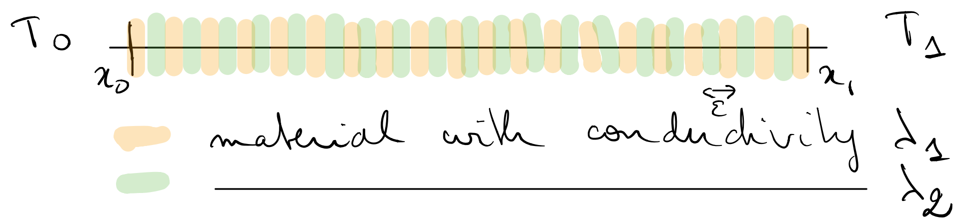
CHAPTER 2: HOMOGENIZATION

Many physical or engineered materials have a small scale structure: e.g. composite materials: alternate, at a microscopic level, two or more materials with $\neq$ properties (e.g. electrical conductivity) to obtain a macroscopic material with "better" properties.
The goal of homogenization theory is to analyze the bulk properties of physical quantities in materials having such a small scale structure.

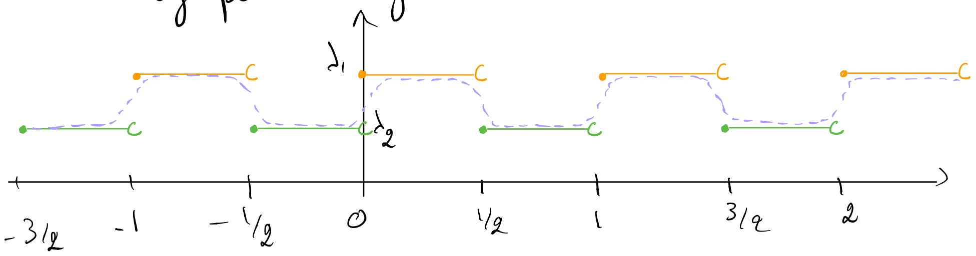
I) An example



$\rightarrow$ Conductivity $\lambda\left(\frac{x}{\varepsilon}\right)$, with

$$\lambda(y) = \begin{cases} \lambda_1 & \text{if } y \in [0, \tfrac{1}{2}[\\ \lambda_2 & \text{if } y \in [\tfrac{1}{2}, 1[\end{cases}$$

extended by periodicity over $\mathbb{R}$:



"Smoothen" λ in order to make it $\mathcal{C}^1$: - - -

Consider the differential equation

$$(H_\varepsilon)\quad \begin{cases} \dfrac{d}{dx}\left(\lambda\left(\dfrac{x}{\varepsilon}\right) T_\varepsilon'(x)\right) = 0 \quad \text{in }]x_0, x_1[\\ T_\varepsilon(x_0) = T_0 ; \quad T_\varepsilon(x_1) = T_1 \end{cases}$$

Question: How does T_ε behave as $\varepsilon \to 0$?
In this particular case, the solution is simple:
since

$$\frac{d}{dx}\left(\lambda\left(\frac{x}{\varepsilon}\right) T_\varepsilon'(x)\right) = 0 \quad \text{in }]x_0, x_1[,$$

we infer that there exists a constant C_ε such that

$$\lambda\left(\frac{x}{\varepsilon}\right) T_\varepsilon'(x) = C_\varepsilon$$

Hence

$$T_\varepsilon'(x) = \frac{C_\varepsilon}{\lambda\left(\frac{x}{\varepsilon}\right)}$$

Integrate between x_0 and x:

$$T_\varepsilon(x) = T_0 + \int_{x_0}^{x} \frac{C_\varepsilon}{\lambda\left(\frac{y}{\varepsilon}\right)}\, dy$$

The constant C_ε is determined by the condition

$$T_0 + \int_{x_0}^{x_1} \frac{C_\varepsilon}{\lambda\left(\frac{y}{\varepsilon}\right)}\, dy = T_1,$$

i.e.

$$\boxed{C_\varepsilon = \frac{T_1 - T_0}{\int_{x_0}^{x_1} \frac{1}{\lambda}\left(\frac{y}{\varepsilon}\right) dy}}$$

Lemma: Let $f:\mathbb{R}\to\mathbb{R}$ be a piecewise continuous and 1-periodic function.

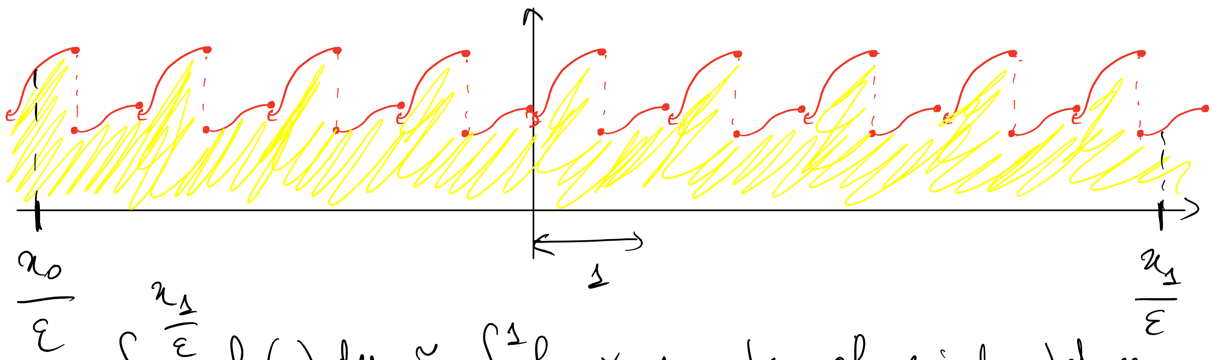
Let $x_0, x_1 \in \mathbb{R}$, $x_0 < x_1$

Then
$$\lim_{\varepsilon\to 0} \int_{x_0}^{x_1} f\left(\frac{x}{\varepsilon}\right) dx = (x_1 - x_0)\langle f\rangle,$$

where $\langle f\rangle = \int_0^1 f$.

Proof: Let $\varepsilon > 0$ be arbitrary.

$$\int_{x_0}^{x_1} f\left(\frac{x}{\varepsilon}\right) dx = \varepsilon \int_{\frac{x_0}{\varepsilon}}^{\frac{x_1}{\varepsilon}} f(y)\,dy$$



$$\int_{\frac{x_0}{\varepsilon}}^{\frac{x_1}{\varepsilon}} f(y)\,dy \approx \int_0^1 f \times \text{number of periods between } \frac{x_0}{\varepsilon} \text{ and } \frac{x_1}{\varepsilon}$$

$$\approx \int_0^1 f \times \left\lfloor \frac{x_1 - x_0}{\varepsilon} \right\rfloor$$

More precisely:

$$\int_{\frac{x_0}{\varepsilon}}^{\frac{x_1}{\varepsilon}} f(y)\,dy = \sum_{k=0}^{\left\lfloor \frac{x_1-x_0}{\varepsilon} \right\rfloor - 1} \int_{\frac{x_0}{\varepsilon}+k}^{\frac{x_0}{\varepsilon}+(k+1)} f(y)\,dy + \int_{\frac{x_0}{\varepsilon}+\left\lfloor \frac{x_1-x_0}{\varepsilon} \right\rfloor}^{\frac{x_1}{\varepsilon}} f(y)\,dy$$

- For any $x \in \mathbb{R}$,

$$\int_x^{x+1} f(y)\,dy = \int_0^1 f(y)\,dy = \langle f \rangle$$

(property of periodic functions)

Hence $\displaystyle\int_{\frac{x_0}{\varepsilon}+k}^{\frac{x_0}{\varepsilon}+(k+1)} f(y)\,dy = \langle f \rangle \quad \forall k \in \mathbb{Z}, \forall \varepsilon > 0.$

- $$\left| \int_{\frac{x_0}{\varepsilon}+\left\lfloor \frac{x_1-x_0}{\varepsilon} \right\rfloor}^{\frac{x_1}{\varepsilon}} f(y)\,dy \right| \leq \sup_{y \in \mathbb{R}} |f(y)| \times \left(\frac{x_1}{\varepsilon} - \left(\frac{x_0}{\varepsilon}\right) - \left\lfloor \frac{x_1-x_0}{\varepsilon} \right\rfloor \right)$$

$$\leq \sup_{y \in \mathbb{R}} |f(y)|$$

by definition of $\lfloor\ \rfloor$ $\left(x - \lfloor x \rfloor \in [0, 1[\ \forall x \in \mathbb{R} \right)$.

Therefore,

$$\int_{\frac{x_0}{\varepsilon}}^{\frac{x_1}{\varepsilon}} f(y)\,dy = \left\lfloor \frac{x_1-x_0}{\varepsilon} \right\rfloor \langle f \rangle + \int_{\frac{x_0}{\varepsilon}+\left\lfloor \frac{x_1-x_0}{\varepsilon} \right\rfloor}^{\frac{x_1}{\varepsilon}} f$$

$$= \frac{x_1 - x_0}{\varepsilon} \langle f \rangle$$

$$+ \left(\left\lfloor \frac{x_1 - x_0}{\varepsilon} \right\rfloor - \frac{x_1 - x_0}{\varepsilon} \right) \langle f \rangle$$

$$+ \int_{\frac{x_0}{\varepsilon} + \left\lfloor \frac{x_1 - x_0}{\varepsilon} \right\rfloor}^{\frac{x_1}{\varepsilon}} f$$

$$= \frac{x_1 - x_0}{\varepsilon} \langle f \rangle + I_\varepsilon$$

where I_ε is such that

$$|I_\varepsilon| \leqslant |\langle f \rangle| + \sup_{y \in [0,1]} |f(y)|$$

$$\leqslant 2 \sup_{y \in [0,1]} |f(y)|$$

Going back to the original expression, we infer

$$\int_{x_0}^{x_1} f\left(\frac{y}{\varepsilon}\right) dy = (x_1 - x_0) \langle f \rangle + \varepsilon I_\varepsilon .$$

and $\quad |\varepsilon I_\varepsilon| \leqslant 2 \varepsilon \sup_{y \in [0,1]} |f(y)|$.

Therefore $\lim_{\varepsilon \to 0} \varepsilon I_\varepsilon = 0$. ∎

Back to (H_ε): we deduce that

$$\lim_{\varepsilon \to 0} C_\varepsilon = \frac{T_1 - T_0}{x_1 - x_0} \quad \frac{1}{\langle \frac{1}{\lambda} \rangle},$$

and for all $x \in [x_0, x_1]$,

$$\lim_{\varepsilon \to 0} T_\varepsilon(x) = T_0 + \frac{x - x_0}{x_1 - x_0} (T_1 - T_0)$$

<u>Conclusion</u>: Let $\overline{T}(x) = T_0 + \frac{x - x_0}{x_1 - x_0} (T_1 - T_0)$

Then: • $\lim_{\varepsilon \to 0} T_\varepsilon(x) = \overline{T}(x) \quad \forall x \in [x_0, x_1]$

(Simple convergence)

In fact, looking more closely at the proof, it can be proved that there exists a constant C, depending on $T_0, T_1, x_0, x_1, \lambda_1, \lambda_2$ such that

$$\sup_{x \in [x_0, x_1]} |T_\varepsilon(x) - \overline{T}(x)| \leq C \varepsilon$$

(Uniform convergence <u>AND</u> error estimate).

• The limit $\overline{T}$ satisfies

$$(H_0) \begin{cases} \overline{T}''(x) = 0 & \forall x \in]x_0, x_1[\\ T(x_0) = T_0, \ T(x_1) = T_1 \end{cases}$$

- The derivative $T_\varepsilon'(x)$ is oscillating!

$$T_\varepsilon'(x) \simeq \frac{\overline{C}}{\lambda\left(\frac{x}{\varepsilon}\right)}, \text{ where } \overline{C} = \lim_{\varepsilon \to 0} C_\varepsilon$$

(H_0) is called the <u>homogenized problem</u> for (H_ε)

<u>Remark</u>: The computations above can be extended to the case when there is a right-hand side: let $g \in \mathcal{C}([x_0, x_1])$, and let S_ε be the solution of

$$(\tilde{H}_\varepsilon) \begin{cases} \frac{d}{dx}\left(\lambda\left(\frac{x}{\varepsilon}\right) S_\varepsilon'(x)\right) = g(x) & \forall x \in]x_0, x_1[\\ S_\varepsilon(x_0) = T_0, \quad S_\varepsilon(x_1) = T_1 \end{cases}$$

Then $\lim_{\varepsilon \to 0} |S_\varepsilon(x) - \overline{S}(x)| = 0$, where $\overline{S}$ is the solution of

$$\begin{cases} \frac{d}{dx}\left(\overline{\lambda}\ \overline{S}'(x)\right) = g(x) \\ \overline{S}(x_0) = T_0, \quad \overline{S}(x_1) = T_1 \end{cases}$$

and $\overline{\lambda} = \dfrac{1}{\langle \frac{1}{\lambda} \rangle}$ (Proof left to the reader).

⚠ $\overline{\lambda} \neq \langle \lambda \rangle$!

Even in this setting, where computations are explicit, non trivial phenomena occur. The homogenized problem for $(\tilde{H}_\varepsilon)$ is not the one one would expect intuitively.

II) The general (linear, 1d) case:

In this paragraph, we consider equations of the form

$$(E_\varepsilon) \begin{cases} -\dfrac{d}{dx}\left(a\left(x,\dfrac{x}{\varepsilon}\right)u_\varepsilon'(x)\right) + c\left(x,\dfrac{x}{\varepsilon}\right)u_\varepsilon(x) = f(x) \quad \text{in }]x_0,x_1[\\ u_\varepsilon(x_0)=0, \quad u_\varepsilon(x_1)=0 \end{cases}$$

where the functions a, c, f satisfy the following assumptions:

$$(H) \begin{cases} \bullet\ a, c \in \mathcal{C}^2([x_0,x_1]\times\mathbb{R}); \\ \bullet\ a, c \text{ are periodic with respect to their second variable: } \begin{aligned} a(x,y+1) &= a(x,y) \\ c(x,y+1) &= c(x,y) \end{aligned} \quad \forall (x,y)\in[x_0,x_1]\times\mathbb{R} \\ \bullet\ \inf\limits_{\substack{x\in[x_0,x_1] \\ y\in\mathbb{R}}} a(x,y) =: a_0 > 0 \\ \quad \inf\limits_{\substack{x\in[x_0,x_1] \\ y\in\mathbb{R}}} c(x,y) =: c_0 > 0 \end{cases}$$

- $f \in \mathcal{C}([x_0, x_1])$.

Remark : To simplify the presentation, we have chosen not to include a term of the form

$$\frac{d}{dx}\left(b\left(x, \frac{x}{\varepsilon}\right) u_\varepsilon(x)\right)$$

but it is possible to do so, under suitable conditions on b (for instance, $b \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R})$, periodic in its second variable, and such that

$$\sup_{\substack{x\in[x_0,x_1]\\ y\in\mathbb{R}}} |b(x,y)| \leq C\, a_0$$

for some constant C depending only on $x_1 - x_0$.

In the case of equation (E_ε), it is hopeless to obtain a general representation formula. Therefore another strategy needs to be developed.

Idea :

- Define rigorously and compute an approximate solution u , such that

$$(E_\varepsilon^{app})\begin{cases} -\dfrac{d}{dx}\left(a\left(x, \dfrac{x}{\varepsilon}\right) u_{app}'(x)\right) + c\left(x, \dfrac{x}{\varepsilon}\right) u_{app}(x) = f(x) + r_\varepsilon(x) \\ \lim\limits_{\varepsilon\to 0} u_{app}(x_0) = \lim\limits_{\varepsilon\to 0} u_{app}(x_1) = 0 \end{cases}$$

with $r_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$ (in a sense to be determined)

- Prove that $u_\varepsilon - u_\varepsilon^{app} \xrightarrow[\varepsilon\to 0]{} 0$ in a suitable sense.

a) Construction of an approximate solution:

Looking at the example of paragraph 1), we recall that: • $T_\varepsilon \to \bar{T}$ uniformly in ε as $\varepsilon \to 0$

• T_ε' is oscillating with respect to ε.

The same type of behavior can be expected for the solution u^ε of (E_ε). Therefore, we propose an "Ansatz" of the form

$$(A) \qquad u_\varepsilon \simeq u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

$$\text{and} \quad u_\varepsilon'(x) \simeq u_0'(x) + \underbrace{(\partial_y u_1)\left(x, \frac{x}{\varepsilon}\right)}_{\text{oscillating part}} + \cdots$$

(see lemma below)

Strategy:

- Plug this Ansatz into equation (E_ε);
- At a formal level, identify the equations on u_0, u_1, etc.;
- Rigorously define u_0, u_1, u_2;
- Compute the remainder term r_ε;

Lemma: Let $u \in \mathcal{C}^1(\mathbb{R}\times\mathbb{R})$
$\varepsilon > 0$.

Then $$\frac{d}{dx}\left(u\left(x,\frac{x}{\varepsilon}\right)\right) = \partial_1 u\left(x,\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}(\partial_2 u)\left(x,\frac{x}{\varepsilon}\right)$$

Proof: We go back to the definition.
Let $x_0 \in \mathbb{R}$ arbitrary, and let $h>0$.

$$\frac{u\left(x_0+h,\frac{x_0+h}{\varepsilon}\right) - u\left(x_0,\frac{x_0}{\varepsilon}\right)}{h}$$

$$= \frac{u\left(x_0+h,\frac{x_0}{\varepsilon}+\frac{h}{\varepsilon}\right) - u\left(x_0+h,\frac{x_0}{\varepsilon}\right)}{\frac{h}{\varepsilon}} \quad \frac{1}{\varepsilon}$$

$$+ \frac{u\left(x_0+h,\frac{x_0}{\varepsilon}\right) - u\left(x_0,\frac{x_0}{\varepsilon}\right)}{h}.$$

By definition,
$$\lim_{h\to 0} \frac{u\left(x_0+h,\frac{x_0}{\varepsilon}\right) - u\left(x_0,\frac{x_0}{\varepsilon}\right)}{h} = \partial_1 u\left(x_0,\frac{x_0}{\varepsilon}\right)$$

Moreover,
$$u\left(x_0+h,\frac{x_0}{\varepsilon}+\frac{h}{\varepsilon}\right) - u\left(x_0+h,\frac{x_0}{\varepsilon}\right) = \int_{\frac{x_0}{\varepsilon}}^{\frac{x_0}{\varepsilon}+\frac{h}{\varepsilon}} \partial_2 u(x_0+h,y)\,dy$$
$$\underset{y=\frac{x_0}{\varepsilon}+t\frac{h}{\varepsilon}}{=} \frac{h}{\varepsilon}\int_0^1 \partial_2 u\left(x_0+h,\frac{x_0}{\varepsilon}+t\frac{h}{\varepsilon}\right)dt$$

Thus

$$\frac{u\left(x_0+h, \frac{x_0}{\varepsilon}+\frac{h}{\varepsilon}\right)-u\left(x_0+h, \frac{x_0}{\varepsilon}\right)}{\frac{h}{\varepsilon}} - \partial_2 u\left(x_0, \frac{x_0}{\varepsilon}\right)$$

$$= \int_0^1 \left(\partial_2 u\left(x_0+h, \frac{x_0}{\varepsilon}+t\frac{h}{\varepsilon}\right) - \partial_2 u\left(x_0, \frac{x_0}{\varepsilon}\right)\right) dt$$

Let $\eta > 0$ be arbitrary.
Since $\partial_2 u$ is a continuous function on $\mathbb{R}^2$, there exists $\delta > 0$ (depending on x_0, ε and η) such that for any $h_1, h_2 \in \mathbb{R}$

$$|h_1| \leq \delta \text{ and } |h_2| \leq \delta \implies \left|\partial_2 u\left(x_0+h_1, \frac{x_0}{\varepsilon}+h_2\right) - \partial_2 u\left(x_0, \frac{x_0}{\varepsilon}\right)\right| \leq \eta.$$

In the present case, if $|h| \leq \delta\varepsilon$, we infer that

$$\left|\frac{u\left(x_0+h, \frac{x_0}{\varepsilon}+\frac{h}{\varepsilon}\right)-u\left(x_0+h, \frac{x_0}{\varepsilon}\right)}{\frac{h}{\varepsilon}} - \partial_2 u\left(x_0, \frac{x_0}{\varepsilon}\right)\right| \leq \eta.$$

Thus $$\lim_{h\to 0} \frac{u\left(x_0+h, x_0+\frac{h}{\varepsilon}\right) - u\left(x_0+h, \frac{x_0}{\varepsilon}\right)}{\frac{h}{\varepsilon}} = \partial_2 u\left(x_0, \frac{x_0}{\varepsilon}\right).$$

Gathering the two terms, we obtain the result announced initially. $\blacksquare$

Now, assume that

$$u_\varepsilon(x) \simeq u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

$$u_\varepsilon'(x) \simeq u_0'(x) + (\partial_y u_1)\left(x, \frac{x}{\varepsilon}\right)$$
$$+ \varepsilon\left[\partial_x u_1\left(x, \frac{x}{\varepsilon}\right) + \partial_y u_2\left(x, \frac{x}{\varepsilon}\right)\right] + \cdots$$

Plugging this asymptotic expansion into the equation and using the above lemma leads to

$$\varepsilon^{-1}\left\{-\frac{\partial}{\partial y}\Big(a(x,y)\big(u_0'(x) + \partial_y u_1(x,y)\big)\Big)\right\}\Bigg|_{y=\frac{x}{\varepsilon}}$$

$$+ \varepsilon^0 \left\{\left\{-\frac{\partial}{\partial x}\Big(a(x,y)\big(u_0'(x) + \partial_y u_1(x,y)\big)\Big)\right.\right.$$
$$-\frac{\partial}{\partial y}\Big(a(x,y)\big(\partial_x u_1(x,y) + \partial_y u_2(x,y)\big)\Big)$$
$$\left.\left.+ c(x,y)\, u_0(x,y) - f(x)\right\}\right\}\Bigg|_{y=\frac{x}{\varepsilon}}$$

$$+ \cdots = 0$$

Idea: choose u_0, u_1, u_2 such that $\{\ \}$ and $\{\{\ \}\}$ are zero.

b) Rigorous construction of u_0, u_1 and u_2:

(*) Consider the equation

$$\frac{\partial}{\partial y}\left(a(x,y)\left(u_0'(x)+\partial_y u_1(x,y)\right)\right)=0 \qquad (C)$$

This is in fact an ODE in y in which x is a parameter. In other words, for any $x\in[x_0,x_1]$, we are solving this ODE in y, whose unknown is u_1.

Remark: (C) is called the corrector problem. Indeed its purpose is to determine u_1 (= the corrector) in terms of u_0.

If u_1 is a solution of (C), then there exists a function $\varphi(x)$ such that

$$a(x,y)\left(u_0'(x)+\partial_y u_1(x,y)\right)=\varphi(x)$$

$$\rightarrow \quad \partial_y u_1(x,y)=\frac{\varphi(x)}{a(x,y)}-u_0'(x)$$

If u_1 is periodic in its second variable, then $\int_0^1 \partial_y u_1(x,y)\,dy=u_1(x,1)-u_1(x,0)=0.$

Therefore $\boxed{\varphi(x) \left\langle \frac{1}{a(x,\cdot)} \right\rangle = u_0'(x)}$

$$\partial_y u_1(x,y) = u_0'(x) \, g(x,y),$$

where $g(x,y) = \frac{1}{a(x,y)} \left(\int_0^1 \frac{1}{a(x,z)} dz \right)^{-1} - 1$

so that $\int_0^1 g(x,y) \, dy = 0 \quad \forall x \in [x_0, x_1]$.

Then we can define

$\boxed{u_1(x,y) = u_0'(x) \; G(x,y)}$, where $G(x,y) = \int_0^y g(x,z) dz$

so that G is periodic.

There remains to define u_0.

⊛ Consider now $\{\{ \quad \}\}$:

we want to define u_0, u_2 so that

$$- \frac{\partial}{\partial x} \Big(a(x,y) \, u_0'(x) \, \big(1 + \partial_y G(x,y)\big)\Big)$$

$$- \frac{\partial}{\partial y} \Big(a(x,y) \, \big(\partial_x u_1(x,y) + \partial_y u_2(x,y)\big)\Big)$$

$$+ \; c(x,y) \, u_0(x) \quad = f(x).$$

First step: Take the average of the equation in y over $[0,1]$:

Then the ∂_y derivative disappears: there remains only

$$(\bar{E}) \quad -\frac{d}{dx}\left(\bar{a}(x)\, u_0'(x)\right) + \bar{c}(x)\, u_0(x) = f(x)$$

where:
$$\bar{a}(x) = \int_0^1 a(x,y)\,(1+\partial_y G(x,y))\,dy$$
$$= \int_0^1 a(x,y)\,(1+g(x,y))\,dy$$
$$= \left(\int_0^1 \frac{1}{a(x,z)}\,dz\right)^{-1}$$

and
$$\bar{c}(x) = \int_0^1 c(x,y)\,dy.$$

Note that $\bar{a}(x) \geq \operatorname{Inf} a \quad \forall x \in [x_0, x_1]$.

Therefore, we obtain the following result:

Proposition: There exists a unique function $u_0 \in \mathcal{C}^2(]x_0,x_1[) \cap \mathcal{C}([x_0,x_1])$ solution of $(\bar{E})$ on $]x_0,x_1[$ and such that $u_0(x_0)=u_0(x_1)=0$.

$(\bar{E})$ is the homogenized problem for (E_ε).

(*) At this stage, u_0 (and therefore u_1) are well-defined. There remains to define u_2. To that end, we notice that if we require that $\{\{\ \}\} = 0$, we obtain an equation of the type

$$\frac{\partial}{\partial y}\left(a(x,y)\frac{\partial u_2}{\partial y}(x,y)\right) = h(x,y),$$

where the right-hand side h is such that

$$\int_0^1 h(x,y)\,dy = 0 \qquad \forall x \in [x_0, x_1]$$

by definition of u_0.

Hence, we define u_2 so that

$$a(x,y)\frac{\partial u_2}{\partial y}(x,y) = \underbrace{\int_0^y h(x,z)\,dz}_{=: H(x,y)} + \Psi(x)$$

and Ψ is determined by the condition

$$\int_0^1 \frac{1}{a(x,y)}\left(H(x,y) + \Psi(x)\right) dy = 0$$

c) Error estimate.

Define $u_{app}(x) = u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right)$.

Then the computations above show that:

- $$-\frac{d}{dx}\left(a\left(x,\frac{x}{\varepsilon}\right)u_{app}'(x)\right)+c\left(x,\frac{x}{\varepsilon}\right)u_{app}(x)$$
$$=f(x)+r^{\varepsilon}(x),$$
where $\sup_{x\in[x_0,x_1]}|r^{\varepsilon}(x)|\leq C\,\varepsilon$
for some constant C, independent of ε.

- $|u_{app}(x_0)|\,,\ |u_{app}(x_1)|\leq C'\varepsilon.$

As a consequence, the difference
$$v_\varepsilon(x)=u_\varepsilon(x)-u_{app}(x)$$
is a solution of
$$\begin{cases}-\frac{d}{dx}\left(a\left(x,\frac{x}{\varepsilon}\right)v_\varepsilon'(x)\right)+c\left(x,\frac{x}{\varepsilon}\right)v_\varepsilon(x)=r_\varepsilon(x)\\ v_\varepsilon(x_0)=-u_{app}(x_0),\quad v_\varepsilon(x_1)=-u_{app}(x_1)\end{cases}$$

Therefore, the theorem from Chapter 1 ensures that
$$\sup_{x\in[x_0,x_1]}|v_\varepsilon(x)-u_{app}(x)|\leq C''\,\varepsilon.$$

As a consequence,

$$\sup_{x\in[x_0,x_1]} |u_\varepsilon(x) - u_0(x)| \leq C''' \varepsilon.$$

(uniform convergence, with rate)

Conclusion:

- Convergence theorem without computing the actual solution.

- Non trivial phenomena occur (the limit is not a solution of

$$-\frac{d}{dx}\left(\langle a(x,\cdot)\rangle\, u_0'(x)\right) + \langle c(x,\cdot)\rangle u_0 = f.$$

Theorem: Assume that a, c satisfy the set of assumptions (H).

Let u_ε be the solution of E_ε, and let u_0 be the solution of the homogenized problem $(\overline{E})$.

Then $$\sup_{x\in[x_0,x_1]} |u_\varepsilon(x) - u_0(x)| \leq \overline{C}\, \varepsilon,$$

where $\overline{C}$ is a constant depending only on the coefficients a and c.