several other relevant equations can be written instead of (5.10). Unfortunately, there does not seem to be any rule which would allow to decide between two limit systems.

Let us illustrate these words by a first series of examples: assume that the flux is divergence free, and let

$$a'(y,\xi) = (a_1(y,\xi), \cdots, a_N(y,\xi)) \in \mathbb{R}^N, \quad (y,\xi) \in Y \times \mathbb{R},$$
$$\mathbb{K} := \{f \in L^1_{\text{loc}}(Y \times \mathbb{R}),\ \text{div}_y(a'f) = 0 \quad \text{in } \mathcal{D}'\}.$$

We denote by P the projection on $\mathbb{K}$ in $L^1_{\text{loc}}(Y \times \mathbb{R})$. Precisely, consider the dynamical system $X(t,y;\xi)$ defined by

$$\begin{cases} \dot{X}(t,y;\xi) = a'(X(t,y;\xi),\xi), & t > 0, \\ X(t=0,y;\xi) = y \in Y. \end{cases}$$

Then for all $\xi \in \mathbb{R}$, the Lebesgue measure on Y is invariant by the semi-group $X(t;\xi)$ because $a'(\cdot,\xi)$ is divergence free for all ξ. Hence, by the ergodic theorem, for all $f \in L^1_{\text{loc}}(Y \times \mathbb{R})$ there exists a function in $L^1_{\text{loc}}(Y \times \mathbb{R})$, denoted by $P(f)(y,\xi)$, such that

$$P(f)(y,\xi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t,y;\xi),\xi)\, dt,$$

and the limit holds a.e. in y, ξ and in $Y \times (-R,R)$ for all $R > 0$.

For $1 \le i \le N$, set $\tilde{a}_i := P(a_i)$. Then if f is a solution of the limit system, f also satisfies

$$\partial_t f + \tilde{a}(y,\xi) \cdot \nabla_x f = \tilde{\mathcal{M}}$$

and f, $\tilde{\mathcal{M}}$ satisfy (5.9) and (5.11) - (5.14). Indeed,

$$\tilde{\mathcal{M}} = \mathcal{M} + [\tilde{a}(y,\xi) - a'(y,\xi)] \cdot \nabla_x f$$

and the term $[\tilde{a}(y,\xi) - a'(y,\xi)] \cdot \nabla_x (f *_x \varphi)(t,x,y,\xi)$ belongs to $\mathbb{K}^\perp$ for all t, x. Of course, uniqueness holds for this limit system (the proof is exactly the same as the one in section 5.3), and thus this constitutes as legitimate a limit system as the one in Definition 5.1.1. In fact, in the separate case, Proposition 5.1.1 indicates that the above system seems to be the relevant one, rather than the one in Definition 5.1.1. Notice that the distribution $\tilde{\mathcal{M}}$ satisfies the additional property

$$\tilde{\mathcal{M}} *_{t,x} \phi(t,x) \in \mathbb{K}^\perp \quad \forall t, x.$$

Let us now go a little further: let $\theta \in \mathcal{C}^1(\mathbb{R})$ such that $0 \le \theta \le 1$, and let

$$a_\theta(y,\xi) = \theta(\xi)a'(y,\xi) + (1 - \theta(\xi))\tilde{a}(y,\xi).$$

Then f is a solution of

$$\partial_t f + a_\theta(y,\xi) \cdot \nabla_x f = \mathcal{M}_\theta,$$

for some distribution $\mathcal{M}_\theta$ satisfying (5.14). Thus this constitutes yet another limit system which has the same structure as the one of Definition 5.1.1. Hence the limit system is highly non unique, and it must be seen as a way of identifying the two-scale limit of the sequence f^ε, rather than as a kinetic formulation of a given conservation law, for instance. Let us also emphasize the following fact: consider the solution $v = v(t, x, y)$ of the scalar conservation law

$$\begin{cases} \partial_t v + \operatorname{div}_x \tilde{A}(y, v) = 0, \\ v(t = 0, x, y) = u_0(x, y), \end{cases}$$

where the flux $\tilde{A}$ is such that $\partial_\xi \tilde{A}_i(y, \xi) = \tilde{a}_i(y, \xi)$. Then, in general, the function $\mathbf{1}_{\xi<v}$ is not a solution of the limit system, except in the so-called "separate case" described in Proposition 5.1.1. Indeed, the function v is not a solution of the cell problem in general, even if u_0 is. In other words, the set $\mathbb{K}$ is not invariant by the evolution equation

$$\partial_t g + \sum_{i=1}^N \tilde{a}_i(y, \xi) \partial_{x_i} g = \partial_\xi m,$$

where m is a nonnegative measure and $g = \mathbf{1}_{\xi<v}$.

Let us now assume that the flux A is not divergence free. Then there are cases where a different notion of limit problem can be given: assume that there exists real numbers $p_1 < p_2$, and a family $\{v(\cdot, p)\}_{p_1 \le p \le p_2}$, which satisfies the following properties:

1. The function $(y, p) \mapsto v(y, p)$ belongs to $L^\infty(Y \times [p_1, p_2])$;

2. For all $p \in [p_1, p_2]$, $v(\cdot, p)$ is an entropy solution of the cell problem; in other words, there exists a nonnegative measure $m(y, \xi; p)$ such that $f(y, \xi; p) = \mathbf{1}_{\xi<v(y,p)}$ is a solution of

$$\sum_{i=1}^N \frac{\partial}{\partial y_i} (a_i(y, \xi) f) + \frac{\partial}{\partial \xi} (a_{N+1}(y, \xi) f) = \frac{\partial}{\partial \xi} m;$$

3. For all $p \in [p_1, p_2]$, $\langle v(\cdot, p) \rangle_Y = 0$;

4. The distribution $\partial_p v$ is a nonnegative function in $L^1(Y \times [p_1, p_2])$; this implies in particular that for all couples $(p, p') \in [p_1, p_2]^2$ such that $p \ge p'$, for almost every $y \in Y$,
$$v(y, p) \ge v(y, p').$$

Under these conditions, a kinetic formulation for equation (5.1) can be derived, based on the family $v(x/\varepsilon, p)$ of stationary solutions of (5.1), rather than on the family of Kruzkov's inequalities. This kind of construction was achieved in [19] in a parabolic setting, following an idea developed by Emmanuel Audusse and Benoît Perthame in [8]; these authors define a new notion of entropy solutions for a heterogeneous conservation law in dimension one, based on the comparison with a family of stationary solutions. Let us explain briefly how the kinetic formulation for entropy solutions of (5.1) is derived: let u^ε be an entropy solution of (5.1). Define the distribution $m^\varepsilon \in \mathcal{D}'((0,\infty)\times\mathbb{R}^N\times(p_1,p_2))$ by

$$\begin{aligned} m^\varepsilon(t,x,p) \quad := \quad & -\left\{\frac{\partial}{\partial t}\left(u^\varepsilon - v\left(\frac{x}{\varepsilon},p\right)\right)_+ + \right. \\ & \left. +\sum_{i=1}^N \frac{\partial}{\partial y_i}\left[\mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon}\left(A_i\left(\frac{x}{\varepsilon},u^\varepsilon\right) - A_i\left(\frac{x}{\varepsilon},v\left(\frac{x}{\varepsilon},p\right)\right)\right)\right]\right\}. \end{aligned} \tag{5.58}$$

Then according to the comparison principle (which was known by Kruzkhov, see [69, 70]), m^ε is a nonnegative measure on $(0,\infty)\times\mathbb{R}^N\times[p_1,p_2]$. Now, set

$$f^\varepsilon(t,x,p) := \mathbf{1}_{v\left(\frac{x}{\varepsilon},p\right)<u^\varepsilon(t,x)} \in L^\infty([0,\infty)\times\mathbb{R}^N\times[p_1,p_2]).$$

Thanks to the regularity assumptions on the family $v(\cdot,p)$, we can differentiate equality (5.58) (which is meant in the sense of distributions) with respect to p, and we are led to

$$\frac{\partial}{\partial t}\left(f^\varepsilon v_p\left(\frac{x}{\varepsilon},p\right)\right) + \frac{\partial}{\partial x_i}\left(f^\varepsilon v_p\left(\frac{x}{\varepsilon},p\right) a_i\left(\frac{x}{\varepsilon},v\left(\frac{x}{\varepsilon},p\right)\right)\right) = \frac{\partial m^\varepsilon}{\partial p}. \tag{5.59}$$

This equation is in fact the appropriate kinetic formulation in the heterogeneous case; its main advantage over equation (5.4) is the absence of the highly oscillating term

$$\frac{1}{\varepsilon}\partial_\xi\left[a_{N+1}\left(\frac{x}{\varepsilon},\xi\right)\mathbf{1}_{\xi<u^\varepsilon}\right].$$

Notice that for all $p\in[p_1,p_2]$,

$$\mathrm{div}_y\left(\frac{\partial v(y,p)}{\partial p}a(y,v(y,p))\right) = 0 \quad \text{in } \mathcal{D}'_{\mathrm{per}}(Y). \tag{5.60}$$

This equation is derived by differentiating equation

$$\mathrm{div}_y A(y,v(y,p)) = 0$$

with respect to p. Thus, if we set

$$\breve{a}(y,p) := \frac{\partial v(y,p)}{\partial p}a(y,v(y,p)),$$

the vector field $\breve{a}\in L^1(Y\times[p_1,p_2])$ is divergence-free, and the same kind of limit system as in the divergence free case can be built. Of course, the interest of such a construction lies in the simplicity of the structure of the limit system in the divergence free case.

Definition 5.6.1. *Let $f\in L^\infty([0,\infty),L^1(\mathbb{R}^N\times Y\times\mathbb{R}))$, $u_0\in L^1\cap L^\infty(\mathbb{R}^N\times Y)$. It is said that f is a generalized kinetic solution of the limit problem associated with the family $v(\cdot,p)$ if there exists a distribution $\mathcal{M}\in\mathcal{D}'_{per}([0,\infty)\times\mathbb{R}^N\times Y\times\mathbb{R})$ such that f and $\mathcal{M}$ satisfy the following properties:*

1. *Compact support in* p*: there exists* $(p_1', p_2') \in [p_1, p_2]^2$ *such that* $p_1 < p_1' \leq p_2' < p_2$*, and*

$$\mathrm{Supp}\mathcal{M} \subset [0, \infty) \times \mathbb{R}^N \times Y \times [p_1', p_2'];$$
$$f(t, x, y, p) = 1 \text{ if } p_1 < p < p_1', \quad f(t, x, y, p) = 0 \text{ if } p_2' < p < p_2.$$

2. *Microscopic equation for* f*:* f *is a solution in the sense of distributions on* $Y \times (p_1, p_2)$ *of*

$$\mathrm{div}_y(\breve{a}(y, p) f(t, x, y, p)) = 0. \tag{5.61}$$

3. *Evolution equation: the couple* $(f, \mathcal{M})$ *is a solution in the sense of distributions on* $[0, \infty) \times \mathbb{R}^N \times Y \times (p_1, p_2)$ *of*

$$\begin{cases} \partial_t(v_p(y, p) f) + \breve{a}(y, p) \cdot \nabla_x f = \mathcal{M}, \\ f(t = 0, x, y, p) = \mathbf{1}_{v(y,p) < u_0(x,y)} =: f_0(x, y, p); \end{cases} \tag{5.62}$$

In other words, for any test function $\phi \in \mathcal{D}_{per}(Q)$ *where* $Q = [0, \infty) \times \mathbb{R}^N \times Y \times (p_1, p_2))$*, setting* $dq = dt\, dx\, dy\, d\xi$*,*

$$\int_Q f(t, x, y, p) v_p(y, p) \left\{ \partial_t \phi(t, x, y, p) + a(y, v(y, p)) \cdot \nabla_x \phi(t, x, y, p) \right\} dq$$
$$= - \langle \phi, \mathcal{M} \rangle_{\mathcal{D}, \mathcal{D}'} - \int_{\mathbb{R}^N \times Y \times \mathbb{R}} \mathbf{1}_{v(y,p) < u_0(x,y)} v_p(y, p) \phi(t = 0, x, y, p)\, dx\, dy\, d\xi.$$

4. *Conditions on* f*:*

$$\partial_p f \leq 0, \tag{5.63}$$
$$0 \leq f(t, x, y, \xi) \leq 1 \quad a.e., \tag{5.64}$$
$$\frac{1}{\tau} \int_0^\tau ||f(s) - f_0||_{L^2(\mathbb{R}^N \times Y \times (p_1, p_2))}\, ds \underset{\tau \to 0}{\longrightarrow} 0. \tag{5.65}$$

5. *Condition on* $\mathcal{M}$*: for all* $\varphi \in \mathcal{D}([0, \infty) \times \mathbb{R}^N$ *such that* $\varphi \leq 0$*, the function* $\mathcal{M} *_{t,x} \varphi$ *belongs to* $\mathcal{C}([0, \infty) \times \mathbb{R}^N, L^1(Y \times \mathbb{R}))$*, and*

$$\begin{cases} \int_{Y \times \mathbb{R}} (\mathcal{M} *_{t,x} \varphi)(t, x, \cdot)\, \psi \leq 0, \\ \forall \psi \in L^\infty_{loc}(Y \times \mathbb{R}),\ \mathrm{div}_y(\breve{a}\psi) = 0, \text{ and } \partial_\xi \psi \geq 0. \end{cases} \tag{5.66}$$

We now state without proof a result analogue to Theorems 7, 8 :

Proposition 5.6.1. *Let* $A \in W^{2,\infty}_{per,loc}(Y \times \mathbb{R})$*. Assume that* $a \in \mathcal{C}^1_{per}(Y \times \mathbb{R})$ *and that* $\tilde{a} \in W^{1,1}(Y \times (p_1, p_2))$*. Let* $u_0 \in L^\infty(\mathbb{R}^N \times Y) \cap L^1_{loc}(\mathbb{R}^N, \mathcal{C}_{per}(Y))$ *such that* $u_0(x, \cdot)$ *is an entropy solution of the cell problem for almost every* $x \in \mathbb{R}^N$*. Assume furthermore that there exists* $p_1' < p_2'$ *in* $(p_1, p_2)^2$ *such that*

$$v(y, p_1') \leq u_0(x, y) \leq v(y, p_2'),$$

and let

$$f_0(x, y, p) := \mathbf{1}_{v(y,p) < u_0(x,y)}$$

Then the following results hold :

1. *There exists a unique generalized kinetic solution f of the limit problem associated with the family $(v(\cdot,p))_{p_1\leq p\leq p_2}$ with initial data f_0. Moreover, there exists a function $u \in L^\infty([0,\infty)\times\mathbb{R}^N\times Y)$ such that*

$$f(t,x,y,p) = \mathbf{1}_{v(y,p)<u(t,x,y)} \quad a.e.$$

2. *Let $u^\varepsilon \in L^\infty([0,\infty)\times\mathbb{R}^N)$ be the entropy solution of* (5.1) *with initial data $u_0(x, x/\varepsilon)$. Let $f(t,x,y,p) = \mathbf{1}_{v(y,p)<u(t,x,y)}$ be the unique solution of the limit problem. Then for all regularization kernels φ^δ of the form*

$$\varphi^\delta(x) = \frac{1}{\delta^N}\varphi\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}^N,$$

with $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\int\varphi = 1$, $0 \leq \varphi \leq 1$, we have, for all compact $K \subset [0,\infty)\times\mathbb{R}^N$,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left|\left|u^\varepsilon(t,x) - u *_x \varphi^\delta\left(t, x, \frac{x}{\varepsilon}\right)\right|\right|_{L^1(K)} = 0. \tag{5.67}$$

Hence a whole variety of limit systems can be given, depending on the choice of the family of solutions of the cell problem. However, it is not obvious that any given system is “better” than another one. But the important result, as far as homogenization is concerned, is that all systems have a unique solution.

Chapitre 6

Homogénéisation d'une équation de transport

On étudie ici l'homogénéisation d'une équation cinétique linéaire, qui modélise l'évolution de la densité d'un ensemble de particules chargées dans un champ électrique aléatoire fortement oscillant. On identifie le profil asymptotique de la densité grâce à des équations microscopiques et macroscopiques couplées ; de plus, on donne des formules explicites permettant de calculer ce profil lorsque la dimension de l'espace est égale à un.

Ce chapitre a été accepté pour publication dans Communications in Partial Differential Equations.

6.1 Introduction

This note is concerned with the homogenization of a linear transport equation in a stationary ergodic setting. The equation studied here describes the evolution of the density of charged particles in a rapidly oscillating random electric potential. This equation can be derived by passing to the semi-classical limit in the Schrödinger equation (see [40], [50], and the presentation in [33]). Our work generalizes a result of E. Frénod and K. Hamdache (see [33]) which was obtained in a periodic setting. The strategy of proof we have chosen here is different from the one of [33], and allows us to retrieve some of the results in [33] in a rather simple and explicit fashion.

Let us mention a few related works on the homogenization of linear transport equations ; we emphasize that this list is by no means exhaustive. In [6], Y. Amirat, K. Hamdache and A. Ziani study the homogenization of a linear transport equation in a periodic setting and give an application to a model describing a multidimensional miscible flow in a porous media. In [23] (see also [35]), Laurent Dumas and François Golse focus on the homogenization of linear transport equations with absorption and scattering terms, in periodic and stationary ergodic settings. And in [24], Weinan E derives strong convergence results for the homogenization of linear and nonlinear transport equations with oscillatory incompressible velocity fields in a periodic setting.

Let us now present the context we will be working in : let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\tau_x)_{x\in\mathbb{R}^N}$ be a group transformation acting on Ω. We assume that τ_x preserves the probability measure P for all $x \in \mathbb{R}^N$, and the group transformation is ergodic, which means

$$\forall A \in \mathcal{F}, \quad \left(\tau_x A = A \;\forall x \in \mathbb{R}^N \Rightarrow P(A) = 0 \text{ or } 1\right).$$

The periodic setting can be embedded the stationary ergodic setting in the following way (see [57]) : take $\Omega = \mathbb{R}^N/\mathbb{Z}^N \simeq [0,1)^N$, and let P be the Lebesgue measure on Ω. Define the group transformation $(\tau_x)_{x\in\mathbb{R}^N}$ by

$$\tau_x y = x + y \mod \mathbb{Z}^N \quad \forall (x,y) \in \mathbb{R}^N \times \Omega.$$

Then it is easily checked that τ_x preserves the measure P for all $x \in \mathbb{R}^N$, and that the group transformation is ergodic. Thus the periodic setting is a particular case of the stationary ergodic setting.

We will denote by $E[\cdot]$ the expectation with respect to the probability measure P; in the periodic case, we will write $\langle f\rangle$ rather than $E[f]$ to refer to the average of f over one period.

We consider a potential function $u = u(y,\omega) \in L^\infty(\mathbb{R}^N \times \Omega)$ which is assumed to be stationary, i.e.

$$u(y+z,\omega) = u(y,\tau_z\omega) \quad \forall (y,z,\omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega.$$

Moreover, we assume that $0 \le u(y,\omega) \le u_{\max} = \sup u \;\forall y \in \mathbb{R}^N, \omega \in \Omega$, and $u(\cdot) \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^N, L^\infty(\Omega))$, so that there exists a set $A \subset \Omega$ such that $P(A) = 0$, and $\nabla_y u(\cdot,\omega)$ is well-defined and locally Lipschitz continuous on $\mathbb{R}^N$, uniformly for $\omega \in \Omega \setminus A$.

Let $f^\varepsilon = f^\varepsilon(t,x,\xi,\omega)$, $(t \geq 0,\ x \in \mathbb{R}^N, \xi \in \mathbb{R}^N, \omega \in \Omega)$ be the solution of the transport equation

$$\begin{cases} \partial_t f^\varepsilon(t,x,\xi,\omega) + \xi \cdot \nabla_x f^\varepsilon(t,x,\xi,\omega) - \dfrac{1}{\varepsilon}\nabla_y u\left(\dfrac{x}{\varepsilon},\omega\right) \cdot \nabla_\xi f^\varepsilon(t,x,\xi,\omega) = 0, \\ f^\varepsilon(t=0,x,\xi,\omega) = f_0\left(x, \dfrac{x}{\varepsilon}, \xi, \omega\right). \end{cases} \tag{6.1}$$

Here, we assume that the initial data $f_0 = f_0(x,y,\xi,\omega)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$ and is stationary in y, i.e.

$$f_0(x, y+z, \xi, \omega) = f_0(x,y,\xi,\tau_z\omega) \quad \text{for all } (x,y,z,\xi,\omega) \in \mathbb{R}^{4N} \times \Omega.$$

We set $F_0(x,\xi,\omega) = f_0(x,0,\xi,\omega)$, and we also assume that f_0 is such that for all $\varepsilon > 0$, the function

$$f^\varepsilon(t=0) : (x,\xi,\omega) \mapsto f_0\left(x, \frac{x}{\varepsilon}, \xi, \omega\right)$$

belongs to $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))$; this fact does not follow directly from the above assumptions because in general, the measurability of $f^\varepsilon(t=0)$ is not clear. However, if $F_0 \in \mathcal{C}_c(\mathbb{R}^N_x, L^1_{\text{loc}}(\mathbb{R}^N_\xi, L^\infty(\Omega)))$, for instance, then $f^\varepsilon(t=0)$ is measurable and belongs to $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))$. We will not comment further on this restriction and we refer to [10] for other sufficient assumptions on F_0. When f_0 satisfies the properties listed above, we say that f_0 is an *admissible initial data*; it can be checked (see [10]) that any linear combination of functions of the type

$$\chi_1(x)\, \chi_2(y,\xi,\omega)$$

with $\chi_1 \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $\chi_2 \in L^1_{\text{loc}}(\mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$, with χ_2 stationary, is an admissible initial data.

It is well-known from the classical theory of linear transport equations that for every $\omega \in \Omega$, there exists a unique solution f^ε of (6.1) in $L^\infty_{\text{loc}}((0,\infty), L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi))$. The goal of this paper is to study the asymptotic behavior of f^ε as $\varepsilon \to 0$. Thus, following [33], we define the constraint space $\mathbb{K}$:

Definition 6.1.1. *Let*

$$\xi \cdot \nabla_y f(y,\xi,\omega) - \nabla_y u(y,\omega) \cdot \nabla_\xi f(y,\xi,\omega) = 0 \tag{6.2}$$

be the constraint equation, and let

$$\mathbb{K} := \{f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega));\ f \text{ satisfies } (6.2) \text{ in } \mathcal{D}'(\mathbb{R}^N_y \times \mathbb{R}^N_\xi) \text{ a.s. in } \omega\}.$$

We also define the projection $\mathbb{P}$ onto the constraint space $\mathbb{K}$, characterised by $\mathbb{P}(f) \in \mathbb{K}$ for $f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega))$ stationary, and

$$\int_{\mathbb{R}^N \times \Omega} (\mathbb{P}(f) - f)(y,\xi,\omega)\, g(y,\xi,\omega)\, d\xi\, dP(\omega) = 0 \quad \text{for a.e. } y \in \mathbb{R}^N$$

for all stationary functions $g \in L^\infty(\mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega) \cap \mathbb{K}$, with compact support in ξ.

(A more precise definition of the projection $\mathbb{P}$ will be given in the second section).

Finally, we define $\mathbb{K}^\perp$ as

$$\mathbb{K}^\perp := \{f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega)); \exists g \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega)), \quad f = \mathbb{P}(g) - g\}.$$

Remark 6.1.1. *Let us indicate that the constraint equation can easily be derived thanks to a formal two-scale Ansatz : indeed, assume that*

$$f^\varepsilon(t,x,\xi,\omega) \approx f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) \quad \text{as } \varepsilon \to 0;$$

inserting this asymptotic expansion in equation (6.1)*, we see that f necessarily satisfies the constraint equation* (6.2)*.*

Remark 6.1.2. *Let $f, g \in L^\infty(\mathbb{R}^N_y, L^2(\mathbb{R}^N_\xi \times \Omega))$ be stationary, and assume that $f \in \mathbb{K}$ and $g \in \mathbb{K}^\perp$. Then for a.e. $y \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N\times\Omega} f(y,\xi,\omega) g(y,\xi,\omega)\, d\xi\, dP(\omega) = 0.$$

This is a characterization of $\mathbb{K}^\perp$ for the class of stationary functions belonging to $L^\infty(\mathbb{R}^N_y, L^2(\mathbb{R}^N_\xi \times \Omega))$.

Here, we provide another proof for the result of E. Frénod and K. Hamdache in [33] in the "non-perturbed case". Our proof is based on the use of the ergodic theorem, and gives a more concrete insight of the projection $\mathbb{P}$ and of the microscopic behavior of the sequence f^ε. Moreover, it allows us to retrieve the explicit formulas of the integrable case.

The first result we prove in this paper is the following theorem :

Theorem 10. *Let $f_0 \in L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y; L^1(\Omega))$ stationary, such that f_0 is an admissible initial data.*

Let $f^\varepsilon = f^\varepsilon(t,x,\xi,\omega)$ be the solution of (6.1)*. Then there exist two functions $f = f(t,x,y,\xi,\omega)$ and $g = g(t,x;\tau,y,\xi,\omega)$, both stationary in y, and a sequence $\{r^\varepsilon(t,x,\xi,\omega)\}_{\varepsilon>0}$ such that for all $\varepsilon > 0$*

$$f^\varepsilon(t,x,\xi,\omega) = f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) + g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) + r^\varepsilon(t,x,\xi,\omega)$$

and :

- $||r^\varepsilon||_{L^1_{loc}((0,\infty)\times\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^1(\Omega))} \to 0$ *as* $\varepsilon \to 0$;
- $f \in L^\infty_{loc}((0,\infty); L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y; L^1(\Omega)))$*, and $f(t,x) \in \mathbb{K}$ for a.e. $t \geq 0$, $x \in \mathbb{R}^N$;*
- *For all $T > 0$, for all compact set $K \subset \mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y$,*

$$\sup_{0\leq t\leq T, 0\leq\tau\leq T} ||g||_{L^1(K\times\Omega)} < \infty.$$

  Moreover, $g(t,x;\tau,\cdot) \in \mathbb{K}^\perp$ for a.e. $(t,x,\tau) \in (0,\infty)\times\mathbb{R}^N\times(0,\infty)$;
- *Microscopic evolution equation for g : for a.e. $t,x \in (0,\infty)\times\mathbb{R}^N$, $g(t,x;\cdot)$ is a solution of*

$$\frac{\partial g}{\partial \tau} + \xi\cdot\nabla_y g - \nabla_y u \cdot \nabla_\xi g = 0. \tag{6.3}$$

Moreover, for all $T > 0$

$$\left\| \int_0^T g\left(t, x; \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) \, dt \right\|_{L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0.$$

- *Macroscopic evolution equation : f and g satisfy*

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} + \xi^\sharp(y, \xi, \omega) \cdot \nabla_x \begin{pmatrix} f \\ g \end{pmatrix} = 0, \tag{6.4}$$

where

$$\xi^\sharp(y, \xi, \omega) := \mathbb{P}(\xi)(y, \xi, \omega);$$

- *Initial data :*

$$f(t = 0, x, y, \xi, \omega) = \mathbb{P}(f_0)(x, y, \xi, \omega),$$
$$g(t = 0, x; \tau = 0, y, \xi, \omega) = [f_0 - \mathbb{P}(f_0)] (x, y, \xi, \omega).$$

Before going any further, we wish to make a few comments on the above results. First, let us stress that it is not obvious that the function g is well-defined : indeed, let $S(t)$ ($t \geq 0$) denote the semi-group associated to the macroscopic evolution equation (6.4), and let $T(\tau)$ ($\tau \geq 0$) be the semi-group associated to the microscopic evolution equation (6.3). Then g is well defined if and only if, for all stationary function $g_0 = g_0(x, y, \xi, \omega)$, for all $t, \tau \geq 0$,

$$T(\tau) [S(t) g_0] = S(t) [T(\tau) g_0] .$$

This identity follows from the fact that the speed $\xi^\sharp(y, \xi, \omega)$ appearing in equation (6.4) is a stationary solution of (6.3) by definition of the projection $\mathbb{P}$, and is thus invariant by the semi-group $T(\tau)$.

Next, let us explain briefly the meaning of theorem 10. The idea is the following : write f_0 as $f_0 = f_{0\|} + f_{0\perp}$, with $f_{0\|}(x, \cdot) \in \mathbb{K}$ and $f_{0\perp}(x, \cdot) \in \mathbb{K}^\perp$ a.e. Assume that $f_{0\|}$ and $f_{0\perp}$ are admissible initial data. Then f^ε can be written as $f^\varepsilon_\| + f^\varepsilon_\perp$, where $f^\varepsilon_\|$ (resp. $f^\varepsilon_\perp$) is the solution of equation (6.1) with initial data $f_{0\|}(x, x/\varepsilon, \xi, \omega)$ (resp. $f_{0\perp}(x, x/\varepsilon, \xi, \omega)$). Theorem 10 states that

$$f^\varepsilon_\| - f\left(t, x, \frac{x}{\varepsilon}, \xi, \omega\right) \to 0$$

strongly in L^1_{loc} norm. In particular, there are no microscopic oscillations in time in this part of f^ε. We wish to emphasize that this result appears to us to be new.

We now focus on the other part, namely $f^\varepsilon_\perp$. An easy consequence of the theorem is

$$\int_0^T f^\varepsilon_\perp(t, x, \xi, \omega) \, dt \to 0$$

in $L^\infty_{\text{loc}}(\mathbb{R}^N_x; L^1_{\text{loc}}(\mathbb{R}^N_\xi; L^1(\Omega)))$ and for all $T > 0$. However, it would be wrong to think that $f^\varepsilon_\perp$ vanishes in $L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))$, for instance. Indeed

$$f^\varepsilon_\perp(t, x, \xi, \omega) \approx g\left(t, x; \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)$$

in L^1_{loc}, and

$$||g(t=0,x;\tau,y)||_{L^1(\mathbb{R}^N_\xi\times\Omega)} = ||f_{0\perp}(x,y)||_{L^1(\mathbb{R}^N_\xi\times\Omega)}$$

as soon as $f_{0\perp}(x,y)\in L^1(\mathbb{R}^N_\xi\times\Omega)$ for almost every x,y. And if $f_{0\perp}$ is stationary, then

$$||f_{0\perp}(x,y)||_{L^1(\mathbb{R}^N_\xi\times\Omega)} = ||f_{0\perp}(x,0)||_{L^1(\mathbb{R}^N_\xi\times\Omega)} \quad \text{for a.e. } x,y.$$

Thus, if $f_{0\perp}\neq 0$, then for all $T>0$, we derive

$$\begin{aligned}
&\int_0^T \left|\left| g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right|\right|_{L^1(\mathbb{R}^N_x\times\mathbb{R}^N_\xi\times\Omega)} dt \\
=& \int_0^T \left|\left| g\left(t,x;\frac{t}{\varepsilon},0,\xi,\omega\right)\right|\right|_{L^1(\mathbb{R}^N_x\times\mathbb{R}^N_\xi\times\Omega)} dt \\
=& \int_0^T \left|\left| g\left(t=0,x;\frac{t}{\varepsilon},0,\xi,\omega\right)\right|\right|_{L^1(\mathbb{R}^N_x\times\mathbb{R}^N_\xi\times\Omega)} dt \\
=& T||f_{0\perp}(x,0)||_{L^1(\mathbb{R}^N_x\times\mathbb{R}^N_\xi\times\Omega)}.
\end{aligned}$$

The same kind of inequality holds if the L^1 norms are replaced with L^1_{loc} norms, but the derivation of the inequality with L^1_{loc} norms involves bounds on the function $\xi^\sharp$ which will be given later. Thus we refer to the proof of lemma 6.3.3 for arguments which yield similar inequalities with L^1_{loc} norms.

Hence $f^\varepsilon_\perp$ does not vanish strongly in general. In other words, there are fast oscillations in time, due to the ill-preparedness of the initial data (i.e. the fact that $f_0(x,\cdot)\notin\mathbb{K}$), but these oscillations do not cancel out as the small parameter ε vanishes.

Remark 6.1.3. *Some of the equations of theorem 10 can be guessed thanks to a two-scale Ansatz. We have already explained how equation* (6.2) *is obtained. The derivation of the evolution equation* (6.3) *is the same as the one of* (6.2)*, except that the Ansatz now involves microscopic oscillations in time. In other words, if*

$$f^\varepsilon(t,x,\xi,\omega) = f^0\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) + \varepsilon f^1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) + \cdots,$$

then $f^0(t,x,\cdot)$ satisfies (6.3).

The derivation of (6.4) *is less obvious, and in fact we have only been able to compute it for the function f, that is, when there are no microscopic oscillations in the time variable. Hence, assume that*

$$f^\varepsilon(t,x,\xi,\omega) = f^0\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) + \varepsilon f^1\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) + \cdots.$$

We insert this expansion in equation (6.1)*, and compute the ε^0 order term. We obtain*

$$\partial_t f^0 + \xi\cdot\nabla_x f^0 + \xi\cdot\nabla_y f^1 - \nabla_y u\cdot\nabla_\xi f^1 = 0$$

Thanks to remark 6.1.2, the term $\varphi^1(y,\xi,\omega) := \xi\cdot\nabla_y f^1 - \nabla_y u\cdot\nabla_\xi f^1$ *belongs to* $\mathbb{K}^\perp$. *Indeed, let* $g\in\mathbb{K}$ *be stationary and smooth, with compact support in* ξ. *According to Birkhoff's ergodic theorem, we have, for all* $y\in\mathbb{R}^N$, *and almost surely in* ω

$$\int_{\mathbb{R}^N\times\Omega}\varphi^1(y,\xi,\omega)g(y,\xi,\omega)\,d\xi\,dP(\omega) = \lim_{R\to\infty}\frac{1}{|B_R|}\int_{B_R\times\mathbb{R}^N_\xi}\varphi^1(z,\xi,\omega)g(z,\xi,\omega)\,dz\,d\xi.$$

And if f^1 *and* g *are smooth,*

$$\begin{aligned}&\int_{B_R\times\mathbb{R}^N_\xi}\varphi^1(z,\xi,\omega)g(z,\xi,\omega)\,dz\,d\xi\\ =\;& -\int_{B_R\times\mathbb{R}^N_\xi}\left(\xi\cdot\nabla_z g - \nabla_z u\cdot\nabla_\xi g\right)(z,\xi,\omega)f^1(z,\xi,\omega)\,dz\,d\xi\\ &+\int_{\partial B_R}\int_{\mathbb{R}^N}\xi\cdot n_{B_R}(y)g(z,\xi,\omega)f^1(z,\xi,\omega)\,dz\,d\xi.\end{aligned}$$

Since $g\in\mathbb{K}$ *and* g *has compact support in* ξ, *we deduce that* $\varphi^1\in\mathbb{K}^\perp$ *thanks to remark 6.1.2. Recall also that* $f^0(t,x,\cdot)$ *belongs to* $\mathbb{K}$ *almost everywhere. Thus projecting the above equation on* $\mathbb{K}$ *yields* (6.4).

Let us now explain how our strategy of proof differs from the one of E. Frénod and K. Hamdache in [33]. The authors of [33] used the concept of two-scale convergence, a notion introduced by Gabriel N'Guetseng in [56], and then formalized and developed by Grégoire Allaire in [3]. We will first explain briefly what are the main arguments of [33], and then we shall expose the great lines of the proof of the present paper.

The notion of two-scale convergence relies on a choice of oscillating test functions; the central result of the theory is the following (see [3]) :

Proposition 6.1.1. *Let* U *be an open set in* $\mathbb{R}^N$, *and let* $(g^\varepsilon)_{\varepsilon>0}$ *be a bounded sequence in* $L^2(U)$. *Then there exists a function* $g^0\in L^2(U\times[0,1]^N)$, *and a subsequence* (ε_n) *such that* $\varepsilon_n\to 0$ *as* $n\to\infty$, *such that*

$$\int_U g^{\varepsilon_n}(x)\psi\left(x,\frac{x}{\varepsilon_n}\right)\,dx \to \int_{U\times[0,1]^N} g^0(x,y)\psi(x,y)\,dx\,dy$$

for all functions $\psi\in L^2(U;\mathcal{C}_{per}([0,1]^N))$.

It is then said that the sequence g^{ε_n} *two-scale converges towards* g_0.

This concept can be generalized with no difficulty to functions depending on t and ξ as well. In [33], the authors pass to the two-scale limit in equation (6.1), after deriving *a priori* bounds on the sequence f^ε. Using test functions of the type

$$\varepsilon\varphi\left(t,x,\frac{x}{\varepsilon},\xi\right),$$

with $\varphi\in\mathcal{D}_{\rm per}([0,\infty)\times\mathbb{R}^N_x\times[0,1]^N\times\mathbb{R}^N_\xi)$, they first prove that the two-scale limit of the sequence f^ε, say $f(t,x,y,\xi)$, satisfies the constraint equation (6.2). Then, taking test functions of the type

$$\varphi\left(t,x,\frac{x}{\varepsilon},\xi\right)$$

such that $\varphi(t,x,\cdot) \in \mathbb{K}$ almost everywhere, they derive the macroscopic evolution equation (6.4). The proof is straight-forward and simpler than the one we present here, but does not include any description of the microscopic oscillations in time. Moreover, since the method of [33] relies on two-scale convergence, the result only provides information on the weak-limit (or two-scale limit) of the sequence f^ε; in other words, the strong convergence result we state here is new, and cannot be derived by two-scale convergence techniques.

However, let us mention that the notion of two-scale convergence has been generalized to stationary settings by Alain Bourgeat, Andro Mikelić and Steve Wright (see [10]); the relevant concept is then called *stochastic two-scale convergence in the mean*. The result of [10] is the following :

Proposition 6.1.2. *Assume that $L^2(\Omega, P)$ is separable.*

Let U be an open set in $\mathbb{R}^N$, and let $(g^\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $L^2(U\times\Omega)$. Then there exists a function $g^0 \in L^2(U\times\Omega)$, and a subsequence (ε_n) such that $\varepsilon_n \to 0$ as $n\to\infty$, such that

$$\int_{U\times\Omega} g^{\varepsilon_n}(x,\omega)\psi\left(x,\tau_{\frac{x}{\varepsilon_n}}\omega\right)\,dx\,dP(\omega) \to \int_{U\times\Omega} g^0(x,\omega)\psi(x,\omega)\,dx\,dP(\omega),$$

for any $\psi\in L^2(U\times\Omega)$ such that the function

$$(x,\omega)\mapsto \psi(x,\tau_x\omega)$$

belongs to $L^2(U\times\Omega)$.

It is likely that the arguments of [33] can be generalized to the present case in order to obtain the same kind of weak convergence results, as long as $L^2(\Omega)$ is separable. We prefer to focus on a different method, which is more explicit and which allows for the derivation of strong convergence results.

The key of our analysis lies in the study of the behavior as $\varepsilon\to 0$ of the Hamiltonian system

$$\begin{cases} \dot{Y}^\varepsilon(t,x,\xi,\omega) = -\Xi^\varepsilon(t,x,\xi,\omega), \quad t>0\\ \dot{\Xi}^\varepsilon(t,y,\xi,\omega) = \frac{1}{\varepsilon}\nabla_y u(Y^\varepsilon(t,x,\xi,\omega),\omega), \quad t>0\\ Y^\varepsilon(t=0,x,\xi,\omega)=x,\ \Xi^\varepsilon(t=0,x,\xi,\omega)=\xi, \quad (x,\xi,\omega)\in\mathbb{R}^{2N}\times\Omega. \end{cases}$$

Indeed, if f_0 is smooth, then

$$f^\varepsilon(t,x,\xi,\omega) = f_0\left(Y^\varepsilon(t,x,\xi,\omega), \frac{Y^\varepsilon(t,x,\xi,\omega)}{\varepsilon}, \Xi^\varepsilon(t,y,\xi,\omega),\omega\right),$$

so that we can deduce the asymptotic behavior of f^ε from the one of $(Y^\varepsilon,\Xi^\varepsilon)$. And it is easily checked that

$$Y^\varepsilon(t,x,\xi,\omega) = \varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),$$
$$\Xi^\varepsilon(t,y,\xi,\omega) = \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),$$

where (Y, Ξ) is the solution of the system

$$\begin{cases} \dot{Y}(t, y, \xi, \omega) = -\Xi(t, y, \xi, \omega), \quad t > 0, \\ \dot{\Xi}(t, y, \xi, \omega) = \nabla_y u(Y(t, y, \xi, \omega), \omega), \quad t > 0, \\ Y(t = 0, y, \xi, \omega) = y, \ \Xi(t = 0, y, \xi, \omega) = \xi, \quad (y, \xi, \omega) \in \mathbb{R}^{2N} \times \Omega. \end{cases} \tag{6.5}$$

Hence, in order to study the limit of f^ε as $\varepsilon \to 0$, we have to investigate the long time behavior of the system (Y, Ξ), and this will be achieved with the help of the ergodic theorem in the second section.

This dynamical system also allows for a better understanding of the function $\xi^\sharp$ appearing in (6.4). Indeed, we shall prove in the second section that

$$\xi^\sharp(y, \xi, \omega) = -\lim_{T\to\infty} \frac{Y(T, y, \xi, \omega)}{T} \quad \text{almost everywhere,}$$

so that $-\xi^\sharp$ is the rotation vector associated with the dynamics (Y, Ξ).

In the case where $N = 1$, we can give explicit formulas for $\xi^\sharp(y, \xi, \omega)$; the proof of this formula in the stationary ergodic case is strongly linked to methods from the Aubry-Mather theory (see [26], [30], [51]), and thus also to the homogenization of Hamilton-Jacobi equations. In the rest of the paper, we set

$$H(y, \xi, \omega) = \frac{|\xi|^2}{2} + u(y, \omega).$$

Let us first recall the definition of the homogenized Hamiltonian $\bar{H}$ (see [49])

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\bar{H}_0(p),$$

$$\bar{H}_0(p) = \begin{cases} 0 & \text{if } |p| < E\left[\sqrt{2(u_{\max} - u)}\right] \\ \\ \lambda & \text{if } |p| \geq E\left[\sqrt{2(u_{\max} - u)}\right], \text{ where } |p| = E\left[\sqrt{2(u_{\max} - u) + \lambda}\right]. \end{cases}$$

Proposition 6.1.3. *Assume that* $N = 1$.

Let $(y, \xi, \omega) \in \mathbb{R} \times \mathbb{R} \times \Omega$ *such that* $H(y, \xi, \omega) > u_{max}$. *Let* $P = P(y, \xi, \omega) \in \mathbb{R}$ *such that* $\bar{H}(P) = H(y, \xi, \omega)$ *and* $\operatorname{sgn}(P) = \operatorname{sgn}(\xi)$. *Then*

$$\xi^\sharp(y, \xi, \omega) = \bar{H}'(P)$$

Moreover, if L *is the dual function of* H, *i.e.*

$$L(y, p, \omega) = \sup_{\xi\in\mathbb{R}} (p\xi - H(y, \xi\, \omega)) = \frac{1}{2}|p|^2 - u(y, \omega),$$

and $\bar{L}$ *is the homogenized Lagrangian (the dual function of* $\bar{H}$*), then*

$$\mathbb{P}(L)(y, \xi, \omega) = \bar{L}(\xi^\sharp(y, \xi, \omega).)$$

In the periodic case, we will give another proof of the above result; the strategy chosen in that case is inspired from techniques and calculations in classical mechanics. It also allows to give a formula for $\xi^\sharp$ for low energies in the periodic setting only:

Proposition 6.1.4. *Assume that $N = 1$ and that the environment is periodic. Let $(y, \xi) \in \mathbb{R}^2$ such that $H(y, \xi) < u_{max}$. Then $\xi^\sharp(y, \xi) = 0$.*

The organisation of this note is the following : in the second section, we derive some preliminary results on the long-time behavior of the system (Y, Ξ) thanks to the ergodic theorem. Those will be useful in the proof of theorem 10, to which is devoted the third section. Eventually, the fourth and last section is concerned with results in the integrable case, both in the periodic and the stationary ergodic settings.

6.2 Preliminaries

This section is largely devoted to the study of the long-time behavior of the Hamiltonian system (Y, Ξ) defined by (6.5). First, notice that the Hamiltonian $H(y, \xi, \omega) := \frac{1}{2}|\xi|^2 + u(y, \omega)$ is constant along the curves of the system (Y, Ξ), and if f is stationary and belongs to $L^\infty(\Omega, \mathcal{C}^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi))$, then

$$f \in \mathbb{K} \iff f\left(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega\right) = f(y, \xi, \omega) \quad \forall (y, \xi, \omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega.$$

Indeed, for all $f \in L^\infty(\Omega, \mathcal{C}^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi))$, we have

$$\frac{\partial}{\partial t} f\left(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega\right) = \{H, f\}\left(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega\right),$$

where $\{H, f\}$ denotes the Poisson bracket of f and H, i.e.

$$\{H, f\}(y, \xi, \omega) = \xi \cdot \nabla_y f(y, \xi, \omega) - \nabla_y u(y, \xi, \omega) \cdot \nabla_\xi f(y, \xi, \omega).$$

Let us mention an easily checked property of the trajectories (Y, Ξ) which will be used extensively in the rest of the article : for all $(y, z, \xi) \in \mathbb{R}^{3N}$, for all $\omega \in \Omega$, $t \geq 0$,

$$\begin{aligned} Y(t, y, \xi, \tau_z\omega) + z &= Y(t, y + z, \xi, \omega), \\ \Xi(t, y, \xi, \tau_z\omega) &= \Xi(t, y + z, \xi, \omega). \end{aligned} \tag{6.6}$$

In the periodic case, this invariance entails that the hamiltonian system (Y, Ξ) can be considered as a dynamical system on the N dimensional torus $[0, 2\pi)^N$. In this periodic setting, it is somewhat natural to introduce the semi-group of transformations $(\mathcal{T}_t)_{t \geq 0}$ on $[0, 2\pi)^N \times \mathbb{R}^N$ given by

$$\mathcal{T}_t(y, \xi) = (Y(t, y, \xi), \Xi(t, y, \xi)), \quad y \in [0, 2\pi)^N, \ \xi \in \mathbb{R}^N.$$

According to Liouville's theorem, this semi-group preserves the Lebesgue measure on $[0, 2\pi)^N \times \mathbb{R}^N$; moreover, we can construct a family of finite invariant measures on $[0, 2\pi)^N \times \mathbb{R}^N$ by setting $m_c(y, \xi) = \mathbf{1}_{H(y,\xi)\leq c}\, dy\, d\xi$ for $c > 0$ (remember that the Hamiltonian is constant along the hamiltonian curves). This construction is the root of the ergodic theorem (see corollary 6.2.1), and thus of the study of the long-time behavior of the system (Y, Ξ).

In the stationary ergodic setting, this construction can be generalized as follows : we define the transformation $T_t : \mathbb{R}^N_\xi \times \Omega \to \mathbb{R}^N_\xi \times \Omega$ by

$$T_t(\xi, \omega) = \big(\Xi(t, 0, \xi, \omega), \tau_{Y(t,0,\xi,\omega)}\omega\big)$$

together with the family of measures

$$\mu_c := \mathbf{1}_{\mathcal{H}(\xi,\omega)\leq c}\, d\xi\, dP(\omega)$$

where $\mathcal{H}(\xi, \omega) := \frac{1}{2}|\xi|^2 + u(0, \omega)$. It is obvious that for all $c \in (0, \infty)$, μ_c is a finite measure on $\mathbb{R}^N_\xi \times \Omega$.

Notice that the "good" generalization to the stationary ergodic setting of the semi-group $(\mathcal{T}_t)$ is a semi-group which acts on $\mathbb{R}^N_\xi \times \Omega$ rather than $\mathbb{R}^N_y \times \mathbb{R}^N_\xi$. Thanks to the group of transformations $(\tau_x)_{x\in\mathbb{R}^N}$, the transformations in Ω can result in transformations in $\mathbb{R}^N_y$, but the definition chosen here allows us to define a family of finite invariant measures, whereas such a construction is rather difficult if one tries to define a semi-group acting on $\mathbb{R}^N_y \times \mathbb{R}^N_\xi$. This will be fundamental in the rest of the proof.

Lemma 6.2.1. *$(T_t)_{t\geq 0}$ is a semi-group on $\mathbb{R}^N_\xi \times \Omega$ and preserves the family of measures μ_c.*

Proof. Let us first prove the semi-group property : let $t, s \in [0, \infty)$, and $(\xi, \omega) \in \mathbb{R}^N \times \Omega$; then

$$\begin{aligned} T_t \circ T_s(\xi, \omega) &= T_t\big(\Xi(s, 0, \xi, \omega), \tau_{Y(s,0,\xi,\omega)}\omega\big) \\ &= \big(\Xi(t, 0, \Xi(s, 0, \xi, \omega), \tau_{Y(s,0,\xi,\omega)}\omega), \omega'\big) \end{aligned}$$

and using the properties (6.6) we deduce

$$\begin{aligned} \Xi(t, 0, \Xi(s, 0, \xi, \omega), \tau_{Y(s,0,\xi,\omega)}\omega) &= \Xi(t, Y(s, 0, \xi, \omega), \Xi(s, 0, \xi, \omega), \omega), \\ &= \Xi(t + s, 0, \xi, \omega) \end{aligned}$$

and

$$\begin{aligned} \omega' &= \tau_{Y(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega)}\tau_{Y(s,0,\xi,\omega)}\omega \\ &= \tau_{Y(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega)+Y(s,0,\xi,\omega)}\omega \\ &= \tau_{Y(t,Y(s,0,\xi,\omega),\Xi(s,0,\xi,\omega),\omega)}\omega \\ &= \tau_{Y(t+s,0,\xi,\omega)}\omega \end{aligned}$$

Thus

$$T_t \circ T_s(\xi, \omega) = \big(\Xi(t + s, 0, \xi, \omega), \tau_{Y(t+s,0,\xi,\omega)}\omega\big) = T_{t+s}(\xi, \omega).$$

Since it is obvious that $T_0 = \mathrm{Id}$, $(T_t)_{t\geq 0}$ is a semi-group on $\mathbb{R}^N \times \Omega$.

We now have to check the invariance property; let $F \in L^1(\mathbb{R}^N \times \Omega; \mu_c)$ arbitrary. We set $f(y, \xi, \omega) := F(\xi, \tau_y\omega)$ for $(y, \xi, \omega) \in \mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega$, and we compute

$$\begin{aligned} & \int_{\mathbb{R}^N \times \Omega} F(T_t(\xi, \omega))\, d\mu_c(\xi, \omega) \\ = \;& E\left[\int_{R^N} f(Y(t, 0, \xi, \omega), \Xi(t, 0, \xi, \omega), \omega)\mathbf{1}_{H(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\le c}\, d\xi\right]. \end{aligned}$$

Since the probability measure P is invariant by the group of transformation τ_y, and

$$f(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega) = f(Y(t, 0, \xi, \tau_y\omega), \Xi(t, 0, \xi, \tau_y\omega), \tau_y\omega),$$

we have, for all $y \in \mathbb{R}^N$

$$\begin{aligned} & E\left[f(Y(t, 0, \xi, \omega), \Xi(t, 0, \xi, \omega), \omega)\mathbf{1}_{H(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\le c}\right] = \\ & \qquad = E\left[f(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega)\mathbf{1}_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\le c}\right]. \end{aligned}$$

Take an arbitrary function $\phi \in L^1(\mathbb{R}^N_y)$, and write

$$\begin{aligned} & \int_{\mathbb{R}^N \times \Omega} F(T_t(\xi, \omega))\, d\mu_c(\xi, \omega) \\ = \;& E\left[\int_{\mathbb{R}^{2N}} dy\, d\xi \phi(y) f(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega)\mathbf{1}_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\le c}\right] \end{aligned}$$

We change variables in the integral by setting $(x, v) = (Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega))$; according to Liouville's theorem, the jacobian of this change of variables is equal to 1, and

$$(x, v) = (Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega)) \iff (y, \xi) = (X(t, x, v, \omega), V(t, x, v, \omega)),$$

where (X, V) is a solution of the Hamiltonian system

$$\begin{cases} \dot{X} = V, \\ \dot{V} = -\nabla u(X, \omega), \\ (X, V)(t = 0, x, v) = (x, v). \end{cases}$$

Observe that in the present case, we have simply

$$X(t, x, v, \omega) = Y(t, x, -v, \omega),$$

so that (X, V) satisfies relations (6.6).

Hence

$$\begin{aligned} & \int_{\mathbb{R}^{2N}} dy\, d\xi\; \phi(y) f(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega)\mathbf{1}_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\le c} \\ = \;& \int_{\mathbb{R}^{2N}} dx\, dv\; \phi(X(t, x, v, \omega)) f(x, v, \omega)\mathbf{1}_{H(x,v,\omega)\le c} \\ = \;& \int_{\mathbb{R}^{2N}} dx\, dv\; \phi(X(t, 0, v, \tau_x\omega) + x) F(v, \tau_x\omega)\mathbf{1}_{\mathcal{H}(v,\tau_x\omega)\le c} \end{aligned}$$

so that

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega)\\
= \ & E\left[\int_{\mathbb{R}^{2N}} dx\, dv\ \phi(X(t,0,v,\tau_x\omega)+x)F(v,\tau_x\omega)\mathbf{1}_{\mathcal{H}(v,\tau_x\omega)\le c}\right]\\
= \ & E\left[\int_{\mathbb{R}^{2N}} dx\, dv\ \phi(X(t,0,v,\omega)+x)F(v,\omega)\mathbf{1}_{\mathcal{H}(v,\omega)\le c}\right]\\
= \ & E\left[\int_{\mathbb{R}^{N}} dv\ \left(\int_{\mathbb{R}^N}\phi(X(t,0,v,\omega)+x)\, dx\right) F(v,\omega)\mathbf{1}_{\mathcal{H}(v,\omega)\le c}\right]\\
= \ & E\left[\int_{\mathbb{R}^{N}} dv F(v,\omega)\mathbf{1}_{\mathcal{H}(v,\omega)\le c}\right] = \int_{\mathbb{R}^N\times\Omega} F\, d\mu_c
\end{aligned}$$

since the integral of ϕ is equal to 1.

Hence, we have proved that for all $F \in L^1(\mathbb{R}\times\Omega;\mu_c)$, for all $t\ge 0$,

$$\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega) = \int_{\mathbb{R}^N\times\Omega} F(\xi,\omega)\, d\mu_c(\xi,\omega),$$

which means exactly that μ_c is invariant by the semi-group $(T_t)_{t\ge 0}$.

□

The following corollary is an immediate consequence of Birkhoff's ergodic theorem:

Corollary 6.2.1. *Let $F\in L^1(\mathbb{R}^N\times\Omega;\mu_c)$. There exists a function $\bar F\in L^1(\mathbb{R}^N\times\Omega;\mu_c)$ such that as $T\to\infty$,*

$$\frac{1}{T}\int_0^T F(T_t(\xi,\omega))\, dt \to \bar F(\xi,\omega)$$

a.e. on $\mathbb{R}^N\times\Omega$ and in $L^1(\mu_c)$. Moreover, $\bar F$ is invariant by T_t for all $t>0$, and

$$\int_{\mathbb{R}^N\times\Omega} F\, d\mu_c = \int_{\mathbb{R}^N\times\Omega} \bar F\, d\mu_c. \tag{6.7}$$

Additionally, if $\bar f = \bar f(y,\xi,\omega)$ is the stationary function associated to $\bar F$, that is, $\bar f(y,\xi,\omega) = \bar F(\xi,\tau_y\omega)$, then $\bar f$ is invariant by the hamiltonian flow (Y,Ξ); precisely, for a.e. $(y,\xi,\omega)\in\mathbb{R}^{2N}\times\Omega$, $t>0$

$$\bar f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) = \bar f(y,\xi,\omega).$$

Proof. We only have to prove the invariance of $\bar f$ by the Hamiltonian flow; first, for $y=0$, we have

$$\begin{aligned}
\bar f(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega) &= \bar F(\Xi(t,0,\xi,\omega),\tau_{Y(t,0,\xi,\omega)}\omega) = \bar F(T_t(\xi,\omega))\\
&= \bar F(\xi,\omega) = \bar f(0,\xi,\omega)
\end{aligned}$$

and the property is proved when $y = 0$.

For $y \in \mathbb{R}^N$ arbitrary,

$$\begin{aligned}\bar{f}(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) &= \bar{f}(Y(t,0,\xi,\tau_y\omega)+y,\Xi(t,0,\xi,\tau_y\omega),\omega)\\ &= \bar{f}(Y(t,0,\xi,\tau_y\omega),\Xi(t,0,\xi,\tau_y\omega),\tau_y\omega)\\ &= \bar{f}(0,\xi,\tau_y\omega) = \bar{f}(y,\xi,\omega)\end{aligned}$$

according to the result in the case $y = 0$.

□

Remark 6.2.1. *We mention here an important but easy consequence of the relations* (6.6) *and the invariance of the measure P with respect to the transformation group τ_y, $y \in \mathbb{R}^N$: for any stationary function $f = f(y,\xi,\omega) = F(\xi,\tau_y\omega)$, $F \in L^\infty(\mathbb{R}^N \times \Omega)$, we have*

$$E[f(Y(t,y,\xi,\cdot),\Xi(t,y,\xi,\cdot),\cdot)] = E[F(T_t(\xi,\cdot))]$$

for all $t > 0$, $y,\xi \in \mathbb{R}^N$; in particular, the left-hand side of the above equality does not depend on y.

This property was used in the proof of lemma 6.2.1

Remark 6.2.2. *Let us precise a little what happens when the function F belongs to $L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$. In that case, $F \in L^1(\mathbb{R}^N_\xi \times \Omega; \mu_c)$ for all $c > 0$. Consequently, for any $c > 0$, we can define the function $\bar{F}_c$ associated to F by corollary 6.2.1.*

It is then easily proved that for any $0 < c < c'$, $\bar{F}_c = \bar{F}_{c'}$, μ_c-almost everywhere. Setting

$$\mathrm{Supp}\mu_n := \{(\xi,\omega), \mathcal{H}(\xi,\omega) \le n\},$$
$$A_n = \{(\xi,\omega) \in \mathrm{Supp}\mu_n;\ \bar{F}_n(y,\xi) \ne \bar{F}_{n+1}(y,\xi)\}$$
$$A := \bigcup_{n=0}^{\infty} A_n,$$

we see that $\mu_c(A) = 0$ for all $c > 0$. Moreover, for all $(\xi,\omega) \in \mathbb{R}^N \times \Omega \setminus A$, for all integers k,l such that $(\xi,\omega) \in \mathrm{Supp}\mu_k \cap \mathrm{Supp}\mu_l$, we have $\bar{F}_k(\xi,\omega) = \bar{F}_l(\xi,\omega)$. We can thus define a function $\bar{F}(\xi,\omega)$ on $\mathbb{R}^N \times \Omega \setminus A$ by

$$\bar{F}(\xi,\omega) = \bar{F}_n(\xi,\omega) \quad \textit{for any } n \in \mathbb{N} \textit{ such that } (\xi,\omega) \in \mathrm{Supp}\mu_n$$

We then now that

$$\frac{1}{T}\int_0^T F(T_t(\xi,\omega))\,dt \to \bar{F}(\xi,\omega) \tag{6.8}$$

as $T \to \infty$, and the convergence holds in $L^1(\mu_c)$ for all $c > 0$, and μ_n almost everywhere for $n \in \mathbb{N}$. Eventually, setting

$$B := \{(\xi,\omega) \in \mathbb{R}^N \times \Omega \setminus A;\ \frac{1}{T}\int_0^T F(T_t(\xi,\omega))\,dt \textit{ does not converge towards } \bar{F}(\xi,\omega)\}$$

it is easily proved that $\mu_c(B) = 0$ for all $c > 0$ (the equality is true for $c \in \mathbb{N}$, and is then deduced for $c > 0$ arbitrary because the family of measures (μ_c) is increasing in c).

Eventually, we have found a function $\bar{F} \in L^1_{loc}(\mathbb{R}^N, L^1(\Omega))$, independent of c, such that (6.8) *holds in $L^1(\mu_c)$ and μ_c-almost everywhere for all $c > 0$.*

Remark 6.2.3. *The construction above allows us to make more precise what we mean by projection $\mathbb{P}$: let $f = f(y,\xi,\omega)$ be a stationary function such that $f \in L^\infty(\mathbb{R}^N_y, L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega)))$, and set $F(\xi,\omega) = f(0,\xi,\omega) \in L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$. We can then associate to F a function $\bar{F} \in L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$ such that* (6.8) *holds in $L^1(\mu_c)$ for all c (see remark 6.2.2). We set*

$$\mathbb{P}(f)(y,\xi,\omega) := \bar{F}(\xi,\tau_y\omega).$$

It follows from corollary 6.2.1 that $\mathbb{P}(f)$ is invariant by the hamiltonian flow (6.5)*, and thus satisfies the constraint equation. From now on, we take this definition for the projection $\mathbb{P}$, instead of the one given in the introduction. Notice that, for all $y \in \mathbb{R}^N$ and μ_c-almost everywhere,*

$$\begin{aligned}\mathbb{P}(f)(y,\xi,\omega) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T F(T_t(\xi,\tau_y\omega))\,dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(Y(t,0,\xi,\tau_y\omega),\Xi(t,0,\xi,\tau_y\omega),\tau_y\omega\right)\,dt\\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T f\left(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega\right)\,dt\end{aligned}$$

And we also give a more precise definition of $\xi^\sharp(y,\xi,\omega)$: let

$$\hat{\xi}:\begin{array}{ccc}\mathbb{R}^N\times\Omega & \to & \mathbb{R}^N\\ (\xi,\omega) & \mapsto & \xi\end{array}.$$

Then $\hat{\xi} \in L^1(\mathbb{R}^N\times\Omega,\mu_c)$ for all $c > 0$, and

$$\xi^\sharp(y,\xi,\omega) = P(\hat{\xi})(y,\xi,\omega) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Xi(t,y,\xi,\omega)\,dt$$

almost everywhere and in $L^1(\mu_c)$ for all $0 < c < \infty$.

Eventually, we mention here a property that will be used in the proof of the theorem; with the same notations as above, let

$$\phi(\tau,y,\xi,\omega) = F\left(T_\tau(\xi,\tau_y\omega)\right),$$

with $F \in L^1_{\text{loc}}(\mathbb{R}^N_\xi, L^1(\Omega))$. Then ϕ is a solution of the evolution equation

$$\partial_\tau\phi + \xi\cdot\nabla_y\phi - \nabla_y u\cdot\nabla_\xi\phi = 0,$$

with initial data $\phi(\tau = 0,y,\xi,\omega) = f(y,\xi,\omega) = F(\xi,\tau_y\omega)$. This is a classical fact if ϕ is $\mathcal{C}^1$ in the variables y,ξ; thanks to a contraction property for the transport equation (6.3) and a density result which will be stated later (see lemma 6.3.2), it is true when F merely belongs to L^1. In order to avoid technical details at this stage, we omit the proof of this result for weak (that is, L^1) solutions.

6.3 The general N-dimensional case

This section is devoted to the proof of theorem 10. The proof is divided in three steps : first, we study the case of an initial data which does not depend on x, then the case when the initial data only depends on x (and not on y, ξ, ω), and eventually, we treat the general case.

6.3.1 First case : f_0 does not depend on x

Here, we assume that $f_0 = f_0(y, \xi, \omega) \in L^1_{\text{loc}}(\mathbb{R}^N_\xi; L^\infty(\mathbb{R}^N_y \times \Omega)) \cap \mathcal{C}^1(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^\infty(\Omega))$. The smoothness assumption will be removed in the third section. Recall that f_0 is stationary, that is, $f_0(y+z, \xi, \omega) = f_0(y, \xi, \tau_z\omega)$ a.s. in ω, for all $(y, z, \xi) \in \mathbb{R}^{3N}$. In the rest of the subsection, we set

$$F_0(\xi, \omega) := f_0(0, \xi, \omega)$$

and

$$\bar{F}_0(\xi, \omega) := \lim_{T\to\infty} \frac{1}{T} \int_0^T F_0(T_t(\xi, \omega))\, dt, \quad \bar{f}_0(y, \xi, \omega) = \bar{F}_0(\xi, \tau_y\omega).$$

Notice that $F_0 \in L^1_{\text{loc}}(\mathbb{R}^N_\xi; L^\infty(\Omega))$, and thus $F_0 \in L^1(\mathbb{R}^N \times \Omega; \mu_c)$ for all $c > 0$.

In that case,

$$\begin{aligned} f^\varepsilon(t, x, \xi, \omega) &= f_0\left(Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right), \Xi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right), \omega\right) \\ &= f_0\left(Y\left(\frac{t}{\varepsilon}, 0, \xi, \tau_{\frac{x}{\varepsilon}}\omega\right), \Xi\left(\frac{t}{\varepsilon}, 0, \xi, \tau_{\frac{x}{\varepsilon}}\omega\right), \tau_{\frac{x}{\varepsilon}}\omega\right) \\ &= F_0\left(T_{\frac{t}{\varepsilon}}\left(\xi, \tau_{\frac{x}{\varepsilon}}\omega\right)\right) \\ &= \bar{f}_0\left(\frac{x}{\varepsilon}, \xi, \omega\right) + \left\{F_0\left(T_{\frac{t}{\varepsilon}}\left(\xi, \tau_{\frac{x}{\varepsilon}}\omega\right)\right) - \bar{F}_0\left(\xi, \tau_{\frac{x}{\varepsilon}}\omega\right)\right\} \end{aligned}$$

In accordance with theorem 10, we set

$$g(\tau, y, \xi, \omega) = \left(F_0 - \bar{F}_0\right)\left(T_\tau\left(\xi, \tau_y\omega\right)\right),$$

and $r^\varepsilon = 0$. Then g satisfies the microscopic evolution equation (6.3) thanks to the remark at the end of the preceding section. Moreover, $g(\tau) \in \mathbb{K}^\perp$ by definition of $\mathbb{K}^\perp$ and because $\mathbb{P}\left(F_0\left(T_\tau\left(\xi, \tau_y\omega\right)\right)\right) = \bar{F}_0(\xi, \tau_y\omega)$. Notice also that $\bar{f}_0 = \mathbb{P}(f_0)$ thanks to remark 6.2.3.

There only remains to check that

$$\int_0^T g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)\, dt \to 0 \quad \text{as } \varepsilon \to 0 \tag{6.9}$$

in $L^1_{\text{loc}}(\mathbb{R}^N_x, L^1(\mathbb{R}^N \times \Omega, \mu_c))$ for all $T > 0$ and $c > 0$.

The invariance of the measure P with respect to the group of transformations $(\tau_x)_{x\in\mathbb{R}^N}$ (see remark 6.2.1) entails that

$$\int_{\Omega\times\mathbb{R}^N_\xi}\left|\frac{1}{\frac{T}{\varepsilon}}\int_0^{\frac{T}{\varepsilon}} f_0\left(Y\left(t,\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(t,\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\,dt-\bar{f}_0\left(\frac{x}{\varepsilon},\xi,\omega\right)\right|\,d\mu_c(\xi,\omega)$$
$$=\int_{\Omega\times\mathbb{R}^N_\xi}\left|\frac{1}{\frac{T}{\varepsilon}}\int_0^{\frac{T}{\varepsilon}} F_0\left(T_t(\xi,\omega)\right)\,dt-\bar{F}_0\left(\xi,\omega\right)\right|\,d\mu_c(\xi,\omega)$$

and the term above goes to 0 as $\varepsilon\to 0$ according to corollary 6.2.1 and is independent of $x\in\mathbb{R}^N$. Thus theorem 10 is proved in the case when f_0 does not depend on the macroscopic variable x.

The following remark will prove to be useful when treating the general case :

Remark 6.3.1. *If $f_0\in L^\infty$ (and f_0 is $\mathcal{C}^1$ in the variables y,ξ), then for any function $a\in L^\infty((0,\infty)\times\mathbb{R}^N_x\times\mathbb{R}^N_y\times\mathbb{R}^N_\xi\times\Omega)$, stationary in y, we have*

$$\int_0^T a\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right)g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\to 0\quad \text{as } \varepsilon\to 0$$

in $L^1_{loc}(\mathbb{R}^N_x,L^1(\mathbb{R}^N_\xi\times\Omega,\mu_c))$ for all $T>0$ and $c>0$.

Indeed, let us first prove the property for $a=a_1(t)a_2(x,y,\xi,\omega)$, with a_1, $a_2\in L^\infty$. If a_1 is an indicator function of the type

$$a_1(t)=\mathbf{1}_{T_1<t<T_2},$$

the property follows from the equality

$$\int_0^T a_1(t)g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt=\int_0^{\inf(T,T_2)}g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt-\int_0^{\inf(T,T_1)}g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt.$$

According to (6.9), the convergence result follows for $a(t,y,\xi,\omega)=a_1(t)a_2(y,\xi\omega)$. Then, let $a_1\in L^\infty([0,\infty))$ be arbitrary, and let $T>0$, $n\in\mathbb{N}^*$. Let $b_n\in L^\infty(([0,\infty))$ such that $||a_1-b_n||_{L^1(0,T)}\le 1/n$, and

$$b_n=\sum_{i=1}^{N_n}\alpha_{i,n}\mathbf{1}_{T_{i,n}<t<T_{i',n}},$$

with $N_n\in\mathbb{N}$, $\alpha_{i,n}\in\mathbb{R}$, $T_{i,n},T'_{i,n}>0$. We have

$$\begin{aligned}&\left|\int_0^T b_n(t)\,a_2\left(x,\frac{x}{\varepsilon},\xi,\omega\right)g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\right|\\ =\ &\left|a_2\left(x,\frac{x}{\varepsilon},\xi,\omega\right)\right|\,\left|\int_0^T b_n(t)g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\right|\\ \le\ &||a_2||_{L^\infty}\left|\int_0^T b_n(t)g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\right|\end{aligned}$$

and the last term in the right-hand side vanishes as $\varepsilon \to 0$ in $L^1_{loc}(\mathbb{R}^N_x, L^1(\mathbb{R}^N \times \Omega, \mu_c))$ by linearity. Moreover,

$$\begin{aligned} & \left| \int_0^T a_1(t)\, a_2\left(x, \frac{x}{\varepsilon}, \xi, \omega\right) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)\, dt \right| \\ \leq \; & \left| \int_0^T b_n(t)\, a_2\left(x, \frac{x}{\varepsilon}, \xi, \omega\right) g\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)\, dt \right| \\ & + \frac{1}{n} ||a_2||_{L^\infty} ||f_0||_{L^\infty}. \end{aligned}$$

Thus the result holds for $a = a_1(t) a_2(y, \xi, \omega)$, with a_1, $a_2 \in L^\infty$ arbitrary.

For a arbitrary, take a sequence a_δ with $\delta > 0$, converging to a in L^1_{loc}, and such that

$$a_\delta = \sum_{k=0}^{n_\delta} a_1^\delta(t) a_2^\delta(x, y, \xi, \omega).$$

with a_1^δ, a_2^δ in L^∞. The property is known for a_δ, and it is thus easily deduced for a using arguments similar to the ones developed above.

6.3.2 Second case : $f_0 = f_0(x)$

Unlike the preceding subsection, we now focus on the case when f_0 only depends on the macroscopic variable x. In order to simplify the analysis, we assume that $f_0 \in W^{1,\infty}(\mathbb{R}^N_x)$ (the case when f_0 is not smooth in x will be treated in the next subsection). In that case,

$$f^\varepsilon(t, x, \xi, \omega) = f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)\right).$$

Hence we have to investigate the behavior as $\varepsilon \to 0$ of

$$\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right).$$

We prove the following

Lemma 6.3.1. *Let $T > 0$ arbitrary. As ε vanishes,*

$$\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \to 0$$

in $L^\infty((0,T) \times \mathbb{R}^N_x; L^1(\mathbb{R}^N_\xi \times \Omega, \mu_c))$.

Proof. Let us write, for $t > 0$

$$\begin{aligned} & \varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \\ = \; & \varepsilon \int_0^{\frac{t}{\varepsilon}} \dot{Y}\left(s, \frac{x}{\varepsilon}, \xi, \omega\right)\, ds + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \\ = \; & -t\frac{\varepsilon}{t} \int_0^{\frac{t}{\varepsilon}} \Xi\left(s, \frac{x}{\varepsilon}, \xi, \omega\right)\, ds + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \\ = \; & -t\left\{\frac{\varepsilon}{t} \int_0^{\frac{t}{\varepsilon}} \hat{\xi}\left(T_s(\xi, \tau_{\frac{x}{\varepsilon}}\omega)\right)\, ds - \xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right\}. \end{aligned}$$

Let $0 < \alpha < T$ arbitrary. For $\alpha \leq t \leq T$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N_\xi \times \Omega} \left| \varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right| \, d\mu_c(\xi,\omega) \\
&= t \int_{\mathbb{R}^N_\xi \times \Omega} \left| \frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \hat{\xi}\left(T_s(\xi,\omega)\right) \, ds - \xi^\sharp\left(0,\xi,\omega\right)\right| \, d\mu_c(\xi,\omega) \\
&\leq T \sup_{\tau \geq \frac{\alpha}{\varepsilon}} \left|\left| \frac{1}{\tau}\int_0^{\tau} \hat{\xi}\left(T_s(\xi,\omega)\right) \, ds - \xi^\sharp\left(0,\xi,\omega\right)\right|\right|_{L^1(\mathbb{R}^N \times \Omega, \mu_c)}
\end{aligned}$$

and the upper-bound vanishes as $\varepsilon \to 0$ for any $\alpha > 0$ thanks to corollary 6.2.1. Notice that the upper-bound does not depend on x, hence the convergence holds in $L^\infty(\mathbb{R}^N_x; L^1(\mu_c))$.

We now have to investigate what happens when t is close to 0; notice that

$$\sup_{x \in \mathbb{R}^N} \left|\left| \xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right|\right|_{L^1(\mathbb{R}^N \times \Omega, \mu_c)} \leq C_0 \tag{6.10}$$

where the constant C_0 only depends on N and c. Indeed, if $\mathcal{H}(\xi,\omega) \leq c$, then for all $t \geq 0$,

$$\frac{1}{2}\left|\Xi(t,0,\xi,\omega)\right|^2 \leq \mathcal{H}(T_t(\xi,\omega)) = \mathcal{H}(\xi,\omega) \leq c.$$

Thus, if $\mathcal{H}(\xi,\omega) \leq c$, then

$$\xi^\sharp(0,\xi,\omega) \leq \sqrt{2cN}.$$

Thus inequality (6.10) holds with $C_0 = \sqrt{2cN}$.

Similarly, for all $t \geq 0$,

$$\sup_{x \in \mathbb{R}^N} \left|\left| \hat{\xi}\left(T_s(\xi, \tau_{\frac{x}{\varepsilon}}\omega)\right)\right|\right|_{L^1(\mathbb{R}^N \times \Omega, \mu_c)} \leq C_0.$$

Hence, if $0 \leq t \leq \alpha$, we have

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N_\xi \times \Omega} \left| \varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + \xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right| \, d\mu_c(\xi,\omega) \leq 2\alpha C_0.$$

Eventually,

$$\begin{aligned}
&\left|\left| \varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right|\right|_{L^\infty((0,T)\times\mathbb{R}^N; L^1(\mu_c))} \leq \\
&\leq \inf_{0<\alpha<T} \left\{ 2C_0\alpha + T \sup_{\tau \geq \frac{\alpha}{\varepsilon}} \left|\left| \frac{1}{\tau}\int_0^{\tau} \hat{\xi}\left(T_s(\xi,\omega)\right) \, ds - \xi^\sharp\left(0,\xi,\omega\right)\right|\right|_{L^1(\mu_c)} \right\}
\end{aligned}$$

and the lemma is proved. □

We easily deduce that theorem 10 is true when $f_0 \in W^{1,\infty}(\mathbb{R}^N)$ with

$$f(t,x,y,\xi,\omega) := f_0(x - t\xi^\sharp(y,\xi,\omega)), \quad g = 0,$$

$$r^\varepsilon(t,x,\xi,\omega) := f^\varepsilon(t,x,\xi,\omega) - f\left(t, x, \frac{x}{\varepsilon}, \xi, \omega\right)$$

and it is easily checked that f satisfies $\mathbb{P}(f) = f$, $f(t=0) = \mathbb{P}(f_0) = f_0$ (since f_0 is independent of y and ξ), and that f is a solution of the evolution equation (6.4).

6.3.3 Third case : f_0 arbitrary

We now tackle the case where the initial data is an arbitrary stationary function belonging to $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$. We begin with the case where

$$f_0(x,y,\xi,\omega) = a(x)b(y,\xi,\omega),$$

with $a \in W^{1,\infty}(\mathbb{R}^N)$ and $b \in L^\infty(\mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega) \cap \mathcal{C}^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi, L^\infty(\Omega))$, b stationary. This case follows directly from the two first subsections. Indeed, let

$$f(t,x,y,\xi,\omega) = a(x - t\xi^\sharp(y,\xi,\omega))\, \mathbb{P}(b)(y,\xi,\omega),$$

and

$$g(t,x;\tau,y,\xi,\omega) = a(x - t\xi^\sharp(y,\xi,\omega))\ (b - \mathbb{P}(b))\,(T_\tau(y,\xi,\omega)).$$

It is already known that f and g satisfy (6.4), that $f(t,x,\cdot) \in \mathbb{K}$, and that g satisfies (6.3) thanks to the preceding subsections and the fact that $\xi^\sharp(y,\xi,\omega)$ is invariant by the Hamiltonian flow (Y,Ξ). Notice that it is capital here that the coefficient $\xi^\sharp(y,\xi,\omega)$ in the transport equation (6.4) belongs to $\mathbb{K}$.

There remains to check that $g(t,x;\tau,\cdot) \in \mathbb{K}^\perp$, that the remainder r^ε goes to 0 strongly in L^1_{loc} and that $g(t,x;t/\varepsilon,x/\varepsilon,\xi,\omega)$ goes weakly to 0 in the sense of theorem 10. First, notice that $a(x-t\xi^\sharp(y,\xi,\omega)) \in \mathbb{K}$ and $(b - \mathbb{P}(b))\,(T_\tau(y,\xi,\omega)) \in \mathbb{K}^\perp$. Thus, $a(x - t\xi^\sharp)\mathbb{P}(b) = \mathbb{P}(a(x - t\xi^\sharp)b)$ almost everywhere (because $a(x - t\xi^\sharp(0,\xi,\omega))$ is invariant by the semi-group T_τ), and consequently

$$g(t,x;\tau,y,\xi,\omega) = \left[a(x - t\xi^\sharp)b - \mathbb{P}\left(a(x - t\xi^\sharp)b\right)\right](T_\tau(\xi,\tau_y\omega)).$$

Hence $g(t,x;\tau,\cdot) \in \mathbb{K}^\perp$ a.e.

Then, setting

$$r^\varepsilon(t,x,\xi,\omega) = f^\varepsilon(t,x,\xi,\omega) - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) - g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),$$

we have to prove that r^ε goes to 0 strongly in L^1_{loc}. We compute the difference

$$\begin{aligned}
& f^\varepsilon(t,x,\xi,\omega) - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) - g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) \\
= \ & a\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right) b\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right) \\
& -a\left(x - t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)\ \mathbb{P}(b)\left(\frac{x}{\varepsilon},\xi,\omega\right) \\
& -a\left(x - t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)[b - \mathbb{P}(b)]\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right) \\
= \ & \left[a\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right) - a\left(x - t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)\right]\ b\left((Y,\Xi)\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)
\end{aligned}$$

The right-hand side of the above equality is bounded by

$$||a||_{W^{1,\infty}}||b||_{L^\infty}\left|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right|$$

and thus converges to 0 as $\varepsilon \to 0$ in $L^\infty((0,T)\times\mathbb{R}^N_x; L^1(\mathbb{R}^N_\xi\times\Omega,\mu_c)))$ according to the second subsection.

Moreover, it is easily proved that as $\varepsilon \to 0$,

$$\int_0^T g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt \to 0$$

strongly in $L^1_{\mathrm{loc}}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^1(\Omega))$ thanks to remark 6.3.1. Hence theorem 10 is proved in that case.

The general case now follows from a density result and a contraction property, which are stated in the two lemmas below. We first explain how the general case can be deduced from the lemmas, and then we prove the lemmas.

The first lemma states that linear combinations of product functions of the type $a(x)b(y,\xi,\omega)$, with a and b smooth, are dense in $L^1_{\mathrm{loc}}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y\times\Omega))$.

Lemma 6.3.2. *Let $f_0 \in L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y\times\Omega))$ arbitrary, and set $F_0(x,\xi,\omega) := f_0(x,0,\xi,\omega)$. Let $R,R'>0$ arbitrary.*

There exists a sequence of functions $F_n \in L^1(\mathbb{R}^N_x\times\mathbb{R}^N_\xi\times\Omega)$ such that

- *$F_n \to F_0$ as $n\to\infty$ in $L^1(B_R\times B_{R'}\times\Omega)$;*
- *For all $n\in\mathbb{N}$, there exist an integer $N_n\in\mathbb{N}$ and functions $a_i^n \in \mathcal{C}^1\cap W^{1,\infty}(\mathbb{R}^N)$, $b_i^n \in L^\infty(\mathbb{R}^N_\xi\times\Omega)$, $1\le i\le N_n$, such that almost surely in ω, for all $(x,\xi)\in\mathbb{R}^{2N}$*

$$F_n(x,\xi,\omega) = \sum_{i=1}^{N_n} a_i^n(x)\, b_i^n(\xi,\omega)\ ;$$

- *For all $n\in\mathbb{N}$, for $1\le i\le N_n$, the function*

$$(y,\xi,\omega)\mapsto b_i^n(\xi,\tau_y\omega)$$

belongs to $\mathcal{C}^1(\mathbb{R}^N_y\times\mathbb{R}^N_\xi, L^\infty(\Omega))$.

The second lemma states a contraction property for equation (6.1) and for equation (6.4).

Lemma 6.3.3. *Let $g_0\in L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y\times\Omega))$ be a stationary admissible data for (6.1). Let g^ε be the solution of (6.1) with initial data $g_0(x,x/\varepsilon,\xi,\omega)$. Then for all $R,R',T>0$, for all $t\in[0,T]$,*

$$E\left[\int_{x\in B_R,\ \xi\in B_{R'}} |g^\varepsilon(t,x,\xi,\omega)|\ dx\,d\xi\right] \le ||G_0||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega, dx\,d\mu_{c_{R'}}(\xi,\omega))} \qquad (6.11)$$

where

$$G_0(x,\xi,\omega) := g_0(x,0,\xi,\omega),$$
$$C_{R,T,R'} := \left\{x\in\mathbb{R}^N,\ |x|\le R+T\sqrt{R'^2+2u_{max}}\right\},$$
$$c_{R'} := \frac{1}{2}R'^2 + u_{max}.$$

Similarly, if g is a solution of (6.4) *with initial data* $g_0 \in L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$*, then*

$$\int_{x\leq R} |g(t,x,y,\xi,\omega)|\ dx \leq \int_{x\leq R+T\sqrt{\xi^2+2u_{max}}} |g_0(x,y,\xi,\omega)|\ dx. \tag{6.12}$$

And if h is solution of (6.3) *with initial data* $h_0 \in L^1_{loc}(\mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$*, then for all* $y \in \mathbb{R}^N,\ \tau \geq 0$,

$$E\left[\int_{\xi\in B_{R'}} |h(\tau,y,\xi,\omega)|\ d\xi\right] \leq \int_{\mathbb{R}^N_\xi\times\Omega} |h_0|(y,\xi\,\omega)\ d\mu_{c_{R'}}(\xi,\omega). \tag{6.13}$$

We postpone the proofs of lemmas 6.3.2 and 6.3.3.

Now, let $R, R' > 0$ abritrary, and let F_n be a sequence converging to F_0 in $L^1(B_R\times B_{R'}\times\Omega)$ as in lemma 6.3.2. Assume that f_0 is an admissible initial data for (6.1), and let f^ε_n (resp. f^ε) be the solution of (6.1) with initial data $F_n\left(x,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)$ (resp. $F_0\left(x,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)$), and let $f_n = f_n(t,x,y,\xi,\omega)$, $g_n = g_n(t,x;\tau,y,\xi,\omega)$ be the functions associated to f^ε_n by theorem 10 for all n.

Let $f(t,x,y,\xi,\omega)$, $g(t,x;\tau,y,\xi,\omega)$ be the solutions of the system

$$\begin{gathered}\mathbb{P}(f)=f,\quad \mathbb{P}(g)=0,\\ \partial_t\begin{pmatrix} f\\ g\end{pmatrix}+\xi^\sharp(y,\xi,\omega)\cdot\nabla_x\begin{pmatrix} f\\ g\end{pmatrix}=0,\\ \partial_\tau g+\xi\cdot\nabla_y g-\nabla_y u(y,\omega)\cdot\nabla_\xi g=0,\\ f(t=0)=\mathbb{P}(f_0),\quad g(t=0,x;\tau=0,y,\xi,\omega)=\left[f_0-\mathbb{P}(f_0)\right](x,y,\xi,\omega).\end{gathered}$$

We have already proved that f_n, g_n satisfy the above system. We denote by $\bar F_0$, $\bar F_n$, the functions associated to F_0, F_n respectively by corollary 6.2.1, so that $\mathbb{P}(f_0)(x,y,\xi,\omega) = \bar F_0(x,\xi,\tau_y\omega)$, and $f_n(t=0,x,y,\xi,\omega) = \bar F_n(x,\xi,\tau_y\omega)$, $g_n(t=0,x,\tau=0,y,\xi,\omega) = (F_n-\bar F_n)(x,\xi,\tau_y\omega)$.

Notice that if $G \in L^1(\mathbb{R}^N_\xi\times\Omega,\mu_c)$, for some $c>0$, then μ_c almost everywhere

$$\begin{aligned}|\bar G(\xi,\omega)| &= \lim_{T\to\infty}\frac{1}{T}\left|\int_0^T G(T_t(\xi,\omega))\,dt\right|\\ &\leq \lim_{T\to\infty}\frac{1}{T}\int_0^T |G(T_t(\xi,\omega))|\ dt = \overline{|G|}\,(y,\xi),\end{aligned}$$

and thus, according to property (6.7),

$$\begin{aligned}||\bar G||_{L^1(\mathbb{R}^N_\xi\times\Omega,\mu_c)} &\leq \int_{\mathbb{R}^N\times\Omega}\overline{|G|}\,(y,\xi)\,d\mu_c(y,\xi)\\ &= \int_{\mathbb{R}^N\times\Omega}|G|(y,\xi)\,d\mu_c(y,\xi) = ||G||_{L^1(\mathbb{R}^N_\xi\times\Omega,\mu_c)}.\end{aligned}$$

Consequently, setting $c := c_{R'}$, we have

$$\begin{aligned}||(f-f_n)(t,\cdot,y,\xi,\omega)||_{L^1(B_R)} &\leq \left|\left|\bar F_0(\cdot,\xi,\tau_y\omega)-\bar F_n(\cdot,\xi,\tau_y\omega)\right|\right|_{L^1(C_{R,T,\xi})}\\ ||(f-f_n)(t,\cdot,y,\cdot,\cdot)||_{L^1(B_R\times B_{R'}\times\Omega)} &\leq \left|\left|\overline{F_0-F_n}\right|\right|_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega,dxd\mu_c(\xi,\omega))}\\ &\leq ||F_0-F_n||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega,dxd\mu_c(\xi,\omega))}.\end{aligned}$$

And similarly, using (6.11), (6.12), (6.13),

$$||f^\varepsilon(t,\cdot) - f_n^\varepsilon(t,\cdot)||_{L^1(B_R\times B_{R'}\times\Omega)} \le ||F_0 - F_n||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega, dx\, d\mu_c(\xi,\omega))},$$
$$||g(t,\cdot;y,\cdot,\cdot) - g_n(t,\cdot;y,\cdot,\cdot)||_{L^1(B_R\times B_{R'}\times\Omega)} \le 2\,||F_0 - F_n||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega, dxd\mu_c(\xi,\omega))}.$$

The above inequalities are true for all $t \in [0,T]$ and for all $\tau \ge 0$.
Set

$$r^\varepsilon(t,x,\xi,\omega) := f^\varepsilon(t,x,\xi,\omega) - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) - g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right).$$

Then for all $t \in [0,T]$, for all $n \in \mathbb{N}$,

$$\begin{aligned}
||r^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)} \le\ & ||f^\varepsilon(t) - f_n^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)} \\
& + ||f(t) - f_n(t)||_{L^\infty(\mathbb{R}^N_y;L^1(B_R\times B_{R'}\times\Omega))} \\
& + \left|\left| g\left(t;\frac{t}{\varepsilon}\right) - g_n\left(t;\frac{t}{\varepsilon}\right)\right|\right|_{L^\infty(\mathbb{R}^N_y;L^1(B_R\times B_{R'}\times\Omega))} \\
& + ||r_n^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)} \\
\le\ & 4\,||F_0 - F_n||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega, dxd\mu_c(\xi,\omega))} \\
& + ||r_n^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)}.
\end{aligned}$$

Thus $r^\varepsilon \to 0$ as $\varepsilon \to 0$ in $L^\infty([0,\infty); L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi; L^1(\Omega))$.

There only remains to check that $\int_0^T g(t,x;t/\varepsilon,x/\varepsilon,\xi,\omega)\,dt$ goes strongly to 0 in L^1_{loc} norm as ε vanishes; this result follows immediately from the same property for g_n and the above inequalities. Therefore, we skip its proof. □

We now tackle the proofs of lemmas 6.3.2 and 6.3.3.

Proof of Lemma 6.3.2. We use the results of chapter 2 in [21]. Since $F_0 \in L^1(B_R \times B_{R'+1}, L^\infty(\Omega))$, $F_0 \in L^1(B_R, L^1(B_{R'+1}\times\Omega))$. Thus, there exists a sequence of functions $(\tilde{F}_n)_{n\in\mathbb{N}}$ such that $\tilde{F}_n \to F_0$ in $L^1(B_R \times B_{R'+1}\times\Omega)$, and for all $n \in \mathbb{N}$,

$$\tilde{F}_n = \sum_{i=1}^{N_n} \mathbf{1}_{A_{i,n}}(x)\phi_{i,n}(\xi,\omega),$$

where $N_n \in \mathbb{N}$, and :

- for all $n \in \mathbb{N}$, $1 \le i \le N_n$, $A_{i,n} \subset B_R$ is a measurable set;
- for all $n \in \mathbb{N}$, $1 \le i \le N_n$, $\phi_{i,n} \in L^1(B_{R'+1}\times\Omega)$.

We shall explain later why we have chosen an approximating sequence in $L^1(B_R \times B_{R'+1}\times\Omega)$ rather than $L^1(B_R\times B_{R'}\times\Omega)$.

The idea is to replace $\tilde{F}_n$ by a function F_n having the same structure, but in which the functions $1_{A_{i,n}}$, $\phi_{i,n}$ have been regularized. Hence we consider a mollifier $\rho \in \mathcal{D}(\mathbb{R}^N)$ such that $\rho \ge 0$ and $\int_{\mathbb{R}^N}\rho = 1$. For $k \in \mathbb{N}$, set

$$\rho_k(x) = k^N\rho(kx), \quad x \in \mathbb{R}^N.$$

The regularization by convolution in the variable x is standard : $\mathbf{1}_{A_{i,n}}(x)$ is replaced by

$$\mathbf{1}_{A_{i,n}} *_x \rho_k(x) = \int_{\mathbb{R}^N} \mathbf{1}_{A_{i,n}}(y)\rho_k(x-y).$$

Concerning the regularization of the functions $\phi_{i,n}$, we first truncate $\phi_{i,n}$ in order to obtain a function in L^∞. For $M > 0$, set

$$\phi_{i,n}^M := \mathrm{sgn}(\phi_{i,n}) \inf(|\phi_{i,n}|, M).$$

Then $\phi_{i,n}^M \in L^\infty(B_{R'+1}\times\Omega)$, and $\phi_{i,n}^M$ converges towards $\phi_{i,n}$ as $M \to \infty$ in $L^1(B_{R'+1}\times \Omega)$. Thus we work with $\phi_{i,n}^M$ instead of $\phi_{i,n}$, and we drop the superscript M in order to avoid too heavy notations.

Now, for $k \in \mathbb{N}$, we set, for $y \in \mathbb{R}^N$, $\xi \in B_{R'}$,

$$\begin{aligned}\varphi_{i,n}(y,\xi,\omega) &= \phi_{i,n}(\xi, \tau_y\omega),\\ \varphi_{i,n}^k(y,\xi,\omega) &= \int_{\mathbb{R}^{2N}} \phi_{i,n}(\xi', \tau_{y'}\omega)\, \rho_k(y-y')\rho_k(\xi-\xi')\, dy'\, d\xi',\\ \phi_{i,n}^k(\xi,\omega) &= \varphi_{i,n}^k(0,\xi,\omega).\end{aligned}$$

This is the part where it is convenient to have $\phi_{i,n} \in L^1(B_{R'+1}, \Omega)$. Indeed, if $\xi \in B_{R'}$ and $|\xi - \xi'| \le 1$, then $\xi' \in B_{R'+1}$; thus the convolution is well-defined on $\mathbb{R}^N \times B_{R'}$ as long as $\phi_{i,n} \in L^1(B_{R'+1}, \Omega)$.

The function $\varphi_{i,n}^k$ belongs to $\mathcal{C}_b^1(\mathbb{R}^N \times B_{R'}, L^\infty(\Omega))$, where $\mathcal{C}_b^1$ denotes the space of $\mathcal{C}^1$ bounded functions with bounded derivatives. Moreover, it is easily checked that $\varphi_{i,n}^k$ is stationary, and $\varphi_{i,n}^k(\cdot,\omega)$ converges towards $\varphi_{i,n}(\omega)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N_y, L^1(B_{R'}))$, almost surely in ω. And if K is any compact set in $\mathbb{R}^N$, then there exists a compact set K' such that $K \subset K'$ and almost surely in ω,

$$||\varphi_{i,n}^k(\cdot,\omega) - \varphi_{i,n}(\cdot,\omega)||_{L^1(K\times B_{R'})} \le 2||\varphi_{i,n}(\cdot,\omega)||_{L^1(K'\times B_{R'})}.$$

Since $||\varphi_{i,n}(\cdot,\omega)||_{L^1(K'\times B_{R'})}$ belongs to $L^1(\Omega)$, using Lebesgue's dominated convergence theorem, we deduce that $\varphi_{i,n}^k$ converges towards $\varphi_{i,n}$ in $L^1(K \times B_{R'} \times \Omega)$, for every compact set $K \subset \mathbb{R}^N$. Thus $\phi_{i,n}^k$ converges towards $\phi_{i,n}$ in $L^1(B_{R'} \times \Omega)$ as $k \to \infty$, due to the invariance of the measure P with respect to the transformation group τ_y.

We set

$$\tilde{F}_{n,k}(x,\xi,\omega) := \sum_{i=1}^{N_n} \mathbf{1}_{A_{i,n}} *_x \rho_k(x)\, \phi_{i,n}^k(\xi,\omega).$$

Then $\tilde{F}_{n,k}$ converges towards $\tilde{F}_n$ as $k \to \infty$ in $L^1(B_R \times B_{R'} \times \Omega)$. Thus there exists an integer k_n such that

$$||\tilde{F}_{n,k_n} - \tilde{F}_n||_{L^1(B_R\times B_{R'}\times\Omega)} \le \frac{1}{n}.$$

Set $F_n := \tilde{F}_{n,k_n}$. Then F_n converges towards F_0 in L^1 as $n \to \infty$, and thus the lemma is true.

□

Proof of Lemma 6.3.3. We first prove the lemma when g_0 is smooth in all variables, namely $g_0 \in \mathcal{C}^1(\mathbb{R}^N_x \times \mathbb{R}^N_y \times \mathbb{R}^N_\xi, L^\infty(\Omega))$. In that case, let us recall that

$$\begin{aligned} & g^\varepsilon(t,x,\xi,\omega) \\ = \;& g_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right) \\ = \;& g_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)+x, Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right), \Xi\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right),\tau_{\frac{x}{\varepsilon}}\omega\right) \\ = \;& G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)+x, T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right), \end{aligned}$$

where $G_0(z,\xi,\omega) := g_0(z,0,\xi,\omega)$ for all $(z,\xi,\omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega$, and the semi-group $(T_t)_{t\geq 0}$ was defined in section 2 (see lemma 6.2.1). Moreover, since

$$\frac{1}{2}\left|\Xi(t,y,\xi,\omega)\right|^2 + u\left(Y(t,y,\xi,\omega)\right) = \frac{1}{2}|\xi|^2 + u(y,\omega)$$

we have

$$|\Xi(t,y,\xi,\omega)| \leq \sqrt{|\xi|^2+2u(y,\omega)} \leq \sqrt{|\xi|^2+2u_{\max}}$$

and almost surely in ω,

$$\left|\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\omega\right)\right| \leq t\sqrt{|\xi|^2+2u_{\max}}.$$

Additionnally, we have, for all $R'>0$, for almost every $\xi \in \mathbb{R}^N$ and almost surely in ω,

$$\mathbf{1}_{|\xi|\leq R'} \leq \mathbf{1}_{\mathcal{H}(\xi,\omega)\leq \frac{1}{2}R'^2+u_{\max}}.$$

(Remember that $\mathcal{H}(\xi,\omega) = \frac{1}{2}|\xi|^2 + u(0,\omega)$). Thus, setting $c_{R'} := \frac{1}{2}R'^2 + u_{\max}$, we have

$$\begin{aligned} & E\left[\int_{x\in B_R,\ \xi\in B_{R'}} |g^\varepsilon(t,x,\xi,\omega)|\ dx\, d\xi\right] \\ = \;& \int_{x\in B_R,\ \xi\in B_{R'}} E\left[\left|G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)+x, T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right)\right|\right] dx\, d\xi \\ = \;& \int_{x\in B_R,\ \xi\in B_{R'}} E\left[\left|G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\omega\right)+x, T_{\frac{t}{\varepsilon}}\left(\xi,\omega\right)\right)\right|\right] dx\, d\xi \\ = \;& E\left[\int_{\xi\in B_{R'}}\left(\int_{x\in B_R}\left|G_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},0,\xi,\omega\right)+x, T_{\frac{t}{\varepsilon}}\left(\xi,\omega\right)\right)\right| dx\right) d\xi\right] \\ \leq \;& E\left[\int_{\xi\in B_{R'}}\left(\int_{|z|\leq R+T\sqrt{R'^2+2u_{\max}}}\left|G_0\left(z, T_{\frac{t}{\varepsilon}}\left(\xi,\omega\right)\right)\right| dz\right) d\xi\right] \\ \leq \;& \int_{|z|\leq R+T\sqrt{R'^2+2u_{\max}}}\int_{\mathbb{R}^N\times\Omega}\left|G_0\left(z, T_{\frac{t}{\varepsilon}}\left(\xi,\omega\right)\right)\right| d\mu_{c_{R'}}(\xi,\omega)\, dz \\ = \;& \int_{|z|\leq R+T\sqrt{R'^2+2u_{\max}}}\int_{\mathbb{R}^N\times\Omega}\left|G_0\left(z,\xi,\omega\right)\right| d\mu_{c_{R'}}(\xi,\omega)\, dz \\ = \;& ||G_0||_{L^1(C_{R,T,R'}\times\mathbb{R}^N_\xi\times\Omega, dx\, d\mu_{c_{R'}}(\xi,\omega))}. \end{aligned}$$

Now, if g_0 is an arbitrary admissible data in $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_y \times \mathbb{R}^N_\xi, L^\infty(\Omega))$, we choose a sequence G_n approaching G_0 as in lemma 6.3.2 (R and R' are fixed). Then inequality (6.11) is true for g^ε_n for all $n \in N$, and for $g^\varepsilon_n - g^\varepsilon_m$ for all $n, m \in N$. Thus the sequence g^ε_n is a Cauchy sequence in $L^\infty_{\text{loc}}((0,\infty), L^1(B_R \times B_{R'} \times \Omega))$. Thus it converges strongly towards a solution of the transport equation (6.1). Thanks to a uniqueness result for the transport equation (6.1), the limit of the sequence g^ε_n as $n \to \infty$ is g^ε. There only remains to pass to the limit as $n \to \infty$ in the inequality (6.11) written for g^ε_n and G^n_0.

The proof of inequalities (6.12) and (6.13) go along the same lines.

□

6.4 The integrable case

In this section, we treat independently the periodic and the stationary ergodic case. Indeed, some results of the periodic case are no longer true in the stationary ergodic setting, and the results which do remain valid are not proved with the same tools.

Let us make precise what we mean about "integrable case" : in the periodic case, we take a function $u(y)$ which has the form

$$u(y) = \sum_{i=1}^{N} u_i(y_i), \tag{6.14}$$

where each function u_i is periodic with period 1 ($1 \leq i \leq N$). The Hamiltonian $H(y,\xi)$ can be written

$$H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y) = \sum_{i=1}^{N} H_i(y_i,\xi_i)$$

where $H_i(y_i,\xi_i) = \frac{1}{2}|\xi_i|^2 + u_i(y_i)$ ($1 \leq i \leq N$). And the Hamiltonian system (6.5) becomes

$$\begin{cases} \dot{Y}_i = -\Xi_i, \\ \dot{\Xi}_i = u_i'(Y_i), \\ Y_i(t=0) = y_i, \quad \Xi_i(t=0) = \xi_i. \end{cases} \tag{6.15}$$

Thus it is enough to investigate the behavior of each one-dimensional Hamiltonian system (6.15) individually, and for most calculations, we can assume without loss of generality that $N = 1$, and we drop all indices i. However, for the calculation of the projection $\mathbb{P}$, a more thorough discussion will be needed, and we will come back to the case where $N > 1$ in the corresponding paragraph.

In the stationary ergodic setting, expression (6.14) can be transposed in the following way : assume that $\Omega = \Pi_{i=1}^N \Omega_i$, where each Ω_i is a probability space, and assume that for $1 \leq i \leq N$, an ergodic group transformation, denoted by $(\tau_{i,y})_{y\in\mathbb{R}}$, acts on each Ω_i.

Then for $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$, and $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$, we set $\tau_y\omega := (\tau_{1,y_1}\omega_1, \cdots, \tau_{N,y_N}\omega_N)$. And we assume that the function u can be written

$$u(y,\omega) = \sum_{i=1}^{N} U_i\left(\tau_{i,y_i}\omega_i\right),$$

where $U_i \in L^\infty(\Omega_i)$ for all $1 \leq i \leq N$. The same remarks as in the periodic case can be made, and thus we will only consider the case $N = 1$; note that in the stationary ergodic case, we are unable to compute the projection $\mathbb{P}$ when $N > 1$.

6.4.1 Periodic setting

The goal of this subsection is to give another proof of the results of K. Hamdache and E. Frénod in [33], based on the study of the system

$$\begin{cases} \dot{Y} = -\Xi, \\ \dot{\Xi} = u'(Y), \\ Y(t=0) = y, \quad \Xi = \xi, \quad y \in \mathbb{R},\ \xi \in \mathbb{R}. \end{cases} \tag{6.16}$$

The Hamiltonian $H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y)$ is constant along the trajectories of the system (6.16), so that

$$\frac{1}{2}|\Xi(t,y,\xi)|^2 + u(Y(t,y,\xi)) = H(y,\xi).$$

We now fix $y, \xi \in \mathbb{R}^N$. Without any loss of generality, we assume $y \in [-\frac{1}{2}, \frac{1}{2})$, and we set $\mathcal{E} := H(y,\xi)$. The above equation describes the movement of a single particle in a periodic potential u, with $0 \leq u \leq u_{\max}$. It is well-known that there are two kinds of behavior, depending on the value of the energy $\mathcal{E}$: if $\mathcal{E} < u_{\max}$, the particle is “trapped” in a well of potential around y, and $Y(t)$ remains bounded as $t \to \infty$. In that case, the trajectories in the phase space are closed curves. If $\mathcal{E} > u_{\max}$, the trajectory of the particle is unconstrained and $|Y(t)| \to \infty$ as $t \to \infty$. We study more precisely these two cases and their consequences on the expression of the projection $\mathbb{P}$ in the following subsections ; we refer for instance to [7] for further calculations and results about Hamiltonian dynamics and ordinary differential equations in general.

A Expression of $\xi^\sharp(y,\xi)$

We begin with the case when $H(y,\xi) < u_{\max}$. In that case, $u(y) \leq H(y,\xi) < u_{\max}$. By continuity of the potential u, there exists $y_- < y$ and $y_+ > y$ such that $H(y,\xi) < u(y_\pm) < u_{\max}$, and the periodicity of u allows us to choose $y_\pm$ such that $|y_+ - y_-| < 1$. Then $y_- < Y(t,y,\xi) < y_+$ for all $t \geq 0$. Indeed, assume that there exists $t > 0$ such that $Y(t,y,\xi) \geq y_+ > y = Y(t=0,y,\xi)$. Since the trajectory Y is continuous in time, there exists $0 < t_0 \leq t$ such that $Y(t=t_0,y,\xi) = y_+$, which is absurd since

$$H(Y(t_0,y,\xi), \Xi(t_0,y,\xi)) = H(y,\xi) \geq u(Y(t_0,y,\xi)) = u(y_+) > H(y,\xi).$$

Thus $Y(t,y,\xi)$ is bounded. Since

$$\xi^\sharp(y,\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Xi(t,y,\xi)\,dt = -\lim_{T\to\infty} \frac{1}{T}\int_0^T \dot{Y}(t,y,\xi)\,dt = \lim_{T\to\infty} \frac{y - Y(T,y,\xi)}{T}$$

we deduce that $\xi^\sharp(y,\xi) = 0$ for all y, ξ such that $H(y,\xi) < u_{\max}$.

We now study the case $H(y,\xi) > u_{\max}$. Since

$$|\dot{Y}(t,y,\xi)|^2 = 2\left(H(y,\xi) - u(Y(t,y,\xi))\right) \geq 2(H(y,\xi) - u_{\max}) > 0,$$

we deduce that $\dot{Y}$ does not vanish for $t \geq 0$. Consequently,

$$\Xi(t,y,\xi) = -\dot{Y}(t,y,\xi) = \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(Y(t,y,\xi))\right)},$$

and since $|Y(t,y,\xi) - y| \geq \sqrt{2\left(H(y,\xi) - u_{\max}\right)}t$, $|Y(t)| \to \infty$ as $t \to \infty$. We immediately deduce that $\Xi(t,y,\xi)$ is periodic in time: indeed, there exists $t_0 > 0$ such that

$$Y(t_0,y,\xi) = y - \operatorname{sgn}(\xi).$$

And

$$\begin{aligned}
\Xi(t=t_0,y,\xi) &= \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(Y(t_0,y,\xi))\right)}\\
&= \operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(y)\right)}\\
&= \xi = \Xi(t=0,y,\xi),
\end{aligned}$$

so that for $s \geq 0$,

$$\begin{aligned}
Y(t_0+s,y,\xi) &= Y(s,y,\xi) - \operatorname{sgn}(\xi),\\
\Xi(t_0+s,y,\xi) &= \Xi(s,y,\xi),
\end{aligned}$$

and Ξ is periodic with period t_0.

Consequently,

$$\xi^\sharp(y,\xi) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Xi(t,y,\xi)\,dt = \frac{1}{t_0}\int_0^{t_0}\Xi(t,y,\xi)\,dt.$$

But

$$\begin{aligned}
\int_0^{t_0}\Xi(t,y,\xi)\,dt &= -\int_0^{t_0}\dot{Y}(t,y,\xi)\,dt\\
&= -\left(Y(t_0,y,\xi) - y\right)\\
&= \operatorname{sgn}(\xi).
\end{aligned}$$

Thus we only have to compute t_0. With this aim in view, we use the change of variables $s = Y(t)$, with Jacobian $ds = \dot{Y}dt$, and we recall that

$$\dot{Y}(t,y,\xi) = -\operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(Y(t,y,\xi))\right)}.$$

Thus

$$\begin{aligned}
t_0 &= \int_0^{t_0} dt\\
&= \int_{Y(t=0)}^{Y(t_0)} \frac{1}{-\operatorname{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(s)\right)}}ds\\
&= -\operatorname{sgn}(\xi)\int_y^{y-\operatorname{sgn}(\xi)} \frac{1}{\sqrt{2\left(H(y,\xi) - u(s)\right)}}ds\\
&= \int_0^1 \frac{1}{\sqrt{2\left(H(y,\xi) - u(s)\right)}}ds
\end{aligned}$$

Eventually, we deduce

$$\xi^\sharp(y,\xi) = \text{sgn}(\xi)\varphi(H(y,\xi)),$$

where

$$\varphi(\mathcal{E}) = \sqrt{2}\mathbf{1}_{\mathcal{E}>u_{\max}} \frac{1}{\left\langle \frac{1}{\sqrt{(\mathcal{E}-u(s))}} \right\rangle}$$

We close this paragraph with a calculation which allows us to express $\xi^\sharp$ in terms of the homogenized Hamiltonian $\bar{H}$. The result we will obtain will be justified in more abstract and theoretical terms in the last subsection, using arguments similar to those of the theory of Aubry-Mather.

First, let us recall the expression of the homogenized Hamiltonian $\bar{H}$ (see [49]) : we have

$$H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y), \quad \text{with} \inf u = 0, \ \sup u = u_{\max},$$

and thus

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\begin{cases} 0 & \text{if } p < \left\langle \sqrt{2(u_{\max}-u)} \right\rangle, \\ \lambda & \text{if } |p| \geq \left\langle \sqrt{2(u_{\max}-u)} \right\rangle, \text{ where } |p| = \left\langle \sqrt{2(u_{\max}-u)+\lambda} \right\rangle. \end{cases}$$

In other words, setting

$$\theta : \begin{array}{ccc} [0,\infty) & \rightarrow & [0,\infty) \\ \lambda & \mapsto & \left\langle \sqrt{2(u_{\max}-u)+\lambda} \right\rangle \end{array}$$

we have

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\mathbf{1}_{|p|\geq\theta(0)}\theta^{-1}(|p|).$$

Hence,

$$\bar{H}'(p) = \text{sgn}(p)\frac{1}{2}\mathbf{1}_{|p|>\theta(0)}\frac{1}{\theta'\left(\theta^{-1}(|p|)\right)};$$

and

$$\begin{gathered} \theta'(\lambda) = \frac{1}{2}\left\langle \frac{1}{\sqrt{2(u_{\max}-u)+\lambda}} \right\rangle, \\ \theta^{-1}(|p|) = 2\left(\bar{H}(p) - u_{\max}\right) \quad \forall |p| \geq \theta(0), \\ |p| > \theta(0) \iff \bar{H}(p) > u_{\max} \quad \forall p. \end{gathered}$$

Gathering all the terms, we are led to

$$\begin{aligned} \bar{H}'(p) &= \text{sgn}(p)\sqrt{2}\mathbf{1}_{\bar{H}(p)>u_{\max}} \frac{1}{\left\langle \frac{1}{\sqrt{\bar{H}(p)-u}} \right\rangle} \\ &= \text{sgn}(p)\varphi\left(\bar{H}(p)\right) \end{aligned}$$

Thus, the final expression is

$$\xi^\sharp(y,\xi) = \bar{H}'(p),$$

where p is such that

$$\bar{H}(p) = H(y,\xi) \vee u_{\max}, \quad \operatorname{sgn}(p) = \operatorname{sgn}(\xi).$$

B Expression of the projection $\mathbb{P}$

We also mention here how to find a general expression of the projection $\mathbb{P}$ in the special case $N=1$, and we explain how to generalize this expression in some particular cases when $N>1$. Recall that if $f = f(y,\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^N_y \times \mathbb{R}^N_\xi)$ is periodic in y, then

$$\mathbb{P}(f)(y,\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt$$

and the limit holds almost everywhere and in $L^1([0,1)\times\mathbb{R}^N, m_c)$, with $dm_c(y,\xi) = \mathbf{1}_{H(y,\xi)\leq c}\,dy\,d\xi$.

We begin with the case $H(y,\xi) > u_{\max}$. We have seen in the previous paragraph that there exists $t_0>0$, which depends only on $H(y,\xi)$ such that for all $t>0$, for all $k\in\mathbb{N}$

$$Y(t+k\,t_0,y,\xi) = Y(t,y,\xi) - k\operatorname{sgn}(\xi), \quad \Xi(t+k\,t_0,y,\xi) = \Xi(t,y,\xi).$$

Thus $f(Y(t),\Xi(t))$ is periodic in time with period t_0, and

$$\mathbb{P}(f)(y,\xi) = \frac{1}{t_0}\int_0^{t_0} f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt.$$

We use once again the change of variables $s = Y(t)$, so that

$$\begin{aligned}
&\int_0^{t_0} f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt\\
&= \int_y^{y-\operatorname{sgn}(\xi)} f\left(s,\operatorname{sgn}(\xi)\sqrt{2\,(H(y,\xi)-u(s))}\right)\frac{1}{-\operatorname{sgn}(\xi)\sqrt{2\,(H(y,\xi)-u(s))}}\,ds\\
&= \left\langle f\left(\cdot,\operatorname{sgn}(\xi)\sqrt{2\,(H(y,\xi)-u(\cdot))}\right)\frac{1}{\sqrt{2\,(H(y,\xi)-u(\cdot))}}\right\rangle.
\end{aligned}$$

And eventually,

$$\mathbb{P}(f)(y,\xi) = \bar{f}(\operatorname{sgn}(\xi),H(y,\xi)) \tag{6.17}$$

with

$$\bar{f}(\eta,\mathcal{E}) := \frac{\left\langle f\left(\cdot,\eta\sqrt{2\,(\mathcal{E}-u(\cdot))}\right)\frac{1}{\sqrt{(\mathcal{E}-u(\cdot))}}\right\rangle}{\left\langle\frac{1}{\sqrt{(\mathcal{E}-u)}}\right\rangle} \quad \eta=\pm1,\ \mathcal{E}>u_{\max}.$$

We now focus on the case $0 < \mathcal{E} < u_{\max}$. In order to simplify the analysis we assume that $\mathcal{E} \notin \{u(y)\,; u \text{ has a local extremum at } y\}$ (this set is finite or countable), and that

$$\forall y \in \mathbb{R}, \quad u'(y) = 0 \Rightarrow u \text{ has a local extremum at } y.$$

In that case, it can be easily proved that $Y(t, y, \xi)$ is periodic in t; this follows directly from the fact that the trajectory in the phase space is closed (see [7]). Indeed, pushing a little further the analysis of the previous paragraph, we construct $z_\pm$ such that

$$\begin{gathered} |z_+ - z_-| < 2\pi, \quad z_- < z_+, \\ u(z_\pm) = \mathcal{E}, \\ z_- \leq y \leq z_+, \\ u(z) < \mathcal{E} \quad \forall z \in (z_-, z_+). \end{gathered}$$

Then the particle starting from y with initial speed $-\xi$ reaches either z_+ or z_- in finite time; the speed of the particle is 0 at that moment since

$$|\dot{Y}|^2 = 2(\mathcal{E} - u(Y)),$$

but its acceleration is $-u'(z_\pm) \neq 0$, so the particle turns around and goes back in the reverse direction. It then reaches the other extremity of the interval (z_-, z_+) in finite time, and the same phenomena occurs. Hence after a finite time t_0, the particle is back at its starting point y with the same speed $-\xi$. Consequently, the movement of the particle is periodic in time with period t_0. Thus, we have

$$\mathbb{P}(f)(y, \xi) = \frac{1}{t_0} \int_{t_1}^{t_1+t_0} f(Y(t, y, \xi), \Xi(t, y, \xi))\, dt,$$

where $t_1 \geq 0$ is arbitrary. It is convenient to choose for t_1 the first time when the particle hits z_-. In that case, it is easily seen that t_0 is twice the time it takes to the particle to go from z_- to z_+, so that

$$\frac{t_0}{2} = \int_{t_1}^{t_1+t_0/2} dt = \int_{z_-}^{z_+} \frac{1}{\sqrt{2\,(\mathcal{E} - u(s))}} ds = \left\langle \mathbf{1}_{u<\mathcal{E}} \frac{1}{\sqrt{2\,(\mathcal{E} - u)}} \right\rangle$$

and

$$\begin{aligned} \int_{t_1}^{t_1+\frac{t_0}{2}} f(Y(t, y, \xi), \Xi(t, y, \xi))\, dt &= \left\langle \mathbf{1}_{u<\mathcal{E}} f(s, -\sqrt{2\,(\mathcal{E} - u)}) \frac{1}{\sqrt{2\,(\mathcal{E} - u)}} \right\rangle, \\ \int_{t_1+\frac{t_0}{2}}^{t_1+t_0} f(Y(t, y, \xi), \Xi(t, y, \xi))\, dt &= \left\langle \mathbf{1}_{u<\mathcal{E}} f(s, \sqrt{2\,(\mathcal{E} - u)}) \frac{1}{\sqrt{2\,(\mathcal{E} - u)}} \right\rangle. \end{aligned}$$

Gathering all the terms, we are led to

$$\mathbb{P}(f)(y, \xi) = \frac{\left\langle \mathbf{1}_{u<\mathcal{E}} \left[f\left(\cdot, \sqrt{2\,(\mathcal{E} - u)}\right) + f\left(\cdot, -\sqrt{2\,(\mathcal{E} - u)}\right) \right] \frac{1}{\sqrt{(\mathcal{E} - u)}} \right\rangle}{2 \left\langle \mathbf{1}_{u<\mathcal{E}} \frac{1}{\sqrt{(\mathcal{E} - u)}} \right\rangle} \tag{6.18}$$

Expressions (6.17) and (6.18) are compatible with the ones in [33].

Let us now come back to the case when $N > 1$, and take a function $\varphi(y,\xi) = \varphi_1(y_1,\xi_1)\cdots\varphi_N(y_N,\xi_N)$, where each φ_i is periodic with period 1. We want to compute the limit

$$\frac{1}{T}\int_0^T \varphi_1(Y_1(t,y_1,\xi_1),\Xi_1(t,y_1,\xi_1))\cdots\varphi_N(Y_N(t,y_N,\xi_N),\Xi_N(t,y_N,\xi_N))\,dt.$$

In general, knowing the behavior of each trajectory (Y_i,Ξ_i) independently is not enough to compute such a product. However, here, we recall that each function $\varphi_i(Y_i(t,y_i,\xi_i),\Xi_i(t,y_i,\xi_i))$ $(1\le i\le N)$ is periodic in time. The period depends only on $H_i(y_i,\xi_i)$ and on the function u_i. More precisely, setting

$$T_i(\mathcal{E}) := \sqrt{2}\int_0^1 \mathbf{1}_{u_i(z)<\mathcal{E}}\frac{1}{\sqrt{\mathcal{E}-u_i(z)}}\,dz \quad \forall\mathcal{E}>0, \mathcal{E}\neq u_{\max},$$

$\varphi_i(Y_i(t,y_i,\xi_i),\Xi_i(t,y_i,\xi_i))$ is periodic in time with period $T_i(H_i(y_i,\xi_i))$.

We can thus use the following result :

Lemma 6.4.1. *Let $f_1,\cdots,f_N\in L^\infty(\mathbb{R})$ such that f_i is periodic with period θ_i, and set $\langle f_i\rangle = \frac{1}{\theta_i}\int_0^{\theta_i} f_i$.*

Assume that

$$\frac{k_1}{\theta_1}+\cdots+\frac{k_N}{\theta_N}\neq 0 \quad \forall(k_1,\cdots,k_N)\in\mathbb{Z}^N\setminus\{0\}. \tag{6.19}$$

Then as $T\to\infty$,

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\,dt \to \langle f_1\rangle\cdots\langle f_N\rangle\,.$$

Sketch of proof. By density, it is enough to prove the lemma for $f_1,\cdot,f_N\in\mathcal{C}^\infty(\mathbb{R})$. Write f_i as a Fourier series (the series converges thanks to the regularity assumption), and use the fact that for all $\alpha\neq 0$,

$$\frac{1}{T}\int_0^T e^{i\alpha t}\,dt\to 0 \quad \text{as } T\to\infty.$$

□

In the present setting, we deduce the following result :

Proposition 6.4.1. *Let $\varphi:(y,\xi)\mapsto\varphi_1(y_1,\xi_1)\cdots\varphi_N(y_N,\xi_N)$, where $\varphi_i\in L^\infty_{per}(\mathbb{R}_y\times\mathbb{R}^\xi)$.*

Let $(y,\xi)\in[0,1)^N\times\mathbb{R}^N$, and let $\theta_i=\theta_i(y_i,\xi_i)=T_i(H_i(y_i,\xi_i))$ for $1\le i\le N$. Assume that $(\theta_1,\cdots,\theta_N)$ satisfy condition (6.19)*. Then*

$$\mathbb{P}(\varphi)(y,\xi)=\mathbb{P}_1(\varphi_1)(y_1,\xi_1)\cdots\mathbb{P}_N(\varphi_N)(y_N,\xi_N) \tag{6.20}$$

where each $\mathbb{P}_i$ is the projection in dimension 1 with potential u_i, given by expressions (6.17) *and* (6.18)*.*

In particular, when the set

$$\{(y,\xi) \in [0,1]^N \times \mathbb{R}^N;\ (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N)) \text{ satisfy condition (6.19)}\}$$

has zero Lebesgue measure, equality (6.20) holds almost everywhere. It can then be generalized to arbitrary functions $\varphi \in L^\infty_{\text{per}}(\mathbb{R}^N \times \mathbb{R}^N)$ (always by linearity and density). The correct expression of the projection $\mathbb{P}$ is then

$$\mathbb{P} = \mathbb{P}_1 \circ \mathbb{P}_2 \circ \cdots \circ \mathbb{P}_N, \tag{6.21}$$

where each projection $\mathbb{P}_i$ acts on the variables (y_i, ξ_i) only. Notice that all projections $\mathbb{P}_i$ thus commute with one another; hence the order in which they are taken is unimportant.

We wish to emphasize that on the open set

$$\{(y,\xi) \in \mathbb{R}^{2N}, \forall i \in \{1,\cdots,N\}\ H_i(y_i,\xi_i) > \max u_i\},$$

the expression (6.21) is true. Indeed, for $k \in \mathbb{Z}^N \setminus \{0\}$, set

$$A_k := \{t \in \mathbb{R}^N; k_1 t_1 + \cdots + k_N t_N = 0\}.$$

Then A_k is a hyperplane, and we have

$$\begin{aligned}&\big\{(y,\xi) \in \mathbb{R}^{2N}, H_i(y_i,\xi_i) > \max u_i\ \forall i \text{ and } (\theta_1,\cdots,\theta_N) \text{ satisfy (6.19)}\big\}\\ =& \bigcup_{k\in\mathbb{Z}^N\setminus\{0\}} \big\{(y,\xi),(y,\xi) \in \mathbb{R}^{2N}, H_i(y_i,\xi_i) > \max u_i\ \forall i \text{ and } (\theta_1,\cdots,\theta_N) \in A_k\big\}.\end{aligned}$$

Above, we have written θ_i instead of $\theta_i(y_i,\xi_i)$ in order to shorten the notation. Hence it suffices to prove that if A is any hyperplane in $\mathbb{R}^N$,

$$\big|\big\{(y,\xi),(y,\xi) \in \mathbb{R}^{2N}, H_i(y_i,\xi_i) > \max u_i\ \forall i \text{ and } (\theta_1,\cdots,\theta_N) \in A\big\}\big| = 0.$$

Without any loss of generality, assume that

$$A = \{t \in \mathbb{R}^N,\ a_1 t_1 + \cdots + a_N t_N = 0\}, \quad \text{with } a_1 \neq 0.$$

Then, using the fact that T_1 is strictly nonincreasing on $(\max u_i, +\infty)$, we can find a open set $C \subset Y \times \mathbb{R}^{N-1}$ and a continuous function $\chi : C \to \mathbb{R}$ such that

$$\begin{aligned}&\big\{(y,\xi),(y,\xi) \in \mathbb{R}^{2N}, H_i(y_i,\xi_i) > \max u_i\ \forall i \text{ and } (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N)) \in A\big\}\\ &= \{(y,\chi(y,\xi'),\xi'),\quad (y,\xi') \in C\} \ \cup\ \{(y,-\chi(y,\xi'),\xi'),\quad (y,\xi') \in C\}.\end{aligned}$$

Above, the set $C \subset Y \times \mathbb{R}^{N-1}$ is defined by

$$\begin{aligned}C := \Big\{&(y,\xi') \in Y \times \mathbb{R}^{N-1}, H_i(y_i,\xi_i) > \max u_i,\ i \geq 2 \text{ and}\\ &\frac{-1}{a_1}\left(a_2\theta_2(y_2,\xi_2) + \cdots + a_N\theta_N(y_N,\xi_n)\right) > 0\Big\},\end{aligned}$$

and the function χ is defined on C by

$$\chi(y_1,\cdots,y_N,\xi_2,\cdots,\xi_N)=$$
$$=\sqrt{2\left[T_1^{-1}\left(\frac{-1}{a_1}\left(a_2\theta_2(y_2,\xi_2)+\cdots+a_N\theta_N(y_N,\xi_n)\right)\right)-u_1(y_1)\right]},$$

where $T_1^{-1}:(0,\infty)\to(\max u_i,+\infty)$ is the inverse function of T_1.

Since χ is a continuous function, the set

$$\{(y,\chi(y,\xi'),\xi'),\quad (y,\xi')\in C\}$$

has zero Lebesgue measure in $\mathbb{R}^{2N}$ (it is the graph of a continuous curve).

As a consequence, the set

$$\{(y,\xi)\in\mathbb{R}^{2N}, H_i(y_i,\xi_i)>\max u_i\ \forall i \text{ and } (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N)) \text{ satisfy } (6.19)\}$$

has zero Lebesgue measure.

However, let us mention here that in general, condition (6.19) cannot be relaxed : indeed, assume for instance that $u_i=u_j:=u$ for $i\neq j$ and assume that the function u is such that

$$\exists y_0>0,\quad u(y)=y^2 \text{ for } |y|<y_0,$$

and $u(y)>y_0^2$ if $y\in[-\frac{1}{2},\frac{1}{2}]\setminus[-y_0,y_0]$.

Then if $|\mathcal{E}|\leq\sqrt{y_0}$, we have

$$T(\mathcal{E})=\int_{-\sqrt{\mathcal{E}}}^{\sqrt{\mathcal{E}}}\frac{1}{\sqrt{\mathcal{E}-y^2}}\,dy=2\int_0^1\frac{1}{\sqrt{1-z^2}}\,dz=:T_0$$

Thus, if $H_i(y_i,\xi_i)\leq\sqrt{y_0}$, then $(Y_i,\Xi_i)(t,y_i,\xi_i)$ is periodic with period T_0. Notice that T_0 does not depend on the energy $H_i(y_i,\xi_i)$

In that case, the function $\varphi(Y(t),\Xi(t))$ is also periodic with period T_0. Thus we have to compute the limit of

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\,dt$$

as $T\to\infty$, where the f_i are arbitrary functions with period T_0. It is then easily proved that

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\,dt\to\sum_{\substack{k\in\mathbb{Z}^N,\\ k_1+\cdots+k_N=0}} a_{1,k_1}\cdots a_{N,k_N}\tag{6.22}$$

where

$$a_{j,l}=\frac{1}{T_0}\int_0^{T_0}f_j(t)e^{-\frac{2il\pi t}{T_0}}\,dt,\quad 1\leq j\leq N,\ l\in\mathbb{Z}.$$

In general, the right-hand side of (6.22) differs from $a_{1,0}\cdots a_{N,0}$, and thus

$$\mathbb{P}\neq\mathbb{P}_1\circ\cdots\circ\mathbb{P}_N$$

for (y,ξ) in a neighbourhood of the origin.

In this regard, let us mention that we believe that there is a slight misprint in [33] concerning the expression of the projection $\mathbb{P}$ when $N=2$ and for low energies. Indeed, when $\mathcal{E}_1 := H_1(y_1,\xi_1) < \max u_1$ and $\mathcal{E}_2 := H(y_2,\xi_2) < u_2$, it is stated in [33] that

$$\begin{aligned}\mathbb{P}(f)(y,\xi) = \frac{1}{2}\int d\nu \Big\{ & f(z_1, z_2, \operatorname{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, \operatorname{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, -\operatorname{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, -\operatorname{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)}\Big\}\end{aligned}$$

where

$$d\nu = d\nu(z_1,z_2) = \frac{1}{C}\mathbf{1}_{u_1(z_1)<\mathcal{E}_1}\frac{1}{\sqrt{\mathcal{E}_1-u_1(z_1)}}\mathbf{1}_{u_2(z_2)<\mathcal{E}_2}\frac{1}{\sqrt{\mathcal{E}_2-u_i(z_2)}}\,dz_1dz_2,$$

and the constant C is such that ν is a probability measure on $[0,1]^2$.

When $\theta_1(y_1,\xi_1)$, $\theta_2(y_2,\xi_2)$ satisfy (6.19), then $\mathbb{P}(f)(y,\xi) = \mathbb{P}_1\circ\mathbb{P}_2(f)(y,\xi)$, and thus in that case, the correct expression is rather

$$\begin{aligned}\mathbb{P}(f)(y,\xi) = \frac{1}{4}\int d\nu \Big\{ & f(z_1, z_2, \operatorname{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, \operatorname{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, \operatorname{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, -\operatorname{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, -\operatorname{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, \operatorname{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)} \\ & + f(z_1, z_2, -\operatorname{sgn}(\xi_1)\sqrt{2(\mathcal{E}_1 - u_1(z_1)}, -\operatorname{sgn}(\xi_2)\sqrt{2(\mathcal{E}_2 - u_2(z_2)}\Big\}.\end{aligned}$$

Let us give an explicit example where $\theta_1(y_1,\xi_1), \theta_2(y_2,\xi_2)$ satisfy (6.19) and $\mathcal{E}_1 < \max u_1$, $\mathcal{E}_2 < \max u_2$. Assume that for $i=1,2$, there exists a_i, y_i^0 such that

$$u_i(y_i) = a_i\,|y_i|^2 \quad \forall |y_i| < y_i^0, \quad \text{and} \quad \left(\frac{a_1}{a_2}\right)^2 \notin \mathbb{Q},$$

and $u_i(y_i) > a_i\,|y_i^0|^2$ if $y_i^0 < |y_i| < 1/2$. Indeed, in that case, $T_i(\mathcal{E}_i) = T_0/\sqrt{a_i}$ if $\mathcal{E}_i < a_i\,|y_i^0|^2$, and the condition $a_1^2/a_2^2 \notin \mathbb{Q}$ ensures that $\theta_1(y_1,\xi_1), \theta_2(y_2,\xi_2)$ satisfy (6.19) for y,ξ in a neighbourhood of the origin.

Thus the expression of [33] is false in that case. In the general case, it is unclear whether a general expression of this kind can be given, considering the discussion around the hypothesis (6.19) above. Nonetheless, we emphasize that this mistake is of no consequence on the rest of the article [33], and that all the other expressions are compatible with ours.

6.4.2 Stationary ergodic setting

In the stationary ergodic setting, some of the expressions or properties above are no longer true. The most significant difference occurs when the energy $H(y,\xi) <$

$u_{\max}$; indeed, in that case the particle is not necessarily trapped, depending on the profile of the potential u. Hence, in the rest of the subsection, we focus on the case $H(y,\xi) > u_{\max}$. In that case, the movement of the particle is unbounded and has many similarities with the periodic case. In particular, the particle sees "all the potential" during its evolution, and this will be fundamental in the use of the ergodic theorem.

A Expression of $\xi^\sharp(y,\xi,\omega)$

This paragraph is devoted to the proof of proposition 6.1.3 in the stationary ergodic setting.

We wish to point out that the expressions in the periodic and in the stationary ergodic case when $H(y,\xi,\omega) > u_{\max}$ are exactly the same (compare proposition 6.1.3 and the end of paragraph A). This expression, and more precisely, the equality $\xi^\sharp = \bar{H}'(P)$ for some P, is in fact strongly linked to Aubry-Mather theory. Indeed,

$$\xi^\sharp(y,\xi,\omega) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Xi(t,y,\xi,\omega)\,dt = -\lim_{T\to\infty} \frac{Y(T,y,\xi,\omega)-y}{T},$$

and $\xi^\sharp(y,\xi,\omega)$ is thus (up to a multiplication by -1) the rotation number associated to the Hamiltonian flow starting at (y,ξ). The interested reader should compare our proposition 6.1.3 to lemma 2.8 in [26] or theorem 4.1 in [30], and our proof to the ones in these articles, and also to the proofs in [51]. We refer to [26, 30, 51] for further references to Aubry-Mather theory and its applications to partial differential equations.

Proof of proposition 6.1.3. In all the proof, we fix y,ξ,ω such that $H(y,\xi,\omega) > u_{\max}$, and we set $P = P(y,\xi,\omega)$.

The proof is in two steps : first, we prove that

$$\mathbb{P}(L)(y,\xi,\omega) \geq \bar{L}\left(\xi^\sharp(y,\xi,\omega)\right), \tag{6.23}$$

which is equivalent to

$$\mathbb{P}(L)(y,\xi,\omega) \geq Q\xi^\sharp(y,\xi,\omega) - \bar{H}(Q) \quad \forall Q\in\mathbb{R},$$

and then we exhibit a particular $Q\in\mathbb{R}$ such that equality holds in the previous inequality.

The proof of (6.23) relies on the following definition of the homogenized Lagrangian (see [67]) : for all $q\in\mathbb{R}$,

$$\bar{L}(q) = \lim_{T\to\infty}\frac{1}{T}\inf\left\{\int_0^T L(\gamma(s),-\dot\gamma(s),\omega)\,ds,\ \gamma\in W^{1,\infty},\ \gamma(0)=0,\ \gamma(T)=Tq\right\}.$$

The clue of inequality (6.23) lies in the following remark : since $Y(T)/T \to -\xi^\sharp$ as $T\to\infty$, we could "almost" choose $\gamma = Y$ in the above definition in order to obtain an upper-bound on $\bar{L}(-\xi^\sharp) = \bar{L}(\xi^\sharp)$. Thus we define a function γ which coincides with Y on a large part of the interval $(0,T)$.

Let $T > 0$ arbitrary, and let $\lambda \in (0,1)$ be fixed. Define γ by

$$\gamma(s) = Y(s, y, \xi, \omega) \quad \text{for } 1 \le s \le \lambda T,$$
$$\gamma(0) = 0, \quad \gamma(T) = -T\xi^\sharp(y, \xi, \omega),$$
$$\gamma \text{ affine between } 0 \text{ and } 1 \text{ and between } \lambda T \text{ and } T.$$

Then as $T \to \infty$,

$$\frac{1}{T}\int_0^1 L(\gamma(s), -\dot\gamma(s), \omega)\, ds \to 0,$$

and

$$\frac{1}{T}\int_1^{\lambda T} L(\gamma(s), -\dot\gamma(s), \omega)\, ds \to \lambda \mathbb{P}(L)(y, \xi, \omega).$$

There remains to evaluate the contribution of the interval $(\lambda T, T)$. On this interval,

$$\begin{aligned}\dot\gamma(s) &= \frac{1}{T - \lambda T}\left(-T\xi^\sharp(y, \xi, \omega) - Y(\lambda T)\right)\\ &= -\frac{1}{1-\lambda}\xi^\sharp(y, \xi, \omega) - \frac{Y(\lambda T)}{\lambda T}\,\frac{\lambda}{1-\lambda}.\end{aligned}$$

Moreover, for all $(y', \xi', \omega') \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega$,

$$L(y', \xi', \omega') = \frac{1}{2}|\xi'|^2 - u(y', \omega) \le \frac{1}{2}|\xi'|^2.$$

Thus

$$\int_{\lambda T}^T L(\gamma(s), -\dot\gamma(s), \omega)\, ds \le \frac{T - \lambda T}{2}\left|-\frac{1}{1-\lambda}\xi^\sharp(y, \xi, \omega) - \frac{Y(\lambda T)}{\lambda T}\,\frac{\lambda}{1-\lambda}\right|^2.$$

We now pass to the limit as $T \to \infty$, with $\lambda \in (0,1)$ fixed; recall that as $\tau \to \infty$,

$$\frac{Y(\tau, y, \xi, \omega)}{\tau} = -\frac{1}{\tau}\left(\int_0^\tau \Xi(t, y, \xi, \omega)\, dt - y\right) \to -\xi^\sharp(y, \xi, \omega).$$

Thus, for all $\lambda \in (0,1)$,

$$\begin{aligned}\bar L\left(\xi^\sharp(y, \xi, \omega)\right) &\le \lambda\mathbb{P}(L)(y, \xi, \omega) + \frac{1-\lambda}{2}\left|-\frac{1}{1-\lambda}\xi^\sharp(y, \xi, \omega) + \frac{\lambda}{1-\lambda}\xi^\sharp(y, \xi, \omega)\right|^2\\ &\le \lambda\mathbb{P}(L)(y, \xi, \omega) + (1-\lambda)\frac{|\xi^\sharp(y, \xi, \omega)|^2}{2}.\end{aligned}$$

Now, letting $\lambda \to 1$, we obtain inequality (6.23).

In order to prove the proposition, we have to find a special $Q_0 \in \mathbb{R}$ such that

$$\mathbb{P}(L)(y, \xi, \omega) = Q_0\xi^\sharp(y, \xi, \omega) - \bar H(Q_0).$$

This will entail that

$$\mathbb{P}(L)(y, \xi, \omega) = \sup_{Q \in \mathbb{R}}\left(Q\xi^\sharp(y, \xi, \omega) - \bar H(Q)\right) = \bar L\left(\xi^\sharp(y, \xi, \omega)\right),$$

and the sup is obtained for $\xi^\sharp(y,\xi,\omega) = \bar{H}'(Q_0)$.

The proof of the equality relies on the use of the cell equation for $Q = P = P(y,\xi,\omega)$. Indeed, notice that

$$v(y,\omega) := \operatorname{sgn}(P)\int_0^y \sqrt{2(\bar{H}(P) - u(z,\omega))}\, dz - Py$$

is a viscosity solution of

$$H(y, P + \nabla_y v, \omega) = \bar{H}(P),$$

and as $y \to \infty$

$$\frac{1}{y}\int_0^y \sqrt{2(\bar{H}(P) - u(z,\omega))}\, dz \to E\left[\sqrt{2(\bar{H}(P) - U)}\right],$$

where $U(\omega) := u(0,\omega)$ for all $\omega \in \Omega$.

By definition of $\bar{H}$,

$$E\left[\sqrt{2(\bar{H}(P) - U)}\right] = |P|;$$

consequently,

$$\frac{1}{1+|y|}\left(\operatorname{sgn}(P)\int_0^y \sqrt{2(\bar{H}(P) - u(z,\omega))}\, dz - Py\right) \to 0 \tag{6.24}$$

as $y \to \infty$, a.s. in ω. Thus v is a corrector, and $v \in L^\infty(\Omega; \mathcal{C}^1(\mathbb{R}^N))$. Thus the method of characteristics, for instance, can be used to prove that for any couple $(y',\xi') = (y', P + \nabla_y v(y',\omega))$, we have

$$\begin{aligned} v(y',\omega) \;&=\; v(Y(T,y',\xi',\omega)) + \bar{H}(P)T \\ &\quad + \int_0^T L(Y(t,y',\xi',\omega), \Xi(t,y',\xi',\omega),\omega)\, dt + P\left[Y(T,y',\xi',\omega) - y'\right]. \end{aligned} \tag{6.25}$$

Before passing to the limit in the above equality, let us prove that we can take $(y',\xi') = (y,\xi)$. First, notice that

$$\nabla_y v(y',\omega) = \operatorname{sgn}(P)\sqrt{2(\bar{H}(P) - u(y',\omega))} - P,$$

and thus $\operatorname{sgn}\left(P + \nabla_y v(y',\omega)\right) = \operatorname{sgn}(\xi') = \operatorname{sgn}(P) = \operatorname{sgn}(\xi)$. Hence, take $y' = y$. Then $|\xi|^2 = |\xi'|^2$ because $H(y,\xi) = \bar{H}(P) = H(y,\xi')$ by definition of ξ'. Thus $\xi = \xi'$, and we can take $(y',\xi') = (y,\xi)$ in (6.25).

Now, we multiply (6.25) by $1/T$, and pass to the limit as $T \to \infty$. Since

$$\frac{1}{2}|\dot{Y}(t,y,\xi,\omega)| = H(y,\xi,\omega) - u \geq H(y,\xi,\omega) - u_{\max} > 0 \quad \forall t > 0,$$

there exist constants $\alpha, \beta > 0$ depending only on $H(y,\xi,\omega)$ and $u_{\max}$, such that

$$0 < \alpha \leq \left|\frac{Y(T,y,\xi,\omega) - y}{T}\right| \leq \beta \quad \forall T > 0.$$

Consequently, $Y(T) \to \infty$ as $T \to \infty$ and

$$\frac{v(y) - v(Y(T, y, \xi, \omega))}{T} = \frac{v(y) - v(Y(T, y, \xi, \omega))}{Y(T, y, \xi, \omega) - y} \frac{Y(T, y, \xi, \omega) - y}{T} \to 0 \quad \text{as } T \to \infty.$$

(remember (6.24)).

Hence, in the limit we infer

$$\mathbb{P}(L)(y, \xi, \omega) = P\xi^\sharp(y, \xi, \omega) - \bar{H}(P),$$

and the proposition follows.

□

Remark 6.4.1. *Notice that the proof of inequality* (6.23) *does not use the fact that the system is integrable, or that* $H(y, \xi, \omega) > u_{max}$*. Thus* (6.23) *remains true for small energies, or when the system* (Y, Ξ) *is not integrable.*

B Expression of the projection $\mathbb{P}$

The same method as in the periodic case can be used in order to find the expression of the projection $\mathbb{P}$ when $H(y, \xi, \omega) =: \mathcal{E} > u_{\max}$; indeed, in that case, remember that

$$\mathbb{P}(f)(y, \xi, \omega) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f\left(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega\right)\, dt$$

and we can use the change of variables

$$dt = \frac{1}{\dot{Y}}\, dY = \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(Y, \omega))}}\, dY$$

in order to obtain

$$\begin{aligned} &\int_0^T f\left(Y(t, y, \xi, \omega), \Xi(t, y, \xi, \omega), \omega\right)\, dt \\ &= \int_y^{Y(T,y,\xi,\omega)} f\left(z, \text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z, \omega))}, \omega\right) \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z, \omega))}}\, dz. \end{aligned}$$

Since the group transformation (τ_x) is ergodic, and $Y(T) \to \infty$ as $T \to \infty$, for all $\mathcal{E} > u_{\max}$, we have, as $T \to \infty$,

$$\begin{aligned} &\frac{1}{Y(T) - y} \int_y^{Y(T,y,\xi,\omega)} f\left(z, \text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z, \omega))}, \omega\right) \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(z, \omega))}}\, dz \\ &\qquad \to E\left[F\left(\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(0, \omega))}, \omega\right) \frac{1}{-\text{sgn}(\xi)\sqrt{2(\mathcal{E} - u(0, \omega))}}\right] \end{aligned}$$

Thus, we obtain

$$\mathbb{P}(f)(y, \xi, \omega) = \xi^\sharp(y, \xi, \omega)\bar{f}(\text{sgn}(\xi), H(y, \xi, \omega)),$$

where

$$\bar{f}(\eta, \mathcal{E}) = E\left[F\left(\eta\sqrt{2(\mathcal{E} - u(0, \omega))}, \omega\right) \frac{1}{\eta\sqrt{2(\mathcal{E} - u(0, \omega))}}\right].$$

Bibliographie

[1] Radjesvarane Alexandre, *Some results in homogenization tackling memory effects*, Asymptot. Anal. **15** (1997), no. 3-4, 229–259.

[2] ———, *Asymptotic behaviour of transport equations*, Appl. Anal. **70** (1999), no. 3-4, 405–430.

[3] Grégoire Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. (1992), no. 23, 1482–1518.

[4] Debora Amadori, *On the homogenization of conservation laws with resonant oscillatory source*, Asymptot. Anal. **46** (2006), no. 1, 53–79.

[5] Debora Amadori and Denis Serre, *Asymptotic behavior of solutions to conservation laws with periodic forcing*, J. Hyperbolic Differ. Equ. **3** (2006), no. 2, 387–401.

[6] Y. Amirat, K. Hamdache, and A. Ziani, *Homogénéisation d'équations hyperboliques du premier ordre et applications aux écoulements miscibles en milieu poreux*, Annales de l'IHP (C) : Analyse non linéaire (1989), no. 6, 397–417.

[7] V. I. Arnold, *Ordinary differential equations*, Cambridge MA, MIT Press, London, 1973.

[8] Emmanuel Audusse and Benoît Perthame, *Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies*, Proc. of the Royal Soc. of Edinburgh (2005), no. 135A, 253–266.

[9] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.

[10] A. Bourgeat, A. Mikelić, and S. Wright, *Stochastic two-scale convergence in the mean and applications*, J. Reine Angew. Math. **456** (1994), 19–51.

[11] F. Castella, P. Degond, and T. Goudon, *Asymptotic problems for laser-matter modeling : quantum and classical models*, Nonlinearity (2007).

[12] Doina Cioranescu and Patrizia Donato, *An Introduction to Homogenization*, Oxford University Press, 1999.

[13] Constantine M. Dafermos, *Hyperbolic conservation laws in Continuum Physics*, Grundlehren Mathematischen Wissenschaften, no. GM 325, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[14] A.-L. Dalibard, *Homogenization of a quasilinear parabolic equation with vanishing viscosity*, J. Math. Pures et Appl. **86** (2006), 133–154.

[15] ———, *Kinetic formulation for heterogeneous scalar conservation laws*, Annales de l'IHP (C) : Analyse non linéaire **23** (2006), 475–498.

[16] ———, *Homogenization of a linear transport equation in a stationary ergodic setting*, Comm. Partial Differential Equations (2007), to appear.

[17] ———, *Homogenization of nonlinear scalar conservation laws*, Arch. Rat. Mech. Anal. (2007), to appear.

[18] ———, *Initial layer for the homogenization of a quasilinear parabolic equation with vanishing viscosity*, Arch. Rat. Mech. Anal. **185** (2007), no. 3, 515–543.

[19] ———, *Kinetic formulation for a parabolic conservation law. Application to homogenization.*, SIAM J. Math. Anal. **39** (2007), 891–915.

[20] Raoul Dautray and Jacques-Louis Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, vol. 5, Masson, 1984.

[21] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, no. 15, American Mathematical Society, Providence, R.I., 1977.

[22] Ron DiPerna, *Measure-valued solutions to conservation laws*, Arch. Rat. Mech. Anal. (1985), no. 88, 223–270.

[23] L. Dumas and F. Golse, *Homogenization of Transport Equations*, SIAM J. of Appl. Math. (2000), no. 60, 1447–1470.

[24] Weinan E, *Homogenization of linear and nonlinear transport equations*, Comm. Pure Appl. Math. (1992), no. 45, 301–326.

[25] ———, *Homogenization of scalar conservation laws with oscillatory forcing terms*, SIAM J. Appl. Math. **4** (1992), no. 52, 959–972.

[26] ———, *Aubry-Mather theory and periodic solutions of the forced Burger's equation*, Comm Pure Applied Math. (1999), no. 52, 811–828.

[27] Weinan E and Denis Serre, *Correctors for the homogenization of conservation laws with oscillatory forcing terms*, Asymptotic Analysis (1992), no. 5, 311–316.

[28] Bjorn Engquist and Weinan E, *Large time behavior and homogenization of solutions of two-dimensional conservation laws*, Comm. Pure Appl. Math (1993), no. 46, 1–26.

[29] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDEs*, Proc. Roy. Soc. Edinburgh Sect. A (1992), no. 120, 245–265.

[30] L. C. Evans and D. Gomes, *Effective Hamiltonians and averaging for Hamiltonian dynamics*, Arch. Rat. Mech. Anal (2001), no. 157, 1–33.

[31] L. C. Evans and D. Gomes, *Effective Hamiltonians and averaging for Hamiltonian dynamics. II*, Arch. Ration. Mech. Anal. **161** (2002), no. 4, 271–305.

[32] L.C. Evans, *Partial differential equations*, Graduate studies in mathematics, vol. 19, AMS, 1998.

[33] E. Frenod and K. Hamdache, *Homogenisation of transport kinetic equations with oscillating potentials*, Proc. of the Royal Soc. of Edinburgh (1996), no. 126A, 1247–1275.

[34] David Gilbarg and Neil Trudinger, *Elliptic partial differential equations of second order*, Springer, 1983, second edition.

[35] F. Golse, *Remarques sur l'homogénéisation des équations de transport*, C. R. Acad. Sci. Paris (1987), no. 305, 801–804, Ser. I.

[36] Thierry Goudon and Antoine Mellet, *Diffusion approximation in heterogeneous media*, Asymptot. Anal. **28** (2001), no. 3-4, 331–358.

[37] ———, *Homogenization and diffusion asymptotics of the linear Boltzmann equation*, ESAIM Control Optim. Calc. Var. **9** (2003), 371–398.

[38] Thierry Goudon and Frédéric Poupaud, *Homogenization of transport equations : weak mean field approximation*, SIAM J. Math. Anal. **36** (2004/05), no. 3, 856–881.

[39] ———, *Homogenization of transport equations : a simple PDE approach to the Kubo formula*, Bull. Sci. Math. **131** (2007), no. 1, 72–88.

[40] P. Gérard, *Mesures semi-classiques et ondes de Bloch*, Séminaire Équations aux dérivées partielles (1990-1991), no. 16, Ecole Polytechnique, Palaiseau.

[41] Paul R. Halmos, *Lectures on ergodic theory*, Chelsea Publishing Co., New York, 1960.

[42] Thomas Y. Hou and Xue Xin, *Homogenization of linear transport equations with oscillatory vector fields*, SIAM J. Appl. Math. **52** (1992), no. 1, 34–45.

[43] P.E. Jabin and A. Tzavaras, *Kinetic decomposition of homogenization problems*, preprint, 2006.

[44] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, Heidelberg, 1994.

[45] M. Krein and M. Rutman, *Linear operator leaving invariant a cone in a Banach space*, AMS Transl. (1962), no. 10, 199–325.

[46] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic type*, American Mathematical Society, 1968.

[47] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, 1968.

[48] P. D. Lax, *Hyperbolic systems of conservation laws. ii*, Comm. Pure Appl. Math. (1957), no. 10, 537–566.

[49] P.-L. Lions, G. Papanicolaou, and S. R. S. Varadhan, *Homogenization of Hamilton-Jacobi equations*, unpublished, 1987.

[50] P.-L. Lions and T. Paul, *Sur les mesures de Wigner*, Rev. Mat. Iberoamericana (1993), no. 9, 553–618.

[51] P.-L. Lions and P. E. Souganidis, *Correctors for the homogenization theory of Hamilton-Jacobi equations in the stationary ergodic setting*, Comm Pure Applied Math. (2003), no. 61, 1501–1524.

[52] Pierre-Louis Lions, Benoît Perthame, and Eitan Tadmor, *Formulation cinétique des lois de conservation scalaires multidimensionnelles*, C.R. Acad. Sci. Paris (1991), no. 312, 97–102, Série I.

[53] ———, *A kinetic formulation of multidimensional conservation laws and related equations*, J. Amer. Math. Soc. (1994), no. 7, 169–191.

[54] A. Mellet and A. Vasseur, *Homogenization of a nonlinear transport equation*, Asymptot. Anal. **51** (2007), no. 2, 157–166.

[55] Philippe Michel, Stéphane Mischler, and Benoît Perthame, *General relative entropy inequality : an illustration on growth models*, J. Math. Pures et Appl. (2005), no. 84, 1235–1260.

[56] Gabriel N'Guetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. (1989), no. 20, 608–623.

[57] G. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Rigorous results in Statistical Mechanics and Quantum Field Theory (J. Fritz, J. L. Lebaritz, and D. Szasz, eds.), Proc. Colloq. Random Fields, vol. 10, Coll. Math. Soc. Janos Bolyai, 1979, pp. 835–873.

[58] Benoît Perthame, *Uniqueness and error estimates in first order quasilinear conservation laws via the kinetic entropy defect measure*, J. Math. Pures et Appl. (1998), no. 77, 1055–1064.

[59] ______, *Kinetic formulation of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, no. 21, Oxford University Press, 2002.

[60] Benoît Perthame and Eitan Tadmor, *A kinetic equation with kinetic entropy functions for scalar conservation laws*, Comm. Math. Phys. (1991), no. 136, 501–517.

[61] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Springer, 1984.

[62] Denis Serre, *Systems of conservation laws. 1. hyperbolicity, entropies, shock waves*, Cambridge University Press, Cambridge.

[63] ______, *Systems of conservation laws. 2. geometric structures, oscillations, and initial-boundary value problems*, Cambridge University Press, Cambridge.

[64] ______, *Systèmes de lois de conservation I et II*, Diderot Editeur Arts et Sciences, 1996.

[65] Denis Serre, *L^1-stability of nonlinear waves in scalar conservation laws*, Evolutionary equations. Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004, pp. 473–553.

[66] Ya. G. Sinai, *Introduction to ergodic theory*, Princeton University Press, Princeton, N.J., 1976, Translated by V. Scheffer, Mathematical Notes, 18.

[67] Panagiotis E. Souganidis, *Stochastic homogenization of Hamilton-Jacobi equations and some applications*, Asymptot. Anal. **20** (1999), no. 1, 1–11.

[68] Luc Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear Analysis and Mechanics : Heriot-Watt Symposium (London), Research Notes in Mathematics, vol. 4, Pitman, 1979, pp. 136–212.

[69] S. N. Kružkov, *Generalized solutions of the Cauchy problem in the large for nonlinear equations of first order*, Soviet Math. Dokl. (1969), no. 10, 785–788.

[70] ______, *First order quasilinear equations in several independent variables*, Math. USSR Sb. (1970), no. 10, 217–243.

Résumé. Cette thèse est consacrée à l'étude du comportement asymptotique de solutions d'une classe d'équations aux dérivées partielles avec des coefficients fortement oscillants. Dans un premier temps, on s'intéresse à une famille d'équations non linéaires, des lois de conservation scalaires hétérogènes, qui interviennent dans divers problèmes de la mécanique des fluides ou de l'électromagnétisme non linéaire. On suppose que le flux de cette équation est périodique en espace, et que la période des oscillations tend vers zéro. On identifie alors les profils asymptotiques microscopique et macroscopique de la solution, et on démontre un résultat de convergence forte ; en particulier, on montre que lorsque la condition initiale ne suit pas le profil microscopique dicté par l'équation, il se forme une couche initiale en temps durant laquelle les solutions s'adaptent à celui-ci. Dans un second temps, on considère une équation de transport linéaire, qui modélise l'évolution de la densité d'un ensemble de particules chargées dans un potentiel électrique aléatoire et très oscillant. On établit l'apparition d'oscillations microscopiques en temps et en espace dans la densité, en réponse à l'excitation par le potentiel électrique. On donne également des formules explicites pour l'opérateur de transport homogénéisé lorsque la dimension de l'espace est égale à un.

Mots clés. Homogénéisation. Loi de conservation scalaire. Formulation cinétique. Équation de transport. Équation parabolique.

Abstract. In this thesis, we study the asymptotic behavior of solutions of a class of partial differential equations with strongly oscillating coefficients. First, we focus on a family of nonlinear evolution equations, namely parabolic scalar conservation laws. These equations are encountered in various problems of fluid mechanics and nonlinear electromagnetism. The flux is assumed to be periodic with respect to the space variable, and the period of the oscillations goes to zero. The asymptotic profiles in the microscopic and macroscopic variables are first identified. Then, we prove a result of strong convergence; in particular, when the initial data does not match the microscopic outline dictated by the equation, it is shown that there is an initial layer in time during which the solution adapts itself to this profile. The other equation studied in this thesis is a linear transport equation, modeling the evolution of the density of charged particles in a highly oscillating random electric potential. It is proved that the density has fast oscillations in time and space, as a response to the excitation by the electric potential. We also derive explicit formulas for the homogenized transport operator when the space dimension is equal to one.

Keywords. Homogenization. Scalar conservation law. Kinetic formulation. Transport equation. Parabolic equation.