Lois de paroi au voisinage de surfaces rugueuses vérifiant une condition de glissement

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Enjeux de Modélisation et Analyse Liés au Problèmes de Surfaces Rugueuses et de Défauts

Plan

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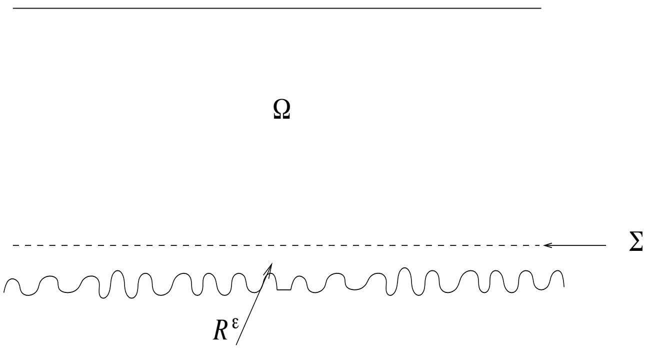
Physical motivations

Consider a fluid in a two-dimensional canal :



Boundary conditions : slip on the rough surface, no-slip on the flat surface.

Question : does the roughness increase/decrease the slip length at the limit ?

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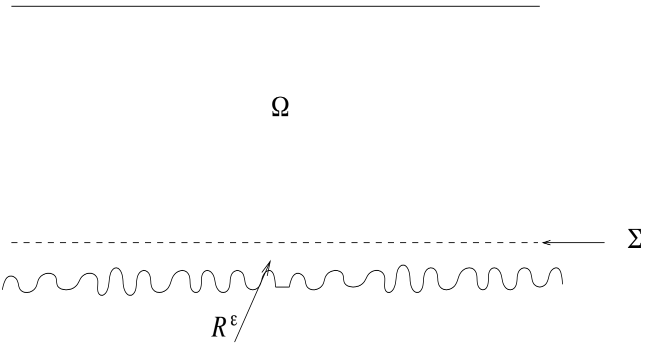

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Physical answers

Experimental results :

- Zhu& Granick, PRL, 2002 : suppression of the slip by increasing surface roughness.
- Watanabe, Udagawa,Y. & Ugadawa, H., J. Fluid Mech. 1999 : achievement of large slip on rough hydrophobic surfaces.

Numerical simulations :

- Cottin-Bizonne *et al.*, Nature Materials, 2003 : significant increase of slip (starting from a system of particles).

Previous mathematical answers

Notation : for $K \subset \mathbb{R}^{d-1}$ compact,

$$\Omega^\varepsilon := \{x = (x', x_d) \in \mathbb{R}^d,\ \varepsilon\eta(x'/\varepsilon) < x_d < 1\},$$
$$\Gamma^\varepsilon := \{x_d = \varepsilon\eta(x'/\varepsilon)\},$$
$$\Omega := \{x = (x', x_d) \in \mathbb{R}^d,\ 0 < x_d < 1\}.$$

No-slip → no slip/small slip :

- Achdou, Pironneau, Valentin, 1998 ;
- Jäger, Mikelic, 2001 ;
- Amirat *et. al.*, 2001 ;
- Gérard-Varet *et. al.*, 2008.

Slip+roughness → no slip :

- Casado-Diaz, Fernandez-Cara, Simon (2002) ;
- Bucur, Feireisl *et. al.*, 2008 ;

→ Different notions of surface roughness.

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Goals of the present talk

Starting point : two-dimensional canal with Navier condition (slip) on the rough boundary.

- Prove error estimates for the Dirichlet wall law.
  Idea : use same kind of error estimates as Jäger&Mikelic.
  Remark : Dirichlet wall law $\rightsquigarrow$ error $O(\varepsilon)$.
  Using a more refined wall law (Navier condition with slip length of order ε), one can obtain error estimates of order $o(\varepsilon)$ (Gérard-Varet et.al.) ($O(\varepsilon^{3/2})$ in the periodic case, see Jäger&Mikelic).
- Derive and prove error estimates for the Navier wall law.
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The Navier-Stokes system

$$\begin{cases} -\Delta u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = 0, \ x \in \Omega^\varepsilon, \\ \mathrm{div}\, u^\varepsilon = 0, \ x \in \Omega^\varepsilon, \\ u^\varepsilon|_{x_2=1} = 0, \ \int_{\sigma^\varepsilon} u_1^\varepsilon = \phi, \\ (I_d - \nu \otimes \nu) u^\varepsilon|_{\Gamma^\varepsilon} \ = \ \lambda_0 (I_d - \nu \otimes \nu) D(u^\varepsilon)\nu|_{\Gamma^\varepsilon}, \ u^\varepsilon \cdot \nu|_{\Gamma^\varepsilon} \ = \ 0. \end{cases} \quad (\mathrm{NS}^\varepsilon)$$

ϕ : prescribed flux across a cross section.
λ_0 : slip length.
ν : outward normal at the rough surface Γ^ε.

Result : There exists $\phi_0 > 0$ s.t. (NS$^\varepsilon$) is well posed for $0 < \phi < \phi_0$ and for all $\varepsilon > 0$.

Main results 1 - Dirichlet wall law

Non-degeneracy assumption on the rough boundary :

$$\text{(P) } \exists \mathcal{C} > 0, \text{ s.t. } \forall u \in C_c^\infty\left(\overline{R}\right) \text{ satisfying } u \cdot \nu|_\Gamma = 0,$$

$$\|u\|_{L^2(R)} \leq \mathcal{C} \, \|\nabla u\|_{L^2(R)}$$

where

$$R := \{y \in \mathbb{R}^2, \eta(y_1) < y_2 < 0\}.$$

Theorem

If $\lambda_0 = 0$ or if (P) holds, one has

$$\|u^\varepsilon - u^0\|_{H^1_{uloc}(\Omega^\varepsilon)} \leq C\,\phi\,\sqrt{\varepsilon}, \quad \|u^\varepsilon - u^0\|_{L^2_{uloc}(\Omega)} \leq C\,\phi\,\varepsilon,$$

where u^0 is the Poiseuille flow, satisfying the Navier-Stokes equations in Ω, with no-slip boundary conditions.

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Main results 2 - Navier wall law

Lack of rotational invariance of the boundary :

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Let η be an ergodic stationary random process, c-Lipschitz almost surely, for some $c > 0$. Assume that (K) holds almost surely, with a uniform $\mathcal{C}$. Then there exists $\alpha > 0$ and $\phi_0 > 0$ such that, for all $|\phi| < \phi_0$, $\varepsilon \leq 1$,

$$\left(\sup_{R \geq 1} \frac{1}{R} \int_{\Omega \cap \{|x_1| < R\}} |u^\varepsilon - u^N|^2\, dx\right)^{1/2} = o(\varepsilon), \quad \textit{almost surely,}$$

where u^N is the solution of the NS system in Ω satisfying

$$u^N|_{x_2=1} = 0,$$

$$u_2^N = 0,\ u_1^N = \alpha\varepsilon\, \partial_2 u_1^N \quad \text{on } x_2 = 0.$$

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Two-scale Ansatz

Rugosity modifies the behavior of the solution at the boundary

$\rightarrow$ Apparition of boundary layer terms.

Ansatz :

$$u^\varepsilon(x) \approx u^0(x) + \varepsilon u_{\mathrm{BL}}\left(\frac{x}{\varepsilon}\right) + \varepsilon u^1(x) + O(\varepsilon^2)$$

where :

- u^0, u^1 are interior terms ;
- u_{BL} is a boundary layer term.

Method : plug asymptotic expansion into (NS$^\varepsilon$) and identify powers of ε.

Remark : the boundary layer terms do not modify the solution in the interior.

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Dirichlet wall law

At the lower (rough) boundary Γ^ε :

$$u^0(x) \cdot \nu^\varepsilon = O(\varepsilon),$$

where $\nu^\varepsilon = \left[1 + (\eta'(x_1/\varepsilon))^2\right]^{-1/2} (\eta'(x_1/\varepsilon), -1)$.

For $y_1 \in \mathbb{R}$, let

$$\nu(y_1) := \frac{1}{\sqrt{1 + (\eta'(y_1))^2}} \left(\eta'(y_1, -1)\right).$$

The Ansatz yields

$$u^0(x_1, 0) \cdot \nu(y_1) = 0 \quad \forall x_1 \in \mathbb{R},\ \forall y_1 \in \mathbb{R}.$$

Non-degenerate rugosity ($\Leftrightarrow$ ν non-constant) :

$$u^0(x)|_{x_2=0} = 0.$$

$\rightarrow$ no-slip boundary condition.

Consequence : u^0 is a Poiseuille flow :

$$u^0(x) = (6\phi x_2(1 - x_2), 0).$$

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Definition of the boundary layer term

Boundary condition at next order on Γ^ε :

$$6\phi(y_2,0)\cdot\nu(y_1)+u^1(x_1,0)\cdot\nu(y_1)+u_{BL}(y)\cdot\nu(y_1)=0 \quad \text{on } \Gamma,$$
$$6\phi(D((y_2,0))\nu)_\tau+(D(u_{BL})\nu)_\tau=0 \quad \text{on } \Gamma.$$

Consequence :

- $u^1|_{x_2=0}=0$;
- The boundary layer term $u_{BL}(y)=6\phi v(y)$ satisfies

$$\begin{cases} -\Delta v+\nabla p=0, & y\in\Omega^{bl},\\ \operatorname{div} v=0, & y\in\Omega^{bl},\\ (D(v)\nu)_\tau=-(D(y_2)\nu)_\tau, & y\in\partial\Omega^{bl},\\ v\cdot\nu=-(y_2,0)\cdot\nu, & y\in\partial\Omega^{bl}, \end{cases} \qquad \text{(BL)}$$

with $\Omega^{bl}=\{y\in\mathbb{R}^2,\ y_2>\eta(y_1)\}$.

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Navier wall law

Lemma

Assume that the setting is stationary ergodic. Then there exists a constant α such that

$$\lim_{y_2 \to \infty} v(y_1, y_2) = (\alpha, 0) \quad a.\ s.$$

Remark : in the periodic setting, the proof is relies on Fourier decomposition and the speed of convergence is exponential.

Fact : $-\sup \eta \leq \alpha \leq -\inf \eta$. (proof in the periodic setting : Achdou, Pironneau, Valentin).

Consequence : if $x_2 \gg \varepsilon$, then

$$u^\varepsilon(x) \approx u^0(x) + 6\phi\varepsilon(\alpha, 0) + \varepsilon u^1(x),$$

so that near $x_2 = 0$,

$$u_2^\varepsilon = 0, \quad u_1^\varepsilon = \varepsilon\alpha\partial_2 u_1^\varepsilon.$$

$\rightarrow$ Navier wall law with slip length $\varepsilon\alpha$.

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Sketch of proof of estimates for the Dirichlet wall law

Remark : restriction to the periodic case.

Energy inequality : $v := u^\varepsilon - u^0 \mathbf{1}_\Omega$ satisfies (with $R^\varepsilon = \Omega^\varepsilon \setminus \Omega$)

$$\int_{\Omega^\varepsilon} |D(v)|^2 + \lambda_0^{-1} \int_{\Gamma^\varepsilon} |v_\tau|^2$$
$$\leq C\,\phi \left(\|v\|_{L^2(\Omega^\varepsilon)} \|D(v)\|_{L^2(\Omega^\varepsilon)} + \sqrt{\varepsilon}\, \|v\|_{L^2(R^\varepsilon)} + \|v|_{x_2=0}\|_{L^2(\mathbb{R})} \right).$$

Standard Korn inequality in Ω^ε :

$$\|\nabla v\|_{L^2(\Omega^\varepsilon)} \leq C \|D(v)\|_{L^2(\Omega^\varepsilon)}.$$

Assumption (P) in the canal R + scaling :

$$\|v\|_{L^2(R^\varepsilon)} \leq C\varepsilon \|\nabla v\|_{L^2(R^\varepsilon)}$$
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A sufficient condition for Poincaré inequalities

Lemma

Let $\eta \in W^{1,\infty}(\mathbb{R})$ *with values in* $(-1, 0)$ *and such that* $\sup \eta < 0$*. Assume that*

$$\exists A > 0, \ \inf_{y_1 \in \mathbb{R}} \int_0^A |\eta'(y_1 + t)|^2 dt > 0. \tag{1}$$

Then assumption (P) is satisfied.

Periodic case and quasi-periodic case :

$$(1) \iff \eta \text{ non constant.}$$

$\rightarrow$ (1) is equivalent to the non-degeneracy assumptions of Bucur *et al.*, Casado-Diaz *et al.*

Stationary ergodic case : (P) seems more stringent than the non-degeneracy assumption of Bucur *et al.* (but no counter example...)

Proof of the sufficiency of condition (1)

Idea : prove that

$$\begin{aligned}\int_R |u(y)|^2 dy &\leq C_B \int_R \int_0^B |u(y_1, y_2) \cdot \nu(y_1 + t)|^2 \, dt \, dy_1 \, dy_2 \\ &\leq C_B \int_R |\nabla u(y)|^2 \, dy\end{aligned}$$

Proof of the sufficiency of condition (1)

Idea : prove that

$$\begin{aligned}\int_R |u(y)|^2 dy &\leq C_B \int_R \int_0^B |u(y_1, y_2) \cdot \nu(y_1 + t)|^2 \, dt \, dy_1 \, dy_2 \\ &\leq C_B \int_R |\nabla u(y)|^2 \, dy\end{aligned}$$

For the first inequality, use assumption (1) and write

$$\begin{aligned} & |u(y_1, y_2) \cdot \nu(y_1 + t)|^2 \\ = \; & \frac{1}{1 + (\eta'(y_1 + t))^2} \Big[u_1(y)^2 + (\eta'(y_1 + t))^2 u_2(y) \\ & \qquad - 2u_1(y)_2(y)\eta'(y_1 + t) \Big] . \end{aligned}$$

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For the second inequality, choose a path $\ell : (0,1) \to R$ such that

$$\ell(1) = y, \quad \ell(0) = (y_1 + t, \eta(y_1 + t)) \in \Gamma.$$

Then

$$\begin{aligned}u(y) \cdot \nu(y_1 + t) &= [u(y) - u(y_1 + t, \eta(y_1 + t))] \cdot \nu(y_1 + t) \\ &= (-t, y_2 - \eta(y_1 + t)) \cdot \int_0^1 \nabla u(\ell(\tau)) \, d\tau.\end{aligned}$$

Integrating with respect to y and t yields the second inequality.

Plan

Construction of the boundary layer term : energy estimates

Remark : restriction to the periodic case.
Let v be a solution of (BL). Then

$$\begin{aligned}\int_{\Omega^{bl}} |D(v)|^2 &= \int_{\mathbb{R}} v_1|_{y_2=0}\\ &\leq C\|v\|_{H^1(R)}\\ &\leq C(\|v\|_{L^2(R)} + \|D(v)\|_{L^2(R)}).\end{aligned}$$

Using assumption (K), we obtain a bound on $\|D(v)\|_{L^2(\Omega^{bl})}$.
Remark : Using assumption (K) and an analyis of the Stokes system in the upper half plane, one retrieves a bound on ∇v.

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A sufficient condition for homogeneous Korn inequalities

Let $\mathcal{R}$ be the set of **rotational invariant and Lipschitz curves.** Notice that $\mathcal{R}$ is closed wrt the weak - $*$ topology in $W^{1,\infty}$.

Lemma

For $A > 0$, $k \in \mathbb{Z}$, let $\gamma_k^A : y_1 \in [0, A] \mapsto (y_1, \eta(y_1 + kA))$. Assume that there exists $A > 0$ such that

$$\overline{\{\gamma_k^A, k \in \mathbb{Z}\}} \cap \mathcal{R} = \emptyset,$$

where the closure is taken wrt the w-$*$ topology in $W^{1,\infty}$. Then (K) holds.

Remark : same kind of assumption in a paper by Desvillettes & Villani.

Comparison between sufficient conditions for Poincaré and Korn inequalities

Remark : (K) $\Rightarrow$ (P).

Periodic setting : both assumptions amount to

$$\eta \text{ non constant.}$$

General case :

Poincaré inequality : measure of rugosity ;

Homogeneous Korn inequality : measure of lack of rotational invariance.

$\longrightarrow$ Different notions.

Open problems :

- Find stationary invariant and non constant curves such that (P) is not satisfied.
- Find quasi-periodic curves such that (P) is satisfied and (K) is not.

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Plan

Summary

- Exhibition of sufficient conditions for the derivation of Dirichlet/Navier wall laws :
  Functional inequalities measuring the rugosity/lack of rotational invariance of the rough boundary.
- Proof of (almost sure) error estimates.
- Slip length : $\alpha\varepsilon$.
- Upper and lower bounds on α (depending on η).

Back to the physical questions

- The slip length obtained with this kind of model is small ($O(\varepsilon)$).
  $\rightarrow$ Rugosity does not enhance slip.
- **Other model** (for super hydrophobic surfaces :)
  flat, periodic lower boundary with alternance of perfect slip/no slip (period : ε).
  Same analysis : the Poincaré and Korn inequalities hold for the rescaled canal.
  $\rightarrow$ Dirichlet wall law at order zero, Navier wall law at first order with slip length $O(\varepsilon)$.
- **Conclusion :** large slip does not seem to be reachable with this type of model (even with hydrophobic surfaces).