Modèles de couche limite en mécanique des fluides : au delà de l'équation de Prandtl ?

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Outline

Plan

Motivation and Ansatz

Goal: understand the behavior of 2d fluids with small viscosity around an obstacle.
Simplification: obstacle=half-plane: consider

$$\begin{aligned} \partial_t \mathbf{u}^\nu + (\mathbf{u}^\nu \cdot \nabla)\mathbf{u}^\nu + \nabla p^\nu - \nu\Delta \mathbf{u}^\nu = 0 \text{ in } \mathbf{R}^2_+, \\ \operatorname{div} \mathbf{u}^\nu = 0 \text{ in } \mathbf{R}^2_+, \\ \mathbf{u}^\nu_{|y=0} = 0, \quad \mathbf{u}^\nu_{|t=0} = \mathbf{u}^\nu_0. \end{aligned} \tag{1}$$

Prandtl, 1904: in the limit $\nu \ll 1$,

$$\mathbf{u}^\nu(x,y) \simeq \begin{cases} \mathbf{u}^E(x,y) \text{ for } y \gg \sqrt{\nu} \text{ (sol. of 2d Euler)}, \\ \left(u^P\left(x, \frac{y}{\sqrt{\nu}}\right), \sqrt{\nu} v^P\left(x, \frac{y}{\sqrt{\nu}}\right)\right) \text{ for } y \lesssim \sqrt{\nu}. \end{cases}$$

Boundary conditions: $\mathbf{u}^E \cdot e_y = 0$ on $\{y=0\}$ (non-penetration), $\mathbf{u}^P_{|Y=0} = 0$.

The Prandtl equation: general remarks

The equation in the boundary layer becomes

$$\begin{aligned} \partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P - \partial_{YY} u^P &= -\frac{\partial p^E}{\partial x}(t,x,0) \\ \partial_x u^P + \partial_Y v^P &= 0, \\ \mathbf{u}^P_{|Y=0} = 0, \ \lim_{Y\to\infty} u^P(t,x,Y) = u_\infty(t,x) &:= u^E(t,x,0), \\ u^P_{|t=0} &= u^P_0 . \end{aligned} \qquad \text{(P)}$$

Comments:

- Nonlocal, scalar equation: write $v^P = -\int_0^Y u^P_x$;
- Pressure is given by Euler flow= data;
- Main source of trouble: nonlocal transport term $v^P \partial_Y u^P$ (loss of one derivative);
- Coupling with Euler is rather weak: no retroaction of the boundary layer on the fluid inside the domain.

Mathematical results: well-posedness in high regularity spaces/monotonic contexts...

WP in high regularity spaces:

- Local well-posedness starting from data that are analytic in x: [Sammartino&Caflisch; Lombardo, Cannone &Sammartino; Kukavica&Vicol; Kukavica, Masmoudi, Vicol&Wong];
- More recently, local well-posedness for Gevrey (in x) initial data: [Gérard-Varet&Masmoudi] .

WP for monotone solutions: [Oleinik; Masmoudi&Wong; Alexandre, Wang, Xu&Yang...]

... and instabilities in Sobolev spaces

- Instabilities develop in short time in Sobolev spaces [Grenier; Gérard-Varet&Dormy; Grenier&Nguyen...]
  Proof relies on computation of an approximate eigenmode with exponential growth.
- Starting from real analytic initial data, for specific outer Euler flow, some solutions display singularities in finite time.
  [Kukavica, Vicol&Wang](van Dommelen-Shen singularity).
  Proof relies on a virial type argument (blow-up of some Sobolev norm in finite time).
  Very recently, quantitative description of this singularity [Collot, Ghoul, Ibrahim&Masmoudi].

Nature of the main instability in the 2d Prandtl equation

(from [Gérard-Varet&Dormy].)

Starting point: linearization around a shear flow $(U_s(Y), 0)$:

$$\partial_t u^P + U_s \partial_x u^P + U_s' v^P - \partial_{YY} u^P = 0, \quad u_x^P + v_Y^P = 0.$$

Cst. coeff. in x $\rightarrow$ Fourier in x, t $\rightarrow$ODE in Y.

Look for an instability $\rightarrow$ high frequency analysis in space&time.

Asymptotic expansion: if U_s has a non-degenerate critical point a, the solution looks like

$$v^P(t,x,Y) \simeq \exp(ik(\omega t + x)) \left(\underbrace{v_a(Y)}_{\text{inviscid sol.}} + \underbrace{\epsilon^{1/2}\tau \mathbf{1}_{y>a} + \epsilon^{1/2}\tau V\left(\frac{y-a}{\epsilon^{1/4}}\right)}_{\text{viscous correction}} \right)$$

where $\epsilon := 1/k \ll 1$, $\omega = -U_s(a) + \epsilon^{1/2}\tau$, where $\tau \in \mathbb{C}$ is such that $\Im(\tau) < 0$.

Conclusion: the k-th mode grows like $\exp(|\Im(\tau)|\sqrt{k}t)$.

Remark: Viscosity induced instability.

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Question

Can one exhibit other boundary layer models, at a possibly higher order in ν, that do not exhibit such bad behavior?

Plan

The displacement thickness approach

Classical approach: $\boxed{\begin{array}{l}\text{Solve Euler}\\ v^E_{|y=0} = 0.\end{array}} \rightarrow \boxed{\begin{array}{c}\text{Trace}\\ u_\infty := u^E_{|y=0}\end{array}} \rightarrow \boxed{\begin{array}{l}\text{Solve}\\ \text{Prandtl}\end{array}}$.

Recall that

$$v^P(x,Y) = -\int_0^Y u^P_x = -Y\partial_x u_\infty - \underbrace{\partial_x \int_0^Y (u^P - u_\infty)}_{=\text{“blowing velocity”}}.$$

Displacement thickness (DT):

$$\delta(t,x) := \int_0^\infty \left(1 - \frac{u^P(t,x,Y)}{u_\infty(t,x)}\right) dY.$$

New approach [Catherall& Mangler, '66]: solve the boundary layer equations in an “inverse” way, i.e. prescribe δ instead of prescribing u_∞.

Coupling at a higher order with Euler: new Ansatz

New asymptotic expansion for $\mathbf{u}^\nu$ in powers of $\sqrt{\nu}$, keeping all terms up to $\sqrt{\nu}$:

$$\mathbf{u}^\nu(\cdot, y) = \mathbf{u}^{E,0}(\cdot, y) + \sqrt{\nu}\mathbf{u}^{E,1}(\cdot, y) + O(\nu) \quad \text{for } y \gg \sqrt{\nu},$$
$$u^\nu(\cdot, y) = u^{P,0}(\cdot, y/\sqrt{\nu}) + \sqrt{\nu}u^{P,1}(\cdot, y/\sqrt{\nu}) + O(\nu) \quad \text{for } y \lesssim \sqrt{\nu},$$

Matching conditions (if $\mathbf{u}^{E,0}$ is irrotational):

$$\lim_{Y\to\infty} u^{P,j}(t, x, Y) = u^{E,j}(t, x, 0), \quad j = 0, 1,$$
$$v^{E,1}(t, x, 0) = \partial_x \int_0^\infty \left(u^{E,0}(t, x, 0) - u^{P,0}(t, x, Y)\right) dY$$
$$= \partial_x \left(u^{E,0}(t, x, 0)\underbrace{\delta(x)}_{\text{DT}}\right).$$

Derivation of the matching conditions

Take $y = K\sqrt{\nu}$, with $1 \ll K \ll \nu^{-1/2}$, and perform a Taylor expansion of $\mathbf{u}^{E,j}(\cdot, y)$, $j = 0, 1$:

$$\begin{aligned} \mathbf{u}^{\nu}(\cdot, y) &= \mathbf{u}^{E,0}(\cdot, y) + \sqrt{\nu}\mathbf{u}^{E,1}(\cdot, y) + O(\nu) \\ &= \mathbf{u}^{E,0}(\cdot, 0) + \sqrt{\nu}K\partial_y \mathbf{u}^{E,0}(\cdot, 0) + \sqrt{\nu}\mathbf{u}^{E,1}(\cdot, 0) + O(\nu). \end{aligned}$$

To be matched with

$$(u^{P,0}(x,\infty), 0) + \sqrt{\nu}(u^{P,1}(x,\infty), -K\partial_x u^{P,0}(x,\infty) + \partial_x(u^{P,0}(x,\infty)\delta(x))).$$

Irrotational flow assumption & divergence free condition:

$$\partial_y \mathbf{u}^{E,0}(\cdot, 0) = (0, -\partial_x u^{E,0}(\cdot, 0)).$$

$\rightarrow$ Matching conditions of previous slide.

The interactive boundary layer (IBL) system

Coupling between an (irrotational) Euler flow and a boundary layer system:

- **Euler flow:** $\partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E + \nabla p^E = 0, \quad \operatorname{div} \mathbf{u}^E = 0.$
- **Boundary layer system:**

$$\begin{aligned}\partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P - \partial_Y^2 u^P = \partial_t u_\infty + u_\infty \partial_x u_\infty,\\ \partial_x u^P + \partial_Y v^P = 0.\end{aligned}$$

- **Coupling conditions:**

$$\begin{aligned}(u^P, v^P)(t,x,0) = 0,\\ u_\infty(t,x) = \lim_{Y\to\infty} u^P(t,x,Y) = u^E(t,x,0),\\ v^E(t,x,0) = \sqrt{\nu}\,\partial_x(u_\infty \delta).\end{aligned}$$

Retroaction of the boundary layer flow on the Euler flow.

Schematic view of the Prandtl vs. IBL system

Construction of the Prandtl boundary layer:

Solve 2d Euler system
in half-space $y > 0$
with boundary data $v^E_{|y=0} = 0$

$\longrightarrow$

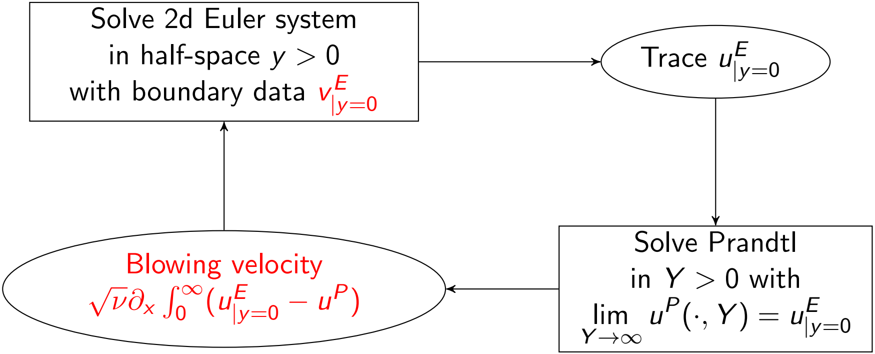
Trace $u^E_{|y=0}$.

$\downarrow$

Solve Prandtl
in $Y > 0$ with
$\lim_{Y \to \infty} u^P(\cdot, Y) = u^E_{|y=0}$

Schematic view of the Prandtl vs. IBL system

Construction of the interactive boundary layer: $\sim$ fixed point



Linearization of the IBL system around a shear flow

Linearize the IBL system around the shear flow $(1,0)$ (for Euler) and $(U_s(Y),0)$ (for the boundary layer part).

- **Linearized Euler system:**

$$\partial_t \mathbf{u}^E + \partial_x \mathbf{u}^E + \nabla p^E = 0, \quad \operatorname{div} \mathbf{u}^E = 0.$$

(and $\mathbf{u}^E$ is irrotational.)
$\Rightarrow \mathbf{u}^E(t,x,y) = \nabla^\perp \psi^E(t,x,y)$, for some harmonic ψ^E.

- **Linearized BL system:**

$$\partial_t u^P + U_s \partial_x u^P + U_s' v^P - \partial_Y^2 u^P = (\partial_t + \partial_x) u_\infty, \quad \partial_x u^P + \partial_Y v^P = 0.$$

- **Coupling condition:**

$$\begin{aligned} \partial_x u_\infty &= \mathcal{DN} v^E(t,x,0) \\ &= \sqrt{\nu}\partial_x \mathcal{DN} \int_0^\infty (u_\infty - u^P). \end{aligned}$$

$\rightarrow$ Closed system for the boundary layer.

Plan

Unconditional instability for the IBL system

Theorem [D., Dietert, Gérard-Varet, Marbach, '17]:
Consider an arbitrary monotone shear flow U_s and the linearized IBL system around $(U_s, 0)$.
Then there exist solutions growing like $\exp(\alpha \nu^{3/4} k^2 t)$ in the regime $|k| \gg \nu^{-3/4}$.

Remarks about the frequency regime

• Formally, the derivation of the IBL model is valid as long as $|k| \lesssim \nu^{-1/2}$. Otherwise, we must keep other terms in the equation, such as $\nu\partial_{xx}$.
The instability depicted here falls outside the physical regime, and is therefore an artefact of the IBL model.

• However, at the numerical level, one should see these instabilities... Projection onto "low" frequencies? Suitable modification of the numerical codes to correct these unconditional instabilities?

• Because of this instability, one does not expect the IBL model to be well-posed in analytic spaces ($\sim$ backwards heat equation).

• Viscosity induced instability (similar to [Gérard-Varet& Dormy]).

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Instabilities for the IBL system within/close to the physical regime

Theorem [D., Dietert, Gérard-Varet, Marbach, '17]:
Consider a monotone shear flow U_s and the linearized IBL system around $(U_s, 0)$. Then:

- *There exist shear flows U_s for which there exist solutions growing like $\exp(\alpha t|k|)$ for some $\alpha > 0$, in the regime $k \gtrsim \nu^{-1/2}$.*
- *If $U_s''(0) > 0$, there exist solutions growing like $\exp(\alpha\nu|k|^3 t)$, in the regime $\nu^{-1/3} \ll |k| \ll \nu^{-1/2}$.*

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Comparison between different types of instabilities

Inviscid instabilities	vs. viscosity-induced instability
Conditions on U_s	Unconditional
Instability is present at inviscid level	Instability disappears from inviscid eq. (oscillations only)
Viscous terms are perturbative	Viscous terms generate the instability through a boundary layer
Transport is paramount	Transport terms are perturbative
$\sim$ Penrose approach for the stability of Vlasov	$\sim$ Prandtl instability of Gérard-Varet&Dormy .

Instabilities for BL models with prescribed displacement thickness

Theorem [D., Dietert, Gérard-Varet, Marbach, '17]:
Consider a monotone shear flow U_s and the linearized boundary layer system around $(U_s, 0)$ with prescribed displacement thickness (PDT).
Under suitable conditions on U_s, there exist solutions of this system growing like $\exp(\alpha t|k|)$ for some $\alpha > 0$, where k is the tangential frequency.

Remarks about the profiles

- Explicit examples of profiles leading to instabilities.
- In most instabilities, we retrieve famous criteria:

Rayleigh: $\exists y_s > 0,\ U_s''(y_s) = 0;$
Fjørtoft: $\exists y \in \mathbf{R},\ U_s''(y)(U_s(y) - U_s(y_s)) < 0.$
Change in concavity is important.

- All profiles are monotone $\rightarrow$ stable for Prandtl!

These models have in fact a worse behaviour than Prandtl (higher growth of the instabilities, fewer stable profiles).

- Inviscid instabilities (viscosity is treated perturbatively).

Plan

Reduction to an ODE

Look for (u^P, v^P, u_∞) in the form

$$(u^P, v^P, u_\infty)(t,x,Y) = e^{ik(\omega t + x)}\left(1-\phi'(Y), ik(\phi(Y)-Y), 1\right).$$

The IBL system becomes

$$\begin{aligned} -\phi'(\omega+U_s) + U_s'\phi - \frac{i}{k}\phi^{(3)} = F(y) := 1 - U_s(y) + yU_s'(y), \\ \phi(0)=0,\ \phi'(0)=1,\ \lim_{Y\to\infty}\phi'(Y)=0 \end{aligned} \tag{2}$$

and the coupling condition is

$$\lim_{Y\to\infty}\phi(Y) = \frac{1}{\sqrt{\nu}|k|}.$$

Reformulation of the problem: for any (ω, k), consider the solution $\phi(\cdot;\omega,k)$ of (2) and define $\Phi(\omega,k) := \lim_{Y\to\infty}\phi(Y;\omega,k)$. The goal is to find sufficient conditions on U_s such that the equation $\Phi(\omega,k) = (\sqrt{\nu}|k|)^{-1}$ has a solution $\omega\in\mathbb{C}_-$ for $|k|\gg 1$.

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A pinch of complex analysis...

- High frequency $|k| \gg 1$: neglect the viscous term in (2).
- The inviscid ODE has an explicit solution ϕ_{inv} such that $\phi_{\mathrm{inv}}(0) = 0$, $\phi'_{\mathrm{inv}}(+\infty) = 0$.
- **New goal:** find solutions of $\Phi_{\mathrm{inv}}(\omega) = (\sqrt{\nu}|k|)^{-1}$ in $\mathbb{C}_-$. (Then, perturbative argument.)
- Sufficient condition: find a closed curve $\mathcal{C}$ embedded in $\mathbb{C}_-$ such that the winding number of $\Phi_{\mathrm{inv}}(\mathcal{C})$ around $\gamma = (\sqrt{\nu}|k|)^{-1}$ is positive.
- **Facts & Tools:**
  - Φ_{inv} is holomorphic on $\mathbb{C} \setminus [0,1]$ (explicit formula);
  - All possible roots $\omega = a + ib$ are within the half-circle $b^2 \leq a(1-a)$, $b \leq 0$.
  - Count number of crossings with the real axis of $\Phi_{\mathrm{inv}}([0,1] - i\eta)$ for some $\eta > 0$. Each crossing corresponds (as $\eta \to 0$) to a cancellation of U_s''.

Summary

- Presentation of alternative boundary layer models, that supposedly have a better behavior than the Prandtl equation...
- But so far, only negative results!
  These systems are actually worse than Prandtl.
- **Tools for the proof:** linearization around a shear flow, spectral analysis. Similar to Penrose approach for the stability of Vlasov.
- For some instabilities, we retrieve well-known necessary criteria for instability. But others are true unconditionally, or under weaker assumptions...
- Slightly different approach from previous works on instabilities for fluid equations: an exact solution of the PDE is constructed.

Perspectives

Hope for positive results?

- Look at stationary versions of the equations, that are claimed to provide a good description of separation/recirculation phenomena.
  Problem: it is unclear how to circumvent the (strong) difficulties that arise for the Prandtl equation with recirculation...
- Salvation may come from (yet) another model?
  The "triple deck" system is also widely used in the physics/engineering literature. In some regimes (that exclude instabilities), the triple deck and the IBL system are equivalent, and the triple deck system seems (at least formally) better behaved...

MERCI POUR VOTRE ATTENTION!