Partially congested propagation fronts in a 1d compressible Navier-Stokes model

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Outline

Plan

About congestion phenomena

Goal: study phenomena in which congested zones are present.
Examples: multi-phase flows; crowd motion; herding problems; etc.
Observation: the dynamics in the congested and non congested zones are very different (e.g. compressible vs. incompressible dynamics).
Issues:

1. Modelization of congestion effects;
2. Description of the transition between congested and non-congested zones.

References: [Maury; Degond, Hua; Berthelin; Bresch, Perrin, Zatorska; ...]

General modelization of congestion phenomena

In present talk, focus on continuous models (no particle systems).

Hard models: [Bouchut; Brenier&Grenier; Cavaletti et al.; Lions & Masmoudi;...] two-phase flows, e.g.

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0,$$
$$(1-\rho)\pi = 0, \quad \pi \geq 0, \quad 0 \leq \rho \leq 1.$$

Soft models: [Bresch, Perrin, Zatorska; Degond et al.;...] one phase flow with singular pressure, e.g.

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$$0 \leq \rho \leq 1, \quad \pi(\rho) = \epsilon \frac{\rho^\alpha}{(1-\rho)^\beta}$$

Transition between soft/hard models: limit $\epsilon \to 0$.

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Presentation of the model

Consider the following 1d compressible Navier-Stokes model

$$\begin{aligned} \partial_t v_\epsilon - \partial_x u_\epsilon &= 0 \\ \partial_t u_\epsilon + \partial_x p_\epsilon(v_\epsilon) - \mu \partial_x \left(\frac{1}{v_\epsilon} \partial_x u_\epsilon \right) &= 0 \end{aligned} \qquad (\mathrm{NS}_\epsilon)$$

with singular pressure $p_\epsilon(v) = \epsilon (v-1)^{-\gamma}, \ \gamma > 1, \ 0 < \epsilon \ll 1$
and endowed with the far-field conditions

$$\lim_{x \to \pm\infty} v_\epsilon(t,x) = v_\pm, \quad \lim_{x \to \pm\infty} u_\epsilon(t,x) = u_\pm.$$

Ref (non singular pressure): [Matsumura, Nishihara; Vasseur, Yao]
Formal limit as $\epsilon \to 0$: congested model

$$\begin{aligned} \partial_t v - \partial_x u &= 0, \\ \partial_t u + \partial_x \pi - \mu \partial_x \frac{\partial_x u}{v} &= 0, \\ \pi \geq 0, \quad (v-1)\pi &= 0. \end{aligned}$$

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Goals of this talk

1. Construct travelling wave solutions of (NS_ϵ), and study their asymptotic behavior;
2. Construct global solutions of (NS_ϵ) in the vicinity of a traveling wave.

Plan

Main result on travelling fronts

Theorem: [D., Perrin, '18] Let $1 < v_- < v_+$, and let $u_\pm \in \mathbb{R}$ such that $(u_+ - u_-)^2 = -(v_+ - v_-)(p_\epsilon(v_+) - p_\epsilon(v_-))$.

1. There exists a unique (up to a shift) solution of (NS_ϵ) of the form $(v_\epsilon, u_\epsilon) = (\mathfrak{v}_\epsilon, \mathfrak{u}_\epsilon)(x - s_\epsilon t)$, where

$$s_\epsilon^2 = -\frac{p_\epsilon(v_+) - p_\epsilon(v_-)}{v_+ - v_-}$$

2. Take $v_- = 1 + \epsilon^{1/\gamma}$, $v_+ \in]1, 2[$ independent of ϵ. Then there exists $\mathfrak{v} \in W^{1,\infty}(\mathbb{R})$ such that

$$\lim_{\epsilon \to 0} \sup_{\xi \in \mathbb{R}} \inf_{C \in \mathbb{R}} \|\mathfrak{v}_\epsilon(\xi + C) - \mathfrak{v}\|_\infty = 0,$$

and $\mathfrak{v}(\xi) = 1$ if $\xi < 0$, $\mathfrak{v}' = \bar{s}\mu^{-1}\mathfrak{v}(v_+ - \mathfrak{v})$ for $\xi > 0$.

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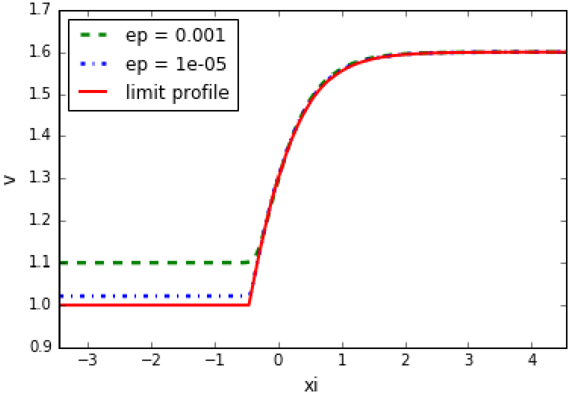
Numerical simulations



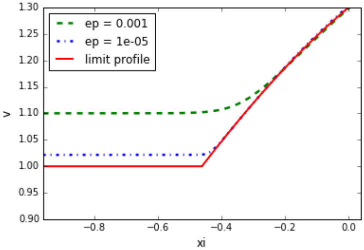
Refined behavior in the transition zone

Ansatz:

$$\mathfrak{v}_\epsilon \simeq \begin{cases} 1 + \epsilon^{1/\gamma} \tilde{v}\left(\dfrac{\xi - \bar{\xi}}{\epsilon^{1/\gamma}}\right) & \text{if} \quad \xi < 0 \\ \bar{v}(\xi + \varphi^\epsilon) & \text{if} \quad \xi \geq 0. \end{cases} \tag{1}$$

Result:

- $\varphi^\epsilon, \bar{\xi} = O(\epsilon^{1/\gamma+1})$;
- $\tilde{v}$ converges towards 1 exponentially fast.



Sketch of proof: ODE for traveling fronts

$$\begin{cases} -s_\epsilon \mathfrak{v}_\epsilon' - \mathfrak{u}_\epsilon' = 0 \\ -s_\epsilon \mathfrak{u}_\epsilon' + \big(p_\epsilon(\mathfrak{v}_\epsilon)\big)'(\xi) - \mu \left(\dfrac{1}{\mathfrak{v}_\epsilon}\mathfrak{u}_\epsilon'\right)' = 0. \end{cases} \tag{2}$$

Rankine-Hugoniot conditions:

$$s_\epsilon(v_+ - v_-) = -(u_+ - u_-), \quad s_\epsilon(u_+ - u_-) = p_\epsilon(v_+) - p_\epsilon(v_-).$$

ODE for $\mathfrak{v}_\epsilon$:

$$\mathfrak{v}_\epsilon' = \frac{\mathfrak{v}_\epsilon}{\mu s_\epsilon}\big(s_\epsilon^2(v_+ - \mathfrak{v}_\epsilon) + p_\epsilon(v_+) - p_\epsilon(\mathfrak{v}_\epsilon)\big).$$

Conclusion: unique solution (up to a shift), strictly increasing from v_- to v_+.
Remark: $\lim_{\epsilon \to 0} s_\epsilon \in]0, +\infty[\iff v_- = 1 + c_0 \epsilon^{1/\gamma} + o(\epsilon^{1/\gamma})$.

Asymptotic behavior

Non-congested zone ($\xi > 0$): corresponds to $p_\epsilon(\mathfrak{v}_\epsilon) \ll 1$.
$\mathfrak{v}_\epsilon \to \mathfrak{v}$, solution of the logistic equation

$$\mathfrak{v}' = \frac{\bar{s}}{\mu}\mathfrak{v}(v_+ - \mathfrak{v}).$$

Take $\mathfrak{v}(0) = 1$. Then explicit solution for $\xi > 0$.
Congested zone ($\xi < 0$): corresponds to $p_\epsilon(\mathfrak{v}_\epsilon) \simeq 1$.
Then $\mathfrak{v}_\epsilon \to 1$.
Remark: the limit profile has a sharp transition!
Refined description of the transition: plug-in Ansatz (1). Then

$$\tilde{v}' = \frac{1}{\mu\bar{s}}\left(1 - \frac{1}{\tilde{v}^\gamma}\right). \tag{3}$$

Hence $\tilde{v}(y) - 1 = O(\exp(y\gamma/(\mu\bar{s})))$ as $y \to -\infty$.

Plan

Effective velocity and linearization

To study stability properties, switch to variables (w_ϵ, v_ϵ), where $w_\epsilon = u_\epsilon - \mu\partial_x \ln(v_\epsilon)$ (see e.g. [Desjardins&Grenier])

$$\begin{aligned} \partial_t w_\epsilon + \partial_x p_\epsilon(v_\epsilon) = 0, \\ \partial_t v_\epsilon - \partial_x w_\epsilon - \mu\partial_{xx} \ln v_\epsilon = 0. \end{aligned} \tag{4}$$

Idea: look at equation on (W_ϵ, V_ϵ), where

$$W_\epsilon := \int_{-\infty}^x (w_\epsilon - \mathfrak{w}_\epsilon), \quad V_\epsilon := \int_{-\infty}^x (v_\epsilon - \mathfrak{v}_\epsilon).$$

Then

$$\begin{aligned} \partial_t W_\epsilon + \overbrace{p_\epsilon(\mathfrak{v}_\epsilon + \partial_x V_\epsilon) - p_\epsilon(\mathfrak{v}_\epsilon)}^{\simeq p_\epsilon'(\mathfrak{v}_\epsilon)\partial_x V_\epsilon} = 0, \\ \partial_t V_\epsilon - \partial_x W_\epsilon - \mu\partial_x \underbrace{\ln \frac{\mathfrak{v}_\epsilon + \partial_x V_\epsilon}{\mathfrak{v}_\epsilon}}_{\simeq \frac{\partial_x V_\epsilon}{\mathfrak{v}_\epsilon}} = 0. \end{aligned}$$

Main tools for the stability analysis

General scheme: Fixed point argument for small data for (W_ϵ, V_ϵ) in a high regularity space.

Tool # 1: weighted L^2 energy estimate for the linearized operator;
Tool #2: commutator estimate (linearized equation is not stable by differentiation);
Tool #3: product laws to control the quadratic terms.

Energy estimate for the linearized operator

Linearized equation:

$$\begin{aligned} \partial_t f + p_\epsilon'(\mathfrak{v}_\epsilon)\partial_x g &= 0, \\ \partial_t g - \partial_x f - \mu \partial_x \left(\frac{\partial_x g}{\mathfrak{v}_\epsilon} \right) &= 0. \end{aligned} \tag{5}$$

Weighted L^2 estimate:

$$\frac{d}{dt}\int_{\mathbb{R}} \left[-\frac{|f|^2}{p_\epsilon'(\mathfrak{v}_\epsilon)} + |g|^2 \right] + s_\epsilon \int_{\mathbb{R}} \frac{p_\epsilon''(\mathfrak{v}_\epsilon)}{(p_\epsilon'(\mathfrak{v}_\epsilon))^2} \partial_x \mathfrak{v}_\epsilon |f|^2 + \mu \int_{\mathbb{R}} \frac{(\partial_x g)^2}{\mathfrak{v}_\epsilon} \leq 0.$$

Remark:

$$-\frac{1}{p_\epsilon'(\mathfrak{v}_\epsilon)} = \frac{(\mathfrak{v}_\epsilon - 1)^{\gamma+1}}{\gamma \epsilon}, \quad \frac{p_\epsilon''(\mathfrak{v}_\epsilon)}{(p_\epsilon'(\mathfrak{v}_\epsilon))^2} = \frac{\gamma}{\gamma} \frac{(\mathfrak{v}_\epsilon - 1)^{\gamma}}{\epsilon}.$$

The control of f is very good in non-congested zones, not as good in very congested zones... Source of loss in the energy estimates.

Sequence of estimates

Let $\mathcal{L}_\epsilon$ be the linearized operator, and rewrite the equation as

$$\partial_t \begin{pmatrix} W_\epsilon \\ V_\epsilon \end{pmatrix} + \mathcal{L}_\epsilon \begin{pmatrix} W_\epsilon \\ V_\epsilon \end{pmatrix} = \mathcal{G}_\epsilon(\partial_x V_\epsilon).$$

$\mathcal{G}_\epsilon$: quadratic term.

Step 1: apply estimates on linearized operator, treating $\mathcal{G}_\epsilon$ perturbatively.

Step 2: differentiate:

$$\partial_t \begin{pmatrix} \partial_x W_\epsilon \\ \partial_x V_\epsilon \end{pmatrix} + \mathcal{L}_\epsilon \begin{pmatrix} \partial_x W_\epsilon \\ \partial_x V_\epsilon \end{pmatrix} = \partial_x \mathcal{G}_\epsilon(\partial_x V_\epsilon) + [\mathcal{L}_\epsilon, \partial_x] \begin{pmatrix} W_\epsilon \\ V_\epsilon \end{pmatrix},$$

etc.

Bound on commutator term?

Commutator result

Lemma: For sufficiently smooth and decaying (f, g),

$$\left| \int_0^T \int_{\mathbb{R}} [\mathcal{L}_\epsilon, \partial_x] \begin{pmatrix} f \\ g \end{pmatrix} \cdot \begin{pmatrix} \dfrac{-\partial_x f}{p'_\epsilon(\mathfrak{v}_\epsilon)} \\ \partial_x g \end{pmatrix} \right| \leq \underbrace{\int_0^T \int_{\mathbb{R}} \partial_x \mathfrak{v}_\epsilon |\partial_x f|^2}_{\text{absorbed in dissipation term}} + C_1 \epsilon^{-2/\gamma} \underbrace{\int_0^T \int_{\mathbb{R}} |\partial_x g|^2}_{\text{lower order dissipation term}} \tag{6}$$

Consequence: Loss of a power $\epsilon^{1/\gamma}$ with each derivative.
Ansatz: Form of energy:

$$E(t) = \sum_k \epsilon^{2k/\gamma} \int_{\mathbb{R}} \left[-\frac{|\partial_x^k W_\epsilon|^2}{p'_\epsilon(\mathfrak{v}_\epsilon)} + |\partial_x^k V_\epsilon|^2 \right]$$

(In practice, because of non-linearity, $k \in \{0, 1, 2\}$.)

WP result in the vicinity of travelling fronts

Theorem [D., Perrin, '18] Let $\alpha > \frac{5}{\gamma}$ and assume that

$$E(0) \leq \epsilon^\alpha.$$

Then the non-linear system has a unique global solution, and $E(t) \leq \epsilon^\alpha$ for all $t \geq 0$.

Corollary: under the same assumptions,

$$\lim_{t\to\infty} (w_\epsilon, v_\epsilon)(t) = (\mathfrak{w}_\epsilon, \mathfrak{v}_\epsilon).$$

Remark: very stringent assumption, but necessary because of the pressure singularity and the loss of powers of ϵ with each derivation.

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Summary

- Presentation of a soft congestion model with singular pressure;
- Travelling fronts for the singular model: description of a sharp transition between congested/non-congested zones;
- Stability of travelling fronts within the singular model, with quantitative analysis.

Perspectives

- Relaxation of assumptions for the stability analysis?
- Rates of convergence as $t \to \infty$?
- Consequences on the modelization of the limit system?

THANK YOU FOR YOUR ATTENTION!