EXISTENCE AND STABILITY OF PLANAR SHOCKS OF VISCOUS SCALAR CONSERVATION LAWS WITH SPACE-PERIODIC FLUX

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Abstract. The goal of this paper is to prove the existence and stability of shocks for viscous scalar conservation laws with space periodic flux, in the multi-dimensional case. Such a result had been proved by the first author in one space dimension, but the extension to a multi-dimensional setting makes the existence proof non-trivial. We construct approximate solutions by restricting the size of the domain and then passing to the limit as the size of the domain goes to infinity. One of the key steps is a "normalization" procedure, which ensures that the limit objects obtained by the approximation scheme are indeed shocks. The proofs rely on elliptic PDE theory rather than ODE arguments as in the 1d case. Once the existence of shocks is proved, their stability follows from classical arguments based on the theory of dynamical systems.

1. Introduction and Main results

In this article, we aim to show the existence and large time stability of multidimensional planar shock fronts of viscous scalar conservation laws with space-periodic flux:

$$\begin{aligned} &\partial_t u + \sum_{i=1}^{N} \partial_{x_i} A_i(x,u) = \Delta_x u, \quad t>0,\ x\in\mathbb{R}^N, \\ &u(0,x) = u_0(x) \end{aligned} \tag{1.1}$$

where the flux functions $A_i : \mathbb{T}^N \times \mathbb{R} \to \mathbb{R}^N$ are assumed to be periodic with respect to the spatial variable x.

The issues in the case of one dimension $N=1$ have been treated by the first author in [5], and therefore our goal is to tackle these issues in the multidimensional case ($N \geq 2$). When the flux A is homogeneous, i.e. when A does not depend on x, a planar shock wave is a special solution of (1.1) of the form $u(t,x) = U(x\cdot\nu - ct)$, for some $c\in\mathbb{R}, \nu\in\mathbb{S}^{N-1}$, $U\in L^\infty(\mathbb{R}^N)$, and with $\lim_{y\to\pm\infty} U(y) = U_\pm$, for some constants $U_+, U_- \in \mathbb{R}$. The profile U is easily found thanks to simple ODE theory together with Rankine-Hugoniot condition. But the stability of planar shock fronts is a challenging issues. Stability for a small perturbation of multidimensional planar shocks has been shown by Goodman [9], Hoff and Zumbrun [10], and the second author, Vasseur and Wang [12]. In one-dimensional case, Freistühler and Serre [8] proved L^1-stability for any L^1-perturbation. Recently, the second author and Vasseur [11] have shown contraction for any L^2-perturbation.
But when A depends on the space variable, the constants are no longer stationary solutions

Date: March 9, 2016.

Acknowledgment. M.-J. Kang was supported by the Foundation Sciences Mathématiques de Paris as a postdoctoral fellowship, and by an AMS-Simons Travel Grant.

of (1.1) in general, and thus cannot be end states of planar shocks. Therefore we first introduce a family of periodic stationary solutions of (1.1), which will play the role of constant solutions in the homogeneous case. These solutions were introduced in [4].

Proposition 1.1 (Existence of periodic stationary solutions of (1.1), see [4])**.** *Let $A \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})^N$. Assume that there exists $C_0 > 0$, and $m \in [0,\infty), n \in [0, \frac{N+2}{N-2})$ when $N > 2$ such that for all $(x,v) \in \mathbb{T}^N \times \mathbb{R}$,*

$$|\partial_v A_i(x,v)| \leq C_0(1+|v|^m), \quad 1 \leq i \leq N, \tag{1.2a}$$

$$|div_x A(x,v)| \leq C_0(1+|v|^n). \tag{1.2b}$$

Assume as well that one of the following three conditions holds:

$$\begin{aligned} &i)\ m = 0 \quad or \\ &ii)\ 0 \leq n < 1 \quad or \\ &iii)\ \Big(n < \min(\frac{N+2}{N}, 2) \quad and \quad \exists p_0\ s.t. \sum_{i=1}^N \partial_{x_i} A_i(x,p_0) \equiv 0\Big). \end{aligned} \tag{1.3}$$

Then for each $p \in \mathbb{R}$, there exists a unique periodic solution $v(\cdot,p) \in H^1(\mathbb{T}^N)$ of the equation

$$-\Delta_x v(x,p) + div_x A(x, v(x,p)) = 0, \quad < v(\cdot,p) >= p. \tag{1.4}$$

In the above proposition and throughout the article, the brackets $\langle\cdot\rangle$ denote the average value of a $\mathbb{T}^N$-periodic function.

We list below further properties of the functions $v(x,p)$ (see Proposition 2.1). We also define the averaged - or homogenized - flux $\bar{A}$ by

$$\bar{A}(p) := \langle A(\cdot, v(\cdot,p))\rangle \quad \forall p \in \mathbb{R}.$$

We are now ready to define stationary (or standing) planar shocks.

Definition 1.1. *A stationary planar viscous shock of* (1.1) *with periodic end states is a function $\bar{U} \in H^1_{loc}(\mathbb{R}^N)$ which is a stationary solution of* (1.1)*, periodic in the variables $x_1, \cdots x_{k-1},\ x_{k+1}, \cdots x_N$ for some $k \in \{1, \cdots, N\}$, and such that there exist $p_+, p_- \in \mathbb{R}$ with $p_+ \neq p_-$ such that*

$$\lim_{x_k \to \pm\infty} \big(\bar{U}(x) - v(x,p_\pm)\big) = 0 \quad in\ L^\infty(\mathbb{T}^{N-1}), \tag{1.5}$$

Such a function is called a stationary shock of (1.1) *with end states $v(\cdot,p_\pm)$, or a stationary shock of* (1.1) *connecting $v(\cdot,p_-)$ to $v(\cdot,p_+)$.*

Remark 1.1. *Notice that because of the periodicity of the flux and of the stationary states, we only consider shocks in the directions $e_1, \cdots, e_N$, (i.e. in the directions of the canonical basis in $\mathbb{R}^N$), and not in any direction $\nu \in \mathbb{S}^{N-1}$ as in the homogeneous case. Indeed, if we take an arbitrary direction ν and look for a shock such that $U(x\cdot\nu, x^\perp) - U_\pm(x) \to 0$ as $x\cdot\nu \to \pm\infty$, where $x^\perp\cdot\nu = 0$, then in general the asymptotic states $U_\pm$ are not periodic solutions of* (1.1)*, but quasi-periodic solutions. Therefore a first step would be to study problems of the type*

$$-\Delta v + \mathrm{div}\tilde{A}(x,v) = 0$$

where the flux $\tilde{A}$ is quasi-periodic in its first variable and the function v is sought as periodic. This is expected to be much more difficult than in the periodic case, due to the lack of compactness and to the non-linearity. Such questions go beyond the scope of this paper, and

thus we focus on periodic end states only.
Moreover, without loss of generality, we focus on the case when $k = 1$ in the rest of the paper.

The stationary shocks in Definition 1.1 can be viewed as a spatial transition front in a space-periodic environment. The spatial transition fronts arising in various (periodic) heterogeneities have also received a lot of attention in the reaction-diffusion community. In particular, the existence of spatial transition waves for one-dimensional space-heterogenous reaction-diffusion equation has been proved by Xin [23] and Berestycki and Hamel [1], and by Nolen and Ryzhik [20] and Mellet, Raquejoffre and Sire [16] for ignition-type equation. These results have been extended by Zlatos [24] to multidimensional case of the cylindrical domain $\mathbb{R} \times \mathbb{T}^{N-1}$. We also refer to [2, 3] for a generalization of the notion of the transition fronts, whereas non-existence of such waves has been studied by Nadin [18] and Nolen *et al.* [19]. *Such transition wave for space-heterogenous reaction-diffusion equation connects two steady states, which are constants, contrary to our case that the stationary shock wave connects two steady states, which are non-constant periodic solutions.*

Our main result is the following:

Theorem 1.1. *(Existence of standing shocks) Assume that $A \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})^N$, and that there exist two periodic solutions $v(\cdot, p_+), v(\cdot, p_-)$ to* (1.4) *with $p_+ \neq p_-$, satisfying the following conditions:*

(1.6a) $$\bar{A}_1(p_-) = \bar{A}_1(p_+) =: \alpha,$$

(1.6b) $$\bar{A}_1(p) < \alpha,\ \forall p \in (p_+, p_-) \text{ if } p_+ < p_-, \quad \bar{A}_1(p) > \alpha,\ \forall p \in (p_-, p_+) \text{ if } p_- < p_+.$$

Then there exists a stationary shock $\bar{V}$ with end states $v(\cdot, p_-)$ and $v(\cdot, p_+)$.

Remark 1.2. *The first assumption* (1.6a) *is an analogue of the Rankine-Hugoniot condition for standing shock waves of homogeneous conservation laws. The second assumption* (1.6b) *is the analogue of the Oleinik condition. It is proved in section 3 that the Rankine-Hugoniot condition is in fact a necessary condition for the existence of a shock wave.*

Theorem 1.1 is proved by passing to the limit in a sequence of approximate problems. In these approximate problems, the domain $\mathbb{R} \times \mathbb{T}^{N-1}$ is replaced by $(-R, R) \times \mathbb{T}^{N-1}$ for some $R > 0$. Standard tools of elliptic theory (Harnack inequality, maximum principle, comparison principle, regularity estimates) are used to prove that the approximate sequence enjoys several nice properties, such as monotony and L^∞ bounds.

From now on, we only handle the first case of (1.6b), i.e.,

(1.7) $$p_+ < p_-, \quad \bar{A}_1(p) < \alpha, \quad \forall p \in (p_+, p_-),$$

the argument for the other case is exactly identical.

Theorem 1.2. *(Stability of standing shocks) Assume the hypotheses of Theorem 1.1, furthermore $A \in W^{3,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})^N$. Let $\bar{U}$ be a stationary shock wave connecting $v(\cdot, p_-)$ to $v(\cdot, p_+)$, and $u_0 \in \bar{U} + L^1(\mathbb{R} \times \mathbb{T}^{N-1})$ be a initial perturbation such that*

(1.8) $$v(x, p_+) \leq u_0(x) \leq v(x, p_-) \quad \text{for a.e. } x \in \mathbb{R} \times \mathbb{T}^{N-1},$$

and $u = u(t, x)$ be the unique entropy solution of (1.1) *with $u_{|t=0} = u_0$.*

- *Assume that $\int_{\mathbb{R} \times \mathbb{T}^{N-1}} (u_0 - \bar{U}) = 0$. Then*

$$\lim_{t \to \infty} \|u(t) - \bar{U}\|_{L^1(\mathbb{R} \times \mathbb{T}^{N-1})} = 0.$$

- *Assume that* $A \in W^{3,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})^N$, *that* $\int_{\mathbb{R}\times\mathbb{T}^{N-1}}(u_0 - \bar{U}) \neq 0$ *and that there exist functions* $\phi, \psi \in L^1(\mathbb{T})$ *such that*

$$\begin{aligned} &\partial_v A_1(x, v(x, p_-)) \geq \phi(x_1), \quad \text{for a.e. } x \in \mathbb{T}^N, \\ &a_- := \int_{\mathbb{T}} \phi dx_1 > 0, \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} &\partial_v A_1(x, v(x, p_+)) \leq \psi(x_1), \quad \text{for a.e. } x \in \mathbb{T}^N, \\ &a_+ := \int_{\mathbb{T}} \psi dx_1 < 0. \end{aligned} \tag{1.10}$$

Then there exists a stationary shock $\bar{V}$ *connecting* $v(\cdot, p_-)$ *to* $v(\cdot, p_+)$ *such that* $u_0 - \bar{V} \in L^1(\mathbb{R} \times \mathbb{T}^{N-1})$ *and*

$$\int_{\mathbb{R}\times\mathbb{T}^{N-1}} (u_0 - \bar{V}) = 0 \quad \text{and} \quad \lim_{t\to\infty} \|u(t) - \bar{V}\|_{L^1(\mathbb{R}\times\mathbb{T}^{N-1})} = 0.$$

Remark 1.3. • *The assumptions* (1.9) *and* (1.10) *are the analogue of the Lax conditions for standing shock waves of homogeneous conservation laws. They are used in the present context to obtain a rate of convergence of stationary shocks towards their end states* $v(\cdot, p_\pm)$. *This rate of convergence yields some* L^1 *compactness for an approximate problem (see* (3.28)*). We refer to the proofs of Lemma 3.2 and Proposition 3.2 below for details.*

• *The proof of Theorem 1.2 uses classical arguments, relying on tools from dynamical system theory. The main difficulty lies in the second part of Theorem 1.2, which requires, for any real number* q *and any shock* $\bar{U}$, *to find a shock* $\bar{V}$ *with the same end states as* $\bar{U}$ *and such that* $\int(\bar{V} - \bar{U}) = q$. *This fact is almost obvious in the homogeneous case, since any spatial translate of a shock is a shock. This statement is still rather easy to prove in the 1d case, since a whole family of shocks depending continuously on a parameter is constructed. In the present case, Theorem 1.1 only gives the existence of a single shock, and therefore the existence of shocks satisfying the above statement for any* $q \in \mathbb{R}$ *is far from trivial, and is proved in Proposition 3.2.*

• *Assumption* (1.8) *is a classical assumption within the framework of shock stability for conservation laws (see* [22] *and the discussion on initial data within the interval* $[u_+, u_-]$ *or outside that interval). In order to remove it, we would typically need to prove the stability of the periodic solutions* $v(\cdot, p_\pm)$ *under zero-mass perturbation in the space* $L^1(\mathbb{R} \times \mathbb{T}^{N-1})$. *However, to our knowledge, the stability of the functions* $v(\cdot, p_\pm)$ *is known in* $L^1(\mathbb{R}^N)$ *and in* $L^1(\mathbb{T}^N)$ *(see respectively* [6] *and* [5]*), but not in* $L^1(\mathbb{R} \times \mathbb{T}^{N-1})$. *Furthermore, the proofs of stability in the whole space* $\mathbb{R}^N$ *and in the torus* $\mathbb{T}^N$ *rely on very different arguments, since in the whole space, dispersive effects take place. It is possible that a hybrid proof could be worked out in spaces of the form* $\mathbb{R}^k \times \mathbb{T}^l$ *with* $k + l = N$, *but such a question goes beyond the scope of this paper and thus we choose to leave it open.*

We now provide some examples of fluxes satisfying assumptions (1.6a)-(1.6b), and (1.9)-(1.10). Let $\Phi : \mathbb{T}^N \to \mathbb{R}^N$ be a divergence-free vector field, $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, and let $A(x, v) := \Phi(x) f(v)$. Then for any constant $p \in \mathbb{R}$, $v(\cdot, p) := p$ is a solution to the elliptic equation (1.4) with $< v(\cdot, p) >= p$. As a consequence,

$$\bar{A}_1(p) = \int_{\mathbb{T}^N} A_1(x, p) dx = f(p) \langle \Phi_1 \rangle .$$

Thus (1.6a) holds if and only if $f(p_+) = f(p_-)$, and (1.6b) holds if and only if $f(p) - f(p_\pm)$ has the same (strict) sign as $\langle \Phi_1 \rangle (p_+ - p_-)$ for $p \in (p_+, p_-)$. For instance, if $\langle \Phi_1 \rangle > 0$ and $f(p) = p^2$, any couple $p_- = -p_+ > 0$ works.
Moreover,

$$\partial_v A_1(x, v(x,p)) = \Phi_1(x) f'(p),$$

and therefore (1.9)-(1.10) are satisfied for instance if there exists $\alpha > 0$ such that $\Phi_1(x) \geq \alpha$ for all x, and if f is strongly convex and such that $f(p_+) = f(p_-)$, with $p_+ < p_-$.

Let us now introduce some notation that will be used throughout the paper. We will often denote the spatial domain by

$$\Omega := \mathbb{R} \times \mathbb{T}^{N-1}.$$

In a similar way, we define, for $R > 0$,

$$\Omega_R := (-R, R) \times \mathbb{T}^{N-1}.$$

We introduce the space $L^1_0(\mathbb{R} \times \mathbb{T}^{N-1})$ of integrable functions with zero mass

$$L^1_0(\mathbb{R} \times \mathbb{T}^{N-1}) := \left\{ f \in L^1(\mathbb{R} \times \mathbb{T}^{N-1}),\ \int_{\mathbb{R} \times \mathbb{T}^{N-1}} f = 0 \right\}.$$

For any integer $k \in \mathbb{Z}$, and any function $f \in L^1_{loc}(\Omega)$, we define

$$\tau_k f(x) := f(x + k e_1), \quad \forall x \in \mathbb{R} \times \mathbb{T}^{N-1}.$$

Let us stress that the main difficulty in this article lies in proving the existence of shock waves. Indeed, shock stability follows from classical arguments in [5] relying on dynamical system theory (see [21]). We recall the arguments in section 4 for the reader's convenience, but the largest part of the paper is devoted to the existence of shocks.

The paper is organized as follows: section 2 is devoted to the proof of Theorem 1.1. In section 3, we review some properties of stationary shocks. Eventually, section 4 is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we construct stationary shocks thanks to an approximation scheme on compact sets, and then pass to the limit. The proof makes an extensive use of the maximum principle and of the Rankine-Hugoniot (1.6a) and Oleinik conditions (1.6b).

Before addressing the proof, we first recall some properties of the functions $v(\cdot, p)$ (see [4]):

Proposition 2.1. *Assume that the hypotheses of Proposition 1.1 are satisfied. The family $(v(\cdot,p))_{p\in\mathbb{R}}$ satisfies the following properties:*
(i) Regularity estimate : For all $p \in \mathbb{R}$, $v(\cdot, p) \in W^{2,q}(\mathbb{T}^N)$ for all $1 < q < \infty$ and

$$\forall R > 0,\ \exists C_R > 0 \quad s.t. \quad \sup_{p \in [-R,R]} \|v(p)\|_{W^{2,q}(\mathbb{T}^N)} \leq C_R.$$

(ii) Growth property : if $p > p'$, then

$$v(x,p) < v(x,p'), \quad x \in \mathbb{T}^N$$

(iii) p-derivative : For all $p \in \mathbb{R}$, $\partial_p v(\cdot, p) \in H^1(\mathbb{T}^N)$ and

$$\forall R > 0,\ \exists C_R > 0 \quad s.t. \quad \sup_{p \in [-R,R]} \|\partial_p v(p)\|_{H^1(\mathbb{T}^N)} \leq C_R.$$

Moreover,

$$\partial_p v(x,p) > 0 \quad a.e.\ (x,p) \in \mathbb{T}^N \times \mathbb{R}. \tag{2.11}$$

(iv) Behavior at infinity : if additionally $\partial_v A_i \in L^\infty(\mathbb{T}^N \times \mathbb{R})$ *for* $1 \le i \le N$, *and*

$$\sup_{v\in\mathbb{R}} \|\partial_v A(\cdot, v)\|_{L^\infty(\mathbb{T}^N)} < \infty,$$

then

$$\lim_{p\to-\infty} \sup_{x\in\mathbb{T}^N} v(x,p) = -\infty, \qquad \lim_{p\to+\infty} \inf_{x\in\mathbb{T}^N} v(x,p) = +\infty. \tag{2.12}$$

2.1. **Construction of approximate solutions.** For any $R > 1$, consider the approximate equation:

$$\begin{aligned} -\Delta \bar{U}_R + \mathrm{div} A(x, \bar{U}_R) = 0 \quad \text{in } (-R,R) \times \mathbb{T}^{N-1}, \\ \bar{U}_R(\pm R, x') = v(\pm R, x', p_\pm) \quad \forall x' \in \mathbb{T}^{N-1}. \end{aligned} \tag{2.13}$$

For the time being, we assume that the flux A satisfies the assumptions of Proposition 1.1 with $m = 0$ and $n < 1$, i.e. A is uniformly Lipschitz with respect to its second variable, and $\mathrm{div}_x A$ has sublinear growth. These assumptions will be removed in Remark 2.1.

In this paragraph, we prove the existence and uniqueness of solutions of (2.13) for any $R > 1$. Using the family $v : \mathbb{T}^N \times \mathbb{R} \to \mathbb{R}$ constructed in Proposition 1.1, we consider a composite function $V(x) := v(x, f(x_1))$ for some $f \in \mathcal{C}^\infty(\mathbb{R})$ with $f(x_1) = p_-$ if $x_1 \le -1$, $f(x_1) = p_+$ if $x_1 \ge 1$. Then we see that (2.13) is equivalent to

$$\begin{aligned} -\Delta U_R + \mathrm{div} B(x, U_R) = S \quad \text{in } \Omega_R, \\ U_R(\pm R, x') = 0, \end{aligned} \tag{2.14}$$

where $U_R := \bar{U}_R - V$ and $S := \Delta V - \mathrm{div} A(x, V)$, $B(x,r) := A(x, V + r) - A(x, V)$. Notice that since A is uniformly Lipschitz with respect to r, there exists a constant C such that

$$|B(x,r)| \le C|r| \quad \forall x \in \Omega,\ \forall r \in \mathbb{R}.$$

Moreover, according to the definition of S and to Proposition 1.1, the support of the function S is included in $[-1,1] \times \mathbb{T}^{N-1}$, and $S \in L^2(\Omega)$.

Therefore, it is enough to prove the existence of (2.14). We want to apply Schaeffer's fixed point theorem. Let us consider the continuous mapping $L_R : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$ such that $W = L_R(U)$ is the unique solution of the linear elliptic equation:

$$\begin{aligned} -\Delta W + \mathrm{div} B(x, U) = S \quad \text{in } \Omega_R, \\ W(\pm R, x') = 0. \end{aligned}$$

We use assumption (1.2a) with $m = 0$ and we obtain

$$\|\nabla W\|^2_{L^2(\Omega_R)} \le \|S\|_{L^2(\Omega_R)} \|W\|_{L^2(\Omega_R)} + C_0 \|U\|_{L^2(\Omega_R)} \|\nabla W\|_{L^2(\Omega_R)}.$$

Using the Poincaré inequality and Young's inequality, we have that

$$\|\nabla W\|_{L^2(\Omega_R)} \le C_R \|S\|_{L^2(\Omega_R)} + C\|U\|_{L^2(\Omega_R)},$$

for some constant C_R depending on R.
Since f is smooth, it follows from Proposition 1.1 that

$$\|\nabla W\|_{L^2(\Omega_R)} \le C_R(\|U\|_{L^2(\Omega_R)} + 1).$$

Thus, using the Rellich-Kondrachov theorem, we infer that the mapping L_R is compact. Now, there remains to prove that the set

$$\{U^\lambda \in H^1_0(\Omega_R) \mid U^\lambda = \lambda L_R(U^\lambda),\ \lambda \in [0,1]\}$$

is bounded. For any $\lambda \in [0,1]$ and for any solution U^λ of $U^\lambda = \lambda L_R(U^\lambda)$, we have

$$\int_{\Omega_R} |\nabla U^\lambda|^2 \le \|S\|_{L^2(\Omega_R)} \|U^\lambda\|_{L^2(\Omega_R)} + \lambda \left| \int_{\Omega_R} B(x, U^\lambda) \cdot \nabla U^\lambda \right|.$$

Let $b : (x,r) \in \mathbb{R}^{N+1} \mapsto \int_0^r B(x,r')\, dr'$. Then

$$\int_{\Omega_R} B(x, U^\lambda) \cdot \nabla U^\lambda = \int_{\Omega_R} \Big(\mathrm{div}(b(x,U^\lambda)) - (\mathrm{div}_x b)(x, U^\lambda) \Big) = - \int_{\Omega_R} (\mathrm{div}_x b)(x, U^\lambda).$$

Notice that

$$\mathrm{div}_x b(x,r) = \int_0^r \Big(\mathrm{div} A(x, V + r') - \mathrm{div} A(x, V) \Big)\, dr',$$

and therefore, using the growth assumption on A, there exists a constant C such that for all $r \in \mathbb{R}$,

$$|\mathrm{div}_x b(x,r)| \le C(1 + |r|^{n+1}) \quad \text{with } n < 1.$$

Using once again the Cauchy-Schwartz and the Poincaré inequality, we infer that

$$\|U^\lambda\|_{H^1(\Omega_R)} \le C_R \quad \forall \lambda \in [0,1].$$

According to Schaeffer's fixed point theorem, L_R has a fixed point in $H^1_0(\Omega_R)$, and therefore (2.13) has a solution in $H^1(\Omega_R)$.

Uniqueness follows for instance from the following argument. Let $\bar U_R, \bar U_R'$ be two solutions of (2.13), and let $W := \bar U_R - \bar U_R'$. Then W solves an elliptic equation of the type

$$-\Delta W + \mathrm{div}(a_R W) = 0 \quad \text{in } \Omega_R,$$
$$W(\pm R, x') = 0,$$

where $a_R \in L^\infty(\Omega_R)$ is defined by

$$a_R(x) := \int_0^1 \partial_v A(x, \tau \bar U_R(x) + (1-\tau) \bar U_R'(x)) d\tau.$$

On the other hand, using the strong form of the Krein-Rutman Theorem (see Appendix), it can be proved that the equation

$$-\Delta w + \mathrm{div}(a_R w) = 0 \quad \text{in } \Omega_R,$$
$$-\partial_1 w + a_{R,1} w = 0 \text{ on } \partial\Omega_R$$

admits a unique positive solution $w \in \mathcal{C}(\bar\Omega_R) \cap H^1(\Omega_R)$ such that $\int_{\Omega_R} w = 1$. A straightforward computation (see [17]) shows that

$$-\Delta \left(\frac{W^2}{w} \right) + \mathrm{div} \left(a_R \frac{W^2}{w} \right) = -2w \left| \nabla \frac{W}{w} \right|^2 \quad \text{in } \Omega_R.$$

Integrating over Ω_R, we deduce that

$$\int_{\Omega_R} w \left| \nabla \frac{W}{w} \right|^2 = 0,$$

which implies that W/w is constant, therefore $W \equiv 0$ due to $W = 0$ at $x_1 = \pm R$.

2.2. **Properties of approximate solutions.** We claim that the approximate solution $\bar{U}_R$ satisfies the following properties.

Lemma 2.1. *For any fixed integer $R > 1$, let $\bar{U}_R$ be the solution of (2.13). Then the following properties holds.*

(1) A priori bound in L^∞: for all $x \in \Omega_R$,

$$v(x, p_+) \leq \bar{U}_R(x) \leq v(x, p_-).$$

(2) Integration constant: there exists a number α_R such that for all $x_1 \in (-R, R)$,

$$-\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} \bar{U}_R(x_1, x')\, dx' + \int_{\mathbb{T}^{N-1}} A_1(x_1, x', \bar{U}_R(x_1, x'))\, dx' = \alpha_R, \tag{2.15}$$

and $\alpha \leq \alpha_R \leq C$ for some constant C independent of R. (Recall $\alpha := \bar{A}_1(p_-) = \bar{A}_1(p_+)$)

(3) Monotony: for all $x \in (-R, R-1) \times \mathbb{T}^{N-1}$,

$$\bar{U}_R(x_1 + 1, x') < \bar{U}_R(x_1, x').$$

(4) Uniform local a priori bound: for any $q \in (1, \infty)$, there exists a constant C_q (independent of R) such that

$$\sup_{k \in \{-R, \cdots R-1\}} \|\bar{U}_R\|_{W^{2,q}((k,k+1)\times\mathbb{T}^{N-1})} \leq C_q.$$

Proof. For the time being, we still assume that the flux A satisfies the assumptions of Proposition 1.1 with $m = 0$ and $n < 1$, which will be removed in Remark 2.1.

(1) A priori bound in L^∞:

First, notice that using elliptic regularity results together with a bootstrap argument, it is easily proved that $\bar{U}_R \in W^{2,q}(\Omega_R)$ for all $q < \infty$, and therefore $\bar{U}_R \in \mathcal{C}(\overline{\Omega_R})$. Thus, thanks to (2.12) in Proposition 2.1 and to the assumption $m = 0$, there exist $\bar{p}_R, \underline{p}_R$ with $\bar{p}_R > \underline{p}_R$ such that

$$v(x, \underline{p}_R) \leq \bar{U}_R(x) \leq v(x, \bar{p}_R) \quad \forall x \in \bar{\Omega}_R.$$

Let us choose $\bar{p}_R$ (resp. $\underline{p}_R$) as the smallest (resp. the largest) real number such that the above inequality is satisfied. Then necessarily, since $\bar{U}_R$ and $v(x, \bar{p}_R)$ are continuous and $\bar{\Omega}_R$ is compact, there exists $x_R \in \overline{\Omega_R}$ such that $\bar{U}_R(x_R) = v(x_R, \bar{p}_R)$. Let us argue by contradiction, and assume that x_R is an interior point of Ω_R. Notice that $g_R := v(x, \bar{p}_R) - \bar{U}_R$ is a non-negative solution of an elliptic equation of the type

$$-\Delta g_R + \operatorname{div}(a g_R) = 0 \quad \text{in } \Omega_R,$$

where $a \in L^\infty(\Omega_R)$ is defined by

$$a(x) := \int_0^1 \partial_v A(x, \tau v(x, \bar{p}_R) + (1 - \tau)\bar{U}_R(x)) d\tau.$$

Since x_R is an interior point and $g_R(x_R) = 0$, by the Harnack inequality, we have that g_R vanishes on any compactly embedded subset of Ω_R. Thus by continuity, $g_R \equiv 0$ on Ω_R, which is in the contradiction with $\bar{U}_R \in \mathcal{C}(\overline{\Omega_R})$ and $p_+ \neq p_-$. Therefore, $x_R \in \partial\Omega_R$, thus $\bar{p}_R \in \{p_+, p_-\}$. Since $p_+ < p_-$, we have $\bar{p}_R = p_-$. Similar arguments lead to $\underline{p}_R = p_+$.

(2) Integration constant:

Integrating equation (2.13) on $\mathbb{T}^{N-1}$ with x_1 fixed, we obtain

$$-\frac{d^2}{dx_1^2}\int_{\mathbb{T}^{N-1}} \bar{U}_R(x_1,x')\,dx' + \frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} A_1(x_1,x',\bar{U}_R(x_1,x'))\,dx' = 0 \quad \forall x_1 \in (-R,R),$$

which provides identity (2.15). Furthermore, notice that since for all $x_1 \in (-R, R-1)$,

$$\alpha_R = \int_{\mathbb{T}^{N-1}} \left(\bar{U}_R(x_1,x') - \bar{U}_R(x_1+1,x')\right) dx' + \int_{x_1}^{x_1+1}\int_{\mathbb{T}^{N-1}} A_1(x,\bar{U}_R(x))\,dx,$$

the boundedness of $\bar{U}_R$ implies that α_R is bounded. Thus, there remains to prove the lower bound $\alpha_R \geq \alpha$. To this end, we consider identity (2.15) at $x_1 = -R$ (notice that (2.15) holds at $x_1 = -R$ because $\bar{U}_R$ is smooth). Using the boundary condition, we have

$$\begin{aligned}\alpha_R = &\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} \left(v(x_1,x',p_-) - \bar{U}_R(x_1,x')\right)\,dx'\Big|_{x_1=-R} \\ &+ \int_{\mathbb{T}^{N-1}} A_1(-R,x',v(-R,x',p_-))\,dx' - \frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} v(x_1,x',p_-)dx'\Big|_{x_1=-R}.\end{aligned}$$

Since $v(x_1,x',p_-) - \bar{U}_R(x_1,x') \geq 0$ for all $(x_1,x') \in \Omega_R$, with equality at $x_1 = -R$, we have that

$$\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} \left(v(x_1,x',p_-) - \bar{U}_R(x_1,x')\right)\,dx'\Big|_{x_1=-R} \geq 0.$$

On the other hand, since

$$-\Delta v(x,p_-) + \operatorname{div} A(x,v(x,p_-)) = 0,$$

we also have that

$$\int_{\mathbb{T}^{N-1}} A_1(x_1,x',v(x_1,p_-))\,dx' - \frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} v(x_1,x'p_-)dx' = \text{constant} \quad \forall x_1 \in \mathbb{R}.$$

Integrating the above identity over $\mathbb{T}$, we deduce that the above constant is $\bar{A}_1(p_-) = \alpha$. Choosing $x_1 = -R$, the inequality $\alpha_R \geq \alpha$ is proved.

(3) Monotony:

Consider the function $\bar{U}_R(x_1+1,x')$ defined on $(-R-1,R-1)\times\mathbb{T}^{N-1}$. Since the flux A is periodic, $\bar{U}_R(x_1+1,x')$ satisfies the same equation as $\bar{U}_R$. Moreover, using the L^∞ a priori estimates and the boundary conditions on $\bar{U}_R$, we have that

$$\bar{U}_R(x_1+1,x') - \bar{U}_R(x_1,x') \leq 0 \quad \text{at } x_1 = -R \text{ and at } x_1 = R-1.$$

Set $H_R := \bar{U}_R(x_1+1,x') - \bar{U}_R(x_1,x')$ and $(H_R)_- := -H_R\mathbf{1}_{H_R\leq 0}$. Since the function $x \mapsto -x\mathbf{1}_{x\leq 0}$ is convex, we have that in $\mathcal{D}'((-R,R-1)\times\mathbb{T}^{N-1})$,

$$-\Delta(H_R)_- + \operatorname{div}_x\left(-\mathbf{1}_{H_R\leq 0}(A(x,\bar{U}_R(x_1+1,x')) - A(x,\bar{U}_R(x_1,x')))\right) \leq 0. \tag{2.16}$$

We denote by $-m_R$ the left hand-side of the above inequality. Then m_R is a non-negative measure. Moreover, straightforward integrations entail

$$\begin{aligned}
&- m_R((-R,R-1)\times\mathbb{T}^{N-1})\\
&\quad=\left[-\partial_1\int_{\mathbb{T}^{N-1}}(H_R)_-dx'\right]_{x_1=-R}^{x_1=R-1}\\
&\qquad+\left[\int_{\mathbb{T}^{N-1}}-\mathbf{1}_{H_R\le 0}\Big(A_1(x,\bar U_R(x_1+1,x'))-A_1(x,\bar U_R(x_1,x'))\Big)dx'\right]_{x_1=-R}^{x_1=R-1}.
\end{aligned}\tag{2.17}$$

Since $H_R\le 0$ at $x_1=-R$ and at $x_1=R-1$, we have

$$\partial_1(H_R)_-=-\mathbf{1}_{H_R\le 0}\partial_1 H_R=-\partial_1 H_R\quad\text{at } x_1=-R \text{ and } R-1.$$

Using (2.15), we have that

$$\begin{aligned}
&-\left(\partial_1\int_{\mathbb{T}^{N-1}}(H_R)_-dx'\right)\Big|_{x_1=R-1}\\
&\quad+\left(\int_{\mathbb{T}^{N-1}}-\mathbf{1}_{H_R\le 0}\Big(A_1(x,\bar U_R(x_1+1,x'))-A_1(x,\bar U_R(x_1,x'))\Big)dx'\right)\Big|_{x_1=R-1}\\
&=\partial_1\int_{\mathbb{T}^{N-1}}\bar U_R(R,x')\,dx'-\int_{\mathbb{T}^{N-1}}(A_1(R,x',\bar U_R(R,x'))dx'\\
&\quad-\partial_1\int_{\mathbb{T}^{N-1}}\bar U_R(R-1,x')\,dx'+\int_{\mathbb{T}^{N-1}}(A_1(R-1,x',\bar U_R(R-1,x'))dx'\\
&=\alpha_R-\alpha_R=0.
\end{aligned}$$

Similarly we have the same result at $x_1=-R$. Therefore, it follows from (2.17) that $m_R((-R,R-1)\times\mathbb{T}^{N-1})=0$, and thus

$$-\Delta(H_R)_-+\operatorname{div}_x\left(-\mathbf{1}_{H_R\le 0}(A(x,\bar U_R(x_1+1,x'))-A(x,\bar U_R(x_1,x')))\right)=0.$$

That is, $(H_R)_-$ is a non-negative solution of an elliptic equation of the type:

$$-\Delta(H_R)_-+\operatorname{div}_x(a_R(H_R)_-)=0,$$

where $a_R\in L^\infty((-R,R-1)\times\mathbb{T}^{N-1})$ is defined by

$$a_R:=\int_0^1\partial_v A(x,\tau\bar U_R(x+e_1)+(1-\tau)\bar U_R(x))d\tau.$$

Let us argue by contradiction and assume that $H_R(x_0)\ge 0$ for some $x_0\in(-R,R-1)\times\mathbb{T}^{N-1}$. Then $(H_R)_-(x_0)=0$,and Harnack's inequality implies that $(H_R)_-=0$ on $(-R,R-1)\times\mathbb{T}^{N-1}$.
In that case, $H_R(x)\ge 0$ for all $x\in(-R,R-1)\times\mathbb{T}^{N-1}$, and since R is an integer, we obtain

$$v(x,p_+)=\bar U_R(R,x')\ge\bar U_R(R-1,x')\ge\cdots\ge\bar U_R(-R,x')=v(x,p_-).$$

This is in contradiction with $p_+<p_-$ as (1.7). Therefore, we deduce that $H_R(x)<0$ for all $x\in(-R,R-1)\times\mathbb{T}^{N-1}$, and

$$\bar U_R(x_1+1,x')-\bar U_R(x_1,x')<0\quad\forall x\in(-R,R-1)\times\mathbb{T}^{N-1}.$$

(4) Uniform bounds in Sobolev spaces:
For any $k \in \{-R+1, \cdots, R-2\}$, $\bar{U}_R$ solves the equation

$$-\Delta \bar{U}_R = -\mathrm{div} A(x, \bar{U}_R) \text{ in } (k, k+1) \times \mathbb{T}^{N-1},$$

with the inherited boundary conditions, which are bounded in $L^\infty(\mathbb{T}^{N-1})$ uniformly in R and k thanks to (1). Using the L^∞ a priori bound together with interior elliptic estimates, we infer that $\bar{U}_R$ is bounded in $H^1((k, k+1) \times \mathbb{T}^{N-1})$ (and even in $W^{1,q}$ for any $q < \infty$), uniformly in k and R. Using a classical bootstrap argument, we then prove that $\bar{U}_R$ is bounded in $W^{2,q}((k, k+1) \times \mathbb{T}^{N-1})$ for any $q < \infty$. Using the fact that the boundary conditions at $-R$ and R are smooth and bounded, we derive similar bounds on $(-R, -R+1) \times \mathbb{T}^{N-1}$ and $(R-1, R) \times \mathbb{T}^{N-1}$. Hence the result follows.

□

Remark 2.1. *We here explain how we can remove the constraints $m = 0$ and $n < 1$ on growth assumptions of the flux. Assume that A belongs to $W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})$ and that there exist two periodic solutions $v(\cdot, p_\pm)$ of* (1.4) *with $p_+ \neq p_-$ satisfying* (1.6a).
Let

$$C_0 := \max(\|v(\cdot, p_+)\|_\infty, \|v(\cdot, p_-)\|_\infty).$$

and $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi(\xi) = 1$ for $|\xi| \leq C_0 + 1$. Define

$$A_\chi(x, \xi) := A(x, \xi)\chi(\xi), \quad x \in \mathbb{T}^N, \ \xi \in \mathbb{R}$$

Then the flux A_χ belongs to $W^{1,\infty}(\mathbb{T}^N \times \mathbb{R})$ and satisfies the growth assumptions of Proposition 1.1 with $m = n = 0$. Therefore, for any $p \in \mathbb{R}$ there exists a unique periodic solution $v_\chi(\cdot, p)$ of

$$-\Delta v_\chi(x) + \mathrm{div} A_\chi(x, v_\chi(x, p)) = 0 \text{ in } \mathbb{T}^N, \quad \langle v_\chi(\cdot, p)\rangle = p.$$

It follows from the uniqueness of v_χ and from the definition of A_χ that $v_\chi(\cdot, p_\pm) = v(\cdot, p_\pm)$.

Now, we can apply the results proved above to the flux A_χ. Thus there exists a unique solution $\bar{U}_R^\chi$ of equation (2.13) *with A replaced by A_χ, and $\bar{U}_R^\chi$ enjoys the properties of Lemma 2.1. In particular, $\|\bar{U}_R^\chi\|_\infty \leq C_0$, and thus*

$$A_\chi(x, \bar{U}_R^\chi) = A(x, \bar{U}_R^\chi) \quad \forall x \in \Omega_R.$$

Hence $\bar{U}_R^\chi$ is also a solution of (2.13) *with the original flux A. Thus we can now drop the χ's, and consider arbitrary fluxes $A \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})$ satisfying the assumptions of Theorem 1.1.*

2.3. **Passing to the limit as $R \to \infty$.**

▷ *First step: Extension to $\mathbb{R} \times \mathbb{T}^{N-1}$ and "normalization".*

We first extend $\bar{U}_R$ to $\mathbb{R} \times \mathbb{T}^{N-1}$ by setting

$$\bar{U}_R(x) = v(x, p_+) \text{ for } x_1 \geq R, \quad \bar{U}_R(x) = v(x, p_-) \text{ for } x_1 \leq -R. \tag{2.18}$$

Thanks to (4) of Lemma 2.1 and to the regularity of v, the above function $\bar{U}_R$ is continuous and bounded uniformly in R in $W^{1,\infty}(\mathbb{R} \times \mathbb{T}^{N-1})$. Moreover $\bar{U}_R(\cdot + e_1) \leq \bar{U}_R$ over the whole space.

Before passing to the limit, one issue is that all integer translations in x_1 of shocks are also shocks. And a shock translated by ke_1, with $|k| \gg 1$, is very close to one of the end states $v(\cdot, p_\pm)$ on compact sets in all Sobolev norms. In order to prevent $\bar{U}_R$ from converging towards $v(\cdot,p_\pm)$, we fix the value (or rather, the mean value) of (a translate of) $\bar{U}_R$ at a given point. We call this step the "normalization" of $\bar{U}_R$.

More precisely, let $\bar{p}\in(p_+,p_-)$ be arbitrary (for instance, take $\bar{p}=\frac{p_++p_-}{2}$). Then since

$$\int_0^1\int_{\mathbb{T}^{N-1}}\bar{U}_R(R+x_1,x')\,dx'\,dx_1=p_+\quad\text{and}\quad\int_0^1\int_{\mathbb{T}^{N-1}}\bar{U}_R(-R-1+x_1,x')\,dx'\,dx_1=p_-,$$

there exists $x_R\in(-R-1,R)$ such that

$$\int_0^1\int_{\mathbb{T}^{N-1}}\bar{U}_R(x_R+x_1,x')\,dx'\,dx_1=\bar{p}.$$

Let $k_R:=\lfloor x_R\rfloor\in\mathbb{Z}$ and define $\bar{V}_R:=\bar{U}_R(x_1+k_R,x')$. Then since A and v are periodic in their first variable, $\bar{V}_R$ solves

$$\begin{aligned}&-\Delta\bar{V}_R+\mathrm{div}A(x,\bar{V}_R)=0\text{ in }(-R-k_R,R-k_R)\times\mathbb{T}^{N-1},\\ &\bar{V}_R(-R-k_R,x')=v(-R-k_R,x',p_-),\\ &\bar{V}_R(R-k_R,x')=v(R-k_R,x',p_+),\quad x'\in\mathbb{T}^{N-1},\end{aligned}\tag{2.19}$$

and there exists $y_R\in[0,1)$ ($y_R=x_R-k_R$) such that

$$\int_0^1\int_{\mathbb{T}^{N-1}}\bar{V}_R(y_R+x_1,x')\,dx'\,dx_1=\bar{p}.$$

Additionally, $\bar{V}_R$ inherits from $\bar{U}_R$ all the properties listed in Lemma 2.1.

$\triangleright$ *Second step: Limit $R\to\infty$.*

Thanks to the bounds listed above and in Lemma 2.1, we can extract a subsequence R_m and find a function $\bar{V}$ such that $\bar{V}_{R_m}\rightharpoonup\bar{V}$ in $W^{1,q}(K)$, $\forall q\in[1,\infty)$ for any compact set $K\subset\mathbb{R}\times\mathbb{T}^{N-1}$, and thus $\bar{V}_{R_m}\to\bar{V}$ strongly in $\mathcal{C}^\alpha(K)$ for some $\alpha>0$. Furthermore, up to a further extraction of a subsequence, there exist some constants $x_+,x_-,\bar{y}$ and $\bar{\alpha}$ such that

$$R-k_R\to x_+\in[0,+\infty],\quad -R-k_R\to x_-\in[-\infty,0],\quad y_R\to\bar{y}\in[0,1],\quad \alpha_R\to\bar{\alpha}\in[\alpha,C].$$

Notice also that $x_+-x_-=+\infty$. Thanks to the strong convergence of $\bar{V}_{R_m}$ in $\mathcal{C}^\alpha(K)$, we have

$$\int_0^1\int_{\mathbb{T}^{N-1}}\bar{V}(\bar{y}+x_1,x')\,dx_1\,dx'=\bar{p}.\tag{2.20}$$

Furthermore, if $x_+<+\infty$ (resp. $x_->-\infty$), then $\bar{V}(x_+,x')=v(x_+,x',p_+)$ (resp. $\bar{V}(x_-,x')=v(x_-,x',p_-)$).

We can also pass to the limit in (2.19), thus $\bar{V}$ is a solution of

$$-\Delta\bar{V}+\mathrm{div}A(x,\bar{V})=0\quad\text{on }(x_-,x_+)\times\mathbb{T}^{N-1}.$$

Eventually, we have further properties on $\bar{V}$ from the properties listed in Lemma 2.1 as follows:

- L^∞ bound : $v(x,p_+)\le\bar{V}(x)\le v(x,p_-)$ for all $x\in\mathbb{R}\times\mathbb{T}^{N-1}$;
- Additional regularity: $\bar{V}_{R_m}\rightharpoonup\bar{V}$ in $W^{2,q}(K)$ for any compact set $K\subset(x_-,x_+)\times\mathbb{T}^{N-1}$ and for all $q<\infty$, and therefore $\bar{V}_{R_m}\to\bar{V}$ strongly in $W^{1,q}(K)$ for such compact sets K and for any $q\in[1,\infty]$. Moreover, $\bar{V}\in W^{1,\infty}(\Omega)$;
- Integration constant:

$$-\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}}\bar{V}(x_1,x')\,dx'+\int_{\mathbb{T}^{N-1}}A_1(x_1,x',\bar{V}(x_1,x'))\,dx'=\bar{\alpha}\quad\forall x_1\in(x_-,x_+).\tag{2.21}$$

- Monotony: $\bar{V}(x+e_1) \le \bar{V}(x)$.

▷ *Third step: Limit states of $\bar{V}$ and value of the integration constant.*

Let us consider the sequence $(u_k)_{k\in\mathbb{Z}}$ defined by

$$u_k : x \in [0,1] \times \mathbb{T}^{N-1} \mapsto \bar{V}(x+ke_1).$$

Thanks to the monotony property and the a priori bounds for $\bar{V}$, the sequence $(u_k)_{k\in\mathbb{Z}}$ is monotonous and bounded in $W^{1,\infty}([0,1]\times\mathbb{T}^{N-1})$. Thus for all $x \in [0,1]\times\mathbb{T}^{N-1}$, $(u_k(x))_{k\in\mathbb{Z}}$ has a finite limit as $k \to \pm\infty$, which we denote as $u_\pm$, and $u_\pm$ is bounded and Lipschitz continuous.

Since $u_k(1,x') = u_{k+1}(0,x')$ for all $k \in \mathbb{Z}$, we deduce that $u_\pm(0,x') = u_\pm(1,x')$ and thus $u_\pm$ is periodic. Let us now prove that $u_\pm = v(\cdot, p_\pm)$. We consider for instance the function u_+, the argument for u_- is strictly identical.

If $x_+ < \infty$, since (2.18) yields

$$\bar{V}_R(x_1,x') = v(x_1,x',p_+), \quad x_1 \ge R - k_R,\ x' \in \mathbb{T}^{N-1}$$

we deduce easily that $\bar{V}(x) = v(x,p_+)$ for all $x \in (x_+,\infty)\times\mathbb{T}^{N-1}$, and as a consequence, $u_+ = v(x,p_+)$.

If $x_+ = +\infty$, extending u_+ by periodicity, we have $u_+ \in \mathcal{C}(\mathbb{T}^N)$ and $\bar{V}(\cdot + ke_1) \to u_+$ locally uniformly as $k \to +\infty$. Since every u_k is a solution of

$$-\Delta u_k + \mathrm{div} A(x,u_k) = 0,$$

taking $k \to \infty$ in the above equation, we deduce that u_+ is a periodic solution of the above equation. Therefore there exists $\bar{p}_+ \in \mathbb{R}$ such that $u_+ = v(\cdot,\bar{p}_+)$. Notice that since $v(\cdot,p_-) \le \bar{V} \le v(\cdot,p_+)$, we have $\bar{p}_+ \in [p_+,\bar{p}_-]$.

In particular,

$$-\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} u_+(x_1,x')\,dx' + \int_{\mathbb{T}^{N-1}} A_1(x_1,x',u_+(x_1,x'))\,dx' = \bar{A}_1(\bar{p}_+) \quad \forall x_1 \in \mathbb{T}.$$

Taking the integral of the above identity over $\mathbb{T}$ and comparing with (2.21), we obtain $\bar{A}_1(\bar{p}_+) = \bar{\alpha}$.

Since $\bar{\alpha} \ge \alpha$ and $\bar{p}_+ \in [p_+,\bar{p}_-]$, the assumption (1.7) leads to $\bar{p}_+ \in \{p_+,p_-\}$ and

$$\bar{\alpha} = \alpha = \bar{A}_1(p_\pm). \tag{2.22}$$

Since $\bar{V}(\bar{y}+x_1+k,x') \le \bar{V}(\bar{y}+x_1,x')$ for all $(x_1,x') \in [0,1]\times\mathbb{T}^{N-1}$ and $k \in \mathbb{N}$,

$$u_+(\bar{y}+x_1,x') \le \bar{V}(\bar{y}+x_1,x') \quad \forall (x_1,x') \in [0,1]\times\mathbb{T}^{N-1}.$$

Taking the average of the above inequality over $[0,1]\times\mathbb{T}^{N-1}$, it follows from (2.20) that $\bar{p}_+ \le \bar{p} < p_-$. Therefore $\bar{p}_+ = p_+$.

Hence we conclude that

$$\bar{V}(x_1,x') - v(x_1,x',p_\pm) \to 0 \quad \text{as } x_1 \to \pm\infty \text{ in } L^\infty(\mathbb{T}^{N-1}).$$

Notice also that since (2.21) is true on $(x_-,x_+)\times\mathbb{T}^{N-1}$, and at least one of the properties $x_- = -\infty$ or $x_+ = +\infty$ always holds, it follows from the argument above that (2.22) always holds.

▷ *Fourth step: Conclusion.*

First of all, if $x_+ = +\infty$ and $x_- = -\infty$, then gathering the properties of the previous steps, $\bar{V}$ is a stationary shock with end states $v(\cdot,p_+)$ and $v(\cdot,p_-)$, thus Theorem 1.1 is proved.

Therefore we now consider the case $x_+ < +\infty$ (the case $x_- > -\infty$ is treated in a similar fashion). In this case, we see that the equation

$$-\Delta \bar{V} + \mathrm{div} A(x, \bar{V}) = 0 \tag{2.23}$$

is satisfied on $(-\infty, x_+) \times \mathbb{T}^{N-1}$. Of course, since $\bar{V}(x) = v(x, p_+)$ for all $x_1 > x_+$, the equation is also satisfied on $(x_+, \infty) \times \mathbb{T}^{N-1}$. Notice that $\bar{V}$ is continuous at the point $x_1 = x_+$, but its derivative in x_1 might have a jump, and therefore there might be a Dirac mass in $\Delta \bar{V}$ at $x_1 = x_+$. We prove that this is not the case.

Thus, using (2.21) and (2.22), we have that

$$-\frac{d}{dx_1} \int_{\mathbb{T}^{N-1}} \bar{V}(x_1, x')\, dx' \Big|_{x_1 = x_+^-} + \int_{\mathbb{T}^{N-1}} A_1(x_1, x', \bar{V}(x_1, x'))\, dx' \Big|_{x_1 = x_+^-} = \bar{A}_1(p_+), \tag{2.24}$$

and recall that

$$-\frac{d}{dx_1} \int_{\mathbb{T}^{N-1}} v(x_1, x', p_+)\, dx' \Big|_{x_1 = x_+} + \int_{\mathbb{T}^{N-1}} A_1(x_1, x', v(x_1, x', p_+))\, dx' \Big|_{x_1 = x_+} = \bar{A}_1(p_+). \tag{2.25}$$

Let $J(x')$ be the jump of $\partial_{x_1} \bar{V}$ at $x_1 = x_+$, i.e.

$$J(x') := \partial_{x_1} v(x_+, x', p_+) - \partial_{x_1} \bar{V}(x_+^-, x').$$

Since $\bar{V}(x_+^-, x') = v(x_+, x', p_+)$, combining (2.24) with (2.25), we get

$$\int_{\mathbb{T}^{N-1}} J(x') dx' = 0.$$

Moreover, since $\bar{V}(x) \geq v(x, p_+)$ for all $x \in \mathbb{R} \times \mathbb{T}^{N-1}$, with equality for $x_1 \geq x_+$, we have $J(x') \geq 0$ for all $x' \in \mathbb{T}^{N-1}$, consequently $J \equiv 0$. Thus $\partial_{x_1} \bar{V}$ has no jump at $x_1 = x_+$, which implies that the equation (2.23) is satisfied over the whole space.
Hence $\bar{V}$ is a stationary shock with end states $v(\cdot, p_+)$ and $v(\cdot, p_-)$, which completes the proof of Theorem 1.1.

Remark 2.2. *In fact, the situation where $x_+ < +\infty$ (resp. $x_- > -\infty$) cannot happen. Indeed, in that case, $w = \bar{V} - v(x, p_+)$ (resp. $w = v(x, p_-) - \bar{V}$) would be the non-negative solution of an elliptic equation of the type*

$$-\Delta w + \mathrm{div}(aw) = 0 \quad \textit{in } \mathbb{R} \times \mathbb{T}^{N-1},$$

with $a \in L^\infty(\mathbb{R} \times \mathbb{T}^{N-1})$, and $w \equiv 0$ for $x_1 \geq x_+$ (resp $x_1 \leq x_-$). Using once again the Harnack inequality, we infer that w has to vanish identically over $\mathbb{R} \times \mathbb{T}^{N-1}$, which leads to a contradiction. Therefore we always have $x_+ = +\infty$ and $x_- = -\infty$.

3. Properties of stationary shocks with periodic end states

We first show that the Rankine-Hugoniot condition (1.6a) is in fact a necessary condition for the existence of shock waves.

Lemma 3.1. *Assume $A_1 \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})$. Let $\bar{U}$ be a stationary shock wave connecting $v(\cdot, p_-)$ to $v(\cdot, p_+)$. Then $\bar{A}_1(p_-) = \bar{A}_1(p_+) =: \alpha$, and $\bar{U}$ satisfies*

$$-\frac{d}{dx_1} \int_{\mathbb{T}^{N-1}} \bar{U}(x_1, x') dx' + \int_{\mathbb{T}^{N-1}} A_1(x_1, x', \bar{U}(x_1, x')) dx' = \alpha.$$

Proof. Since the shock wave $\bar{U}$ is a solution of

$$-\Delta\bar{U} + \mathrm{div} A(x,\bar{U}) = 0,$$

there exists a constant $\bar{C}$ such that

$$-\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} \bar{U}(x)dx' + \int_{\mathbb{T}^{N-1}} A_1(x,\bar{U}(x))dx' = \bar{C}, \quad \forall x_1 \in \mathbb{R}.$$

Moreover, since

$$-\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} v(x,p_+)dx' + \int_{\mathbb{T}^{N-1}} A_1(x,v(x,p_+))dx' = \bar{A}_1(p_+),$$

we have

(3.26)

$$-\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} \Big(\bar{U}(x)-v(x,p_+)\Big)dx' + \int_{\mathbb{T}^{N-1}} \Big(A_1(x,\bar{U}(x))-A_1(x,v(x,p_+))\Big)dx' = \bar{C}-\bar{A}_1(p_+).$$

Notice that (1.5) and $A_1 \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})$ yield that for any $\varepsilon \in (0,1)$, there exists $m > 0$ such that for all $x_1 > m$,

$$|\bar{U}(x) - v(x,p_+)| \le \varepsilon, \quad |A_1(x,\bar{U}(x)) - A_1(x,v(x,p_+))| \le \varepsilon \|\partial_v A_1\|_{L^\infty(\mathbb{T}^N\times(-K,K))},$$

where the constant K is such that $K \ge \|v(\cdot,p_+)\|_\infty + 1$. Thus, integrating (3.26) over $[m, m+1]$, we have that

$$|\bar{C} - \bar{A}_1(p_+)| \le C\varepsilon \quad \forall \varepsilon \in (0,1),$$

which implies that $\bar{C} = \bar{A}_1(p_+)$. Similarly, applying the above argument to $v(\cdot,p_-)$, we have $\bar{C} = \bar{A}_1(p_-)$. □

If we impose additional conditions on the flux A at the two end states, the shock wave exponentially converges towards the end states:

Proposition 3.1. *Let $\bar{U}$ be a stationary shock wave connecting $v(\cdot,p_-)$ to $v(\cdot,p_+)$ satisfying $v(\cdot,p_+) \le \bar{U} \le v(\cdot,p_-)$. Assume that $A_1 \in (W^{1,\infty}_{loc} \cap \mathcal{C}^1)(\mathbb{T}^N \times \mathbb{R})$ and that there exist periodic functions $\phi \in L^1(\mathbb{T})$ and $\psi \in L^1(\mathbb{T})$ such that the Lax conditions* (1.9), (1.10) *are satisfied.*

Then there exist positive constants R and C_R such that for all $\pm x_1 > R$,

$$\int_{\mathbb{T}^{N-1}} |\bar{U}(x_1,x') - v(x_1,x',p_\pm)|dx' < C_R\, e^{a_\pm x_1/2} \int_{\mathbb{T}^{N-1}} |\bar{U}(\pm R,x') - v(\pm R,x',p_\pm)|dx'.$$

Proof. We show the convergence towards the left end state $v(\cdot,p_-)$. First of all, we see that Lemma 3.1 yields

$$\begin{aligned}
&\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} |\bar{U}(x) - v(x,p_-)|dx' \\
&\quad= \frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} (v(x,p_-) - \bar{U}(x))dx' \\
&\quad= \int_{\mathbb{T}^{N-1}} (A_1(x,v(x,p_-)) - A_1(x,\bar{U}(x)))dx' \\
&\quad= \int_{\mathbb{T}^{N-1}} \left(\int_0^1 \partial_v A_1(x,\tau\bar{U} + (1-\tau)v(x,p_-))d\tau\right) (v(x,p_-) - \bar{U}(x))dx'.
\end{aligned}$$

It follows from (1.5) that

$$\forall\varepsilon > 0,\ \exists R > 0 \quad \text{s.t.} \quad |\bar{U}(x) - v(x,p_-)| < \varepsilon \quad \text{for all } x_1 < -R,\ x' \in \mathbb{T}^{N-1}.$$

Since the function $\partial_v A_1$ is continuous, using (1.9), we infer that there exists a constant C such that

$$\partial_v A_1(x, \tau\bar{U} + (1-\tau)v(x,p_-)) \geq \phi(x_1) - C\varepsilon \quad \text{for all } x_1 < -R,\ x' \in \mathbb{T}^{N-1},\ \tau \in [0,1].$$

Thus for all $x_1 < -R$,

$$\frac{d}{dx_1}\int_{\mathbb{T}^{N-1}} |\bar{U}(x) - v(x,p_-)|dx' \geq (\phi(x_1) - C\varepsilon)\int_{\mathbb{T}^{N-1}} |\bar{U}(x) - v(x,p_-)|dx'.$$

This inequality implies that for all $x_1 < -R$

$$\int_{\mathbb{T}^{N-1}} |\bar{U}(x) - v(x,p_-)|dx' < \int_{\mathbb{T}^{N-1}} |\bar{U}(x) - v(x,p_-)|dx'\Big|_{x_1=-R} \times \exp\Big(\int_{-R}^{x_1} (\phi(s) - C\varepsilon)ds\Big).$$

Since the periodicity of ϕ implies that there exists a positive constant C such that

$$\int_{-R}^{x_1} \phi(s)ds \leq (x_1 + R)\int_{\mathbb{T}} \phi(s)ds + C,$$

we have the desired estimate for the case of $v(x,p_-)$, choosing ε so that $a_- - C\varepsilon \geq a_-/2$. The same arguments also lead to the convergence towards $v(\cdot,p_+)$ as $x_1 \to +\infty$. □

Lemma 3.2. *Assume $A \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})^N$ satisfying the assumptions of Theorem 1.1, together with (1.7). Let $\bar{U}$ and $\bar{V}$ be two stationary shock waves connecting $v(\cdot,p_-)$ to $v(\cdot,p_+)$. Then $\bar{U}$ and $\bar{V}$ enjoy the following properties:*

- *The function $\bar{U} - \bar{V}$ keeps a constant sign;*
- *$\bar{U} - \tau_1\bar{U} \geq 0$, $\bar{V} - \tau_1\bar{V} \geq 0$;*
- *There exist $k_-, k_+ \in \mathbb{Z}$, with either $k_+ = 0$ or $k_- = 0$, such that*

$$\tau_{k_-}\bar{U} \leq \bar{V} \leq \tau_{k_+}\bar{U};$$

- *$\bar{U} - \bar{V} \in L^1(\mathbb{R} \times \mathbb{T}^{N-1})$.*

Remark 3.1. *The first statement of Lemma 3.2 implies that stationary shocks are ordered in the sense that any two shocks $\bar{U}$ and $\bar{V}$ satisfy only one of $\bar{U} = \bar{V}$, $\bar{U} < \bar{V}$ and $\bar{U} > \bar{V}$.*

Proof. Let us first prove that $\bar{U} - \bar{V}$ keeps a constant sign. Assume for instance that $\bar{U}(0) \leq \bar{V}(0)$. Using the same argument as the one developed from (2.16), we have that $W := \bar{U} - \bar{V}$ satisfies

$$-\Delta|W| + \operatorname{div}_x\Big(\operatorname{sgn}(W)\left(A(x,\bar{U}) - A(x,\bar{V})\right)\Big) \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{T}^{N-1}).$$

We denote by $-m$ the left hand-side of the above inequality. Then m is a non-negative measure. But since

$$\lim_{x_1 \to \pm\infty} \|\bar{U}(x_1,\cdot) - \bar{V}(x_1,\cdot)\|_{L^\infty(\mathbb{T}^{N-1})} = 0,$$

we have that $m(\mathbb{R} \times \mathbb{T}^{N-1}) = 0$. Thus,

$$-\Delta|W| + \operatorname{div}_x\Big(\operatorname{sgn}(W)\left(A(x,\bar{U}) - A(x,\bar{V})\right)\Big) = 0.$$

That is, $|W|$ is a non-negative solution of an elliptic equation of the type:

$$-\Delta|W| + \operatorname{div}_x(a|W|) = 0,$$

where

$$a := \int_0^1 \partial_v A(x, \tau\bar{U} + (1-\tau)\bar{V})d\tau \in L^\infty(\mathbb{R} \times \mathbb{T}^{N-1}).$$

As a consequence, Harnack's inequality implies that either W is identically zero, or W never vanishes. Thus there are two possibilities:

- If $\bar U(0)=\bar V(0)$, then $W\equiv 0$ and $\bar U=\bar V$;
- If $\bar U(0)<\bar V(0)$, then W never vanishes and $\bar V-\bar U$ remains strictly positive. In that case

$$v(x,p_+)<\bar U(x)<\bar V(x)<v(x,p_-),\quad \forall x\in\mathbb{R}\times\mathbb{T}^{N-1}.$$

Hence the first statement of the Lemma is proved.

Concerning the second statement, observe that $\tau_1\bar U$ and $\tau_1\bar V$ are also stationary shock waves connecting $v(\cdot,p_-)$ to $v(\cdot,p_+)$. As a consequence, according to the first statement, $\bar U-\tau_1\bar U$ and $\bar V-\tau_1\bar V$ keep a constant sign. It follows that the sequences of functions $(\tau_k\bar U)_{k\in\mathbb{Z}},(\tau_k\bar V)_{k\in\mathbb{Z}}$ are monotonous, and using assumption (1.7), we infer that these sequences are necessarily non-increasing. Hence $\bar U-\tau_1\bar U\geq 0$, $\bar V-\tau_1\bar V\geq 0$.

We now address the third statement. Once again, without loss of generality, we assume that $\bar U\leq\bar V$, so that $k_-=0$. Moreover, since the sequence $(\bar U(ke_1))_{k\in\mathbb{Z}}$ is monotonous, we have

$$\tau_k\bar U(x)\to v(x,p_-)\quad\text{as } k\to-\infty \text{ in } L^\infty([0,1]\times\mathbb{T}^{N-1})$$

Thus there exists $k_+\in\mathbb{Z}$, $k_+\leq 0$, such that for all $x\in[0,1]\times\mathbb{T}^{N-1}$,

$$\bar V(x)\leq\tau_{k_+}\bar U(x).$$

Using the first statement and the fact that $\tau_{k_+}\bar U$ is a standing shock, we infer that $\bar V\leq\tau_{k_+}\bar U$.

Eventually, still working under the assumption $\bar U\leq\bar V$, we have, for any $K>0$,

$$\begin{aligned}\int_{-K}^{K}\int_{\mathbb{T}^{N-1}}|\bar U-\bar V| &\leq \int_{-K}^{K}\int_{\mathbb{T}^{N-1}}(\tau_{k_+}\bar U-\bar U)\\ &= \int_{-K+k_+}^{-K}\int_{\mathbb{T}^{N-1}}\bar U-\int_{K+k_+}^{K}\int_{\mathbb{T}^{N-1}}\bar U\leq 2|k_+|\|\bar U\|_\infty,\end{aligned}$$

and therefore $\bar U-\bar V\in L^1(\Omega)$. □

Proposition 3.2. *Assume that the assumptions of Theorem 1.1 are satisfied, together with the Lax assumptions* (1.9)-(1.10). *Assume furthermore that* $\partial_v A,\partial_v^2 A\in W^{1,\infty}_{loc}(\mathbb{T}^N\times\mathbb{R})$.

Let $\bar U$ *be a stationary shock wave connecting* $v(\cdot,p_-)$ *to* $v(\cdot,p_+)$. *Let* $q\in\mathbb{R}$ *be arbitrary. Then there exists a unique shock* $\bar V\in\bar U+L^1(\mathbb{R}\times\mathbb{T}^{N-1})$ *such that* $\int_{\mathbb{R}\times\mathbb{T}^{N-1}}(\bar V-\bar U)=q$.

Remark 3.2. *The sole purpose of assumptions* (1.9)-(1.10) *is to ensure that the family* $(p_R)_{R>0}$ *defined by* (3.28) *below is equi-integrable, and therefore compact with respect to* R. *If this compactness property can be retrieved in another way, then assumptions* (1.9)-(1.10) *can be removed from the statement of Proposition 3.2.*

Proposition 3.2 has the following immediate consequence:

Corollary 3.1. *Let* p_-,p_+ *be constants such that the assumptions of Theorem 1.1 are satisfied. Assume that* (1.9)-(1.10) *hold, and let* $\bar U$ *be a stationary shock wave connecting* $v(\cdot,p_-)$ *to* $v(\cdot,p_+)$. *If* $u\in\bar U+L^1(\mathbb{R}\times\mathbb{T}^{N-1})$, *then there exists a unique standing shock* $\bar V$ *such that* $u\in\bar V+L^1_0(\mathbb{R}\times\mathbb{T}^{N-1})$.

We now turn to the proof of Proposition 3.2. The proof relies heavily on properties of the function $\bar U(\cdot+e_1)-\bar U$, which we list in the following Lemma:

Lemma 3.2. *Assume the hypotheses of Theorem 1.1, together with* (1.7)*, futhermore* $\partial_v A \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})$*. Let* $\bar U$ *be a stationary shock wave connecting* $v(\cdot, p_-)$ *to* $v(\cdot, p_+)$*, and let*

$$p := \bar U - \bar U(\cdot + e_1).$$

Then p *satisfies the following properties:*

- *Setting*

$$b(x) = \int_0^1 \partial_v A(x, s\bar U(x) + (1-s)\bar U(x+e_1))\, ds \in W^{1,\infty}(\Omega),$$

  the function p *is a non-negative solution of*

$$-\Delta p + \operatorname{div}(bp) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^{N-1}; \tag{3.27}$$

  Moreover, $p \in L^1 \cap W^{1,\infty}(\mathbb{R} \times \mathbb{T}^{N-1})$.
- *For any* $R > 1$*, consider the approximate problem*

$$\begin{aligned} -\Delta p_R + \operatorname{div}(bp_R) = 0 \quad &\text{in } (-R,R) \times \mathbb{T}^{N-1},\\ -\partial_1 p_R + b_1 p_R = 0 \ &\text{at } x_1 = \pm R,\\ \int_{\Omega_R} p_R = \int_\Omega p&. \end{aligned} \tag{3.28}$$

  Then equation (3.28) *has a unique solution* $p_R \in H^1(\Omega_R)$*. Moreover,* $p_R(x) > 0$ *for all* $x \in \Omega_R$*, and if* p_R *is extended by zero outside* Ω_R*, the family* $(p_R)_{R>0}$ *is uniformly bounded in* $L^q(\Omega)$ *for all* $1 \le q < \infty$*.*
- *Assume that the Lax conditions* (1.9)-(1.10) *are satisfied. Then*

$$p_R \to p \quad \text{as } R \to \infty, \text{ in } L^1(\mathbb{R} \times \mathbb{T}^{N-1}).$$

Proof. • *Properties of* p*:* the integrability, sign and regularity properties of p follow from Lemma 3.2 and from the regularity properties of $\bar U$. The equation on p simply follows from making the difference between the equations on $\tau_1 \bar U$ and $\bar U$.

• *Properties of* p_R*:* existence, uniqueness and positivity are a consequence of the Krein-Rutman theorem (see Appendix). The uniform L^1 bound follows from the normalization and the positivity. We then obtain H^1 bounds by multiplying (3.28) by p_R and integrating by parts. We obtain

$$\int_{\Omega_R} |\nabla p_R|^2 \le \frac{1}{2} \|b\|_{W^{1,\infty}} \left(\int_{\mathbb{T}^{N-1}} p_R^2(R,x')\, dx' + \int_{\mathbb{T}^{N-1}} p_R^2(-R,x')\, dx' + \int_{\Omega_R} p_R^2 \right).$$

Using first a trace inequality and then the Gagliardo-Nirenberg interpolation inequality, we infer that for all $\nu > 0$, there exists a constant C_ν, independent of R, and such that

$$\begin{aligned} \|p_R(\pm R, \cdot)\|_{L^2(\mathbb{T}^{N-1})} &\le C\|p_R\|_{H^{1/2}(\Omega_R)} \le C_\nu \|p_R\|_{L^2} + \nu \|\nabla p_R\|_{L^2} \\ &\le C_\nu \|p_R\|_{L^1}^{1-\alpha} \|\nabla p_R\|_{L^2}^{\alpha} + \nu \|\nabla p_R\|_{L^2} \\ &\le C_\nu \|p_R\|_{L^1} + 2\nu \|\nabla p_R\|_{L^2}, \end{aligned}$$

where $\alpha = N/(N+2)$. Taking ν sufficiently small, we infer that

$$\int_{\Omega_R} |\nabla p_R|^2 \le C \|b\|_{W^{1,\infty}} \|p_R\|_{L^1}^2 \le C.$$

Using once again the uniform L^1 bound together with the Gagliardo-Nirenberg interpolation inequality, we obtain

$$\sup_{R>0} \|p_R\|_{H^1(\Omega_R)} < \infty.$$

Likewise, we have uniform L^q bounds, i.e., for any $1 < q < \infty$,

$$\|p_R\|_{L^q(\Omega_R)} \leq \|p_R\|_{L^1}^{1-\alpha'} \|\nabla p_R\|_{L^2}^{\alpha'} \leq C, \quad \alpha' = \frac{2N(q-1)}{(N+2)q} \in (0,1).$$

• *Asymptotic behaviour of p_R when the Lax conditions are satisfied:* we first obtain estimates on the rate of decay in x_1 in the following way. Integrating equation (3.28) on $\mathbb{T}^{N-1}$ leads to

$$-\frac{d^2}{dx_1^2} \int_{\mathbb{T}^{N-1}} p_R + \frac{d}{dx_1} \int_{\mathbb{T}^{N-1}} b_1 p_R = 0,$$

and thus

$$-\frac{d}{dx_1} \int_{\mathbb{T}^{N-1}} p_R + \int_{\mathbb{T}^{N-1}} b_1 p_R = \text{cst. on } [-R, R].$$

The boundary conditions imply that the constant has to be zero, and therefore

$$-\frac{d}{dx_1} \int_{\mathbb{T}^{N-1}} p_R + \int_{\mathbb{T}^{N-1}} b_1 p_R = 0 \text{ on } [-R, R].$$

Now, since Proposition 3.1 yields that for all $\pm x_1 > R$,

$$\int_{\mathbb{T}^{N-1}} |p| \leq \int_{\mathbb{T}^{N-1}} |\bar{U}(x) - v(x, p_\pm)| + \int_{\mathbb{T}^{N-1}} |v(x + e_1, p_\pm) - \bar{U}(x + e_1)| < C_R\, e^{a_\pm x_1/2},$$

there exists $x_0' \in \mathbb{T}^{N-1}$ such that $p(x_1, x_0') \leq C_R\, e^{a_\pm x_1/2}$ for all $\pm x_1 > R$, which implies together with Harnack inequality that $p(x_1, \cdot)$ converges exponentially fast towards zero in $L^\infty(\mathbb{T}^{N-1})$ as $x_1 \to \pm\infty$, and b_1 therefore converges exponentially fast towards $\partial_v A(x, v(x, p_\pm))$ in $L^\infty(\mathbb{T}^{N-1})$ as $x_1 \to \pm\infty$. The Lax conditions (1.9)-(1.10) imply that for any $\varepsilon < \max(|a_+|, |a_-|)/2$, there exists $K > 0$ such that

$$b_1(x) \geq \phi(x_1) - \varepsilon \quad \text{for } x_1 < -K, \quad b_1(x) \leq \psi(x_1) + \varepsilon \quad \text{for } x_1 > K.$$

Thus, if $K < x_1 < R$, we obtain, since $p_R > 0$,

$$0 \leq -\frac{d}{dx_1} \int_{\mathbb{T}^{N-1}} p_R + (\psi(x_1) + \varepsilon) \int_{\mathbb{T}^{N-1}} p_R.$$

As a consequence, there exists a constant C such that

$$\int_{\mathbb{T}^{N-1}} p_R(x_1, x')\, dx' \leq C \exp(x_1 a_+/2).$$

We also obtain similar estimates on $(-R, -K)$. Using the Harnack inequality, we deduce eventually that there exists a constant C (independent of R) such that

$$C^{-1} \exp(a_- x_1/2) \leq p_R(x) \quad \text{if } x_1 < 0, \quad p_R(x) \leq C \exp(a_+ x_1/2) \quad \text{if } x_1 > 0.$$

Furthermore, if we consider

$$\varphi_+ := \inf_{x' \in \mathbb{T}^{N-1}} \partial_v A_1(x_1, x', v(x_1, x', p_+)), \quad \varphi_- := \sup_{x' \in \mathbb{T}^{N-1}} \partial_v A_1(x_1, x', v(x_1, x', p_-)),$$

then (1.9)-(1.10) imply that $b_+ := \int_{\mathbb{T}} \varphi_+ dx_1 < 0$ and $b_- := \int_{\mathbb{T}} \varphi_- dx_1 > 0$, and using the above arguments, we obtain that

$$p_R(x) \leq C \exp(b_- x_1/2) \quad \text{if } x_1 < 0, \quad C^{-1} \exp(b_+ x_1/2) \leq p_R(x) \quad \text{if } x_1 > 0.$$

Hence, the sequence $(p_R)_{R>1}$ is equi-integrable. Using the uniform H^1 estimate, we deduce that $(p_R)_{R>1}$ is compact in $L^1(\Omega)$. By uniqueness (up to a multiplicative constant) of the solutions of (3.27), it follows that $p_R \to p$ in $L^1(\Omega)$.

□

Remark 3.3. *Obviously, the same statements hold for $p_k := |\bar{U}(\cdot + ke_1) - \bar{U}|$ for any $k \in \mathbb{Z}$, replacing every occurrence of $\bar{U}(\cdot + e_1)$ by $\bar{U}(\cdot + ke_1)$.*

We are now ready to prove Proposition 3.2:

3.1. **Proof of Proposition 3.2.** Let $q \in \mathbb{R}$ be fixed. Notice first that if there exist two shocks $\bar{V}_1, \bar{V}_2$ with $\int(\bar{V}_1 - \bar{U}) = \int(\bar{V}_2 - \bar{U}) = q$, then $\bar{V}_1 - \bar{V}_2 \in L^1_0(\Omega)$ and $\bar{V}_1 - \bar{V}_2$ keeps a constant sign according to Lemma 3.2. Hence $\bar{V}_1 = \bar{V}_2$. The uniqueness of $\bar{V}$ follows. We therefore focus on the existence of $\bar{V}$ in the rest of the proof.

First, there exists an integer $k \in \mathbb{Z}$ such that q has the same sign as $\bar{U}(\cdot + ke_1) - \bar{U}$, and

$$|q| \le \|\bar{U}(\cdot + ke_1) - \bar{U}\|_{L^1} = \|p_k\|_{L^1}.$$

In order to fix ideas, we work with $q > 0$, so that $k < 0$ and $p_k = \bar{U}(\cdot + ke_1) - \bar{U}$. In the sequel, we set

$$b_k(x) = \int_0^1 \partial_v A(x, (1-s)\bar{U}(x) + s\bar{U}(x + ke_1))\, ds \in W^{1,\infty}(\Omega).$$

The goal is to prove that for all $q \in \mathbb{R}$, the following equation has at least one solution

$$-\Delta W + \mathrm{div} B(x, W) = 0 \quad \text{in } \Omega, \quad W \in L^1(\Omega), \ \int_\Omega W = q, \tag{3.29}$$

where $B(x, r) = A(x, \bar{U} + r) - A(x, \bar{U})$. Setting $W = \bar{V} - \bar{U}$, this is strictly equivalent to the statement of Proposition 3.2.

In order to require that $\int_\Omega W = q$, we slightly modify the form of equation (3.29) and rather look for solutions of the equation

$$-\Delta W + \mathrm{div}(b_k W) + \mathrm{div}\tilde{B}_k(x, W) = 0, \quad W \in L^1(\Omega), \ \int_\Omega W = q \tag{3.30}$$

where $\tilde{B}_k(x,r) = \tilde{A}(x, \bar{U}+r) - \tilde{A}(x, \bar{U}) - b_k(x) r$. Here, $\tilde{A}$ is defined by $\tilde{A}(x,r) := A(x,r)\chi(r)$, where $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi(r) = 1$ for $|r| \le r_0$ for some large constant r_0 with $r_0 > 2\|v(\cdot, p_\pm)\|_\infty$, thus $\tilde{A} \in W^{1,\infty}(\mathbb{T}^N \times \mathbb{R})$. It is clear that if $W \in L^1$ is a solution of (3.30), then $\bar{V} = W + \bar{U}$ is a standing shock for the flux $\tilde{A}$, with periodic end states $v(\cdot, p_\pm)$. As a consequence, $v(\cdot, p_+) \le \bar{V} \le v(\cdot, p_-)$, and thus $\tilde{A}(x, \bar{V}(x)) = A(x, \bar{V}(x))$. Whence $\bar{V}$ is a standing shock for the flux A such that $\int(\bar{V} - \bar{U}) = q$.

Notice also that there exists a constant C such that

$$|\tilde{B}_k(x,r)| \le C|r| \quad \forall r \in \mathbb{R}, \ \forall x \in \Omega,$$

and that for all $r \in \mathbb{R}, x \in \Omega$, since $\partial_v^2 A \in L^\infty_{loc}$,

$$\begin{aligned} \tilde{B}_k(x,r) &= r\int_0^1 \left(\partial_v \tilde{A}(x, \bar{U} + sr) - \partial_v \tilde{A}(x, \bar{U} + sp_k(x))\right) ds \\ &= r(r - p_k)\int_0^1 \int_0^1 s\partial_v^2 \tilde{A}(x, \bar{U} + s\tau r + s(1-\tau)p_k(x))\, d\tau\, ds. \end{aligned}$$

As a consequence, for all $r \in \mathbb{R}, x \in \Omega$,

$$|\tilde{B}_k(x,r)| \le C|r||r - p_k(x)|. \tag{3.31}$$

We prove the existence of solutions of (3.30) by using approximate problems on Ω_R and passing to the limit as $R \to \infty$. Using Lemma 3.2, we first introduce the function $p_{k,R}$ which solves

$$
\begin{aligned}
&- \Delta p_{k,R} + \operatorname{div}(b_k p_{k,R}) = 0 \text{ in } \Omega_R, \\
&- \partial_1 p_{k,R} + b_{k,1} p_{k,R} = 0 \text{ for } x_1 = \pm R, \quad \int_{\Omega_R} p_{k,R} = \int_{\Omega} p_k.
\end{aligned} \tag{3.32}
$$

We recall that $p_{k,R} > 0$ in Ω_R. We define

$$
\tilde{B}_{k,R}(x,r) = \chi_R(x_1) r (r - p_{k,R}) \int_0^1 \int_0^1 s \partial_v^2 \tilde{A}(x, \bar{U} + s\tau r + s(1-\tau) p_k(x))\, d\tau\, ds
$$

for some cut-off function χ_R such that $\chi_R \equiv 1$ on $(-R+1, R-1)$ and Supp $\chi_R \subset (-R + 1/2, R - 1/2)$.

We now prove that for all $R > 1$, there exists a solution $W_R \in H^1(\Omega_R)$ of the equation

$$
\begin{aligned}
-\Delta W_R + \operatorname{div}(b_k W_R) + \operatorname{div} \tilde{B}_{k,R}(x, W_R) &= 0 \text{ in } \Omega_R, \\
-\partial_1 W_R + b_{k,1} W_R &= 0 \text{ for } x_1 = \pm R, \\
\int_{\Omega_R} W_R &= q.
\end{aligned} \tag{3.33}
$$

Let us solve equation (3.33) by using Schaefer's fixed point theorem. Let $W_1 \in H^1(\Omega_R)$ be arbitrary. We use the Fredholm alternative to solve the equation

$$
\begin{aligned}
-\Delta W_2 + \operatorname{div}(b_k W_2) + \operatorname{div} \tilde{B}_{k,R}(x, W_1) &= 0 \text{ in } \Omega_R, \\
-\partial_1 W_2 + b_{k,1} W_2 &= 0 \text{ for } x_1 = \pm R, \\
\int_{\Omega_R} W_2 &= q.
\end{aligned} \tag{3.34}
$$

Indeed, according to Lemma A.1, the solutions of the homogeneous equation

$$
-\Delta w + \operatorname{div}(b_k w) = 0 \text{ in } \Omega_R, \quad -\partial_1 w + b_{k,1} w = 0 \text{ for } x_1 = \pm R \tag{3.35}
$$

are the functions $w = c p_{k,R}$ where $p_{k,R} > 0, c \in \mathbb{R}$. Since the dual problem of (3.35) is

$$
-\Delta q - b_k \cdot \nabla q = 0 \quad \text{in } \Omega_R, \quad \partial_1 q = 0 \text{ for } x_1 = \pm R,
$$

and a simple computation gives

$$
\int_{\Omega_R} p_{k,R} |\nabla q|^2 dx = 0,
$$

the solutions of the dual problem are the constants. Thus, the inhomogeneous term $-\operatorname{div} \tilde{B}_{k,R}(x, W_1)$ of (3.34) is orthogonal to the constants thanks to the cut-off function χ_R. This ensures the existence of solutions of the first two lines of (3.34); these solutions are defined up to a function of the form $c p_{k,R}$, for $c \in \mathbb{R}$, and the third line of (3.34) fixes the value c and ensures uniqueness of solutions of (3.34). Hence we can define the operator $L_R : W_1 \in L^2(\Omega_R) \mapsto W_2 \in L^2(\Omega_R)$. Notice that in fact, the operator L_R is continuous from $L^2(\Omega_R)$ to $H^1(\Omega_R)$, and therefore L_R is compact for all $R > 0$. Now, let $\lambda \in [0,1]$ be arbitrary, and let W^λ be such that $\lambda L_R(W^\lambda) = W^\lambda$. We first observe that since $\tilde{B}_{k,R}(x, 0) = 0$,

$(W^\lambda)_+ := W^\lambda \mathbf{1}_{W^\lambda \geq 0}$ satisfies

$$-\Delta (W^\lambda)_+ + \mathrm{div}(b_k (W^\lambda)_+) + \lambda \mathrm{div}\left(\mathbf{1}_{W^\lambda>0}\tilde{B}_{k,R}(x,W^\lambda)\right) \leq 0 \quad \text{in } \Omega_R,$$
$$-\partial_1 (W^\lambda)_+ + b_{k,1}(W^\lambda)_+ = 0 \text{ for } x_1 = \pm R.$$

Using once again an argument similar to the one developed form (2.16), we deduce that W^λ keeps a constant sign on Ω_R. Thus

$$\|W^\lambda\|_{L^1(\Omega_R)} = \int W^\lambda = q > 0. \tag{3.36}$$

We derive an uniform H^1 bound on W^λ in the following way: we have

$$\int_{\Omega_R} |\nabla W^\lambda|^2 - \int_{\Omega_R} b_k W^\lambda \cdot \nabla W^\lambda = \lambda \int_{\Omega_R} \tilde{B}_{k,R}(x,W^\lambda)\cdot \nabla W^\lambda.$$

Using trace estimates together with the Gagliardo-Nirenberg interpolation as in the proof of Lemma 3.2, we have that for any $\nu > 0$ there exists $C_\nu > 0$ such that

$$\begin{aligned}\int_{\Omega_R} b_k W^\lambda \cdot \nabla W^\lambda &= -\frac{1}{2}\int_{\Omega_R} \mathrm{div}(b_k)|W^\lambda|^2 + \frac{1}{2}\int_{\mathbb{T}^{N-1}} b_{k,1}(R,x')|W^\lambda(R,x')|^2\,dx' \\ &\quad -\frac{1}{2}\int_{\mathbb{T}^{N-1}} b_{k,1}(R,x')|W^\lambda(R,x')|^2\,dx' \\ &\leq C_\nu \|b_k\|_{W^{1,\infty}} \|W^\lambda\|_{L^1}^{2-\alpha}\|\nabla W^\lambda\|_2^\alpha + \nu \|\nabla W^\lambda\|_2^2\end{aligned}$$

for some $\alpha \in (0,2)$. On the other hand, setting

$$\beta_{k,R}(x,r) := \int_0^r \tilde{B}_{k,R}(x,r')\,dr', \quad x \in \Omega,\ r \in \mathbb{R},$$

we have (notice that since $\partial_v^2 A \in W^{1,\infty}_{loc}$, we also have $\tilde{B}_{k,R} \in W^{1,\infty}$)

$$\tilde{B}_{k,R}(x,W^\lambda)\cdot \nabla W^\lambda = \mathrm{div}\left(\beta_{k,R}(x,W^\lambda)\right) - (\mathrm{div}_x \beta_{k,R})(x,W^\lambda).$$

Since $\beta_{k,R}(\pm R, x', r) = 0$ for all x', r,

$$\begin{aligned}\int_{\Omega_R} \tilde{B}_{k,R}(x,W^\lambda)\cdot \nabla W^\lambda dx &= -\int_{\Omega_R} (\mathrm{div}_x \beta_{k,R})(x,W^\lambda)dx \\ &= -\int_{\Omega_R}\int_0^{W^\lambda} (\mathrm{div}_x \tilde{B}_{k,R})(x,r')\,dr'dx \\ &\leq \int_{\Omega_R}\int_0^{W^\lambda} C\,dr'dx \\ &\leq C\|W^\lambda\|_{L^1(\Omega_R)}.\end{aligned}$$

Using Young's inequality together with the L^1 bound (3.36) on W^λ, we infer that there exists a constant C independent of λ and R, such that

$$\|W^\lambda\|_{H^1(\Omega_R)} \leq C.$$

Therefore, it follows from Schaefer's theorem that (3.33) has a solution $W_R \in H^1 \cap L^1(\Omega_R)$. Moreover, using the estimates above for $\lambda = 1$, we deduce that the family $(W_R)_{R>0}$ is bounded in $H^1 \cap L^1(\Omega_R)$ uniformly in R.

Furthermore, we claim that

$$0 \le W_R \le p_{k,R} \quad \forall R. \tag{3.37}$$

The positivity of W_R has been proved above. As for the upper-bound, we notice that by definition of $\tilde{B}_{k,R}$, $\tilde{B}_{k,R}(x, p_{k,R}) \equiv 0$, and thus it follows from (3.32) that $p_{k,R}$ is a solution of

$$-\Delta p_{k,R} + \operatorname{div}(b_k p_{k,R}) + \operatorname{div}\tilde{B}_{k,R}(x, p_{k,R}) = 0 \text{ in } \Omega_R,$$
$$-\partial_1 p_{k,R} + b_{k,1} p_{k,R} = 0 \text{ for } x_1 = \pm R.$$

Using the same argument as the one leading to the positivity of W_R, we deduce that $W_R - p_{k,R}$ keeps a constant sign over Ω_R. By definition of k,

$$q = \int_{\Omega_R} W_R \le \int_{\Omega} p_k = \int_{\Omega_R} p_{k,R},$$

we deduce that $W_R - p_{k,R} \le 0$.

We can now pass to the limit in (3.33) as $R \to \infty$. According to the uniform $H^1 \cap L^1$ bounds, we deduce that there exists $W \in H^1 \cap L^1(\Omega)$ such that $W_R \rightharpoonup W$ in $H^1(\Omega)$, and $W_R \to W$ in $L^2_{loc}(\Omega)$ up to a subsequence. Since $p_{k,R} \to p_k$ in L^1 according to Lemma 3.2, we deduce that W is a solution of

$$-\Delta W + \operatorname{div}(b_k W) + \operatorname{div}\tilde{B}_k(x, W) = 0.$$

Eventually, using inequality (3.37) together with the convergence in L^1 of the functions $p_{k,R}$, we deduce that $(W_R)_{R>0}$ is uniformly equi-integrable, and therefore compact in L^1. Hence, up to a further extraction of subsequences, $W_R \to W$ in L^1 and

$$\int W = \lim_{R \to \infty} \int W_R = q.$$

Thus the existence of solutions of (3.30) is proved, which completes the proof of Proposition 3.2.

4. Stability of stationary shocks

The goal of this section is to prove Theorem 1.2. Throughout the section, we denote by $(S_t)_{t \ge 0}$ the semi-group associated with equation (1.1). We recall (see for instance [22]) that S_t is well-defined in $L^1(\Omega) + L^\infty(\Omega)$, is order preserving and satisfies conservation and contraction principles in $L^1(\Omega)$: if $u, v \in L^1(\Omega) + L^\infty(\Omega)$ are such that $u - v \in L^1(\Omega)$, then $S_t u - S_t v \in L^1(\Omega)$ for all $t \ge 0$ and

$$\int_\Omega (S_t u - S_t v) = \int_\Omega (u - v), \quad \|S_t u - S_t v\|_{L^1} \le \|u - v\|_{L^1} \quad \forall t \ge 0.$$

First of all, Corollary 3.1 allows us to restrict the proof of Theorem 1.2 to the case of zero-mass perturbation $u_0 \in \bar{U} + L^1_0(\mathbb{R} \times \mathbb{T}^{N-1})$.

On the other hand, the following lemma allows us to replace the inequality (1.8) by an inequality where the upper bound and lower bounds are standing shocks.

Lemma 4.1. *Let $\bar{U}$ be a stationary shock wave connecting $v(\cdot, p_-)$ to $v(\cdot, p_+)$. Assume that $u \in \bar{U} + L^1_0(\mathbb{R} \times \mathbb{T}^{N-1})$ satisfies $v(x, p_+) \le u(x) \le v(x, p_-)$ for a.e. $x \in \mathbb{R} \times \mathbb{T}^{N-1}$.*

Then, for any $\varepsilon > 0$, there exist a function $u^\varepsilon \in \bar{U} + L^1_0(\mathbb{R}\times\mathbb{T}^{N-1})$ and standing shocks $U^\varepsilon_\pm$ connecting $v(\cdot,p_-)$ to $v(\cdot,p_+)$ such that

$$\|u - u^\varepsilon\|_{L^1(\mathbb{R}\times\mathbb{T}^{N-1})} \le \varepsilon, \quad U^\varepsilon_+ \le u^\varepsilon \le U^\varepsilon_-.$$

The case of $N = 1$ above (i.e., $\mathbb{R}$ instead of $\mathbb{R}\times\mathbb{T}^{N-1}$) has been shown in [5, Lemma 3.6], whose proof can be directly extended to the above lemma, because other variables x' are in $\mathbb{T}^{N-1}$. The idea is to take $u^\varepsilon = \bar{U}$ outside of a compact set $[-A_\varepsilon, A_\varepsilon]\times\mathbb{T}^{N-1}$ and then to perturb slightly u on the compact set $[-A_\varepsilon, A_\varepsilon]\times\mathbb{T}^{N-1}$ in order to be strictly between the two end states. We leave the details of the proof to the reader since they are identical to [5, Lemma 3.6].

Now, thanks to Lemma 4.1 together with the L^1-contraction principle, it is enough to prove Theorem 1.2 for the class of initial data $u_0 \in \bar{U} + L^1_0(\mathbb{R}\times\mathbb{T}^{N-1})$ such that

$$U_+ \le u_0 \le U_-, \quad \text{for some standing shocks } U_\pm. \tag{4.38}$$

Indeed, assume that $\lim_{t\to\infty}\|S_t v_0 - \bar{U}\|_{L^1(\mathbb{R}\times\mathbb{T}^{N-1})} = 0$ for any $v_0 \in \bar{U} + L^1_0(\mathbb{R}\times\mathbb{T}^{N-1})$ satisfying (4.38). By Lemma 4.1, for any $u_0 \in \bar{U} + L^1_0(\mathbb{R}\times\mathbb{T}^{N-1})$ satisfying (1.8), and $\varepsilon > 0$, there exists a function $u^\varepsilon_0 \in \bar{U} + L^1_0(\mathbb{R}\times\mathbb{T}^{N-1})$ such that $\|u_0 - u^\varepsilon_0\|_{L^1(\mathbb{R}\times\mathbb{T}^{N-1})} \le \varepsilon$ and (4.38). Then the L^1-contraction principle yields that for all $t \ge 0$,

$$\|S_t u_0 - \bar{U}\|_{L^1(\Omega)} \le \|S_t u_0 - S_t u^\varepsilon_0\|_{L^1(\Omega)} + \|S_t u^\varepsilon_0 - \bar{U}\|_{L^1(\Omega)} \le \varepsilon + \|S_t u^\varepsilon_0 - \bar{U}\|_{L^1(\Omega)}.$$

Since $t \mapsto \|S_t u_0 - \bar{U}\|_{L^1}$ is non-increasing, and thus has a finite limit as $t \to \infty$,

$$\lim_{t\to\infty}\|S_t u_0 - \bar{U}\|_{L^1(\mathbb{R}\times\mathbb{T}^{N-1})} \le \varepsilon,$$

which implies that $\lim_{t\to\infty}\|S_t u_0 - \bar{U}\|_{L^1(\mathbb{R}\times\mathbb{T}^{N-1})} = 0$.

Therefore, there remains to prove Theorem 1.2 for the initial data $u_0 \in \bar{U} + L^1_0(\mathbb{R}\times\mathbb{T}^{N-1})$ satisfying (4.38). We follows the same arguments as [5], which is based on the dynamical system theory due to Osher and Ralston [21]. The strategy is to prove that the ω-limit set of the trajectory $S_t u_0$ is reduced to $\{\bar{U}\}$ using the L^1-contraction principle. Thus, we need to first show that the ω-limit set is non-empty.

▷ *First step : Structure of the ω-limit set.*

We begin by noticing that the comparison principle together with (4.38) imply that for all $t \ge 0$,

$$U_+ \le S_t u_0 \le U_-,$$

and thus, setting $w(t) = S_t u_0 - \bar{U}$,

$$U_+ - \bar{U} \le w(t) \le U_- - \bar{U}.$$

Since $U_+ - \bar{U}$ and $U_- - \bar{U}$ are in $L^1 \cap L^\infty(\mathbb{R}\times\mathbb{T}^{N-1})$ by Lemma 3.2, the family $(w(t))_{t\ge 0}$ is equi-integrable in $L^1(\mathbb{R}\times\mathbb{T}^{N-1})$ and uniformly bounded in $L^\infty([0,\infty)\times\Omega)$. Moreover, since w solves a linear parabolic equation of the type

$$\partial_t w + \mathrm{div}_x\Big(a(t,x)w\Big) - \Delta w = 0,$$

where $a := \int_0^1 \partial_v A(x, \tau S_t u_0 + (1-\tau)\bar{U})d\tau \in L^\infty([0,\infty)\times\Omega)$, it follows from [14, Theorem 10.1] that there exists $\alpha > 0$ such that for all $t_0 \ge 1$ and $R > 1$,

$$\|w\|_{H^{\alpha/2,\alpha}((t_0,t_0+1)\times(-R,R)\times\mathbb{T}^{N-1})} < \infty.$$

Thus, $(w(t))_{t\geq 0}$ is also equi-continuous in L^1.
Therefore, it follows from the Riesz-Fréchet-Kolmogorov theorem that $(w(t))_{t\geq 0}$ is relatively compact in L^1. Thus the ω-limit set

$$\mathcal{B} := \Big\{ W \in \bar{U} + L^1_0(\Omega) \mid \exists (t_n)_{n\in\mathbb{N}},\ t_n \to \infty,\ S_{t_n} u_0 \to W \text{ in } L^1(\Omega) \Big\},$$

is non-empty. Notice that $\mathcal{B} \subset \bar{U} + L^1_0(\Omega)$ because of $u_0 \in \bar{U} + L^1_0(\Omega)$ and the conservation of mass.
By the definition of ω-limit set, $\mathcal{B}$ is forward and backward invariant by the semi-group S_t, i.e., $S_t\mathcal{B} = \mathcal{B}$ for all t. Moreover, thanks to parabolic regularity, all functions in $\mathcal{B}$ are smooth, for example $\mathcal{B} \subset H^1_{loc}(\Omega)$. As a consequence, for any $W \in \mathcal{B}$, it follows from [14, Theorem 6.1] that $S_t W \in L^2(0,T; H^2_{loc}(\Omega)) \cap H^1(0,T; L^2_{loc}(\Omega))$.

We take advantage of LaSalle invariance principle [15] with a suitable Lyapunov functional $F[u] := \|u - \bar{U}\|_{L^1(\Omega)}$. Since $t \mapsto F[S_t W]$ is non-increasing by the L^1-contraction principle, F takes a constant value on $\mathcal{B}$, which we denote by C_0.

$\triangleright$ *Second step :* $\mathcal{B} = \{\bar{U}\}$. We now prove $\mathcal{B} = \{\bar{U}\}$. For any $W_0 \in \mathcal{B}$, we set $W(t) = S_t W_0$. Notice that $W(t) \in \Omega$ for all $t \geq 0$. Since $W(t) - \bar{U}$ satisfies

$$\partial_t(W - \bar{U}) + \text{div}_x\Big(A(x, W) - A(x, \bar{U})\Big) - \Delta(W - \bar{U}) = 0,$$

we have

$$\partial_t|W - \bar{U}| + \text{div}_x\Big(b(t,x)|W - \bar{U}|\Big) - \text{sgn}(W - \bar{U})\Delta(W - \bar{U}) = 0, \tag{4.39}$$

where $b(t,x) = \int_0^1 \partial_v A(x, \tau W + (1-\tau)\bar{U})d\tau$.
In order to show that $\text{sgn}(W - \bar{U})\Delta(W - \bar{U}) = \Delta|W - \bar{U}|$, we use the following lemma.

Lemma 4.2. *Let $f \in L^1 \cap L^\infty(\mathbb{R} \times \mathbb{T}^{N-1})$ such that $\nabla f \in L^2(\mathbb{R} \times \mathbb{T}^{N-1})$ and $\Delta f \in L^1_{loc}(\mathbb{R} \times \mathbb{T}^{N-1})$. Assume that f satisfies*

$$\lim_{R\to\infty} \int_\Omega \text{sgn}(f)\Delta f\, \theta\Big(\frac{x_1}{R}\Big) dx = 0, \tag{4.40}$$

for all $\theta \in \mathcal{C}^\infty_0(\mathbb{R})$ such that $\theta \equiv 1$ in a neighborhood of the origin. Then

$$\lim_{\delta\to 0} \frac{1}{\delta} |\nabla f|^2 \mathbf{1}_{|f|<\delta} = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

therefore,

$$\text{sgn}(f)\Delta f = \Delta|f| \quad \text{in } \mathcal{D}'(\Omega).$$

The case of $N = 1$ above has been shown in [5, Lemma B.1], whose proof can be directly extended to the above lemma. Now, in order to show that the condition (4.40) is satisfied in our case, we recall from the previous step that $F[W(t)] = \|W(t) - \bar{U}\|_{L^1(\Omega)} = C_0$ for all $t \geq 0$. For any $t' > t \geq 0$, since

$$\begin{aligned} \int_t^{t'} \int_\Omega \partial_t |W - \bar{U}| \theta(\frac{x_1}{R}) dx ds &\leq \int_\Omega |W(t') - \bar{U}| dx - \int_{|x_1|\leq CR} |W(t) - \bar{U}| dx \\ &= \int_{|x_1|\geq CR} |W(t) - \bar{U}| dx \to 0 \quad \text{as } R \to \infty, \end{aligned}$$

and

$$\int_t^{t'}\int_\Omega \operatorname{div}_x\Big(b(t,x)|W-\bar U|\Big)\theta(\frac{x_1}{R})dxds \;\le\; \|A\|_{W^{1,\infty}}\|\theta'\|_\infty \frac{1}{R}\int_t^{t'}\int_\Omega |W(t)-\bar U|dxds$$
$$= \frac{C(t'-t)}{R}\to 0 \quad \text{as } R\to\infty,$$

we have

$$\int_t^{t'}\int_\Omega \operatorname{sgn}(W-\bar U)\Delta(W-\bar U)\theta(\frac{x_1}{R})dxds \to 0 \quad \text{as } R\to\infty.$$

Thus, a slightly modified version of Lemma 4.2 implies that

$$\operatorname{sgn}(W-\bar U)\Delta(W-\bar U)=\Delta|W-\bar U|.$$

Therefore, $|W-\bar U|$ is a non-negative solution of a parabolic equation of the type

$$\partial_t|W-\bar U| + \operatorname{div}_x\Big(b(t,x)|W-\bar U|\Big) - \Delta|W-\bar U| = 0,$$

where $b\in L^\infty([0,\infty)\times\Omega)$. Thanks to the Harnack inequality for the parabolic equations, for any compact set K in Ω, there exists C_K such that

$$\sup_{x\in K}|(W_0-\bar U)(x)| \le C_K \inf_{x\in K}|(W(1)-\bar U)(x)|. \tag{4.41}$$

Moreover, using the fact that $W(1)-\bar U\in L^1_0\cap H^1_{loc}(\Omega)$, there exists $x_1\in\Omega$ such that

$$(W(1)-\bar U)(x_1)=0,$$

which implies together with (4.41) that $W_0\equiv V$. Hence we have $\mathcal{B}=\{\bar U\}$, and thus complete the proof of Theorem 1.2.

Appendix A. Use of the Krein-Rutman theorem to prove the positivity of solutions of some elliptic equations

In this Appendix, we prove the following result, which has been used in several instances in the paper:

Lemma A.1. *Let $R>0$ be arbitrary, and let $b\in L^\infty(\Omega_R)$. Consider the equation*

$$\begin{aligned} -\Delta w + \operatorname{div}(bw) = 0 \quad &\text{in } \Omega_R,\\ -\partial_1 w + b_1 w = 0 \ &\text{for } x_1=\pm R. \end{aligned} \tag{A.1}$$

Then the vector space of solutions of equation (A.1) is $\mathbb{R}w_1$, where $w_1\in H^1(\Omega_R)\cap\mathcal{C}(\bar\Omega_R)$ is a strictly positive solution of (A.1) such that $\int_{\Omega_R} w_1 = 1$.

Proof. The dual of problem (A.1) is

$$\begin{aligned} -\Delta q - b\cdot\nabla q = 0 \quad &\text{in } (-R,R)\times\mathbb{T}^{N-1},\\ \partial_1 q = 0 \ &\text{at } x_1=\pm R, \end{aligned}$$

of which the constant function equal to one is a strictly positive solution.

Let us introduce the operator $F: u\in L^2(\Omega_R)\mapsto v\in L^2(\Omega_R)$ where $v=F(u)$ is the unique solution of the equation

$$-\Delta v - b\cdot\nabla v + \alpha v = \alpha u \text{ in } \Omega_R, \quad \partial_1 v = 0 \text{ at } x_1=\pm R,$$

and α is a positive constant chosen so that the bilinear form associated to F is coercive (e.g. $\alpha=\frac{||b||^2_\infty}{2}+\frac12$). With that choice of α, F is a strictly positive operator.

Next, using regularity results for linear elliptic equations, we show that F maps $L^q(\Omega_R)$ into $W^{2,q}(\Omega_R)$ for all $q \geq 2$. Hence, the restriction of F to $\mathcal{C}(\bar{\Omega}_R)$, still denoted by F, is a compact operator from $\mathcal{C}(\bar{\Omega}_R)$ into itself. The last step consists in using the strong form of the maximum principle together with Hopf's Lemma: if $u \in \mathcal{C}(\bar{\Omega}_R)$, $u \geq 0$, $u \neq 0$ and $v = F(u)$, then $v(y) > 0$ for all $y \in \bar{\Omega}_R$.

Hence, $F : \mathcal{C}(\bar{\Omega}_R) \to \mathcal{C}(\bar{\Omega}_R)$ is a strongly positive operator. We conclude by using the strong form of the Krein-Rutman theorem (see [7, 13]): since $F(\bar{1}) = \bar{1}$, the spectral radius of F is equal to 1 and 1 is a simple eigenvalue of F^*, the adjoint of F, with a positive eigenvector. We infer that (A.1) has a unique non-negative solution w_1 normalized in L^1.

□

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