Energy equality and weak-strong uniqueness in dimension 3

The purpose of this document is to provide some guidelines for the proof of the energy equality satisfied by the strong solutions (of Fujita-Kato type) of the Navier-Stokes system in dimension 3, and for the rigorous justification of the weak-strong uniqueness principle.

Useful facts/results :

— If $f \in L^2(\mathbb{R}^3)$, then $\nabla f \in H^{-1}(\mathbb{R}^3)$ and

$$\|\nabla f\|_{H^{-1}(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)}$$

— Sobolev embedding : $\dot{H}^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$

1 About the Stokes system

In this section, we work in dimension N, where $N = 2$ or 3. We consider the Stokes system

$$\begin{aligned} \partial_t u + \nabla p - \Delta u &= f, \\ \text{div } u &= 0, \\ u_{|t=0} &= u_0, \end{aligned} \tag{1}$$

with $u_0 \in L^2(\mathbb{R}^N)$ divergence free, $f \in L^2([0,T], H^{-1}(\mathbb{R}^N))$.

1. Prove that if a solution u of (1) exists in $L^\infty([0,T], L^2(\mathbb{R}^N)) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^N))$ then u must satisfy, for a.e. $t \in [0,T]$

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{u}_0(\xi) + \int_0^t e^{-(t-s)|\xi|^2} M(\xi)\hat{f}(s,\xi)\, ds, \tag{2}$$

where $M(\xi) = \text{Id} - (\xi \otimes \xi)/|\xi|^2$. In particular, if a solution exists, it is necessarily unique.

2. Now, for $u_0 \in L^2(\mathbb{R}^N)$ such that div $u_0 = 0$, and for $f \in L^2([0,T], H^{-1}(\mathbb{R}^N))$, define u by (2). Check that $u \in L^\infty([0,T], L^2(\mathbb{R}^N)) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^N))$ and that

$$\|u\|_{L^\infty([0,T],L^2(\mathbb{R}^N))} + \|\nabla u\|_{L^2([0,T]\times\mathbb{R}^N)} \leq C\left(\|u_0\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^2([0,T],H^{-1}(\mathbb{R}^N))}\right).$$

3. Let $u \in L^\infty([0,T], L^2(\mathbb{R}^N)) \cap L^2([0,T], \dot{H}^1(\mathbb{R}^N))$ be the unique solution of the Stokes system (1). Show that $\partial_t u \in L^2([0,T], H^{-1})$. Using the identity

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^N)} = 2\langle \partial_t u(t), u(t)\rangle_{H^{-1},H^1},$$

prove that u satisfies the energy equality

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} + 2\int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{R}^N)}\, ds = \|u_0\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \langle f(s), u(s)\rangle_{H^{-1},H^1}\, ds. \tag{3}$$

2 Energy equality

The goal of this section is to prove the following

Proposition 1. *Let $u_0 \in L^2(\mathbb{R}^3)$ be divergence free, and let $u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ be a Leary solution of the Navier-Stokes system. Assume that $u \in L^4([0,T], \dot{H}^1(\mathbb{R}^3))$. Then for all $t \in [0,T]$, u satisfies the energy equality*

$$\|u(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2(\mathbb{R}^3)}^2 \, ds = \|u_0\|_{L^2(\mathbb{R}^3)}^2. \tag{4}$$

Throughout this section, C denotes a universal constant, which may change from line to line.

1. Let $v \in L^\infty([0,T], L^2(\mathbb{R}^3)) \cap L^4([0,T], \dot{H}^1(\mathbb{R}^3))$. Prove that for a.e. $t \in [0,T]$,

$$\|v(t) \otimes v(t)\|_{L^2(\mathbb{R}^3)} \le C \|v(t)\|_{L^2(\mathbb{R}^3)}^{1/2} \|v(t)\|_{\dot{H}^1(\mathbb{R}^3)}^{3/2},$$

   and deduce that

$$\|\text{div } (v \otimes v)\|_{L^2([0,T], H^{-1}(\mathbb{R}^3))} \le C T^{1/8} \|v\|_{L^\infty([0,T], L^2(\mathbb{R}^3))} \|v\|_{L^4([0,T], \dot{H}^1(\mathbb{R}^3))}^{3/2}$$

2. (a) Let $\psi, \phi \in \mathcal{C}^\infty \cap H^1(\mathbb{R}^3)^3$ be divergence free. Consider a cut-off $\chi_R := \chi(\cdot/R)$, where $R > 0$, $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ is such that $\chi(z) = 1$ if $|z| \le 1$. Prove that

$$\int_{\mathbb{R}^3} \chi_R \psi \otimes \phi : \nabla \phi = -\frac{1}{2} \int_{\mathbb{R}^3} (\nabla \chi_R \cdot \psi) |\phi|^2.$$

   Passing to the limit as $R \to \infty$, deduce that

$$\int_{\mathbb{R}^3} \psi \otimes \phi : \nabla \phi = 0. \tag{5}$$

   (b) Let $\psi, \phi \in H^1(\mathbb{R}^3)^3$ be divergence free. Using a regularization by convolution, prove that (5) is still satisfied.

3. Let $u \in L^\infty([0,T], L^2(\mathbb{R}^3)) \cap L^4([0,T], \dot{H}^1(\mathbb{R}^3))$ be divergence free, and set $f = -\text{div } (u \otimes u)$. Recalling that $f \in L^2([0,T], H^{-1}(\mathbb{R}^3))$, show that for almost every $t \in (0,T)$,

$$\langle f(t), u(t) \rangle_{H^{-1}, H^1} = \int_{\mathbb{R}^3} u(t) \otimes u(t) : \nabla u(t) = 0.$$

4. Using the section on the Stokes system, prove Proposition (1).

3 Weak-strong uniqueness principle

The goal of this section is to prove the following

Théorème 2. *Let u, v be two Leary solutions of the Navier-Stokes equation in dimension 3, with respective initial data u_0, v_0. Assume that $u \in L^4([0,T], \dot{H}^1(\mathbb{R}^3))$. Then for a.e. $t \in [0,T]$,*

$$\|u(t) - v(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla(u - v)\|_{L^2(\mathbb{R}^3)}^2 \le \|u_0 - v_0\|_{L^2(\mathbb{R}^3)}^2 \exp\left(C \int_0^t \|u(s)\|_{\dot{H}^1}^4 ds \right).$$

Throughout this section, we set $w = u - v$ and

$$\delta(t) := \|w(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\int_0^t \|\nabla w(s)\|_{L^2}^2 ds.$$

1. Using the fact that u and v are Leray solutions of the Navier-Stokes system, prove that

$$\delta(t) \leq \|v_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 - 2D(t),$$

where

$$D(t) = \int_{\mathbb{R}^3} u(t) \cdot v(t) + 2 \int_0^t \int_{\mathbb{R}^3} \nabla u(s) : \nabla v(s)\, ds.$$

2. Define $u_\epsilon := u *_t \rho_\epsilon$, where $\rho_\epsilon = \epsilon^{-1}\rho(\cdot/\epsilon)$, where ρ is a standard mollification kernel such that Supp $\rho \subset]-\infty, 0]$. Using u_ϵ as a test function in the weak formulation for v together with the equation on u, prove that for all $t \in [0, T]$,

$$\int_{\mathbb{R}^3} u_\epsilon(t) \cdot v(t) + 2 \int_0^t \int_{\mathbb{R}^3} \nabla v(s) : \nabla u_\epsilon(s)\, ds$$
$$= \int_{\mathbb{R}^3} v_0 u_\epsilon(0) + \int_0^t \int_{\mathbb{R}^3} (u \otimes u) * \rho_\epsilon : \nabla v + \int_0^t \int_{\mathbb{R}^3} (v \otimes v) : \nabla u_\epsilon.$$

3. Pass to the limit as $\epsilon \to 0$ and prove that

$$D(t) = \int_{\mathbb{R}^3} v_0 u_0 + \int_0^t \int_{\mathbb{R}^3} (u \otimes u) : \nabla v + \int_0^t \int_{\mathbb{R}^3} (v \otimes v) : \nabla u.$$

4. Using property (5), show that

$$D(t) = \int_{\mathbb{R}^3} v_0 u_0 + \int_0^t \int_{\mathbb{R}^3} [(w \cdot \nabla) u] \cdot w.$$

5. Deduce that

$$\delta(t) \leq \delta(0) + C \int_0^t \|\nabla w\|_{L^2(\mathbb{R}^3)}^{3/2} \|w\|_{L^2(\mathbb{R}^3)}^{1/2} \|\nabla u\|_{L^2(\mathbb{R}^3)}$$

and conclude using Young's inequality and a Gronwall argument.