CHAPTER 3: PENALIZATION PROBLEMS

Formal definition of penalization problems:
diff. eq. involving a small parameter $\varepsilon > 0$, of order $k \geq 1$ (i.e. derivatives up to order at most k) and that "converge" towards a differential equation of lower order.

<u>Example</u>:
$$\varepsilon u_\varepsilon'(x) + u_\varepsilon^2(x) = f(x)$$
$$u_\varepsilon(0) = 0$$
As $\varepsilon \to 0$, formally, we retrieve the equation $\bar{u}^2(x) = f(x)$

<u>More rigorously</u>:

<u>Definition</u>: Let $(\mathcal{L}_\varepsilon)_{\varepsilon>0}$ be a differential operator of order $k \geq 1$ involving a parameter $\varepsilon > 0$.
Assume that there exists a differential operator $\bar{\mathcal{L}}$ of order $\ell < k$ with the following property:
for any family $(u_\varepsilon)_{\varepsilon>0}$, $u_\varepsilon \in \mathcal{C}^k([a,b])$ such that

- $\sup\limits_{\varepsilon>0} \max\limits_{\substack{0 \leq j \leq k \\ x \in [a,b]}} |u_\varepsilon^j(x)| \leq C$
- $u_\varepsilon^{(j)}(x) \xrightarrow[\varepsilon \to 0]{} \bar{u}^{(j)}(x)$ (simple convergence) $\forall j \in \{0, \dots, k-1\}$

we have $\lim\limits_{\varepsilon \to 0} \mathcal{L}_\varepsilon u_\varepsilon = \bar{\mathcal{L}} \bar{u}$.
Then we say that $(\mathcal{L}_\varepsilon)_{\varepsilon>0}$ is a family of penalization operators.

Back to the example:

$$u_\varepsilon'(x) + \frac{1}{\varepsilon}\underbrace{\left(u_\varepsilon^2(x) - f(x)\right)}_{\text{penalization}} = 0$$

Questions:

- What is the expected behavior of solutions of differential equations with a penalization?
- How does it depend on the penalization?
- Difference between initial value problems and boundary value problems?
- Can methods be developed to analyze the solutions?

Schematic answers:

Generically, the penalization will create fast variations: blow-up in a very short time, initial or boundary layers, time oscillations. The nature these variations, and the way to investigate them, will depend on the type of the penalization (e.g., does the differential equation associated with the penalization increase/dissipate/conserve energy?)

In this chapter, we will consider 2 examples illustrating each of these behaviors.

I) Fast blow-up, initial layer: filtering methods

Consider the equation

$$(P_\varepsilon^\pm) \quad \begin{cases} u_\varepsilon'(x) \pm \frac{1}{\varepsilon} u_\varepsilon(x) - u_\varepsilon^2(x) = 0 \\ u_\varepsilon(0) = 1 \end{cases}$$

and the associated solution on $[0, T_\varepsilon[$.

Observation: u_ε remains positive (consequence of the C-L theorem)

Remark: In this case, explicit solutions can be computed easily... but if we change $(P_\varepsilon^\pm)$ into

$$u_\varepsilon'(x) \pm \frac{\alpha(x)}{\varepsilon} u_\varepsilon(x) + \beta(x) u_\varepsilon^q(x) = f(x)$$

this is no longer the case. Hence we rather explain some robust methods in order to study $(P_\varepsilon^\pm)$.

Heuristic analysis: the solution of $u' - u^q = 0, u(0)=1$ exhibits blow-up in finite time. On the other hand, the solutions of $v_\varepsilon'(t) \pm \frac{1}{\varepsilon} v_\varepsilon(t) = 0$ are $v_\varepsilon(t) = \exp\left(\mp \frac{t}{\varepsilon}\right)$. Therefore:

- for P_ε^+, the penalization makes the solution exponentially small, and the nonlinear term $u_\varepsilon(t)^2$ is not strong enough to create blow up
- for P_ε^-, the penalization makes the solution exponentially large, and blow-up will occur much faster.

<u>More rigorously</u>: at least at first, the penalization is expected to dominate. Hence we "filter out" the evolution associated with the penalization, and we define

$$\tilde{u}_\varepsilon(t) = u_\varepsilon(t) \overbrace{\exp\left(\pm \frac{t}{\varepsilon}\right)}^{\text{filtering}} \qquad \forall t \in [0, T_\varepsilon[$$

[Note that, if there were no term u_ε^2 in the equation, $\tilde{u}_\varepsilon$ would just be constant...]

Then

$$\tilde{u}_\varepsilon'(t) = \left[u_\varepsilon'(t) \pm \frac{1}{\varepsilon} u_\varepsilon(t)\right] \exp\left(\pm \frac{t}{\varepsilon}\right)$$

$$= u_\varepsilon^2(t) \exp\left(\pm \frac{t}{\varepsilon}\right)$$

$$\tilde{u}_\varepsilon'(t) = \tilde{u}_\varepsilon^2(t) \exp\left(\mp \frac{t}{\varepsilon}\right) \qquad (\tilde{P}_\varepsilon^\pm)$$

<u>Remark</u>: It is easy to compute the solutions of $(\tilde{P}_\varepsilon^\pm)$ explicitly. Exercise: prove that on its interval of definition, the solution of $(\tilde{P}_\varepsilon^\pm)$ with $\tilde{u}_\varepsilon(0) = 1$ is given by $\tilde{u}_\varepsilon(t) = \dfrac{1}{1 \pm \varepsilon\left(\exp\left(\mp \frac{t}{\varepsilon}\right) - 1\right)}$.

Infer that blow-up occurs for $(\tilde{P}_\varepsilon^-)$ at time

$$T_\varepsilon = \varepsilon \log\left(1 + \frac{1}{\varepsilon}\right)$$

whereas the solution of $(\tilde{P}_\varepsilon^+)$ is globally defined.

Here, the goal is to make a qualitative analysis of $(\tilde{P}_\varepsilon^\pm)$ <u>without</u> computing the solution explicitely.

- $(\tilde{P}_\varepsilon^+)$: let us first prove that for all $t \in [0, T_\varepsilon[$, if $\varepsilon \leq \frac{1}{8}$, $\quad |\tilde{u}_\varepsilon(t)| \leq 2$.

Define $\Theta_\varepsilon = \sup\{T \in [0,T_\varepsilon[\,;\ |\tilde{u}_\varepsilon(t)| \leq 2\ \forall t \in [0,T]\}$

Then since $\tilde{u}_\varepsilon(0) = 1$, $\Theta_\varepsilon > 0$.
Furthermore, by definition of Θ_ε, for all $t \in [0, \Theta_\varepsilon[$, $|\tilde{u}_\varepsilon(t)| \leq 2$
Using $(\tilde{P}_\varepsilon^+)$, we infer that

$$0 \leq \tilde{u}_\varepsilon'(t) \leq 4 e^{-\frac{t}{\varepsilon}}$$

$$0 \leq \tilde{u}_\varepsilon(t) - 1 \leq 4\varepsilon(1 - e^{-t/\varepsilon}) \quad (\exp' = \exp)$$

$$\leq 4\varepsilon$$

$$1 \leq \tilde{u}_\varepsilon(t) \leq 1 + 4\varepsilon \leq \frac{3}{2}.$$

Hence, for all $t \in [0, \Theta_\varepsilon[$, $|\tilde{u}_\varepsilon(t)| \leq \frac{3}{2}$

We deduce that $\Theta_\varepsilon = T_\varepsilon$: indeed, if $\Theta_\varepsilon < T_\varepsilon$, then $|\tilde{u}_\varepsilon(\Theta_\varepsilon)| \leq \frac{3}{2}$, and by continuity, there exists $\delta_\varepsilon > 0$ such that $\Theta_\varepsilon + \delta_\varepsilon < T_\varepsilon$ and $|\tilde{u}_\varepsilon(t)| \leq 2\ \forall t \in [\Theta_\varepsilon, \Theta_\varepsilon + \delta_\varepsilon]$. Hence $\Theta_\varepsilon + \delta_\varepsilon \leq \Theta_\varepsilon$: contradiction.

Hence $\Theta_\varepsilon = T_\varepsilon$, and $\tilde{u}_\varepsilon(t) \in [1, \frac{3}{2}]\ \forall t \in [0, T_\varepsilon[$
As a consequence, there is no blow-up $\to$ <u>$T_\varepsilon = +\infty$</u>.

Furthermore, $1 \leq \tilde{u}_\varepsilon(t) \leq 1 + 4\varepsilon \quad \forall t \in \mathbb{R}_+$.

Hence $$0 \leq u_\varepsilon(t) - \underbrace{e^{-\frac{t}{\varepsilon}}} \leq 4\,\varepsilon\, e^{-\frac{t}{\varepsilon}}$$

solution of
the linear equation $v_\varepsilon' + \frac{1}{\varepsilon} v_\varepsilon = 0$

$$\sup_{t \geq 0} \left(u_\varepsilon(t) - e^{-\frac{t}{\varepsilon}}\right) \leq 4\,\varepsilon$$

⇝ Uniform convergence

⇝ After an initial layer of size ε, this error becomes much smaller (exponentially small)

$$\sup_{t \geq \varepsilon K} \left(u_\varepsilon(t) - e^{-t/\varepsilon}\right) \leq 4\,\varepsilon \exp(-K)$$

- $(\tilde{P}_\varepsilon^-)$: First, observe that $\tilde{u}_\varepsilon(t) \geq \tilde{u}_\varepsilon(t)^2$.

Therefore, blow-up occurs in finite time, and $T_\varepsilon < 1$ ($=$ blow-up time for the differential equation $u' = u^2$, $u(0) = 1$).

<u>Question</u>: how does T_ε behave with ε?

<u>Exercise</u>: without computing the solution, show that
$$\tilde{u}_\varepsilon(t) \geq 1 + \varepsilon\left(e^{\frac{t}{\varepsilon}} - 1\right) \quad \forall t \in [0, T_\varepsilon[.$$
Deduce that
$$T_\varepsilon \leq 2\,\varepsilon \ln \frac{1-\varepsilon}{\varepsilon}$$

Conclusion: The main idea here is:

- filter out the behavior linked to the penalization
- investigate the asymptotic behavior of the filtered system (explicit computations, homogenization techniques, etc.)

II) Boundary layers:

Generically, boundary layers occur within boundary value problems that involve a penalization operator.

For $\varepsilon > 0$, consider the BVP

$$(E_\varepsilon) \begin{cases} -\varepsilon u_\varepsilon''(x) + c(x)\, u_\varepsilon(x) = f(x) \text{ in }]0,1[\\ u_\varepsilon(0) = u_\varepsilon(1) = 0 \end{cases}$$

where $c, f \in \mathcal{C}^2([0,1])$, and $\operatorname{Inf} c := \gamma_0 > 0$.

The theorem of Chapter 1 ensures that a solution of (E_ε) exists for all $\varepsilon > 0$

Question: Behavior of u_ε as $\varepsilon \to 0$?

Idea: as in Chapter 2, build an approximate solution.

⊛ Interior part of the solution: we expect that as $\varepsilon \to 0$, $u_\varepsilon(x) \to \bar{u}(x)$, where $\bar{u}$ satisfies

$$c(x)\,\bar{u}(x) = f(x),$$

i.e. $$\bar{u} = \frac{f}{c}$$

Then $\bar{u} \in \mathcal{C}^2([0,1])$ by assumption, and

$$\sup_{x\in[0,1]} |\varepsilon\, \bar{u}''(x)| \leq C\,\varepsilon$$

where the constant C depends on γ_0 and on $\sum_{k=0}^{2} \|c^{(k)}\|_\infty + \|f^{(k)}\|_\infty$

However, $\bar{u}(0) \neq 0$ and $\bar{u}(1) \neq 0$ a priori...
We need to add correctors close to the points $x=0$, $x=1$ to compensate these traces.

$\rightsquigarrow$ Look for an approximate solution in the form
$$u_{app}(x) = \bar{u}(x) + u_0^{BL}\left(\frac{x}{\eta_0}\right) + u_1^{BL}\left(\frac{1-x}{\eta_1}\right)$$

$0 < \eta_0, \eta_1 \ll 1$: sizes of the boundary layers close to $x=0$, $x=1$.

Assumption: $\bullet \lim_{\xi \to +\infty} u_i^{BL}(\xi) = 0$

$\bullet\ u_i^{BL}(0) = -\bar{u}(i)$, $i = 0, 1$.

Then

$$
\begin{aligned}
&-\varepsilon u_{app}'' + c(x)\, u_{app}(x) - f(x) \\
&= \varepsilon \bar{u}''(x) \quad \leftarrow \text{small error term} \\
&\quad - \frac{\varepsilon}{\eta_0^2} u_0^{BL\,''}\left(\frac{x}{\eta_0}\right) + c(0)\, u_0^{BL}\left(\frac{x}{\eta_0}\right) \\
&\quad - \frac{\varepsilon}{\eta_1^2} u_1^{BL\,''}\left(\frac{1-x}{\eta_1}\right) + c(1)\, u_1^{BL}\left(\frac{1-x}{\eta_1}\right) \\
&\quad \left.\begin{aligned} &+ \big(c(x) - c(0)\big)\, u_0^{BL}\left(\frac{x}{\eta_0}\right) \\ &+ \big(c(x) - c(1)\big)\, u_1^{BL}\left(\frac{1-x}{\eta_1}\right) \end{aligned}\right\} \text{small if } u_i^{BL}(\xi) \text{ has a strong decay as } \xi \to \infty.
\end{aligned}
$$

This leads us to <u>choose</u> η_i so that

$$\frac{\varepsilon}{\eta_i^2} = c(i) \quad \rightsquigarrow \quad \eta_i = \sqrt{\frac{\varepsilon}{c(i)}}$$

and to define u_i^{BL} as the solution of

$$\begin{cases} -u_i^{BL\,''}(\xi) + u_i^{BL}(\xi) = 0 , & \xi > 0 \\ u_i^{BL}(0) = -\bar{u}(i) \end{cases}$$

We obtain

$$u_i^{BL}(\xi) = -\bar{u}(i)\exp(-\xi) \quad \xi > 0$$

As a consequence,

$$\left|\left(c(x) - c(0)\right) u_0^{BL}\left(\frac{x}{\eta_0}\right)\right|$$

$$= \left|\left(c(x) - c(0)\right) \bar{u}(0) \exp\left(-\frac{x\sqrt{c(0)}}{\sqrt{\varepsilon}}\right)\right|$$

$$\leqslant |\bar{u}(0)| \sup_{y\in[0,1]} |c'(y)| \left| x \exp\left(-\frac{x\sqrt{c(0)}}{\sqrt{\varepsilon}}\right)\right|$$

$$\leqslant C_0\sqrt{\varepsilon},$$

where $C_0 = |\bar{u}(0)| \left(\sup_{y\in[0,1]} |c'(y)|\right) \frac{1}{\sqrt{c(0)}} \sup_{\xi\geqslant 0}\left(\xi e^{-\xi}\right)$

Similarly,

$$\left|\left(c(x) - c(1)\right) u_1^{BL}\left(\frac{1-x}{\eta_1}\right)\right| \leqslant C_1\sqrt{\varepsilon}.$$

We deduce that $w_\varepsilon = u_\varepsilon - u_{app}$ is a solution of:

$$\begin{cases} -\varepsilon w_\varepsilon''(x) + c(x) w_\varepsilon = r_\varepsilon \\ w_\varepsilon(0) = \alpha_\varepsilon^0, \quad w_\varepsilon(1) = \alpha_\varepsilon^1, \end{cases}$$

where $\sup_{x\in[0,1]} |r_\varepsilon(x)| \leq \varepsilon \sup \bar{u}'' + C_0\sqrt{\varepsilon} + C_1\sqrt{\varepsilon}$

$$\leq C\sqrt{\varepsilon}$$

and $\alpha^0_\varepsilon = u^{BL}_1\left(\frac{1}{\eta_1}\right)$, $\alpha^1_\varepsilon = u^{BL}_0\left(\frac{1}{\eta_0}\right)$

$\rightsquigarrow$ $|\alpha^i_\varepsilon| \leq C_i' \exp\left(-\frac{\gamma_0}{\sqrt{\varepsilon}}\right)$

Consequence:

$$|w_\varepsilon(x)| \leq \max\left(\frac{|r_\varepsilon(x)|}{\gamma_0}, |\alpha^0_\varepsilon|, |\alpha^1_\varepsilon|\right)$$

$$\leq C\sqrt{\varepsilon}$$

$u_\varepsilon - u_{app}$ converges uniformly towards zero.