Separation for the stationary Prandtl equation

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Outline

Plan

What is boundary layer separation?

Flow with low viscosity around an obstacle, in the presence of an adverse pressure gradient (=the outer flow is decreasing along the obstacle):

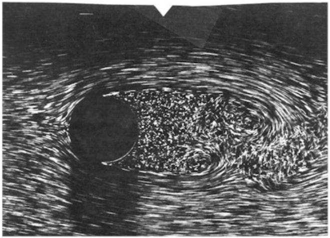

Figure: Cross-section of a flow past a cylinder (source: ONERA, France)

After separation: widely open problem (turbulence?)
Mathematical interest: understand the vanishing viscosity limit in the Navier-Stokes equation.
Industrial interest: reduce drag.

Derivation of the (stationary) Prandtl system

Starting point: stationary 2d Navier-Stokes with small viscosity $\nu \ll 1$ in a half-plane. Velocity field $\mathbf{u}^\nu : \mathbf{R}^2_+ \to \mathbf{R}^2$ with

$$\begin{aligned}(\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu + \nabla p^\nu - \nu\Delta\mathbf{u}^\nu &= \mathbf{f} \text{ in } \mathbf{R}^2_+,\\ \mathrm{div}\,\mathbf{u}^\nu &= 0 \text{ in } \mathbf{R}^2_+,\\ \mathbf{u}^\nu_{|y=0} &= 0.\end{aligned} \tag{1}$$

Ansatz: $\mathbf{u}^\nu(x,y) \simeq \begin{cases} \mathbf{u}^E(x,y) \text{ for } y \gg \sqrt{\nu} \text{ (sol. of 2d Euler)},\\ \left(u\left(x, \frac{y}{\sqrt{\nu}}\right), \sqrt{\nu}v\left(x, \frac{y}{\sqrt{\nu}}\right)\right) \text{ for } y \lesssim \sqrt{\nu}. \end{cases}$

Stationary Prandtl system: with $Y = y/\sqrt{\nu}$,

$$\begin{aligned}u\partial_x u + v\partial_Y u - \partial_{YY} u &= f_1(x,0) - \frac{\partial p^E(x,0)}{\partial x}\\ \partial_x u + \partial_Y v &= 0,\\ u_{|Y=0} = 0, \quad v_{|Y=0} = 0, \quad \lim_{Y\to\infty} u &= u^E(x,0).\end{aligned}$$

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The stationary Prandtl equation: general existence result

Stationary Prandtl system:

$$\begin{aligned} u\partial_x u + v\partial_Y u - \partial_{YY} u = g(x) \\ \partial_x u + \partial_Y v = 0, \quad u_{|x=0} = u_0 \\ u_{|Y=0} = 0, \quad v_{|Y=0} = 0, \quad \lim_{Y\to\infty} u = u^E(x,0). \end{aligned} \qquad \text{(P)}$$

$\sim$ Nonlocal, scalar evolution eq. in x.

Locally well-posed as long as $u > 0$:

Theorem [Oleinik, 1962]: Let $u_0 \in \mathcal{C}_b^{2,\alpha}(\mathbf{R}_+)$, $\alpha > 0$. Assume that $u_0(Y) > 0$ for $Y > 0$, $u_0'(0) > 0$, and that

$$\partial_{YY} u_0 + g(0) = O(Y^2) \quad \text{for } 0 < Y \ll 1.$$

Then there exists $x^* > 0$ such that (P) has a unique strong $\mathcal{C}^2$ solution in $\{(x,Y) \in \mathbf{R}^2,\ 0 \le x < x^*,\ 0 \le Y\}$. If $g(x) \ge 0$, then $x^* = +\infty$.

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The stationary Prandtl equation: monotonicity, comparison principle

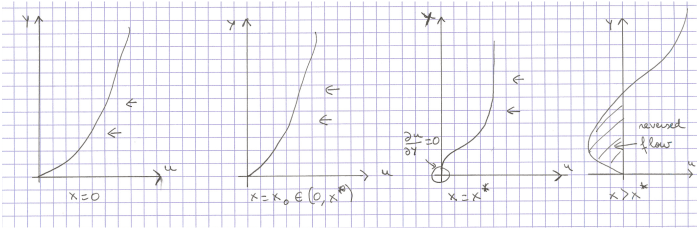
Nonlinear change of variables [von Mises]: transforms (P) into a local diffusion equation (porous medium type).
→ Maximum principle holds for the new eq. by standard tools and arguments.

- **Monotonicity** is preserved by (P).
- **Comparison principle for the Prandtl equation:**
  - Consider a super-solution $\bar{u}$ for Prandtl;
  - Von Mises $\rightsquigarrow$ sub/super solution $\bar{w}$ for the eq. in new variables;
  - Maximum principle for the new eq.: $w \leq \bar{w}$.
  - ODE arguments: $u \leq \bar{u}$.

Remark: we will consider strictly increasing solutions only: ensures that separation happens at the boundary.

General mechanism behind separation

Setting: monotone solutions with adverse pressure gradient ($g \leq 0$).



In general there exists x^* such that

$$\frac{\partial u}{\partial Y}_{|x=x^*, Y=0} = 0.$$

For $x > x^*$, $Y \lesssim 1$, u takes negative values: reversed flow near the boundary.

$\rightarrow$ There exists a curve $Y = F(x)$ such that $u(x, F(x)) = 0$: separation of the boundary layer.

Definition: x^* is called the separation point.

Questions

1. Does separation really happen?
   Can you cook-up solutions of (P) such that $\partial_Y u_{|Y=0}(x) \to 0$ as $x \to x^*$ for some finite x^*?
2. If you can, what is the rate at which $\partial_Y u_{|Y=0}(x)$ vanishes?

Related results

Instability results (time dependent version):
Local well-posedness in high regularity spaces (analytic, Gevrey) [Sammartino& Caflisch; Gérard-Varet& Masmoudi...] or for monotonic data [Oleinik; Masmoudi&Wong; Alexandre, Wang, Xu& Yang...]
BUT instabilities develop in short time in Sobolev spaces [Grenier; Gérard-Varet&Dormy...]

Formation of singularities (time dependent version):
[Kukavica, Vicol, Wang](van Dommelen-Shen singularity)
Starting from real analytic initial data, for specific outer Euler flow, some solutions display singularities in finite time .

Justification of the Prandtl Ansatz when $\nu \ll 1$: [Guo& Nguyen, Iyer] Starting from stationary Navier-Stokes above a moving plate (non-zero boundary condition on the wall), local convergence/global convergence for small data ($\rightarrow$ no singularity).

Plan

Formal derivations of the self similar rate

• Formal computations of an exact solution to (P) by [Goldstein '48, Stewartson '58] thanks to **Taylor expansions in self-similar variables.**

Self similar change of variables: rely on the observation that (P) is **invariant by the scaling**

$$\begin{aligned} u(x,Y) &\to \frac{1}{\sqrt{\mu}} u(\mu x, \mu^{1/4} Y), \\ v(x,y) &\to \mu^{1/4} v(\mu x, \mu^{1/4} Y), \end{aligned} \quad \text{with } \mu > 0.$$

Remark: the coefficients of the asymptotic expansion are never entirely determined (dependence on initial data?)

• Heuristic argument by Landau: $\partial_Y u_{|Y=0}(x) \sim \sqrt{x^* - x}$. (same as Goldstein& Stewartson.)

Statement by Luis Caffarelli and Weinan E

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Theorem [Caffarelli, E, 1995]: *Assume that $g(x) = -1$, and that u_0 satisfies*

$$u_0^2 - \frac{3}{2}\partial_Y u_0 \int_0^Y u_0 \geq 0.$$

Then:

- *There exists $x^* > 0$ such that the solution cannot be extended beyond x^*;*
- *The family $u_\mu := \frac{1}{\sqrt{\mu}} u(\mu(x^* - x), \mu^{1/4} Y)$ is compact in $\mathcal{C}(\mathbf{R}_+^2)$.*

The author also states two (non-trivial...) Lemmas playing a key role in the proof, relying on the maximum principle.

Unfortunately the complete proof was never published...

Goal of the present talk: propose an alternate proof, relying on different techniques, and giving a more quantitative result.

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Main result

Theorem [D., Masmoudi, '16]:
Consider the equation (P) with $g(x) = -1$. Then for a class of initial data $u_0 = u_{|x=0}$ satisfying

- u_0 is strictly increasing with respect to Y;
- $u_0(Y) \simeq \lambda_0 Y + \frac{Y^2}{2}$ for $Y \ll 1$ and for some $\lambda_0 \ll 1$;

separation occurs at a finite distance $x^* = O(\lambda_0^2)$.

Moreover for all $x \in (0, x^*)$,

$$\lambda(x) := \partial_Y u_{|Y=0}(x) \sim C\sqrt{x^* - x}$$

and for some weight $w = w(x, Y)$,

$$\|u - u^{app}\|_{L^2(w)} = o(\|u^{app}\|_{L^2(w)}) \text{ as } x \to x^*,$$

where for $Y \lesssim (x^* - x)^{1/4}$

$$u^{app}(x, Y) = \lambda(x)Y + \frac{Y^2}{2} - \alpha Y^4 - \beta\lambda(x)^{-1}Y^7.$$

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Remarks

- Two results: asymptotic behavior of $\partial_Y u_{|Y=0}$ and error estimate between u and u^{app}.
- The self-similar rate is the one predicted by Landau, Goldstein and Stewartson.
- Existence of other (unstable) rates?
- Comparison with result by Caffarelli and E:
  - Encompasses their result;
  - More stringent assumptions;
  - Quantitative result; the limit is identified.
- Tools and scheme of proof:
  - Inspired by study of blow-up rates for NLS [Zakharov, Sulem& Sulem; Merle& Raphaël]; successfully applied to wave and Schrödinger maps, Keller-Segel system, harmonic heat flow [Merle, Raphaël, Rodnianski, Schweyer...]
  - Perform a self-similar change of variables; approximate solution, energy estimates in rescaled variables;
  - Use techniques based on modulation of variables to find the self-similar rate $\lambda(x) = \partial_Y u_{|Y=0}$.

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The self-similar change of variables

Let $\lambda(x) := \partial_Y u_{|Y=0}(x)$. Define $\tilde{u} = \tilde{u}(x,\xi)$ by

$$\tilde{u}(x,\xi) = \lambda^{-2}(x) u(x, \lambda(x)\xi).$$

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$$\lambda^4 \left(\tilde{u}\tilde{u}_x - \tilde{u}_\xi \int_0^\xi \tilde{u}_x \right) + \lambda_x \lambda^3 \left(2\tilde{u}^2 - 3\tilde{u}_\xi \int_0^\xi \tilde{u} \right) - \tilde{u}_{\xi\xi} = -1.$$

Define s, b, U such that

$$b = -2\lambda_x \lambda^3, \quad \frac{dx}{ds} = \lambda^4, \quad U(s,\xi) = \tilde{u}(x(s),\xi).$$

Then U satisfies

$$UU_s - U_\xi \int_0^\xi U_s - bU^2 + \frac{3b}{2} U_\xi \int_0^\xi U - U_{\xi\xi} = -1. \qquad \text{(R)}$$

Remark: $x \to x^*$ corresponds to $s \to \infty$ provided $\int_0^{x^*} \lambda^{-4} = \infty$.

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Strategy

From now on, work only on equation on U:

$$UU_s - U_\xi \int_0^\xi U_s - bU^2 + \frac{3b}{2} U_\xi \int_0^\xi U - U_{\xi\xi} = -1.$$

⚠At this stage, b is an unknown. The asymptotic behavior of b dictates the self-similar rate $\lambda(x)$.

Scheme of proof:

1. Construct an approximate solution;
2. Choose the approximate solution with the “least possible growth” at infinity: heuristics for the modulation rate b;
3. Energy estimate on the remainder of the solution.

Rule of thumb: the expected rate $\lambda(x) = C\sqrt{x^* - x}$ corresponds to

$$b(s) = \frac{1}{s} \Leftrightarrow b_s + b^2 = 0.$$

Plan

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Construction of an approximate solution

$$\underbrace{UU_s - U_\xi \int_0^\xi U_s - bU^2 + \frac{3b}{2} U_\xi \int_0^\xi U}_{=:\mathcal{A}(U)} - U_{\xi\xi} = -1.$$

Boundary conditions at $\xi = 0$: by definition of λ,

$$U_{|\xi=0} = 0, \;\; \partial_\xi U_{|\xi=0} = 1.$$

Case $b = 0$: exact stationary solution

$$U_0 := \xi + \frac{\xi^2}{2} \; (\sim \text{“ground state”}).$$

Case $b \neq 0$: Look for asymptotic expansion in the form

$$U = U_0 + bT_1 + b^2 T_2 + \cdots$$

Then T_1 is given by $b\partial_{\xi\xi} T_1 = \mathcal{A}(U_0) + 1 \Rightarrow T_1 = -\frac{\xi^4}{48}$.
What about T_2?

Finding the ODE on b

Rule: choice of the approx. solution with the least possible growth.
Remainder for $U_1 := U_0 + bT_1$:

$$\begin{aligned} &\mathcal{A}(U_1) - \partial_{\xi\xi} U_1 + 1 \\ = \;& -\alpha\left(\frac{4}{5}b_s + \frac{13}{10}b^2\right)\xi^5 - \frac{3}{10}\alpha\left(b_s + b^2\right)\xi^6 + \alpha^2\frac{b}{5}\left(b_s + b^2\right)\xi^8. \end{aligned}$$

"Choice" of b such that the ξ^6 term disappears:

$$b_s + b^2 = 0.$$

Ansatz: in the algorithm defining T_N, replace every occurrence of b_s by $-b^2$.
Consequence: setting $U_N := U_0 + bT_1 + \cdots + b^N T_N$,
$U - U_2 \sim (b_s + b^2)(c_7\xi^7 + c_8\xi^8)$ near $\xi = 0$.
$\Rightarrow \|U - U_N\| \gtrsim |b_s + b^2|$ for $N \geq 2$.

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Obtaining stability estimates

General idea: control $(b_s + b^2)^2$ via an appropriate energy $E(s) := \|U - U_N\|^2$.

Goal: prove that

$$E(s) = O(s^{-4-\eta}) \text{ for some } \eta > 0. \qquad (2)$$

Starting point: write eq. on $U - U_N$ for N "large" ($N = 3$).
$\rightarrow$ of the form

$$\partial_s(U - U_N) + \cdots = \text{remainder terms.}$$

Error estimate: prove that

$$\frac{dE}{ds} + \frac{\alpha}{s}E(s) \leq \rho(s).$$

In order to achieve (2), one needs:

- $\rho(s) = O(s^{-5-\eta})$: "good" approximate solution;
- $\alpha > 4$: algebraic manipulations on the equation (R).

A transport-diffusion equation for U

Define, for $W \in L^\infty(\mathbf{R}_+)$,

$$L_U W := UW - U_\xi \int_0^\xi W = \left(\frac{\int_0^\xi W}{U} \right)_\xi U^2,$$

so that, if $W(\xi) = O(\xi^2)$ near $\xi = 0$,

$$L_U^{-1} W = \left(U \int_0^\xi \frac{W}{U^2} \right)_\xi .$$

Remark: $L_U^{-1} \sim$ division by $U \simeq \xi + \xi^2/2$.

Then (R) can be written as

$$\partial_s U - bU + \frac{b}{2}\xi \partial_\xi U - L_U^{-1}(\partial_{\xi\xi} U - 1) = 0.$$

Define $\mathcal{L}_U := L_U^{-1}\partial_{\xi\xi}$: diffusion operator. Then, with $V = U - U_N$

$$\partial_s V - bV + \frac{b}{2}\xi \partial_\xi V - \mathcal{L}_U V = \mathcal{R}_N.$$

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so that, if $W(\xi) = O(\xi^2)$ near $\xi = 0$,

$$L_U^{-1} W = \left(U \int_0^\xi \frac{W}{U^2} \right)_\xi .$$

Remark: $L_U^{-1} \sim$ division by $U \simeq \xi + \xi^2/2$.
Then (R) can be written as

$$\partial_s U - bU + \frac{b}{2} \xi \partial_\xi U - L_U^{-1}(\partial_{\xi\xi} U - 1) = 0.$$

Define $\mathcal{L}_U := L_U^{-1} \partial_{\xi\xi}$: diffusion operator. Then, with $V = U - U_N$

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A transport-diffusion equation for U

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Energy and dissipation terms

$$V = U - U_N, \quad \partial_s V - bV + \frac{b}{2}\xi\partial_\xi V - \mathcal{L}_U V = \mathcal{R}_N.$$

Facts:

1. Estimates are "almost" linear (up so some commutators...)
2. $V = U - U_N \sim (b_s + b^2)(c_1\xi^7 + c_2\xi^8)$ for $\xi \ll 1$;
3. $\mathcal{L}_U$ is a diffusion operator ($\mathcal{L}_U \sim \frac{1}{U}\partial_{\xi\xi}$).

Ideas:

- Differentiate equation/use weights/apply operator $\mathcal{L}_U$ to make the zero-order + transport term positive:

$$\partial_\xi^k \left(\partial_s V - bV + \frac{b}{2}\xi\partial_\xi V\right) = \left(\partial_s + \frac{k-2}{2}b + \frac{b}{2}\xi\partial_\xi\right)\partial_\xi^k V.$$

- Compromise between control of $(b_s + b^2)^2$ by energy/small remainder term/positivity of transport and diffusion...
- Energy $E(s) := \|(\mathcal{L}_U^2 V)_\xi\|_{H^1(w)}^2$ for some weight w.

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Tools for the proof

- Weighted L^2 estimates;
- Commutator estimates;
- L^∞ estimates coming from maximum principle (sub-super solutions);
- Bootstrap argument.

Plan

Summary

- Proof of separation for the stationary Prandtl equation in the case of adverse pressure gradient ($g(x) = -1$);
- Computation of a self-similar rate compatible with Landau's predictions:
$$\partial_Y u_{|Y=0} \sim \sqrt{x^* - x};$$
- Quantitative error estimates between true solution and approximate solution (in weighted H^s spaces);
- Construction of an approximate solution, ODE on the separation rate: relies on arguments close to singularity formation for the nonlinear Schrödinger equation.
- Energy estimates rely heavily on the structure of the equation, and need to be combined with maximum principle techniques.

Perspectives

- Other (unstable) separation rates?
- Better description of the solution in the zone $Y \gtrsim (x^* - x)^{1/4}$ $(\Leftrightarrow \xi \gtrsim s^{1/4})$;
- Higher dimensions?
- What happens after separation?
    1. ⚠Both turbulent and laminar regimes are possible... But turbulent regimes are out of reach for the time being.
    2. ⚠The validity of the Prandtl system after separation is far from clear...

THANK YOU FOR YOUR ATTENTION!