Étude mathématique de fluides en rotation rapide avec forçage en surface

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Main assumptions in the interior

- **Starting point:** Ocean = homogeneous, incompressible fluid in a rotating frame.
  → 3D Navier-Stokes equations with Coriolis force $\Omega \wedge u$.
- **Coriolis acceleration:**
  → *f*-plane approximation: $f = 2|\Omega|\sin(\theta)$ homogeneous ("small" geographical zone, midlatitudes);
  → effect of horizontal component of Ω is neglected.
- **Frictional forces $\mathcal{F}$:** notion of "turbulent viscosity":

$$\mathcal{F} = A_v \partial_z^2 u + A_h \Delta_h u, \quad A_h, A_v > 0, \; A_h \neq A_v.$$

- **Conclusion:** the velocity u of currents inside the ocean is described by

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + f e_3 \wedge u - A_v \partial_z^2 u - A_h \Delta_h u + \nabla p = 0, \\ \nabla \cdot u = 0. \end{aligned} \tag{1}$$

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Boundary conditions

- **Bottom of the ocean:** flat ($h_B \equiv 0$).
  Homogeneous Dirichlet boundary condition (no-slip):
  $$u_{|z=0} = 0.$$
- **Surface of the ocean:** rigid lid approximation: $h \equiv D$.
  Description of wind-stress:
  $$\partial_z u_{h|z=D} = \sigma_h,$$
  $$u_{3|z=D} = 0.$$
- **Horizontal boundaries:** box $\rightarrow$ horizontal domain:
  $[0, La_1) \times [0, La_2)$ with periodic boundary conditions.

Scaling assumptions

- **High rotation limit:** Rossby number $\varepsilon := \dfrac{U}{f|L|} \ll 1.$
- **Horizontal and vertical viscosities:**

$$\frac{A_h}{UL} \approx 1, \quad \nu := \frac{LA_v}{UD^2} \ll 1.$$

- **Amplitude of wind stress:** $\alpha := \dfrac{D\sigma_0}{U} \gg 1.$

Ω	Earth rotation vector	A_h	Turbulent horizontal viscosity
L	Horizontal length scale	A_v	Turbulent vertical viscosity
U	Horizontal velocity scale	σ_0	Amplitude of wind velocity
D	Vertical length scale		

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- **Conclusion:** the system in rescaled variables becomes

$$\partial_t u + u \cdot \nabla u + \frac{1}{\varepsilon} e_3 \wedge u + \nabla p - \Delta_h u - \nu \partial_z^2 u = 0,$$

$$\begin{aligned} \mathrm{div} u &= 0, \qquad & \partial_z u_{h,z=a} &= \alpha\sigma, \\ u_{|z=0} &= 0, & u_{3,z=a} &= 0. \end{aligned}$$

New domain: $V = [0, a_1) \times [0, a_2) \times [0, a);$

Modelization of the wind stress

- Full atmosphere/ocean coupled model is out of reach...
  $\rightarrow$ Effect of a given wind stress on ocean dynamics.
- **Time dependance** of wind stress:
  Coriolis op. $\rightsquigarrow$ fast oscillations in time (freq. $\sim 1/\varepsilon$).
  $\rightarrow$ Interesting scaling: $\sigma = \sigma\left(t, \frac{t}{\varepsilon}, x_h\right)$.
- **First choice:** σ almost-periodic: [Masmoudi, 2000]

$$\sigma(t, \tau, x_h) = \sum_{\mu \in M} \sum_{k_h \in \mathbb{Z}^2} \hat{\sigma}(t, \mu, k_h) e^{ik_h \cdot x_h} e^{i\mu\tau}$$

- **Second choice:** σ stationary:

$$\sigma(t, \tau, x_h; \omega) = S(t, x_h, \theta_\tau \omega),$$

  where
  - $\omega \in E$, and $(E, \mathcal{A}, \mu)$ is a probability space,
  - $(\theta_\tau)_{\tau \in \mathbb{R}}$ is a measure preserving transformation group acting on E.

  Interest: introduce some randomness in the equation.

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Brief review of results on rotating fluids

Ref: Chemin, Desjardins, Gallagher, Grenier.

- **Dominant process:** Coriolis operator:

$$L = \mathbb{P}(e_3 \wedge \cdot);$$

$$\text{Spectrum } \{\lambda_k := -i\frac{k_3'}{|k'|},\ k \in \mathbb{Z}^3 \setminus \{0\}\ \}.$$

$\rightarrow$ Creation of waves propagating at speed ε^{-1}.

- **Filtering method** [Grenier; Schochet]:
  Equation for $u_L = \exp\left(\frac{t}{\varepsilon}L\right) u$.
  $\rightarrow$ Passage to the limit as $\varepsilon, \nu \rightarrow 0$: envelope equations;
  $\rightarrow$ Problem: u_L does not match the boundary conditions.
- **Construction of boundary layers**[Colin-Fabrie; Desjardins-Grenier; Grenier-Masmoudi; Masmoudi ...]
  $\rightarrow$ Creation of source terms (Ekman pumping) in envelope equation.

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Coupling between interior and boundary layer terms

Consider the following **Ansatz**

$$u(t,x,y,z) \approx u_{\text{int}}\left(t, \frac{t}{\varepsilon}, x, y, z\right) + u_{\text{BL}}\left(t, \frac{t}{\varepsilon}, x, y, z\right),$$

where

- $u_{\text{int}}(t,\tau) = \exp(-\tau L)\, u_L(t) + \delta u_{\text{int}}(t,\tau)$, $\delta u_{\text{int}} = o(1)$;
  Role: $u_{\text{int}}(t, t/\varepsilon)$ satisfies the evolution equation (up to $o(1)$);
- $u_{\text{BL}}(\cdot, z) = u_T(\cdot, (a-z)/\eta) + u_B(\cdot, z/\eta)$, $\eta \ll 1$.
  Role: u_{BL} matches the horizontal boundary conditions.

Remarks:

- The horizontal BC for u_{BL} depend on u_{int};
- The vertical BC for δu_{int} depends on u_{BL}, and creates a source term (Ekman pumping) in equation for u_L.

$\longrightarrow$ Coupling between u_{int} and u_{BL}.

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Method of resolution

Idea: define a **boundary layer operator $\mathcal{B}$:**

- Input: arbitrary horizontal boundary conditions.
- Output: divergence-free boundary layer term, matching the horizontal BC and equation at leading order.

and an **interior operator $\mathcal{U}$:**

- Input: arbitrary initial data and vertical boundary conditions.
- Output: interior term matching the vertical boundary conditions and equation at leading order.

Elementary step: adapt inputs of $\mathcal{U}$ and $\mathcal{B}$ such that BC and eq. are satisfied (at leading order).

Question: when should the construction stop ?

$\longrightarrow$ Answer: when all remaining boundary terms are $o(\varepsilon)$ in L^2.

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Convergence result

Theorem:[D., Saint-Raymond, 2008]
Let $u = u^{\varepsilon,\nu}$ be the solution of

$$\begin{cases} \partial_t u + \frac{1}{\varepsilon} e_3 \wedge u - \nu \partial_z^2 u - \Delta_h u + \nabla p = 0, \\ \mathrm{div} u = 0, \\ u_{|z=0} = 0, \\ u_{3|z=a} = 0, \quad \partial_z u_{h|z=a}(t) = \frac{1}{(\varepsilon\nu)^\kappa} \sum_{\mu, k_h} \hat{\sigma}(\mu, k_h) e^{i\mu\frac{t}{\varepsilon}} e^{ik_h \cdot x_h}. \end{cases}$$

Let w be the solution of the envelope equation. There exists a function u^{sing}, of order $(\varepsilon\nu)^{-\kappa}$ in L^∞, and a constant $\kappa_0 > 0$, such that if $\varepsilon, \nu \to 0$ with $\nu = \mathcal{O}(\varepsilon)$ and $\kappa < \kappa_0$, then

$$u^{\varepsilon,\nu} - \left(\exp\left(-\frac{t}{\varepsilon} L \right) w(t) + u^{\text{sing}} \right) \to 0,$$

in $L^\infty_{\text{loc}}(0, \infty; L^2(V)) \cap L^2_{\text{loc}}(0, \infty; H^1_h(V))$.

Remarks on the convergence result

- No *a priori* bounds for $u^{\varepsilon,\nu}$.
- In general, $u^{\varepsilon,\nu}$ does not remain bounded: destabilization of the whole fluid inside the domain.
- The **singular profile** u^{sing} is explicit. Linear response to forcing on the mode

$$k_h = 0, \mu = \pm 1.$$

In particular, u^{sing} does not depend on x_h and $u_3^{\text{sing}} \equiv 0$.
$\rightarrow$ No singular Ekman transpiration velocity.

- No asymptotic expansion for $u^{\varepsilon,\nu}$ with this method.

In the sequel:

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- Focus on uncommon behaviour: apparition of atypical boundary layers, singular profile.

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General setting

Ansatz:

$$u_{\mathrm{BL}} = u_B\left(t, \frac{t}{\varepsilon}, x_h, \frac{z}{\sqrt{\varepsilon\nu}}\right) + u_T\left(t, \frac{t}{\varepsilon}, x_h, \frac{a-z}{\sqrt{\varepsilon\nu}}\right),$$

and

$$u_T/u_B = \sum_{k_h,\mu} \hat{u}_T/\hat{u}_B(t,k_h,\mu)e^{i\mu\tau}e^{ik_h\cdot x_h}\exp(-\lambda z).$$

Linearity: work with fixed k_h and μ ($\lambda = \lambda(k_h,\mu)$).

Equation in rescaled variables:

$$\begin{aligned} i\mu\hat{u}_1 - \lambda^2\hat{u}_1 - \hat{u}_2 + \varepsilon k_h^2\hat{u}_1 + \varepsilon\nu\frac{k_1k_2\hat{u}_1 - k_1^2\hat{u}_2}{\lambda^2 - \varepsilon\nu k_h^2} &= 0, \\ i\mu\hat{u}_2 - \lambda^2\hat{u}_2 + \hat{u}_1 + \varepsilon k_h^2\hat{u}_2 + \varepsilon\nu\frac{-k_1k_2\hat{u}_2 + k_2^2\hat{u}_1}{\lambda^2 - \varepsilon\nu k_h^2} &= 0, \\ \sqrt{\varepsilon\nu}(ik_1\hat{u}_1 + ik_2\hat{u}_2) \pm \lambda\hat{u}_3 &= 0. \end{aligned} \tag{2}$$

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General setting - 2

Question: find $\lambda \in \mathbb{C}$ such that $\det A_\lambda = 0$, where

$$A_\lambda = \begin{pmatrix} i\mu - \lambda^2 + \varepsilon k_h^2 + \dfrac{\varepsilon\nu k_1 k_2}{\lambda^2 - \varepsilon\nu k_h^2} & -1 - \dfrac{\varepsilon\nu k_1^2}{\lambda^2 - \varepsilon\nu k_h^2} \\ 1 + \dfrac{\varepsilon\nu k_2^2}{\lambda^2 - \varepsilon\nu k_h^2} & i\mu - \lambda^2 + \varepsilon k_h^2 - \dfrac{\varepsilon\nu k_1 k_2}{\lambda^2 - \varepsilon\nu k_h^2} \end{pmatrix}$$

Different cases:

- $\mu \neq \pm 1$: eigenvalues of $\begin{pmatrix} i\mu & -1 \\ 1 & i\mu \end{pmatrix}$ are non zero.

  $\rightarrow$ Stability by small linear perturbations.

  Conclusion: $\lambda = \mathcal{O}(1)$ (bounded away from 0).

- $\mu = \pm 1$: one of the eigenvalues of $\begin{pmatrix} i\mu & -1 \\ 1 & i\mu \end{pmatrix}$ is zero.

  $\rightarrow$ Two sub-cases:

  - $k_h \neq 0$: atypical boundary layer ($\lambda = \mathcal{O}(\sqrt{\varepsilon} + (\varepsilon\nu)^{1/4})$).
  - $k_h = 0$: $\lambda = 0$ is a solution! $\rightarrow$ singular profile (bifurcation).

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$$A_\lambda = \begin{pmatrix} i\mu - \lambda^2 + \varepsilon k_h^2 + \dfrac{\varepsilon\nu k_1 k_2}{\lambda^2 - \varepsilon\nu k_h^2} & -1 - \dfrac{\varepsilon\nu k_1^2}{\lambda^2 - \varepsilon\nu k_h^2} \\ 1 + \dfrac{\varepsilon\nu k_2^2}{\lambda^2 - \varepsilon\nu k_h^2} & i\mu - \lambda^2 + \varepsilon k_h^2 - \dfrac{\varepsilon\nu k_1 k_2}{\lambda^2 - \varepsilon\nu k_h^2} \end{pmatrix}$$

Different cases:

- $\mu \neq \pm 1$: eigenvalues of $\begin{pmatrix} i\mu & -1 \\ 1 & i\mu \end{pmatrix}$ are non zero.

  $\rightarrow$ Stability by small linear perturbations.
  Conclusion: $\lambda = \mathcal{O}(1)$ (bounded away from 0).
- $\mu = \pm 1$: one of the eigenvalues of $\begin{pmatrix} i\mu & -1 \\ 1 & i\mu \end{pmatrix}$ is zero.

  $\rightarrow$ Two sub-cases:
  - $k_h \neq 0$: atypical boundary layer ($\lambda = \mathcal{O}(\sqrt{\varepsilon} + (\varepsilon\nu)^{\frac{1}{4}})$).
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Classical Ekman layers: $\mu \neq \pm 1$

At first order,

$$A_\lambda = \begin{pmatrix} i\mu - \lambda^2 & -1 \\ 1 & i\mu - \lambda^2 \end{pmatrix}.$$

Eigenvalues: $\lambda_\pm^2 = i(\mu \pm 1) + o(1)$;
Eigenvectors: $w_\pm = (1, \pm i) + o(1)$.
Conclusion: $\{w_+, w_-\}$ basis of $\mathbb{C}^2$.

Method: decompose the boundary condition δ_h (input of $\mathcal{B}$) onto basis $\{w_+, w_-\}$:

$$\hat{\delta}_h(k_h, \mu) = \alpha_+ w_+ + \alpha_- w_-.$$

Horizontal part of the boundary layer term is given by

$$u_{B,h} = \left(\alpha_+ w_+ e^{-\lambda_+ z} + \alpha_- w_- e^{-\lambda_- z}\right) e^{i\mu\tau} e^{ik_h \cdot x_h}$$

$$u_{T,h} = (\varepsilon\nu)^{\frac{1}{2}-\kappa} \left(\frac{\alpha_+}{\lambda_+} w_+ e^{-\lambda_+ z} + \frac{\alpha_+}{\lambda_-} w_- e^{-\lambda_- z}\right) e^{i\mu\tau} e^{ik_h \cdot x_h}.$$

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Atypical boundary layers: $\mu = \pm 1$, $k_h \neq 0$

$$\det A_\lambda = 0 \Rightarrow \begin{cases} \lambda_+^2 = 2\mu i + o(1) \\ \text{or } \lambda_-^2 = \mathcal{O}(\varepsilon + \sqrt{\varepsilon\nu}). \end{cases}$$

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Apparition of a singular profile: $\mu = \pm 1$, $k_h = 0$

Choosing for example $\mu = 1$, we derive

$$A_\lambda = \begin{pmatrix} i-\lambda^2 & -1 \\ 1 & i-\lambda^2 \end{pmatrix}.$$

Eigenvalues: $\lambda_-^2 = 2i$, $\lambda_+^2 = 0$;
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Remark: define $\bar{u}^{\text{sing}} := \frac{z}{(\varepsilon\nu)^\kappa} e^{i\frac{t}{\varepsilon}} \begin{pmatrix} w_+ \\ 0 \end{pmatrix}$. Then

$$\bar{u}^{\text{sing}}_{|z=0} = 0, \quad \partial_z \bar{u}^{\text{sing}}_{h|z=a} = \frac{1}{(\varepsilon\nu)^\kappa} e^{i\frac{t}{\varepsilon}} w_+.$$

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Singular part of the "boundary layer" term is given by

$$u_{\text{BL},h} = \left(\alpha_{B,+} + \frac{\alpha_{T,+} z}{(\varepsilon\nu)^\kappa}\right) w_+ e^{i\frac{t}{\varepsilon}}.$$

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Plan

Decomposition of u_{int} for $k_h \neq 0$

Explicit construction:

$$u_{\text{int}} = \mathcal{U}[v_B, v_T, u_0]$$

such that u_{int} is a solution of the evolution equation and satisfies

$$u_{\text{int}}(t=0) = u_0 + o(1),\ \ u_{\text{int},3|z=0} = \sqrt{\varepsilon\nu}\, v_B,\ \ u_{\text{int},3|z=a} = \sqrt{\varepsilon\nu}\, v_T.$$

Decomposition: $u_{\text{int}} = \exp\left(\frac{t}{\varepsilon}L\right) w(t) + v_{\text{int}} + u_{\text{int}}^{\text{osc}}$ where

- $w(t)$: preponderant term; matches initial data u_0;
- v_{int}: known explicitely;

$$v_{\text{int}} := \sqrt{\varepsilon\nu} \begin{pmatrix} \nabla_h \Delta_h^{-1}(v_B - v_T) \\ v_T z + v_B(1-z) \end{pmatrix};$$

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Derivation of equations for w and $u_{\text{int}}^{\text{osc}}$

Functional preliminaries: define

$$F_0 := \{u \in L^2(V),\ \operatorname{div} u = 0,\ u_{3|z=0} = u_{3|z=a} = 0\}.$$

$\mathbb{P}$: projection on F_0;
$(N_k)_{k\geq 0}$: hilbertian basis of F_0, such that $LN_k = \lambda_k N_k$.
Set

$$\Sigma := \partial_t v_{\text{int}} + \frac{1}{\varepsilon} e_3 \wedge v_{\text{int}} - \nu \partial_z^2 v_{\text{int}} - \Delta_h v_{\text{int}}.$$

Then $w_{\text{int}} := \exp(-t/\varepsilon L)\, w + u_{\text{int}}^{\text{osc}}$ is such that

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Rules:

- All terms in $\langle N_k, \Sigma\rangle$ oscillating at frequencies λ_k/ε become source terms in equation on w;
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Singular profile for $k_h = 0$

Problem: recall singular profile

$$\bar{u}^{\text{sing}} = \sum_{\pm} \left(\alpha_{B,\pm} + \frac{\alpha_{T,\pm} z}{(\varepsilon\nu)^{\kappa}} \right) w_{\pm} e^{\pm i \frac{t}{\varepsilon}}.$$

Does not match initial condition !

Idea: build $u^{\text{sing}} := \bar{u}^{\text{sing}} + u^{\text{sing}}_{\text{osc}}$, where

$$\partial_t u^{\text{sing}}_{\text{osc}} + \frac{1}{\varepsilon} L u^{\text{sing}}_{\text{osc}} - \nu \partial_z^2 u^{\text{sing}}_{\text{osc}} = 0$$

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Idea: build $u^{\text{sing}} := \bar{u}^{\text{sing}} + u^{\text{sing}}_{\text{osc}}$, where

$$\partial_t u^{\text{sing}}_{\text{osc}} + \frac{1}{\varepsilon} L u^{\text{sing}}_{\text{osc}} - \nu \partial_z^2 u^{\text{sing}}_{\text{osc}} = 0$$

$$u^{\text{sing}}_{\text{osc}}(t=0) = -\bar{u}^{\text{sing}}(t=0),$$

$$u^{\text{sing}}_{\text{osc},h|z=0} = 0, \quad \partial_z u^{\text{sing}}_{\text{osc},h|z=a} = 0 \ (t>0),$$

$$u^{\text{sing}}_{\text{osc},3} \equiv 0.$$

Remark: no stabilization.

Plan

Explicit construction of the approximate solution

- **First step:** define the singular profile u^{sing} and the solution of the envelope equation w, given by

$$\begin{cases} \partial_t w - \Delta_h w + \sqrt{\frac{\nu}{\varepsilon}} S_{\text{Ekman}}[w] = 0, \\ w_{|t=0} = u_{|t=0}, \end{cases}$$

  where $S_{\text{Ekman}} : F_0 \to F_0$, $S_{\text{Ekman}} \geq 0$.
- **Second step:** define a BL term $\mathcal{B}(\delta_{0,h}, \delta_{1,h})$, where
  - $\delta_{0,h}$: trace of w on $z = 0$;
  - $\delta_{1,h} = \sigma$ (wind forcing).
- **Third step:** define the rest of the interior term (of order $o(1)$ in L^2): $v^{\text{int}} + u^{\text{int}}_{\text{osc}}$.
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  At this stage: evolution eq. satisfied up to $\mathcal{O}(1)$ terms, horizontal BC are satisfied, and vertical BC are satisfied up to $\mathcal{O}((\varepsilon\nu)^{\frac{1}{2}-\kappa})$ terms.
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Conclusion of the almost-periodic case

Linear problem:

- Apparition of atypical boundary layers due to resonant forcing ($\mu = \pm 1$) on the non-homogeneous modes ($k_h \neq 0$).
- Singular profile ($\mu = \pm 1, k_h = 0$) which destabilizes the whole fluid for arbitrary initial data.
- Linearity of the equation enables explicit calculations.

Nonlinear problem:

Recent result [D., Saint-Raymond, '07]: stability of singular profile in H^s norm and when the amplitude of the wind-stress is not too large.

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The stationary setting

Recall that

$$\sigma = S\left(t, x_h, \theta_{\frac{t}{\varepsilon}}\omega\right).$$

Assumption of non-resonance: (avoid singular profile)
Define approximate Fourier transform: for $\gamma > 0$,

$$\hat{\sigma}_\gamma(\lambda, \omega) := \frac{1}{2\pi}\int_{\mathbb{R}} \exp(-\gamma|\tau|)e^{-i\lambda\tau}\sigma(\tau, \omega)\, d\tau.$$

Assume that

(H1) $\forall\gamma > 0$, $\hat{\sigma}_\gamma \in L^\infty(E, L^1(\mathbb{R}))$, and

$$\sup_{\gamma>0} ||\hat{\sigma}_\gamma||_{L^\infty(E,L^1(\mathbb{R}))} < +\infty.$$

(H2) $\exists$ neighbourhoods $V_\pm$ of ± 1, independent of $\gamma > 0$, such that

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Convergence result in the nonlinear stationary case

Theorem:[D., 2007] Let $u = u^{\varepsilon,\nu}$ be the solution of

$$\begin{cases} \partial_t u + \frac{1}{\varepsilon} e_3 \wedge u + u \cdot \nabla u - \nu \partial_z^2 u - \Delta_h u + \nabla p = 0, \\ \mathrm{div} u = 0, \\ u_{|z=0} = 0, \\ u_{3|z=a} = 0, \quad \partial_z u_{h|z=a}(t) = \frac{1}{(\varepsilon\nu)^{\frac{1}{2}}} \sigma\left(t, \frac{t}{\varepsilon}, x_h, \omega\right). \end{cases}$$

Let $w \in L^\infty(0, T^*; H^s)$ ($s > 5/2$) be the solution of the envelope equation, and assume that **(H1)-(H2)** are satisfied.
Then as $\varepsilon, \nu \to 0$ with $\nu = \mathcal{O}(\varepsilon)$,

$$u^{\varepsilon,\nu} - \left(\exp\left(\frac{t}{\varepsilon} L\right) w(t)\right) \to 0,$$

in $L^\infty(0, T; L^2(V \times E)) \cap L^2((0, T) \times E, H^1_h(V))$ for all $T < T^*$.

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Elements of the proof

Same strategy as in almost-periodic case. Main features:

- No atypical boundary layer terms (non-resonance);
- Boundary layer terms are random stationary in time;
- Filtering methods $\rightarrow$ need to investigate average behaviour of oscillating functions.
  Variant of ergodic Theorem:

Lemma

Let $\phi \in L^1(E, \mu)$, and let $\lambda \in \mathbb{R}$. Then $\exists \bar{\phi}^\lambda \in L^1(E)$,

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The envelope equation

The function w is a solution of

$$\begin{cases} \partial_t w + \bar{Q}(w,w) - \Delta_h w + \underbrace{\bar{S}_B(w) + \bar{S}_T(\omega)}_{\text{Ekman pumping}} = 0, \\ w(t=0) = w_0 \in H^s, \quad \operatorname{div} w_0 = 0, \\ \operatorname{div} w = 0, \\ w_{3|z=0} = 0, \ w_{3|z=a} = 0, \end{cases}$$

In general, w **is random**... However, $\bar{w} = 1/a \int_0^a w$ is not!

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Limit system in the case of non-resonant torus

If the torus is non-resonant, then

$$\bar{Q}(w,w) = \bar{Q}(\bar{w},\bar{w}) + \underbrace{\bar{Q}(\bar{w}, w-\bar{w}) + \bar{Q}(w-\bar{w},\bar{w})}_{=:q(\bar{w},w-\bar{w})}.$$

→ **The limit equation decouples:** $w = \bar{w} + \tilde{w}_1 + \tilde{w}_2$, where

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Perspectives

- Include treatment of singular profile in the random case (avoid non-resonance assumptions);
- Use β-plane instead of f-plane model (variations of Coriolis parameter): modification of the weak limit, apparition of vertical boundary layers on the western boundaries.
- Consider more general boundaries (different types of boundary layers are expected).
- Work with density-dependent models.