The Galerkin method

October 29, 2019

Abstract

The strategy of the Galerkin method is the projection of a PDE onto a finite dimensional basis. This allows the use of finite dimensional tools (such as the Cauchy-Lipschitz theorem for ODEs) to find an sequence of approximate solutions. Here, we present this method in the case of a transport-diffusion equation in a bounded domain.

In this note, we consider the equation

$$\begin{aligned} \partial_t u + b\cdot\nabla u + cu - \Delta u = 0 \quad \text{in } (0,T)\times\Omega, u_{|\partial\Omega} = 0,\\ u_{|t=0} = u_0, \end{aligned} \tag{TD}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded $\mathcal{C}^1$ domain, $b\in L^\infty((0,T)\times\Omega)^N$, $c\in L^\infty((0,T)\times\Omega)$, and $u_0\in L^2(\Omega)$.

We introduce the following

Definition 1. A function $u\in L^2((0,T),H^1_0(\Omega))$ is a weak solution of (TD) if the following properties are satisfied:

(i) $\partial_t u\in L^2([0,T],H^{-1}(\Omega))$;

(ii) $u_{|t=0} = u_0$;

(iii) For any $v\in H^1_0(\Omega)$, for almost every $t\in(0,T)$,

$$\langle \partial_t u(t), v\rangle_{H^{-1},H^1_0} + \int_\Omega \left(b(t,x)\cdot\nabla u(t,x) + c(t,x)u(t,x)\right)v(x)\,dx + \int_\Omega \nabla u(t,x)\cdot\nabla v(x)\,dx = 0$$

Remark 2. Let $w\in L^2([0,T],H^1_0(\Omega))$ such that $\partial_t w\in L^2([0,T],H^{-1}(\Omega))$.

Then $w\in\mathcal{C}([0,T],L^2(\Omega))$ and for all $0\le s\le t\le T$,

$$\|w(t)\|^2_{L^2(\Omega)} - \|w(s)\|^2_{L^2(\Omega)} = 2\int_s^t \langle\partial_t w(\tau), w(\tau)\rangle_{H^{-1},H^1_0(\Omega)} d\tau.$$

(See Theorem 3 in Section 5.9.2 of Evans.) As a consequence, any weak solution of (TD) belongs to $\mathcal{C}([0,T],L^2(\Omega))$, and point (ii) of the above definition holds in $L^2(\Omega)$.

The goal of the note is to prove the following

Theorem 3. *Let $T>0$, $b\in L^\infty((0,T)\times\Omega)^N$, $c\in L^\infty((0,T)\times\Omega)$, and $u_0\in L^2(\Omega)$. There exists a unique weak solution of* (TD).

1 A priori estimate

Assume in this section that $u \in \mathcal{C}^1([0,T], \mathcal{C}^2(\bar{\Omega}))$ is a classical solution of (TD).

(a) Show that for all $t \in [0,T]$,

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u(t,x)|^2 dx \leq (2\|c\|_\infty + \|b\|_\infty)\, \|u(t)\|^2_{L^2(\Omega)}.$$

(b) Let $C := 2\|c\|_\infty + \|b\|_\infty$. Show that

$$\|u\|^2_{L^\infty([0,T],L^2(\Omega))} + \int_0^T \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt \leq \|u_0\|^2_{L^2(\Omega)} e^{CT}.$$

(c) Deduce that

$$\|\partial_t u\|_{L^2([0,T],H^{-1}(\Omega))} \leq \left(\|b\|_\infty + \|c\|_\infty \sqrt{T+1}\right) \|u_0\|_{L^2(\Omega)} e^{\frac{CT}{2}}.$$

2 Construction of an approximating sequence

Let $(w_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ which is orthogonal[1] in $H^1_0(\Omega)$. For any $n \in \mathbb{N}$, let $E_n := \mathrm{Vect}(w_0, \cdots, w_n)$, and let $\mathbb{P}_n$ be the orthogonal projection onto E_n in $L^2(\Omega)$

(a) Let $n \in \mathbb{N}$ be arbitrary. Show that there exists a unique $u_n \in W^{1,\infty}([0,T], E_n)$ such that for a.e. $t \in [0,T]$, for all $k \in \{0, \cdots, n\}$,

$$\int_\Omega \partial_t u_n(t,x) w_k(x)\, dx + \int_\Omega \left(b(t,x)\cdot \nabla u_n(t,x) + c(t,x) u_n(t,x)\right) w_k(x)\, dx + \int_\Omega \nabla u_n(t,x)\cdot \nabla w_k(x)\, dx = 0, \quad \text{(TDn)}$$

$$u_n(0) = \mathbb{P}_n(u_0) \in E_n.$$

(Indication: write $u_n(t) = \sum_{k=0}^n d^n_k(t) w_n$ and rewrite (TDn) as a linear ODE on the coefficients $(d^n_0, \cdots, d^n_n)$.) Is u_n a solution of (TD)? Why?

(b) Show that for all $n \in \mathbb{N}$,

$$\|u_n\|^2_{L^\infty([0,T],L^2(\Omega))} + \int_0^T \|\nabla u_n(t)\|^2_{L^2(\Omega)}\, dt \leq \|u_0\|^2_{L^2(\Omega)} e^{CT}.$$

(c) Let $v \in H^1_0(\Omega)$ be arbitrary, and let $v_n = \mathbb{P}_n v \in E_n$, $\tilde{v}_n = v - v_n$. Show that

$$\|v\|^2_{L^2} = \|v_n\|^2_{L^2} + \|\tilde{v}_n\|^2_{L^2}, \quad \|v\|^2_{H^1} = \|v_n\|^2_{H^1} + \|\tilde{v}_n\|^2_{H^1}.$$

Deduce that

$$\|v_n\|_{H^1} \leq \|v\|_{H^1}.$$

(d) Let $v \in H^1_0(\Omega)$ be arbitrary. Prove that for a.e. $t \in [0,T]$,

$$\left|\langle \partial_t u_n(t), v\rangle_{H^{-1},H^1_0}\right| \leq \|v\|_{H^1_0}\left(\|b\|_\infty \|\nabla u_n(t)\|_{L^2(\Omega)} + \|c\|_\infty \|u_n(t)\|_{L^2(\Omega)} + \|\nabla u_n(t)\|_{L^2(\Omega)}\right).$$

Deduce that

$$\|\partial_t u_n\|_{L^2([0,T],H^{-1}(\Omega))} \leq \left(\|b\|_\infty + \|c\|_\infty \sqrt{T+1}\right) \|u_0\|_{L^2(\Omega)} e^{\frac{CT}{2}}.$$

[1]Take the eigenfunctions of the laplacian.

3 Passing to the limit

(a) Show that there exists an increasing sequence of integers $(n_k)_{k\in\mathbb{N}}$ and a function $u \in L^2([0,T], H_0^1(\Omega))$ such that $\partial_t u \in L^2([0,T], H^{-1})$ and such that

$$u_{n_k} \underset{k\to\infty}{\rightharpoonup} u \quad \text{in } w - L^2([0,T], H_0^1(\Omega)),$$
$$\partial_t u_{n_k} \underset{k\to\infty}{\rightharpoonup} \partial_t u \quad \text{in } w - L^2([0,T], H^{-1}).$$

(b) Prove that u satisfies (iii) of Definition 1.

(c) By using a test function $v \in \mathcal{C}^1([0,T], E_m)$ such that $v(T) = 0$ in (TDn) (with $m \leq n$) and in (iii) of Definition 1 and performing integration by parts, show that $u(0) = u_0$.

(d) Conclude that u is a weak solution of (TD), and therefore the existence part of Theorem 3 is proved.

4 Uniqueness

Let $u \in L^2([0,T], H_0^1(\Omega))$ be a weak solution of (TD) such that $u(0) = 0$. Prove that the inequality

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} \leq C\|u(t)\|^2_{L^2(\Omega)}$$

holds for almost every $t \in [0,T]$ (both sides of the inequality belong to $L^1([0,T])$. Conclude.