CHAPTER 1: Generalities about differential equations

I) Notion of derivative, $\mathcal{C}^1$ function:

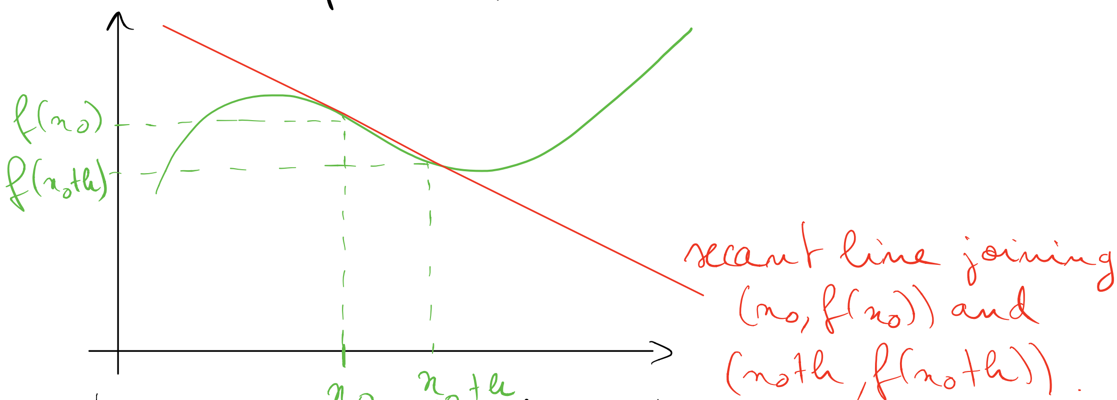
Definition: (Derivative)

Let $a < b$ be real numbers,
$f \in \mathcal{C}([a,b], \mathbb{R})$
$x_0 \in [a,b]$.

Difference quotient

Assume that the limit $\lim\limits_{\substack{h \to 0 \\ h \neq 0, x_0+h \in [a,b]}} \dfrac{f(x_0+h)-f(x_0)}{h}$ exists in $\mathbb{R}$. Then this limit is denoted by $f'(x_0)$, and called the derivative of f at x_0.
The function f is said to be differentiable at x_0.

Picture: Graph of f:



$f'(x_0)$ = limit of the slope of the secant line as $h \to 0$
= slope of the tangent line.

Examples:

- If f is a polynomial, then f is differentiable everywhere:

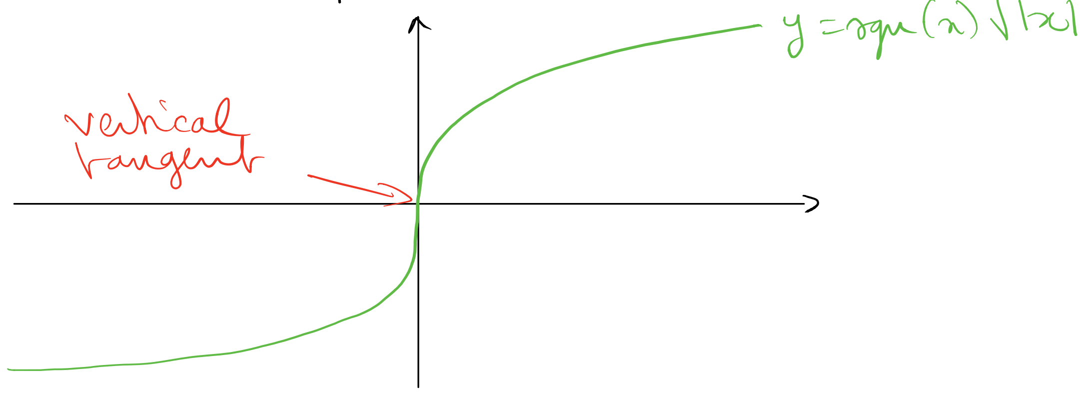
$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \forall x \in \mathbb{R}$$

$n \geq 1$

$$f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} \quad \forall x \in \mathbb{R}$$

- The function $x \mapsto |x|$ is not differentiable at $x=0$ (the difference quotients on the left and on the right have different limits).

- The function $x \mapsto \mathrm{sgn}(x)\sqrt{|x|}$ is not differentiable at $x=0$ (the limit of the difference quotient is $+\infty$).



Definition: ($\mathcal{C}^1$ function).
Let $a<b$ be real numbers
$f \in \mathcal{C}([a,b])$

Then $f \in \mathcal{C}^1([a,b])$ iff f is differentiable everywhere on $[a,b]$, and if the function

$x \in [a,b] \mapsto f'(x)$ is continuous.

Higher order derivatives: define by induction

Let $k \geqslant 1$; assume that the derivative of order k of f, denoted by $f^{(k)}$, exists on $[a,b]$ and that it is continuous on this interval.
Let $x_0 \in [a,b]$. Assume that $f^{(k)}$ is differentiable at x_0. Then

$$f^{(k+1)}(x_0) \hat{=} \left(f^{(k)}\right)'(x_0).$$

Definition: ($\mathcal{C}^k$ functions, $\mathcal{C}^\infty$ functions)

- let $k \geqslant 1$;
  $a < b$ real numbers

  Let $f \in \mathcal{C}([a,b])$
  Then f is a $\mathcal{C}^k$ function on $[a,b]$ iff all derivatives up to order k (included) exist on $[a,b]$, and are continuous.

- f is a $\mathcal{C}^\infty$ function on $[a,b]$ iff $f \in \mathcal{C}^k([a,b])$ for all $k \geqslant 1$.

Examples:

- Polynomials are $\mathcal{C}^\infty$ functions on $\mathbb{R}$: if
  $$f(x) = a_n x^n,$$
  then for $k \leq n$, $x \in \mathbb{R}$

$f^{(k)}(x) = a_n n(n-1) \cdots (n-k+1) x^{n-k}$
and for $k \geqslant n+1$, $f^{(k)}(x) = 0$.

- exp is a $\mathcal{C}^\infty$ function on $\mathbb{R}$, ln is a $\mathcal{C}^\infty$ function on $]0,+\infty[$.

Functions of several variables:

<u>Definition</u>: Let $d \geqslant 1$, and let $f : \mathbb{R}^d \to \mathbb{R}$.

- (Partial derivative): let $i \in \{1, \dots, d\}$, and let $y_0 \in \mathbb{R}^d$. Assume that the limit

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R} \setminus \{0\}}} \frac{f(y_0 + h e_i) - f(y_0)}{h} \qquad \Big(e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, 0 \dots 0)\Big)$$

exists. Then we say that f admits an i-th partial derivative at y_0, and we denote it by $\frac{\partial f}{\partial x_i}(y_0)$ or $\partial_i f(y_0)$

- ($\mathcal{C}^1$ function): $f \in \mathcal{C}^1(\mathbb{R}^d)$ iff $\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x)$ exist for all $x \in \mathbb{R}^d$ <u>and</u> if all these partial derivatives are continuous.

<u>Example</u>: Let $f_1, f_2 \in \mathcal{C}^1(\mathbb{R})$. Define
$f : \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto f_1(x) + f_2(y) \end{array}$, $g : \begin{array}{l} \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto f_1(x) f_2(y) \end{array}$
Then $f, g \in \mathcal{C}^1(\mathbb{R}^2)$.

Integral:

Definition: Let $f \in \mathcal{C}([a,b])$. Then $\int_a^b f$ $= \int_a^b f(x)\,dx$ (integral of f from a to b) is the algebraic area between the graph of f, the axis $y=0$, and the lines $x=a$, $x=b$.

$$\int_a^b f = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{b-a}{n} f\left(a + k\,\frac{b-a}{n}\right)$$

Remark: If $f \in \mathcal{C}^1([a,b])$, $\int_a^b f' = f(b) - f(a)$

II) Crash course on differential equations:

A differential equation is an equation relating a function f and its derivatives:

(DE) $\quad F(t, f(t), \dots, f^{(n)}(t)) = 0,$

where $F : [a,b] \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

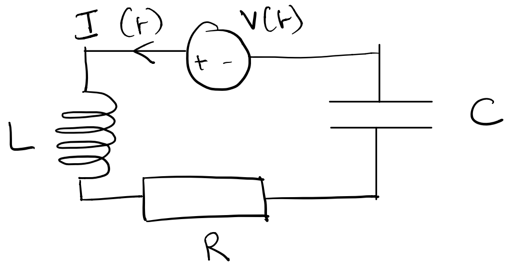
Such equations are ubiquitous in physics, biology, social sciences, ... Here are a few examples.

⊛ Electronics: Consider an electronic RLC circuit

R = resistor

L = inductor

C = capacitor



Then the electric current $I(t)$ obeys the differential equation

$$L\, I''(t) + R\, I'(t) + \frac{1}{C} I(t) = V'(t)$$

Here, $n = 2$ and

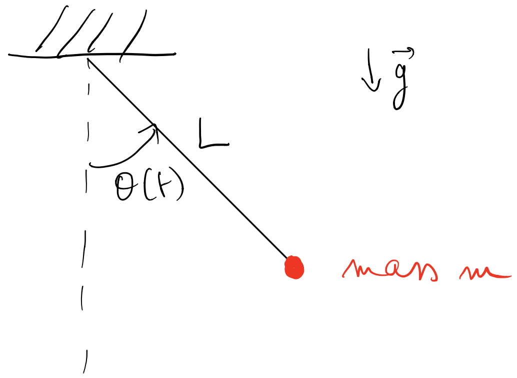
$$F(t, X, Y, Z) = LZ + RY + \frac{1}{C} X - V'(t).$$

⊛ Mechanics: many examples are provided by the application of Newton's law. Consider a particle of mass m moving on a straight line (or a circle ...) and let $x(t)$ be the position of its center of mass on the line at time t. Then

$$m\, x''(t) = \mathcal{F} \qquad (N)$$

where $\mathcal{F}$ is the sum of all forces applied to the particle at time t. In particular, if $\mathcal{F}$ depends only on t, $x(t)$ and $x'(t)$ (speed of the particle), then (N) becomes a differential equation.

Example: Pendulum:



The equation governing the evolution of θ is

$$\theta''(t) + \frac{g}{L} \sin\theta = 0 \qquad (P)$$

<u>Remark</u>: This equation is non linear: if θ_1, θ_2 are solutions of (P), then in general, $\theta_1 + \theta_2$ is not a solution.

⊛ Ecology (population dynamics): a simple model for the evolution of a population of size $N(t)$ is the logistic equation (Verhulst)

$$N'(t) = r N(t) \left(1 - \frac{N(t)}{K}\right)$$

$r > 0$: growth parameter
$K > 0$: accounts for the limited resources of the environment: if $N(t) > K$, then $N'(t) < 0$: N decreases

Questions (from the mathematical point of view):

1) Solve the differential equation (DE).

But what does "solve" mean? Proving that there exists a unique solution of (DE)? Finding an explicit formula for the solutions?

a) In general, there is <u>no</u> formula for solutions of non linear differential equations:

Example: Liouville:

$$(*) \quad u'(t) = t + u^2(t)$$

Solutions of (*) cannot be expressed as a combination of a finite number of usual functions (polynomials, exponential, logarithm ...)

b) However, it is possible to prove that solutions exist and are unique (without computing them)

<u>Theorem</u> (Cauchy-Lipschitz): Let $n \geq 1$.

Let $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $G \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n)$.

$u_0, u_1, \dots, u_{n-1} \in \mathbb{R}$

Consider the differential equation

(CL) $u^{(n)}(t) = G(t, u(t), \dots, u^{(n-1)}(t))$

together with the "initial condition"

(IC) $u(0) = u_0, \dots, u^{(n-1)}(0) = u_{n-1}$.

Then there exist $T_+ > 0$, $T_- < 0$, such that (CL)–(IC) has a unique solution on the interval $]T_-, T_+[$.

Examples of application: all examples above!

<u>Remark</u>: This theorem provides a <u>local</u> ($\neq$ global) solution: the solution is not defined on $\mathbb{R}$ in general. Blow-up phenomena may occur.

Example: $\begin{cases} u'(t) = u(t)^2 \\ u(0) = 1 \end{cases}$

exercise: check that the unique solution is given by the formula

$$u(t) = \frac{1}{1-t}, \quad t \in]-\infty, 1[,$$

and therefore $\lim_{t \to 1^-} u(t) = +\infty$: the solution cannot be extended beyond $t = 1$.

2) Investigate the "qualitative" properties of the solution:

- Is the solution global (i.e. defined on $\mathbb{R}$)?
- Does it blow-up?
- Is it increasing / decreasing?
- Is it periodic?
- Are there equilibrium points? (i.e. initial data $(u_0, \dots, u_{m-1})$ such that the associated solution is constant)
- If there are equilibrium points, are they "stable"? (This may have different meanings: starting from an initial data close to the equilibrium point, the associated solution is global and
  - (i) converges as $t \to \infty$ to the eq. point
  - (ii) remains close to the eq. point for all times.

etc.

3) When the exact solution is not known (i.e. most of the time ...), compute (either with a computer or by hand ...) an approximate solution, and prove that the exact and the approximate solutions are indeed close.

⇝ Numerical analysis (not discussed in these lectures)

⇝ Asymptotic analysis, when the equation involves a small parameter → subject of lectures 2 and 3.

<u>Remark</u>: The Cauchy-Lipschitz theorem is an <u>initial value problem</u>: given the initial data $(u_0, \cdots, u_{m-1})$, find the value of $u(t)$ for $t \in]T_-, T_+[$.
In other cases (see in particular lecture 2) it will be relevant to look at <u>boundary value problems</u>:

Example: consider a 1d heat conducting material, with thermal conductivity $\lambda(x)$, $x \in]0,1[$. Assume that the temperature at $x=0$ and $x=1$ is fixed by thermostats, and that a stationary regime has been reached (the temperature does not evolve with time).
Then the temperature within the material obeys Fourier's law:

$$\begin{cases} \frac{d}{dx}\left(\lambda(x)\, T'(x)\right) = 0 \\ T(0) = T_0, \quad T(1) = T_1 \quad \text{(thermostats)} \end{cases}$$

The existence of solutions of such problems is not provided by the Cauchy-Lipschitz theorem. For such equations, we will rather use the following result:

<u>Theorem</u>: Let $x_0, x_1 \in \mathbb{R}$, $x_0 < x_1$

$u_0, u_1 \in \mathbb{R}$

$f \in \mathcal{C}([x_0, x_1])$

$a, b \in \mathcal{C}^1([x_0, x_1])$, $c \in \mathcal{C}([x_0, x_1])$

Consider the boundary value problem

$$(BVP) \begin{cases} -(a u')' + (b u)' + c u = f & \text{in }]x_0, x_1[\\ u(x_0) = u_0, \quad u(x_1) = x_1. \end{cases}$$

Assume that $\operatorname{Inf}_{[x_0,x_1]} a > 0$ and that one of the following assumptions is satisfied:

- $c + \frac{1}{2} b' \geqslant 0$

or

- $\operatorname{Inf}_{[x_0,x_1]} c \geqslant 0$ and $\operatorname{Sup}_{[x_0,x_1]} |b| \leqslant C_1 (\operatorname{Inf} c)^{1/2} (\operatorname{Inf} a)^{1/2}$

or

- $\operatorname{Inf}_{[x_0,x_1]} c \geqslant 0$ and $\operatorname{Sup}_{[x_0,x_1]} |b| \leqslant C_2 \operatorname{Inf} a$

or

- $\operatorname{Sup}_{[x_0,x_1]} (|b| + |c|) \leqslant C_3 \operatorname{Inf} a$

where C_1, C_2, C_3 are universal constants depending only on $x_1 - x_0$.

Then (BVP) has a unique solution u on (x_0, x_1) such that $u \in \mathcal{C}([x_0, x_1]) \cap \mathcal{C}^2(]x_0, x_1[)$ and $\int_{x_0}^{x_1} u'(x)^2 \, dx < +\infty$.

Furthermore, if $b = 0$ and $\operatorname{Inf} c > 0$, then

$$\sup_{x \in [x_0, x_1]} |u(x)| \leq \max\left(\sup_{x \in [0,1]} \left| \frac{f(x)}{c(x)} \right|, |u_0|, |u_1| \right)$$

III) Measuring the distance between functions:

Consider a differential equation

$$(DE^\varepsilon) \quad \mathcal{L}^\varepsilon[u^\varepsilon] = 0,$$

possibly depending on a small parameter $\varepsilon > 0$.

Let us assume that:

- We know that a solution of (DE^ε) exists (for instance thanks to one of the two theorems above), BUT we are not able to compute it.

- We are able to compute, either numerically or theoretically, an approximate solution u^ε_{app}, in the sense that

$$\mathcal{L}^\varepsilon[u^\varepsilon_{app}] = r^\varepsilon$$

where r^ε is "small" (in a sense to be made precise)

Questions:

- What does "r^ε is small" mean?

- Are we able to prove that $u^\varepsilon - u^\varepsilon_{app}$ is small, in order to have a good description of u^ε? If yes, in which sense?

Mathematically, claiming that "f^ε is small when the parameter $\varepsilon > 0$ is small" means that f^ε converges towards zero as $\varepsilon \to 0$. Therefore, one of the central questions of these lectures is: how does one characterize the convergence towards zero of a family of functions?

Different notions of convergence:

In what follows, $(f_\varepsilon)_{\varepsilon > 0}$ is a family of continuous functions on an interval $[a,b] \subset \mathbb{R}$.

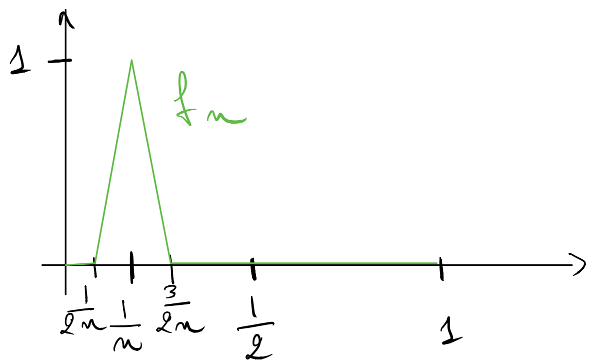
Definition: (Simple convergence) The family $(f_\varepsilon)_{\varepsilon>0}$ simply converges towards zero iff $\lim_{\varepsilon \to 0} f_\varepsilon(x) = 0$ for all $x \in [a,b]$.

Definition: (Uniform convergence) The family $(f_\varepsilon)_{\varepsilon>0}$ uniformly converges towards zero iff

$$\lim_{\varepsilon \to 0} \sup_{x\in[a,b]} |f_\varepsilon(x)| = 0$$

<u>Remark</u>: Uniform convergence $\Rightarrow$ Simple convergence, but the converse is not true.

Example: on $[0, 1]$:



It is easily checked that $(f_n)_{n\in\mathbb{N}}$ converges simply, but not uniformly towards zero.

In differential equations, it is also useful to look at convergences in "average", for example:

<u>Definition</u>:

- (L^1 convergence): the family $(f_\varepsilon)_{\varepsilon>0}$ converges towards zero in L^1 iff

$$\lim_{\varepsilon\to 0} \int_a^b |f_\varepsilon(x)|\, dx = 0$$

- (L^2 convergence): the family $(f_\varepsilon)_{\varepsilon>0}$ converges towards zero in L^2 iff

$$\lim_{\varepsilon\to 0} \int_a^b f_\varepsilon(x)^2\, dx = 0$$

Remark:

- Uniform convergence $\Rightarrow L^2$ convergence $\overset{\text{if } b-a \in \mathbb{R}}{\Rightarrow} L^1$ convergence

  (But the converse is not true, see example above)

- L^1 convergence $\Rightarrow$ simple convergence "almost every where" for a subsequence.

Remark: We can extend these notions to $\mathcal{C}^k$ functions for $k \geq 1$ by taking into account the convergence of the derivatives.

For instance, for $\mathcal{C}^1$ functions, one can look at the following (different) notions of convergence:

- $$\lim_{\varepsilon \to 0} \sup_{x \in [a,b]} \left(|f_\varepsilon(x)| + |f_\varepsilon'(x)| \right) = 0$$

- $$\lim_{\varepsilon \to 0} \left(\sup_{x \in [a,b]} |f_\varepsilon(x)| + \int_a^b |f_\varepsilon'(y)| \, dy \right) = 0$$

- $$\lim_{\varepsilon \to 0} \int_a^b \left(f_\varepsilon(x)^2 + f_\varepsilon'(x)^2 \right) dx = 0$$

etc.

When you look at a given problem, it is not obvious at first what is the good notion of

convergence (and more generally, what is the good way of measuring the size of the solution)
Determining the appropriate notion of convergence is a crucial part of the work of mathematicians working in PDEs.