

**Mathematical study
of degenerate boundary layers:
A large scale ocean circulation problem**

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ABSTRACT. This paper is concerned with a complete asymptotic analysis as $E \rightarrow 0$ of the Munk equation $\partial_x \psi - E \Delta^2 \psi = \tau$ in a domain $\Omega \subset \mathbf{R}^2$, supplemented with boundary conditions for ψ and $\partial_n \psi$. This equation is a simple model for the circulation of currents in closed basins, the variables x and y being respectively the longitude and the latitude. A crude analysis shows that as $E \rightarrow 0$, the weak limit of ψ satisfies the so-called Sverdrup transport equation inside the domain, namely $\partial_x \psi^0 = \tau$, while boundary layers appear in the vicinity of the boundary.

These boundary layers, which are the main center of interest of the present paper, exhibit several types of peculiar behaviour. First, the size of the boundary layer on the western and eastern boundary, which had already been computed by several authors, becomes formally very large as one approaches northern and southern portions of the boundary, i.e. pieces of the boundary on which the normal is vertical. This phenomenon is known as geostrophic degeneracy. In order to avoid such singular behaviour, previous studies imposed restrictive assumptions on the domain Ω and on the forcing term τ . Here, we prove that a superposition of two boundary layers occurs in the vicinity of such points: the classical western or eastern boundary layers, and some northern or southern boundary layers, whose mathematical derivation is completely new. The size of northern/southern boundary layers is much larger than the one of western boundary layers ($E^{1/4}$ vs. $E^{1/3}$). We explain in detail how the superposition takes place, depending on the geometry of the boundary.

Moreover, when the domain Ω is not connex in the x direction, ψ^0 is not continuous in Ω , and singular layers appear in order to correct its discontinuities. These singular layer are concentrated in the vicinity of horizontal lines, and therefore penetrate the interior of the domain Ω . Hence we exhibit some kind of boundary layer separation. However, we emphasize that we remain able to prove a convergence theorem, so that the boundary layers somehow remain stable, in spite of the separation.

Eventually, the effect of boundary layers is non-local in several aspects. On the first hand, for algebraic reasons, the boundary layer equation is radically different on the west and east parts of the boundary. As a consequence, the Sverdrup equation is endowed with a Dirichlet condition on the East boundary, and no condition on the West boundary. Therefore western and eastern boundary layers have in fact an influence on the whole domain Ω , and not only near the boundary. On the second hand, the northern and southern boundary layer profiles obey a propagation equation, where the space variable x plays the role of time, and are therefore not local.

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CHAPTER 1

Introduction

The present paper is mainly concerned with mathematical methods investigating singular behaviours on the boundary of a bounded domain Ω , when the size of the boundary layer depends strongly on its localization, and more precisely when it becomes degenerate on some part of the boundary.

Such a situation has been depicted in many former works, but never really dealt with insofar as additional assumptions were often made to guarantee that boundary terms vanish in the vicinity of the singularity. This is the case for instance of our study [5] of the β -plane model for rotating fluids in a thin layer where we suppose that the wind forcing vanishes at the equator. The same type of assumption was used in the paper [19] of F. Rousset which investigates the behaviour of Ekman-Hartmann boundary layers on the sphere. This holds also true for the work of Desjardins and Grenier [6] on Munk and Stommel layers where it is assumed that the Ekman pumping (which is directly related to the wind forcing) is zero in the vicinity of the Northern and Southern coasts. The difficulty was pointed out by D. Gérard-Varet and T. Paul in [9].

We intend here to get rid of this non physical assumption on the forcing, and to obtain a mathematical description of the singular boundary layers. More generally, we would like to understand how to capture the effects of the geometry in such problems of singular perturbations on domains with possibly characteristic boundaries.

1.1. Munk boundary layers

The equation we will consider, the so-called Munk equation, can be written as follows when the domain Ω is simply connected:

$$(1.1) \quad \begin{aligned} \partial_x \psi - E \Delta^2 \psi &= \tau \text{ in } \Omega, \\ \psi|_{\partial\Omega} &= 0, \quad (n \cdot \nabla \psi)|_{\partial\Omega} = 0. \end{aligned}$$

This model comes from large-scale oceanography and is expected to provide a good approximation of the stream function of oceanic currents, assuming that

- the motion of the fluid is purely two-dimensional $\Omega \subset \mathbf{R}^2$ (shallow-water approximation),
- the wind forcing is integrated as a source term $\tau \in W^{4,\infty}(\bar{\Omega})$ (Ekman pumping),
- the Coriolis parameter depends linearly on latitude (β -plane approximation),
- nonlinear effects are negligible,
- the Ekman pumping at the bottom is negligible.

We refer to Appendix D for a derivation of the equation together with a discussion of the physical approximations involved. When Ω is not simply connected, the boundary conditions are slightly different. We refer to paragraph 1.2.3 for more details.

1.1.1. State of the art. Desjardins and Grenier studied in [6] a time-dependent and nonlinear version of (1.1). Their result implies in particular the following statement for the linear Munk equation:

THEOREM 1. *Let Ω be a smooth domain, defined by*

$$(1.2) \quad \Omega := \{(x, y) \in \mathbf{R}^2, x_W(y) < x < x_E(y), y_{min} < y < y_{max}\},$$

where $x_E, x_W \in \mathcal{C}^2(y_{min}, y_{max})$.

Assume that the wind forcing $\tau = \nabla^\perp \cdot \mathcal{T} \in H^s(\Omega)$ vanishes identically in the vicinity of the North and of the South

$$(A0) \quad \exists \lambda > 0, \quad \tau(x, y) \equiv 0 \text{ if } y \leq y_{min} + \lambda \text{ or } y \geq y_{max} - \lambda.$$

Denote by $u_E = \nabla^\perp \psi_E$ any solution to the vorticity formulation of the 2D Stokes-Coriolis system

$$(1.3) \quad \begin{aligned} E \partial_t \Delta \psi + \partial_x \psi - E \Delta^2 \psi &= \tau, \\ \psi|_{\partial\Omega} &= 0, \quad (n \cdot \nabla \psi)|_{\partial\Omega} = 0. \end{aligned}$$

Then, for all $N \in \mathbb{N}$, if s is sufficiently large, there exists

$$u_{app}^N = \sum_{i=0}^N E^{i/3} (u_i^{int} + u_i^{BL})$$

such that

$$\|u_E - u_{app}^N\|_{L^\infty((0,T), L^2(\Omega))} \leq C E^{(N-1)/6}.$$

The main order term $u_0^{int} = \nabla^\perp \psi^0$ is the weak limit of u_E in L^2 as $E \rightarrow 0$ and satisfies the Sverdrup relation

$$(1.4) \quad \partial_x \psi^0 = \tau, \quad \psi^0|_{\Gamma_E} = 0,$$

where Γ_E is the East boundary

$$\Gamma_E = \{(x, y) \in \partial\Omega, x = x_E(y)\}.$$

The boundary layers u_i^{BL} are located in a band of width $E^{1/3}$ in the vicinity of the East and West components of $\partial\Omega$.

The approximate solution u_{app} is computed starting from asymptotic expansions in terms of the parameter E .

- At main order in the interior of the domain, we get the Sverdrup relation

$$\partial_x \psi^0 = \tau.$$

We can then prescribe only one boundary condition, either on the Eastern coast or on the Western coast.

- Since u_0^{int} does not vanish on the boundary, we then introduce boundary layer corrections, which are given by the balance between $\partial_x \psi^{BL}$ and $E \Delta^2 \psi^{BL}$. Because the space of admissible (localized) boundary corrections is of dimension 2 on the Western boundary, and of dimension 1 on the Eastern boundary, we can recover all the boundary conditions except

$\psi|_{\Gamma_E}^0 = 0$, which is chosen as the boundary condition for the Sverdrup equation.

The proof relies then on a standard energy method. The assumption on τ guarantees that ψ^0 is smooth, and that the coefficients arising in the definition of the boundary layer terms are uniformly bounded, and actually small enough to be absorbed in the viscosity term. Desjardins and Grenier actually study a nonlinear version of (1.3), for which a smallness condition on τ is required to deal with the nonlinear convection term.

1.1.2. Boundary layer degeneracies.

Our main goal in this paper is to get rid of the assumption (A0) on the forcing τ , and to treat more general domains Ω (e.g. domains with islands, or which are not convex in the x direction). This problem related to singular boundary layers has already been studied (at the formal level) by De Ruijter [20] for some simplified geometries involving typically rectangles.

The main difficulty to get rid of assumption (A0) is that the incompatibility between the formal limit equation and the boundary conditions

$$\psi|_{\partial\Omega} = 0, \quad (n \cdot \nabla\psi)|_{\partial\Omega} = 0$$

is not of the same nature depending on the part of the boundary to be considered: at the North and at the South, the transport term is tangential to the boundary and therefore is not expected to be singular even in boundary layers.

Introducing boundary layer terms to restore the boundary conditions, we will find typically

$$\partial_x \psi_{E,W} - E \partial_x^4 \psi_{E,W} = 0,$$

on the lateral boundaries, and

$$\partial_x \psi_{N,S} - E \partial_y^4 \psi_{N,S} = 0,$$

on the horizontal boundaries. This implies in particular that western and eastern boundary layers should be of size $E^{1/3}$ while northern and southern boundary layers should be much larger, of size $E^{1/4}$. Of course there are superposition zones, which have to be described rather precisely if we want to get an accurate approximation, and prove a rigorous convergence result.

This is actually the first step towards the understanding of more complex geometries. Additional difficulties will be discussed in the next section.

1.1.3. Stability of the stationary Munk equation. Another important difference with [6] comes from the fact that (1.1) is a stationary equation, so that classical energy methods ([12, 22]) are irrelevant.

Denote by ψ_{app} the solution to the following approximate equation

$$(1.5) \quad \begin{aligned} \partial_x \psi_{app} - E \Delta^2 \psi_{app} &= \tau + \delta\tau \text{ on } \Omega, \\ \psi_{app}|_{\partial\Omega} &= 0, \quad (n \cdot \nabla \psi_{app})|_{\partial\Omega} = 0. \end{aligned}$$

In particular, denoting $\delta\psi = \psi - \psi_{app}$, we have

$$(1.6) \quad \begin{aligned} \partial_x \delta\psi - E \Delta^2 \delta\psi &= \delta\tau \text{ on } \Omega, \\ \delta\psi|_{\partial\Omega} &= 0, \quad (n \cdot \nabla \delta\psi)|_{\partial\Omega} = 0. \end{aligned}$$

The stability estimates used in this paper rely on the use of **weighted spaces** as suggested by Bresch and Colin [1]. From the identities

$$-\int_{\Omega} (\partial_x \delta\psi) e^x \delta\psi = \frac{1}{2} \int_{\Omega} (\delta\psi)^2 e^x,$$

$$\int_{\Omega} (\Delta^2 \delta\psi) e^x \delta\psi = \int_{\Omega} (\Delta \delta\psi)^2 e^x + 2 \int_{\Omega} e^x \Delta \delta\psi \partial_x \delta\psi + \int_{\Omega} e^x \Delta \delta\psi \delta\psi,$$

together with Cauchy-Schwarz inequality, we indeed deduce that

$$\frac{1-E}{2} \int_{\Omega} \delta\psi^2 e^x + \frac{E}{2} \int_{\Omega} (\Delta \delta\psi)^2 e^x \leq - \int_{\Omega} \delta\tau \delta\psi e^x - 2E \int_{\Omega} \Delta \delta\psi \partial_x \delta\psi e^x.$$

Assume that $\delta\tau$ can be decomposed into

$$\delta\tau = \delta\tau_1 + \delta\tau_2,$$

with $\delta\tau_1 \in L^2(\Omega)$, $\delta\tau_2 \in H^{-2}(\Omega)$. Then, using the Poincaré inequality,

$$\left| \int_{\Omega} \delta\tau \delta\psi e^x \right| \leq C(\|\delta\tau_1\|_{L^2} \|\delta\psi\|_{L^2} + \|\delta\tau_2\|_{H^{-2}} \|\Delta \delta\psi\|_{L^2}),$$

while

$$\left| \int_{\Omega} e^x \Delta \delta\psi \partial_x \delta\psi \right| \leq C \|\Delta \delta\psi\|_{L^2} \|\partial_x \delta\psi\|_{L^2} \leq C \|\Delta \delta\psi\|_{L^2}^{3/2} \|\delta\psi\|_{L^2}^{1/2}.$$

It comes finally, assuming that $E \ll 1$,

$$(1.7) \quad \|\delta\psi\|_{L^2}^2 + E \|\Delta \delta\psi\|_{L^2}^2 \leq C \left(\|\delta\tau_1\|_{L^2}^2 + \frac{\|\delta\tau_2\|_{H^{-2}}^2}{E} \right).$$

We then define some relevant notion of approximate solution.

DEFINITION 1.1.1. *A function $\psi_{app} \in H^2(\Omega)$ is an approximate solution to (1.1) if it satisfies the approximate equation (1.5) for some $\delta\tau \in H^{-2}(\Omega)$ such that $\delta\tau = \delta\tau_1 + \delta\tau_2$ with*

$$(1.8) \quad \lim_{E \rightarrow 0} \frac{\|\delta\tau_1\|_{L^2}^2 + E^{-1} \|\delta\tau_2\|_{H^{-2}}^2}{\|\psi_{app}\|_{L^2}^2 + E \|\Delta \psi_{app}\|_{L^2}^2} = 0.$$

Plugging (1.8) in (1.7) we obtain

$$\|\psi - \psi_{app}\|_{L^2}^2 + E \|\Delta(\psi - \psi_{app})\|_{L^2}^2 = o(\|\psi_{app}\|_{L^2}^2 + E \|\Delta \psi_{app}\|_{L^2}^2)$$

meaning that $\psi_{app} \sim \psi$.

By such a method, we exhibit the *dominating phenomena in terms of their contribution to the energy balance*. In particular, for the oceanic motion, we expect to justify the crucial role of boundary currents as they account for a macroscopic part of the energy.

In view of the energy estimate (1.7) and of the expected sizes of boundary layers (see (1.21) and (1.22) below), the idea is to prove that ψ_{app} satisfies equation (1.5) with an error term $\delta\tau = \delta\tau_1 + \delta\tau_2$ such that

$$(1.9) \quad \|\delta\tau_1\|_{L^2(\Omega)} = o(E^{1/8}), \quad \|\delta\tau_2\|_{H^{-2}(\Omega)} = o(E^{5/8}).$$

Notice that the energy estimate (1.7) allows us to capture both types of boundary layers and the interior term, as we will explain in Theorem 2.

We therefore use the following notion:

DEFINITION 1.1.2. *In the rest of the paper, we say that $\delta\tau = \delta\tau_1 + \delta\tau_2$ is an admissible error term if it satisfies (1.9).*

Our main result, Theorem 2, is the construction of an explicit approximate solution with an admissible error term in the sense of Definition 1.1.2.

1.2. Geometrical preliminaries

The goal of this section is to state the precise assumptions regarding the domain, under which we are able to derive a nice approximation for the Munk problem.

As we will see in the course of the proof, error estimates depend strongly on the geometry of the domain, and especially on the flatness of the boundary near horizontal parts.

We will therefore consider

- the generic case, when the slope of the tangent vector vanishes polynomially;
- a special case of flat boundary, for which the exponential decay is prescribed;
- the case of a corner between the East/West boundary and the horizontal part (as considered by De Ruijter in [20]).

The specific assumptions in the case of corners will be considered separately in subsection 1.2.4, in an attempt to keep the presentation as simple as possible. Therefore the regularity and geometry assumptions detailed in subsections 1.2.1, 1.2.2, and 1.2.3 apply to the first two cases above.

1.2.1. Regularity and flatness assumptions. First of all we assume, without loss of generality, that Ω is bounded and connected¹. We further require that (H1)

the boundary is a compact C^4 manifold, described by a finite number of charts.

In particular, the perimeter L is finite, and the boundary of Ω can be parametrized by the arc-length s . If $\partial\Omega$ is connected, we will often write $\partial\Omega = \{(x(s), y(s)) / 0 \leq s \leq L\}$; if $\partial\Omega$ has several connected components, such a representation holds on every connected component of $\partial\Omega$. We further choose the orientation of the arc-length in such a way that the local frame (t, n) with n the exterior normal is direct.

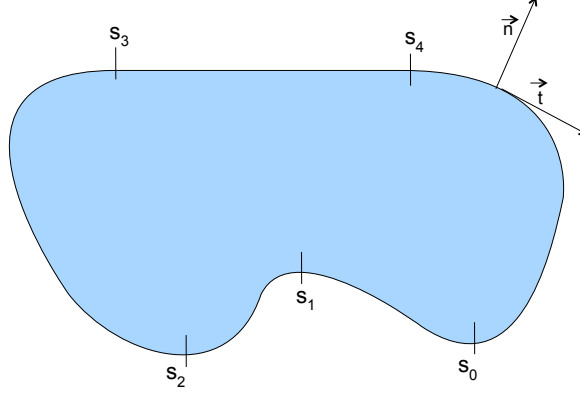
Furthermore, by the local inversion theorem, there exists $\delta > 0$ such that any point $(x, y) \in \Omega$ at a distance z smaller than δ from $\partial\Omega$ has a unique projection on the boundary. In other words, on such a tube, the arc-length s and the distance to the boundary z form a nice system of coordinates.

We thus introduce some truncation function $\chi_0 \in C_c^\infty([0, +\infty), [0, 1])$ such that

$$(1.10) \quad \text{Supp}\chi_0 \subset [0, \delta) \text{ and } \chi_0 \equiv 1 \text{ on } [0, \delta/2].$$

As mentioned in the previous heuristic study, we expect the boundary layers to be singular on the horizontal parts of the boundary. In order to understand the connection between both types of boundary layers and to get a nice approximation of the solution to the Munk problem, we therefore need precise information on the profile of the boundary near horizontal parts.

¹If Ω is not connected, since (1.1) is a local equation, we can perform our study on every connected component, and we therefore obtain a result on the whole domain Ω .

FIGURE 1.1. Positions of the abscissa s_i

We assume that the horizontal part $\Gamma_N \cup \Gamma_S$ of the boundary consists in a finite number of intervals (possibly reduced to points where the tangent to the boundary is horizontal). Denote by $\theta(s)$ the oriented angle between the horizontal vector e_x and the exterior normal n . By definition, we set

$$\begin{aligned}\Gamma_N &:= \{s \in \partial\Omega, \cos \theta(s) = 0 \text{ and } \sin \theta(s) = 1\}, \\ \Gamma_S &:= \{s \in \partial\Omega, \cos \theta(s) = 0 \text{ and } \sin \theta(s) = -1\}.\end{aligned}$$

We also introduce the following notation (see figure 1.1): let $s_1 < s_2 < \dots < s_k$ such that

$$\cos(\theta(s_i)) = 0 \quad 1 \leq i \leq k$$

and for all $i \in \{1, \dots, k\}$, either $\cos(\theta(s))$ is identically zero or $\cos(\theta(s))$ does not vanish between s_i and s_{i+1} . Throughout the article, we use the conventions $s_{k+1} = s_1$, $s_0 = s_k$. We further define some partition of unity $(\rho_i)_{1 \leq i \leq k}$

$$(1.11) \quad \begin{aligned}\rho_i &\in C_c^\infty(\partial\Omega, [0, 1]), \quad \sum_{i=1}^k \rho_i(s) = 1, \\ \text{Supp } \rho_i &\subset]s_{i-1}, s_{i+1}[.\end{aligned}$$

We denote by Γ_E, Γ_W the East and West boundaries of the domain:

$$\begin{aligned}\Gamma_E &:= \{s \in \partial\Omega, \cos \theta(s) > 0\}, \\ \Gamma_W &:= \{s \in \partial\Omega, \cos \theta(s) < 0\}.\end{aligned}$$

Eventually, let

$$(1.12) \quad I_+ := \{i \in \{1, \dots, k\}, s_i \in \partial\Gamma_E\}.$$

The profile assumption states then as follows: for any s_i , for $\sigma = \pm$ such that $\cos \theta(s) \neq 0$ on $[s_i, s_{i\sigma 1}]$,

- (i) either there exists $n \geq 1$ and $C \neq 0$ (depending on i and σ) such that as $s \rightarrow s_i$, $s \in (s_i, s_{i\sigma 1})$,

$$(H2(i)) \quad \begin{aligned} \cos \theta(s) &\sim \frac{C}{n!} (s - s_i)^n, \\ \text{and } \theta^{(l)}(s) &\sim \frac{C}{(n-l)!} (s - s_i)^{n-l} \text{ for } 1 \leq l \leq \inf(3, n); \end{aligned}$$

- (ii) or there exists $C \neq 0$ and $\alpha > 0$ (also depending on i and σ) such that as $s \rightarrow s_i$, $s \in (s_i, s_{i\sigma 1})$,

$$(H2(ii)) \quad \begin{aligned} \cos \theta(s) &\sim C e^{-\alpha/|s-s_i|}, \\ \text{and } \theta'(s) &\sim \frac{C\alpha}{(s-s_i)^2} e^{-\alpha/|s-s_i|}. \end{aligned}$$

The first situation corresponds to the generic case when the cancellation is of finite order. The second one is an example of infinite order cancellation: in that case, which is important since it is the archetype of C^∞ boundary with flat parts, we prescribe the exponential decay because there is no general formula for error estimates. Notice that we do not require the behaviour of θ to be the same on both sides of s_i , provided the function θ belongs to $\mathcal{C}^3(\partial\Omega)$, so that $(x(s), y(s)) \in \mathcal{C}^4(\partial\Omega)$.

This profile assumption will essentially guarantee that, up to a small truncation, we will be able to lift boundary conditions either by East/West boundary layers, or by North/South boundary layers at any point of the boundary. This is therefore the main point to get rid of assumption (A0).

1.2.2. Singularity lines. In the case when the domain Ω is not convex in

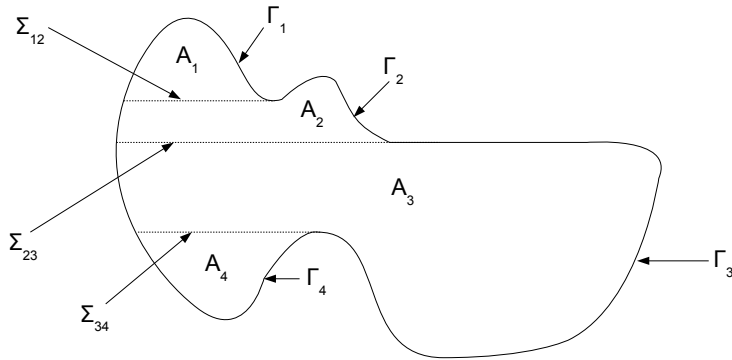


FIGURE 1.2. Discontinuity lines of a general domain Ω

the x direction, we will see that the asymptotic picture is much more complex, especially because the solution ψ^0 to the Sverdrup equation (1.4) is discontinuous as soon as $\bar{\Gamma}_E$ has more than two connected components (notice that if $\cos \theta$ has an isolated point of cancellation in the interior of $\bar{\Gamma}_E$, so that Γ_E has two connected components but $\bar{\Gamma}_E$ is connected, then there is no discontinuity in ψ^0 .)

We therefore introduce the lines Σ_{ij} , across which the main order term will be discontinuous, which will give rise to boundary layer singularities: we set

$$\bar{\Gamma}_E := \bigcup_{j=1}^M \Gamma_j,$$

where $\Gamma_1, \dots, \Gamma_M$ are the closed connected components of $\bar{\Gamma}_E$. For $j \in \{1, \dots, M\}$, we set

$$A_j := \{(x, y) \in \Omega, \exists x' \in \mathbf{R}, (x', y) \in \Gamma_j \text{ and } (tx + (1-t)x', y) \in \Omega \forall t \in]0, 1[\}.$$

We have clearly

$$\Omega = \bigcup_{j=1}^M A_j.$$

We also define (see Figure 1.2, and also Figures 3.2 and 3.3 in [20])

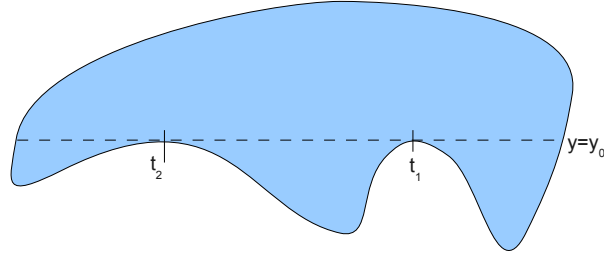
$$(1.13) \quad \Sigma_{ij} := \bar{A}_i \cap \bar{A}_j \quad \text{for } i \neq j, \quad \Sigma := \bigcup_{i,j} \Sigma_{ij}.$$

It can be easily checked that every set Σ_{ij} is either empty or a horizontal line with ordinate y_{ij} such that there exist $x_i, x_j \in \mathbf{R}$ with $(x_i, y_{ij}) \in \Gamma_i$, $(x_j, y_{ij}) \in \Gamma_j$, and either $(x_i, y_{ij}) \in \partial\Gamma_i$ or $(x_j, y_{ij}) \in \partial\Gamma_j$.

Eventually, we parametrize every set Γ_j by a graph x_E^j , namely

$$\Gamma_j = \{(x_E^j(y), y), y_{min}^j \leq y \leq y_{max}^j\}.$$

We will therefore need to build singular correctors which are not localized in the vicinity of the boundary. This construction is rather technical, and for the sake of simplicity, we will use two additional assumptions, which are quite general:



The set $\{s, y(s)=y_0 \text{ and } \cos \theta(s)=0\} = \{t_1\} \cup \{t_2\}$
is not connected.

FIGURE 1.3. Example of a domain when assumption (H3) is not satisfied

(H3) $\forall y_0 \in \mathbf{R}$, the set

$$\{s \in \partial\Omega, y(s) = y_0, \cos\theta(s) = 0\}$$

is a connected set (see figure 1.3).

(H4) Let $s_j \in \{s_1, \dots, s_k\}$ be a boundary point such that

- $s_j \in \partial\Gamma_E \cap \partial\Gamma_W$;
- and Ω is not convex in x in a neighbourhood of $(x(s_j), y(s_j))$.

Then $\cos\theta(s) = O(|s - s_j|^4)$ for $s \rightarrow s_j$.

Assumption (H4) will be discussed in Remark 3.4.7.

1.2.3. Domains with islands.

When the domain Ω is not simply connected, the boundary conditions on $\partial\Omega$ are slightly different. This case has been studied in particular in [2], where the authors investigate the weak limit of the Munk equation in a domain with islands. Let $\Omega_1, \dots, \Omega_K$ be \mathcal{C}^4 simply connected domains of \mathbf{R}^2 , such that

- $\Omega_i \Subset \Omega_1$ for $i \geq 2$;
- $\Omega_i \cap \Omega_j = \emptyset$ for $2 \leq i < j \leq K$;
- $\Omega := \Omega_1 \setminus \cup_{i \geq 2} \Omega_i$ satisfies (H1)-(H4).

Let $C_i = \partial\Omega_i$ for $i \geq 1$. Notice that of course, the presence of islands gives rise to discontinuity lines Σ as described in the preceding paragraph.

Then the Munk equation can be written as

$$(1.14) \quad \begin{aligned} \partial_x \psi - E\Delta^2 \psi &= \tau \text{ in } \Omega, \\ \partial_n \psi &= 0 \text{ on } \partial\Omega, \\ \psi|_{C_1} &= 0, \quad \psi|_{C_i} = c_i \text{ for } i \geq 2. \end{aligned}$$

The constants c_i are different from zero in general: indeed, the condition $u \cdot n|_{\partial\Omega} = 0$, where $u = \nabla^\perp \psi$ is the current velocity, becomes $\psi = \text{constant}$ on every connected component of $\partial\Omega$. However, the constants are not required to be all equal. In fact, the values of c_2, \dots, c_K are dictated by compatibility conditions, namely

$$(1.15) \quad \forall j \geq 2, \quad E \int_{C_j} \partial_n \Delta \psi - \int_{C_j} \mathcal{T}^\perp \cdot n = 0,$$

where $\tau = \text{curl } \mathcal{T}$.

We explain in Appendix D where condition (1.15) comes from. The constants c_2, \dots, c_K are then uniquely determined, as shows the following

LEMMA 1.2.1. • *Following [2], define ψ_1, \dots, ψ_K by*

$$(1.16) \quad \begin{aligned} \partial_x \psi_1 - E\Delta^2 \psi_1 &= \tau, \\ \psi_1|_{\partial\Omega} &= 0, \quad \partial_n \psi_1|_{\partial\Omega} = 0, \end{aligned}$$

and for $i \geq 2$,

$$(1.17) \quad \begin{aligned} \partial_x \psi_i - E\Delta^2 \psi_i &= 0, \\ \partial_n \psi_i|_{\partial\Omega} &= 0, \\ \psi_i|_{C_j} &= \delta_{ij} \quad \forall j \in \{1, \dots, K\}. \end{aligned}$$

Then ψ satisfies equation (1.14) for some constants c_1, \dots, c_K if and only if

$$(1.18) \quad \psi = \psi_1 + \sum_{i \geq 2} c_i \psi_i.$$

- Define the matrix M^E and the vector D^E by

$$M^E := \left(E \int_{C_i} \partial_n \Delta \psi_j \right)_{2 \leq i, j \leq K},$$

$$D^E := \left(-E \int_{C_j} \partial_n \Delta \psi_1 + \int_{C_j} \mathcal{T}^\perp \cdot n \right)_{2 \leq j \leq K}.$$

Then the following facts hold:

- M^E is invertible;
- ψ is a solution of (1.14) satisfying the compatibility condition (1.15) if and only if ψ is given by (1.18) with $M^E c = D^E$.

The proof of Lemma 1.2.1 is postponed to Appendix D.

- The formula (1.18) shows that it is sufficient to understand the asymptotic behaviour of the functions ψ_i . Indeed, the coefficients c_i (which depend on E) are obtained as the solutions of a linear system involving the functions ψ_i . It is proved in [2] that for any domain $V \subset \Omega$ such that $\bar{V} \cap \Sigma = \emptyset$, where Σ is defined by (1.13)

$$\begin{aligned} \psi_1 &\rightharpoonup \psi^0 && \text{in } L^2(\Omega), \\ \psi_1 &\rightarrow \psi^0 && \text{in } L^2(V), \end{aligned}$$

where ψ^0 is the solution of the Sverdrup equation (1.4), and

$$\begin{aligned} \psi_i &\rightharpoonup \mathbf{1}_{B_i} && \text{in } L^2(\Omega), \\ \psi_i &\rightarrow \mathbf{1}_{B_i} && \text{in } L^2(V), \end{aligned}$$

where B_i is defined by

$$B_i := \{(x, y) \in \Omega, \exists x' > x, (x', y) \in C_i \cap \bar{\Gamma}_E \text{ and } (tx + (1-t)x', y) \in \Omega \forall t \in (0, 1)\}.$$

This result is in fact sufficient to compute the asymptotic limit of the coefficients c_i , which converge towards some constants \bar{c}_i . We refer to section 2.5 for more details. We will go one step further in the present paper, since we are able to compute an asymptotic development for the functions ψ_i , and therefore give a rate of convergence for the coefficients c_i and the function ψ .

1.2.4. Periodic domains and domains with corners.

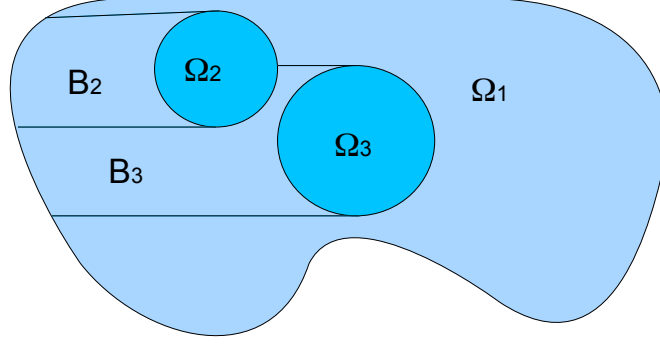
The case when the connection between the horizontal part of the boundary and the East or West part is a corner, which is typically the case of rectangles considered by De Ruijter, is actually easier to deal with, because there is no superposition zone for the boundary layers.

This corresponds to have a parametrization of the boundary by a function which is piecewise \mathcal{C}^4 , with some jumps for the angle $\theta(s)$

$$\cos \theta(s) \equiv 0 \text{ on } \Gamma_N \cup \Gamma_S, \quad \inf_{\Gamma_E \cup \Gamma_W} |\cos \theta| > 0.$$

We need only to suppose that angles in West corners are obtuse.

Our arguments also allow us to investigate domains of the type $\Omega = \mathbb{T} \times (y_-, y_+)$, where $\mathbb{T} = \mathbf{R}/\mathbf{Z}$. The interest for such domains stems from the analysis of circumpolar currents: indeed, realistic ocean basins consist of a part of a spherical surface, and for some latitudes $y \in]y_c^-, y_c^+[$ there may be no continent. The latter

FIGURE 1.4. Definition of the domains B_i

part of the fluid domain is called the “circumpolar” component. More precisely, De Ruijter considers domains of the form

$$(1.19) \quad \Omega = \Omega_{circ} \cup \Sigma_c^+ \cup \Sigma_c^- \cup \Omega_c^+ \cup \Omega_c^-,$$

where (see Figure 1.5)

$$\begin{aligned} \Omega_{circ} &= \mathbb{T} \times]y_c^-, y_c^+[, \\ \Omega_c^+ &= \{(x, y) \in \mathbb{T} \times \mathbf{R}, y_c^+ < y < \gamma_c^+(x)\}, \\ \Omega_c^- &= \{(x, y) \in \mathbb{T} \times \mathbf{R}, y_c^- > y > \gamma_c^-(x)\}, \\ \Sigma_c^\pm &= \mathbb{T} \times \{y_c^\pm\}, \end{aligned}$$

where γ_c^\pm are smooth periodic functions such that $\gamma_c^\pm(0) = y_c^\pm$, and $\gamma_c^+(y) > y_c^+$ for all $y \in (0, 1)$, $\gamma_c^-(y) < y_c^-$ for all $y \in (0, 1)$.

It is very likely that the analysis of the case (1.19) is in fact a combination of the arguments for “standard” smooth domains in \mathbf{R}^2 , which are the main concern of this article, and periodic domains of the type $\mathbb{T} \times (y_-, y_+)$. However, because of strong singularities near the junction points $(0, y_c^\pm)$, the construction and the energy estimates become very technical, without seemingly exhibiting any new mathematical behaviour or ideas. Therefore we will focus on the periodic case and explain, within this simplified geometry, why circumpolar currents appear.

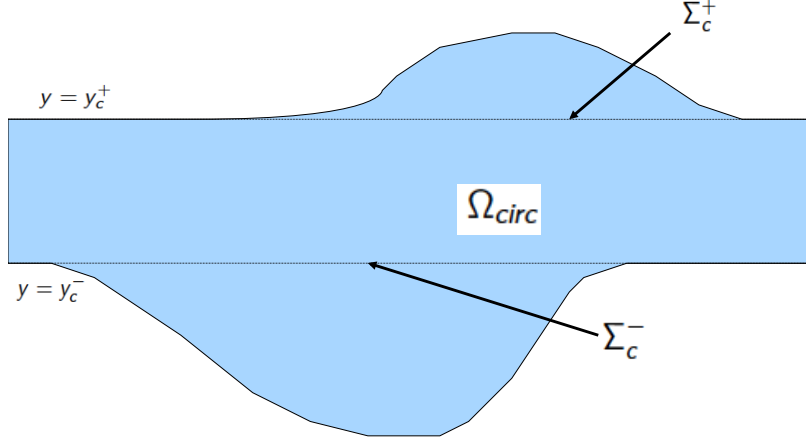


FIGURE 1.5. Circumpolar domain

1.3. Main approximation results

1.3.1. General case. We first describe our result in a domain $\Omega \subset \mathbf{R}^2$ satisfying (H1)-(H4), and then explain how our result can be extended to other types of domains.

We will prove the existence of approximate solutions in the form

$$(1.20) \quad \psi_{app} = \psi^{int} + \psi_{E,W} + \psi_{N,S} + \psi^\Sigma$$

where

- ψ^{int} is a regularization of ψ^0 (or of $\psi^0 + \sum_{i=2}^K \bar{c}_i \mathbf{1}_{B_i}$ when the domain Ω has islands), and ψ^0 is the solution to the Sverdrup equation (1.4). Let us emphasize that this regularization includes boundary layer correctors located in horizontal bands of width $E^{1/4}$ in the vicinity of every singular line Σ_{ij} . Furthermore, $\|\psi^{int} - \psi^0\|_{L^2} = o(\|\psi^{int}\|_{L^2})$;
- $\psi_{E,W}$ groups together eastern and western boundary terms, which decay on a distance of order $E^{1/3}$;
- $\psi_{N,S}$ is the contribution of southern and northern boundary layers terms, which decay on a distance of order $E^{1/4}$;
- and ψ^Σ is an additional boundary layer term, located in horizontal bands of width $E^{1/4}$ in the vicinity of every singular line Σ_{ij} .

Notice that ψ^{int} , $\psi_{E,W}$, $\psi_{N,S}$ and ψ^Σ do not have the same sizes in L^2 and H^2 : typically, if the boundary condition to be lifted is of order 1, the size in L^2 of a boundary layer type term is $\lambda^{-1/2}$, where λ^{-1} is the size of the boundary layer,

while its size in H^2 is $\lambda^{3/2}$. As a consequence, we roughly expect that

$$(1.21) \quad \underbrace{\|\psi^{int}\|_{L^2}}_{\sim 1} \gg \underbrace{\|\psi_{N,S}\|_{L^2} \sim \|\psi^\Sigma\|_{L^2}}_{\sim E^{1/8}} \gg \underbrace{\|\psi_{E,W}\|_{L^2}}_{\sim E^{1/6}},$$

while

$$(1.22) \quad \underbrace{\|\psi^\Sigma\|_{H^2} \sim \|\psi_{N,S}\|_{H^2} \sim \|\psi^{int}\|_{H^2}}_{\sim E^{-3/8}} \ll \underbrace{\|\psi_{E,W}\|_{H^2}}_{\sim E^{-1/2}},$$

Our result is the following;

THEOREM 2. *Assume that the domain $\Omega \subset \mathbf{R}^2$ satisfies assumptions (H1)-(H4). Consider a non trivial forcing $\tau = \text{curl } \mathcal{T} \in W^{4,\infty}(\Omega)$, and let ψ be the solution of the Munk equation (1.1) (or (1.14)-(1.15) when Ω has islands).*

Then there exists a function ψ_{app} of the form

$$\psi_{app} = \psi^{int} + \psi_{E,W} + \psi_{N,S} + \psi^\Sigma$$

satisfying the approximate equation (1.5) with an admissible remainder in the sense of (1.9).

More specifically, ψ^{app} is a good approximation of ψ , in the following sense: let $V \subset \Omega$ be a non-empty open set. Then, for a generic forcing term τ , the following properties hold:

- *Approximation of the interior term: if $V \Subset \Omega$ is such that $\bar{V} \cap \Sigma = \emptyset$, then*

$$\|\psi - \psi^0\|_{L^2(V)} = o(\|\psi^0\|_{L^2(V)});$$

- *Approximation of the Σ boundary layers: if $V \Subset \Omega$ is such that $V \cap \Sigma_{ij} \neq \emptyset$ for some $i \neq j \in \{1, \dots, M\}$, then*

$$\|\psi - (\psi^{int} + \psi^\Sigma)\|_{L^2(V)} = o(\|\psi^\Sigma\|_{L^2(V)}),$$

$$\|\psi - (\psi^{int} + \psi^\Sigma)\|_{H^2(V)} = o(\|\psi^{int} + \psi^\Sigma\|_{H^2(V)}).$$

- *Approximation of the North and South boundary layers: if $\bar{V} \cap \dot{\Gamma}_{N,S} \neq \emptyset$ and $\bar{V} \cap \Gamma_{E,W} = \emptyset$, $\bar{V} \cap \Sigma = \emptyset$, then*

$$\|\psi - (\psi^0 + \psi_{N,S})\|_{L^2(V)} = o(\|\psi_{N,S}\|_{L^2(V)}),$$

$$\|\psi - \psi_{N,S}\|_{H^2(V)} = o(\|\psi_{N,S}\|_{H^2(V)}).$$

- *Approximation of the West boundary layer: if $\bar{V} \cap \Gamma_W \neq \emptyset$ and $\bar{V} \cap \Gamma_{N,S} = \emptyset$, $\bar{V} \cap \Sigma = \emptyset$, $\bar{V} \cap \Gamma_E = \emptyset$,*

$$\|\psi - \psi_W\|_{H^2(V)} = o(\|\psi_W\|_{H^2(V)}).$$

In the above Theorem, the term ‘‘generic forcing’’ is necessarily unprecise at this stage. It merely ensures that the terms constructed in the approximate solution are not identically zero. We will give a more precise assumption in the next chapter (see Definition 2.4.1). The interior of $\Gamma_{N,S}$, namely $\dot{\Gamma}_{N,S}$, is to be understood through the induced topology on $\partial\Omega$. If the forcing is generic, some rough estimates on the sizes of the different terms are given in the table page 107 in the Appendix.

REMARK 1.3.1. *Note that the present result does not say anything about the validity of East boundary correctors. Indeed, their amplitude is typically $O(E^{1/3})$ on the zones where $\cos\theta$ is bounded away from zero, and therefore they are too small to be captured by the energy estimate. We will comment more on this point*

in section 3.2 (see Remark 3.2.3). Note however that in non-degenerate settings, a solution can be built at any order (see [6]), and therefore the energy estimates may capture the east corrector in this case. Moreover, the east boundary layer equation is somewhat indirectly justified by the fact that the interior term vanishes on the east boundary.

The construction of an approximate solution relies on a local asymptotic expansion of the form

$$\psi \simeq \psi^0 + \psi^{BL}(s, \lambda(s)z),$$

where ψ^0 is the interior term, which solves the Sverdrup equation (1.4), s is the arc-length, z is the distance to the boundary, and $\lambda(s) \gg 1$ is the inverse of the boundary layer size. The boundary layer term $\psi^{BL}(s, Z)$ is also assumed to vanish as $Z \rightarrow \infty$. Plugging this expansion into equation (1.1), we find an equation for ψ^{BL} , in which both ψ^{BL} and λ are unknown. The idea is then to choose λ in a clever way, depending on the zone of the boundary under consideration, and then to solve the corresponding equation on ψ^{BL} . As a consequence, different equations for the boundary layer are obtained on different zones of the boundary, and a matching between these zones must be performed: this is the difficult part of the mathematical analysis, which is absent from [20]. Eventually, once the approximate solution is defined, we check that the corresponding error terms are all admissible in the sense of Definition 1.1.2.

REMARK 1.3.2 (About the geometric assumptions). *The geometric assumptions are mostly used in the zones of transition between the different types of boundary layers. Note that they may not all be necessary, but we found no systematic way of dealing with all possible cases simultaneously.*

- (H1) can be weakened since we are able to deal with corners (corners involving only East or West boundaries are handled by a simple truncation while corners with horizontal boundaries will be discussed in chapter 3);
- (H2) is used to have explicit estimates for the transition zone between East/West boundary layers and North/South boundary layers. In particular, the rate of cancellation of $\cos \theta$ determines:
 - The size of the truncation of the forcing τ in singular zones, see (3.4), (3.5);
 - The domain of validity of the east and west boundary layers, see (2.8), (3.15), (3.16);
 - The zone on which energy is injected in the north and south boundary layers, see (3.23)
- (H3) allows to avoid the connection between some Σ layer and a horizontal boundary. It is not completely clear whether or not we would be able to handle such a singular transition;
- (H4) is a technical assumption (comparable to (H2)) to get good estimates on the Σ layer, and in particular, on the impact of the Σ layer on the west boundary. It is discussed in detail in Remark 3.4.7.

REMARK 1.3.3 (Comparison with previous results). • *One of the main features of our construction lies in the precise description of the connection between boundary layers. In particular, we prove that the sizes and profiles of the North/South and East/West boundary layers are unrelated; the transition between both types of boundary layers occurs through their amplitude only. Notice that this result is rather*

unexpected: indeed, most works on boundary layers (see for instance [9]) assume that an asymptotic expansion of the form

$$\psi \sim \psi^{int} + \psi^{BL}(s, \lambda(s)z)$$

holds, where the size of the boundary layer, λ^{-1} , is defined on the whole boundary and is continuous. Here, we exhibit a different type of asymptotic expansion, which shows that a superposition of two types of boundary layers occurs on the transition zone. Also, on this transition zone, the ratio between the sizes of the two boundary layers is very large, so that the asymptotic expansion above cannot hold.

• The analysis we present in this paper can be extended without difficulty to very general anisotropic degenerate elliptic equations, in particular to the convection-diffusion equation

$$\partial_x \psi - E \Delta \psi = \tau$$

which has been studied for special domains by Eckhaus and de Jager in [8], followed by Grasman in [11], and more recently by Jung and Temam [13]. Note that in the case of the convection-diffusion equation, the maximum principle (which does not hold anymore in our case) can be used to prove convergence in L^∞ . In the paper [8], the authors exhibit parabolic boundary layers on the North and South boundaries, but only treat the case when the domain Ω is a rectangle, which turns out to be easier, as explained in the next chapter.

Note that the tools we develop in the present paper allow to consider more general geometries, for which

- there is a continuous transition between lateral and horizontal boundaries,
- there are singular interfaces Σ_{ij} , and even islands.

1.3.2. Periodic and rectangle cases. Our arguments also allow to consider the cases when the domain Ω is a rectangle $(x_-, x_+) \times (y_-, y_+)$ or an x -periodic domain $\mathbb{T} \times (y_-, y_+)$, which are in fact much less involved than the case of smooth domains.

In the case of a rectangle, the approximate solution is given by

$$(1.23) \quad \psi_{app} = \psi_t^0 + \psi_{E,W} + \psi_{N,S},$$

exactly as in the case of a smooth domain. It turns out that the interaction between east/west boundary layers and north/south boundary layers in the corners is rather simple.

In the periodic case, the definition of approximate solution has to be changed a little bit because the circumpolar current ψ^{circ} (see definition below) is generically very large in all norms, so that comparing the size of the error with ψ^{circ} does not give any precise information on the asymptotic expansion of the solution ψ . Moreover, the weight $\exp(x)$ (or, more generally, any increasing function of x whose derivatives are bounded from below) cannot be used in the energy estimates, since only periodic weights are allowed. Therefore the energy estimate (1.7) for equation (1.6) becomes

$$(1.24) \quad E \|\Delta \delta \psi\|_{L^2(\mathbb{T} \times (y_-, y_+))} \leq \|\delta \tau\|_{H^{-2}(\mathbb{T} \times (y_-, y_+))}.$$

We therefore replace Definition 1.1.1 by the following:

DEFINITION 1.3.4. A function $\psi_{app} \in H^2(\mathbb{T} \times (y_-, y_+))$ is an approximate solution to (1.1) if it satisfies the approximate equation (1.5) for some $\delta \tau \in H^{-2}(\mathbb{T} \times$

(y_-, y_+) with

$$(1.25) \quad \lim_{E \rightarrow 0} \frac{\|\delta\tau\|_{H^{-2}}}{E\|\Delta(\psi_{app} - \psi^{circ})\|_{L^2}} = 0.$$

In this case, the approximate solution is defined by

$$(1.26) \quad \psi_{app} = \psi^{circ} + \psi_{per}^0 + \psi_{N,S}^{BL}$$

where

- $\psi^{circ} = \psi^{circ}(y)$ is the circumpolar current, due to the average forcing by the wind, namely

$$\begin{aligned} -E\partial_y^4 \psi^{circ} &= \langle \tau \rangle (y), \quad y \in (y_-, y_+), \\ \psi|_{y=y_{\pm}} &= 0, \\ \partial_y \psi|_{y=y_{\pm}} &= 0, \end{aligned}$$

where $\langle f \rangle := \int_{\mathbb{T}} f$ for any periodic function f .

- ψ_{per}^0 is the classical Sverdrup current in Ω_{circ} , defined by

$$\partial_x \psi_{per}^0(x, y) = \tau(x, y) - \langle \tau \rangle (y), \quad x \in \mathbb{T}, y \in (y_-, y_+),$$

with $\langle \psi_{per}^0 \rangle (y) = 0$.

- $\psi_{N,S}^{BL}$ are periodic North and South boundary layers, whose definition differ slightly from usual North and South boundary layers.

We then have the following results:

- PROPOSITION 1.3.5. (1) Assume that $\Omega = (x_-, x_+) \times (y_-, y_+)$ and that $\tau \in W^{4,\infty}(\Omega)$. Then ψ_{app} defined by (1.23) satisfies (1.5) with an admissible remainder in the sense of (1.9), and is therefore an approximate solution in the sense of Definition 1.1.1.
- (2) Assume that $\Omega = \mathbb{T} \times (y_-, y_+)$ and that $\tau \in H^s(\Omega)$ for s sufficiently large. Then ψ_{app} defined by (1.26) satisfies (1.5) with a remainder $\delta\tau$ satisfying

$$\|\delta\tau\|_{H^{-2}} = O(E),$$

so that for generic forcing τ ,

$$\|\psi - (\psi^{circ} + \psi_{N,S}^{BL})\|_{H^2} = o(\|\psi_{N,S}^{BL}\|_{H^2}).$$

In particular, ψ_{app} is an approximate solution in the sense of Definition 1.3.4.

REMARK 1.3.6. Notice that because of the lack of an L^2 estimate, the Sverdrup part of the solution in the periodic case is not captured by the energy estimate. However, if τ is sufficiently smooth, the construction can be iterated and an approximate solution can be built at any order, so that the existence of every term in the expansion can be justified. Indeed, there is no degeneracy in the problem, and therefore no singularity. The construction of North and South boundary layers only costs a finite, quantifiable number of derivatives on τ .

1.3.3. Outline of the paper. Since the proof of Theorem 2 is very technical, we have chosen to separate as much as possible the construction of ψ_{app} and the proof of convergence. Therefore the organization of the paper is the following.

In Chapter 2, we expose the main lines of the construction of the boundary layer type terms, namely $\psi_{E,W}$, $\psi_{N,S}$ and ψ^Σ , without going into the technicalities. An important point is that, while the East and West boundary layers $\psi_{E,W}$ are defined by some local operator, $\psi_{N,S}$ and ψ^Σ are obtained as the solutions of some parabolic equations, which accounts for the terminology of parabolic boundary layers used by De Ruijter in his book [20]: “when the southern boundary of the continent A coincides with a characteristic the first approximation in the free shear layer is not only dependent on the matching conditions but also on the initial condition at the rim of the parabolic boundary layer. In this way the information about the processes along the southern coast of A is reflected in the interior of the basin. ”

In Chapter 3, we give all the necessary details for the construction insisting on the connection between the different types of boundary layers, and we estimate the sizes of the four terms defining ψ_{app} . Even though this part can seem essentially technical, there are two important features of the construction to be noted. The first one is that the connection between boundary layers has to be understood as a superposition: amplitudes are matched, but not profiles. The second ingredient is that the order for the construction of the different correctors is prescribed and this has something to do with the disymmetry between East and West: the construction is essentially Westwards, as are the transport by the Sverdrup equation and the diffusion in the parabolic layers. We also explain how rectangular or periodic domains can be handled.

Eventually, in Chapter 4, we prove the estimate on the error term (1.9), which entails that ψ_{app} is an approximate solution. Moreover, as a very large number of notations are introduced throughout the paper, an index of notations is available after the Appendix. We also included a table summarizing the sizes of the different parameters and terms.

CHAPTER 2

Multiscale analysis

Searching as usual an approximate solution to the Munk equation (1.1)

$$\partial_x \psi - E \Delta^2 \psi = \tau$$

in the form

$$\begin{aligned} \psi_{app} &= \psi^0 + \psi^{BL} && \text{if } \Omega \text{ is simply connected (no islands),} \\ \text{and } \psi_{app} &= \psi^0 + \sum_{i=2}^K \bar{c}_i \mathbf{1}_{B_i} + \psi^{BL} && \text{if } \Omega \text{ has islands,} \end{aligned}$$

with ψ^0 satisfying the Sverdrup relation

$$\partial_x \psi^0 = \tau \text{ in } \Omega, \quad \psi^0|_{\Gamma_E} = 0,$$

we see that both ψ^0 and ψ^{BL} present singularities near the North and South boundaries of the domain, i.e. as $\cos \theta$ vanishes, as well as on the interfaces Σ_{ij} .

- First, the main term of the approximate solution, ψ^0 , is singular near all smooth “North-East” and “South-East corners”. More precisely, combining the integral definition of ψ^0 together with equivalents for the coordinates of boundary points given in Appendix B, we see that the y derivatives of ψ^0 explode near such corners.
- Moreover, as we explained in paragraph 1.1.1, the size of the boundary layer becomes much larger as $\cos \theta \rightarrow 0$, going from $E^{1/3}$ to $E^{1/4}$. This also creates strong singularities in the boundary layer terms, which are completely independent from the singularity described above. In fact, there is a small zone in which both boundary layers coexist and are related to one another through their amplitude.
- Finally, for complex domains, i.e. for domains where the closure of the East boundary $\bar{\Gamma}_E$ is not connected, the solution of the interior problem ψ^0 has discontinuities across the horizontal lines Σ_{ij} . These discontinuities gives rise to a “**boundary layer singularity**”, which is apperanted to the North and South boundary terms, the size of which is therefore $E^{1/4}$.

Because of these three types of singularities, the construction of the solution is quite technical. We therefore start with a brief description of all kinds of boundary type terms (see Figure 2.1).

2.1. Local coordinates and the boundary layer equation

As usual in linear singular perturbation problems, we build boundary layer correctors as solutions to the homogeneous linear equation

$$(2.1) \quad \partial_x \psi - E \Delta^2 \psi = 0,$$

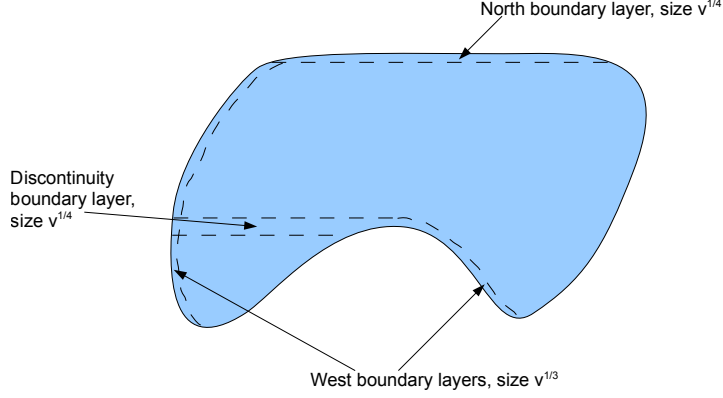


FIGURE 2.1. Overall view of the boundary layers

localized in the vicinity of the boundary (therefore depending in a singular way of the distance z to the boundary)

$$\psi \equiv \psi^{BL}(s, \lambda z) \text{ with } \partial_z \psi^{BL} = \lambda \partial_Z \psi^{BL} \gg \partial_s \psi^{BL}$$

$$\psi^{BL} \rightarrow 0 \text{ as } Z \rightarrow \infty,$$

and lifting boundary conditions

$$\psi^{BL}(s, 0) = -\psi^{int}(s, 0), \quad \partial_Z \psi^{BL}(s, 0) = -\partial_n \psi^{int}(s, 0).$$

Note that the parameter $\lambda \gg 1$ is expected to measure the inverse size of the boundary layer, it can therefore depend on s .

It is then natural to rewrite the homogeneous Munk equation in terms of the local coordinates (z, s) . We have by definition of θ , s and z ,

$$(2.2) \quad \begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial s} \end{pmatrix} = \begin{pmatrix} -\cos \theta & (1 + z\theta') \sin \theta \\ -\sin \theta & -(1 + z\theta') \cos \theta \end{pmatrix}$$

so that

$$\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{pmatrix} = \begin{pmatrix} -\cos \theta & -\sin \theta \\ (1 + z\theta')^{-1} \sin \theta & -(1 + z\theta')^{-1} \cos \theta \end{pmatrix}.$$

We therefore deduce that the jacobian of the change of variables $(x, y) \rightarrow (z, s)$ is equal to $(1 + z\theta')^{-1}$. Furthermore,

$$\partial_x \psi^{BL} = -\cos \theta \lambda \partial_Z \psi^{BL} + \frac{\sin \theta}{1 + \lambda^{-1} Z \theta'} \left[\partial_s \psi^{BL} + \frac{\lambda'}{\lambda} Z \partial_Z \psi^{BL} \right].$$

On most part of the boundary the second term is negligible compared to the first one. Nevertheless we see immediately that on horizontal parts the first term is zero, so that we have to keep the second one. Since $\lambda \gg 1$, we approximate the jacobian term $(1 + \lambda^{-1} Z \theta')^{-1}$ by 1.

Similar computations allow to express the bilaplacian in terms of the local coordinates (which involves more or less twenty terms). However in the boundary layers, we expect that the leading order term is the fourth derivative with respect to z

$$\Delta^2 \psi^{BL} = \lambda^4 \partial_Z^4 \psi^{BL} + O(\lambda^3).$$

We will thus consider only this term and check *a posteriori* in Chapter 4 that the contribution of other terms is indeed negligible. Note that, as we want to estimate the H^{-2} norm of the remainder, we will only need to express the laplacian (rather than the bilaplacian) in local coordinates.

We will therefore define boundary layer correctors as (approximate) solutions to the equation

$$(2.3) \quad -\lambda \cos \theta \partial_Z \psi^{BL} + \sin \theta \left(\partial_s \psi^{BL} + \frac{\lambda'}{\lambda} Z \partial_Z \psi^{BL} \right) - E \lambda^4 \partial_Z^4 \psi^{BL} = 0.$$

This equation remaining still complicated, we will actually consider two regimes depending on the precise localization on the boundary. For each one of these regimes, we will neglect one of the first two terms in (2.3) (i.e. part of the ∂_x derivative), so that the size of λ as well as the profile of ψ^{BL} will be different. Of course we will need to check *a posteriori* that the term which has been neglected can be dealt with as a remainder in the energy estimate.

2.2. East and West boundary layers

In this section, we construct the boundary layers on the lateral sides of the domain. We retrieve rigorously the result announced in the introduction, namely the intensification of Western boundary currents and the dissymetry between the East and West coasts.

2.2.1. The scaled equation.

Along the East and West coasts of the domain, on intervals on which $\cos \theta$ remains bounded away from zero, it can be expected that the main terms in the boundary layer equation (2.3) are $\lambda \cos \theta \partial_Z f(s, \lambda z)$ and $E \lambda^4 \partial_Z^4 f(s, \lambda z)$. Hence we take λ such that

$$(2.4) \quad \lambda_{E,W} = \left(\frac{|\cos \theta|}{E} \right)^{1/3},$$

and f such that

$$(2.5) \quad \partial_Z^4 f = -\text{sign}(\cos(\theta)) \partial_Z f.$$

We recall that Z is the rescaled boundary layer variable ($Z = \lambda z$), so that $Z \in (0, \infty)$.

We look for solutions of the above equation which decay as $Z \rightarrow \infty$. Consequently, the dimension of the vector space of solutions of the simplified boundary layer equation (2.5) depends on the sign of $\cos(\theta)$:

- If $\cos \theta > 0$ (East coast), decaying solutions of equation (2.5) are of the form

$$f(s, Z) = A(s) \exp(-Z)$$

- If $\cos \theta < 0$ (West coast), decaying solutions of equation (2.5) are of the form

$$f(s, Z) = A_+(s) \exp(-e^{i\pi/3}Z) + A_-(s) \exp(-e^{-i\pi/3}Z).$$

Notice that we retrieve the dissymmetry between the East and West coasts: indeed, only one boundary condition can be lifted on the East boundary, whereas two boundary conditions (namely, the traces of ψ^0 and $\partial_n \psi^0$) can be lifted on the West boundary. As a consequence, ψ^0 must vanish at first order on the East coast, so that the role of ψ^{BL} on Γ_E is merely to correct the trace of $\partial_n \psi^0$.

Note that, in order that the trace and the normal derivative of $\psi^0 + \psi^{BL}$ are exactly zero on Γ_E , we will actually need an additional corrector, which is built in the next chapter.

In first approximation, the boundary layer terms on the East and West coasts, denoted respectively by ψ_E and ψ_W , are thus defined by

$$\begin{aligned} \psi_E(s, Z) &= A(s) \exp(-Z), \\ \psi_W(s, Z) &= \sum_{\pm} A_{\pm}(s) \exp(-e^{\pm i\pi/3}Z), \end{aligned}$$

where the coefficients A, A_+, A_- ensure that the trace and the normal derivative of $\psi^0(s, z) + \psi^{BL}(s, \lambda_{E,W}z)$ vanish at main order on the East and West coasts. This leads to

$$\begin{aligned} (2.6) \quad A(s) &= \lambda_E^{-1} \partial_n \psi|_{\partial\Omega}^0(s), \\ \begin{pmatrix} A_+(s) \\ A_-(s) \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ \lambda_W e^{i\pi/3} & \lambda_W e^{-i\pi/3} \end{pmatrix}^{-1} \begin{pmatrix} -\psi|_{\partial\Omega}^0(s) \\ \partial_n \psi|_{\partial\Omega}^0(s) \end{pmatrix} \\ (2.7) \quad &= \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\pi/6} & -i\lambda_W^{-1} \\ e^{-i\pi/6} & i\lambda_W^{-1} \end{pmatrix} \begin{pmatrix} -\psi|_{\partial\Omega}^0(s) \\ \partial_n \psi|_{\partial\Omega}^0(s) \end{pmatrix} \end{aligned}$$

Let us emphasize that the precise value of A on East boundaries is in fact irrelevant in energy estimates, since equation (2.6) implies that $A = O(E^{1/3})$ on zones where $\cos \theta$ does not vanish. Therefore the East boundary layer itself is not captured by energy estimates. But its incidence on the interior term, through the fact that $\psi|_{\Gamma_E}^0 = 0$, is clearly seen in the L^2 estimate.

2.2.2. Domain of validity.

Simplifying (2.3) into (2.5), we have neglected the terms $\lambda' \lambda^{-1} Z \partial_Z f$ and $\partial_s f$. As f is an exponential profile, the term corresponding to $\lambda' \lambda^{-1} Z \partial_Z f$ is smaller than $E \lambda^4 \partial_Z^4 f$ and $\lambda \cos(\theta) \partial_Z f$ as long as

$$\left| \frac{\lambda'}{\lambda} \right| \ll |E \lambda^4| = \frac{|\cos \theta|^{4/3}}{E^{1/3}},$$

which leads to

$$(2.8) \quad E^{1/3}|\theta'| |\cos \theta|^{-7/3} \ll 1.$$

As for the term $\partial_s f$, using (2.6) and (2.7), we obtain that, if $\psi|_{\partial\Omega}^0$ and $\partial_n \psi|_{\partial\Omega}^0$ are smooth with respect to s , the corresponding error terms can be neglected as long as

$$\lambda |\cos(\theta)| \gg 1, \text{ i.e. } |\cos(\theta)| \gg E^{1/4}.$$

This last condition is less stringent than (2.8). Hence we only keep (2.8) in order to determine the interval of validity of the construction.

The intervals on which the East and West boundary layers are defined follow from the validity condition (2.8). More precisely, if $\cos \theta \neq 0$ on (s_i, s_{i+1}) (West or East coasts), we set

$$s_i^+ := \sup \left\{ s \in \left(s_i, \frac{s_i + s_{i+1}}{2} \right), E^{1/3}|\theta'| |\cos \theta|^{-7/3} \geq 1 \right\},$$

$$s_{i+1}^- = \inf \left\{ s \in \left(\frac{s_i + s_{i+1}}{2}, s_{i+1} \right), E^{1/3}|\theta'| |\cos \theta|^{-7/3} \geq 1 \right\}.$$

Notice that s_i^+, s_{i+1}^- depend on E and are well-defined as long as $\theta \in \mathcal{C}^1([0, L])$. By definition,

$$E^{1/3}|\theta'| |\cos \theta|^{-7/3} < 1 \quad \forall s \in (s_i^+, s_{i+1}^-),$$

and it is easily proved that $\cos(\theta(s_i^\pm))$ vanish as $E \rightarrow 0$, so that

$$\lim_{E \rightarrow 0} s_i^+ = s_i, \quad \lim_{E \rightarrow 0} s_{i+1}^- = s_{i+1}.$$

REMARK 2.2.1. *Notice that (2.8) is not satisfied in the vicinity of s_i^+, s_{i+1}^- . Moreover, the derivatives of $\psi|_{\partial\Omega}^0$ and $\partial_n \psi|_{\partial\Omega}^0$ with respect to s may become very large as s approaches s_i and s_{i+1} . Chapter 4 is devoted to the estimation of the corresponding error terms and to the proof of their admissibility.*

2.3. North and South boundary layers

In this section, we construct the boundary layer terms near the intervals where $\cos \theta$ vanishes. Notice that these intervals may in fact be reduced to single points. On the horizontal parts of the boundary, we get a parabolic equation of order 4 with constant coefficients, as obtained by De Ruijter in [20]. However to account for the part of the boundary which is ‘‘almost horizontal’’ we will need to consider a more complicated equation with variable coefficients.

2.3.1. The scaled equation. We consider in this section an interval (s_i, s_{i+1}) on which $\cos \theta$ is identically zero and $\sin \theta \equiv -1$ (South boundary), so that equation (2.3) becomes

$$(2.9) \quad \partial_s f + \frac{\lambda'}{\lambda} Z \partial_Z f + E \lambda^4 \partial_Z^4 f = 0.$$

As in section 2.2, we have to choose λ and f so that the above equation is satisfied. The simplest choice is to take

$$(2.10) \quad \lambda_{N,S} := E^{-1/4}$$

so that $\lambda' = 0$, and the boundary layer equation becomes a diffusion-like equation, with the arc-length s playing the role of the time variable. Note that such degenerate parabolic boundary layers have been exhibited in [8] for instance.

This raises several questions:

- What is the direction of propagation of “time” in (2.9)? In other words, what is the initial data for (2.9)?
- Is equation (2.9) well-posed?
- What is the domain of definition (in s) of the South boundary layer term?
- How are the South, East and West boundary layers connected?

We will prove in this section that equations of the type

$$\begin{aligned} \partial_s f + \partial_Z^4 f &= 0, \quad s \in (0, T), \quad Z > 0, \\ f|_{s=0} &= f_{in} \\ f|_{Z=0} &= f_0, \quad \partial_Z f|_{Z=0} = f_1, \end{aligned}$$

with $f_{in} \in L^2(0, \infty)$, $f_0, f_1 \in W^{1, \infty}(0, T)$, are well-posed, and we will give some energy estimates on the solutions of such equations. In the present context, this means that on South boundary layers, equation (2.9) is a forward equation (in s), while on North boundary layers it becomes a backward equation. This is consistent with the definition of the interior term ψ^0 : in all cases, the boundary condition in s is prescribed on the East end of the interval.



FIGURE 2.2. The arc-lengths $\sigma_i^-, s_i^-, s_i, s_{i+1}, s_{i+1}^+, \sigma_{i+1}^+$ on a South boundary

Let us also recall that the domain of validity of the West and East boundary layer terms does not reach the zone where $\cos \theta = 0$ (see (2.8)). As a consequence, the South boundary layer term must be defined for $s \in (\sigma_i^-, s_i)$ and $s \in (s_{i+1}, \sigma_{i+1}^+)$ where the arc-lengths $\sigma_i^-, \sigma_{i+1}^+$ will be defined later on and should satisfy (see Figure 2.2)

$$\sigma_i^- < s_i^-, \quad \sigma_{i+1}^+ > s_{i+1}^+.$$

This requires to slightly modify the boundary layer equation. From now on, we define the South boundary layer term ψ_S as the solution of

$$(2.11) \quad \begin{aligned} \partial_s \psi_S - E^{-1/4} \frac{\cos \theta}{\sin \theta} \partial_Z \psi_S - \frac{1}{\sin \theta} \partial_Z^4 \psi_S &= 0, \quad s > \sigma_i^-, \quad Z > 0, \\ \psi_S|_{s=\sigma_i^-} &= 0, \\ \psi_S|_{Z=0} &= \Psi_0, \quad \partial_Z \psi_S|_{Z=0} = \Psi_1, \end{aligned}$$

with Ψ_0, Ψ_1 to be defined later on (see (2.14)). Notice that we choose $\lambda_{N,S} = E^{-1/4}$ even in the zones where $\cos \theta \neq 0$, i.e. outside the interval (s_i, s_{i+1}) .

2.3.2. Study of the boundary layer equation (2.11):

In this paragraph, we give a well-posedness result for equation

$$(2.12) \quad \begin{aligned} \partial_s f + \gamma(s) \partial_Z f + \mu(s) \partial_Z^4 f &= 0, \quad s \in (0, T), \quad Z > 0, \\ f|_{s=0} &= f_{in}, \\ f|_{Z=0} &= f_0, \quad \partial_Z f|_{Z=0} = f_1. \end{aligned}$$

We first note that, up to lifting the boundary conditions, we are brought back to the study of the same parabolic equation with homogeneous boundary conditions and with a source term: indeed f is a solution of (2.12) if and only if

$$g(s, Z) := f(s, Z) - f_0(s)(Z+1) \exp(-Z) - f_1(s)Z \exp(-Z)$$

is a solution of

$$(2.12) \quad \begin{aligned} \partial_s g + \gamma(s) \partial_Z g + \mu(s) \partial_Z^4 g &= S(s, Z), \quad s \in (0, T), \quad Z > 0, \\ g|_{s=0} &= g_{in}, \\ g|_{Z=0} &= 0, \quad \partial_Z g|_{Z=0} = 0, \end{aligned}$$

where

$$g_{in} := f_{in} - f_0(0)(Z+1) \exp(-Z) - f_1(0)Z \exp(-Z)$$

and

$$(2.12) \quad \begin{aligned} S(s, Z) &:= -f'_0(s)(Z+1) \exp(-Z) - f'_1(s)Z \exp(-Z) \\ &\quad + \gamma(s)f_0(s)Z \exp(-Z) - \gamma(s)f_1(s)(1-Z) \exp(-Z) \\ &\quad - \mu(s)f_0(s)(Z-3) \exp(-Z) - \mu(s)f_1(s)(Z-4) \exp(-Z). \end{aligned}$$

Our precise result is then the following:

LEMMA 2.3.1. *Let $T > 0$, and let $\mu \in L^\infty(0, T)$, $\gamma \in L^\infty(0, T)$, $f_0, f_1 \in \mathcal{C}^1([0, T])$, $g_{in} \in L^2(0, \infty)$, and $S \in L^1((0, T), L^2(\mathbf{R}_+))$.*

Assume that there exists $\mu_0 > 0$ such that

$$\mu(s) \geq \mu_0 \quad \text{for a.e. } s \in (0, T).$$

Then the equation

$$(2.13) \quad \begin{aligned} \partial_s g + \gamma(s) \partial_Z g + \mu(s) \partial_Z^4 g &= S(s, Z), \quad s \in (0, T), \quad Z > 0, \\ g|_{s=0} &= g_{in}, \\ g|_{Z=0} &= 0, \quad \partial_Z g|_{Z=0} = 0, \end{aligned}$$

has a unique solution $g \in \mathcal{C}([0, T], L^2(\mathbf{R}_+)) \cap L^2((0, T), H^2(\mathbf{R}_+))$ which satisfies the energy estimate

$$\begin{aligned} &\frac{1}{2} \|g(s)\|_{L^2(\mathbf{R}_+)}^2 + \int_0^s \mu(s') \|\partial_Z^2 g(s')\|_{L^2(\mathbf{R}_+)}^2 ds' \\ &\leq \|g_{in}\|_{L^2(\mathbf{R}_+)}^2 + \left(\int_0^s \|S(s')\|_{L^2(\mathbf{R}_+)} ds' \right)^2. \end{aligned}$$

PROOF. Let

$$V := \{v \in H^2(0, \infty), v|_{Z=0} = 0, \partial_Z v|_{Z=0} = 0\}.$$

For $u, v \in V$, $s \in (0, T)$, define the quadratic form

$$a(s, u, v) = \int_0^\infty \gamma(s) \partial_Z u(Z) v(Z) dZ + \mu(s) \int_0^\infty \partial_Z^2 u(Z) \partial_Z^2 v(Z) dZ.$$

Then for all $u, v \in V$, the map

$$s \mapsto a(s, u, v)$$

is measurable, and for almost every $s \in (0, T)$

$$\begin{aligned} |a(s, u, v)| &\leq \left(\|\gamma\|_{L^\infty(0, T)} + \|\mu\|_{L^\infty(0, T)} \right) \|u\|_{H^2(\mathbf{R}_+)} \|v\|_{H^2(\mathbf{R}_+)} \\ a(s, u, u) &\geq \mu_0 \|\partial_Z^2 u\|_{L^2(\mathbf{R}_+)}^2 \\ &\geq \mu_0 (\|u\|_{H^2(\mathbf{R}_+)}^2 - \|\partial_Z u\|_{L^2(\mathbf{R}_+)}^2 - \|u\|_{L^2(\mathbf{R}_+)}^2). \end{aligned}$$

Notice that if $u \in V$,

$$\begin{aligned} \|\partial_Z u\|_{L^2}^2 &= \int_0^\infty (\partial_Z u)^2 = - \int_0^\infty u \partial_Z^2 u \\ &\leq \frac{\|u\|_{L^2}^2}{2} + \frac{\|\partial_Z^2 u\|_{L^2}^2}{2}. \end{aligned}$$

We infer eventually that

$$a(s, u, u) \geq \frac{\mu_0}{2} \|u\|_{H^2(\mathbf{R}_+)}^2 - \frac{3\mu_0}{2} \|u\|_{L^2(\mathbf{R}_+)}^2.$$

Using Theorem 10.9 by J.L. Lions in [3], we infer that there exists a unique solution $g \in L^2((0, T), V) \cap \mathcal{C}([0, T], L^2(\mathbf{R}_+))$ of equation (2.13) such that $\partial_s g \in L^2((0, T), H^{-2}(\mathbf{R}_+))$.

Multiplying equation (2.13) by g and integrating on \mathbf{R}_+ , we infer

$$\frac{1}{2} \frac{d}{ds} \int_0^\infty |g(s)|^2 + \mu(s) \int_0^\infty |\partial_Z^2 g(s)|^2 = \int_0^\infty S(s, Z) g(s, Z) dZ.$$

Therefore

$$\begin{aligned} &\frac{d}{ds} \left[\frac{1}{2} \|g(s)\|_{L^2}^2 + \int_0^s \mu(s') \|\partial_Z^2 g(s')\|_{L^2}^2 ds' \right] \\ &\leq \|S(s)\|_{L^2} \|g(s)\|_{L^2} \\ &\leq \sqrt{2} \|S(s)\|_{L^2} \left[\frac{1}{2} \|g(s)\|_{L^2}^2 + \int_0^s \mu(s') \|\partial_Z^2 g(s')\|_{L^2}^2 ds' \right]^{1/2}. \end{aligned}$$

Integrating with respect to s leads then to the desired inequality. \square

We deduce easily that (2.11) has a unique solution on any interval of the form $(\sigma_i^-, \sigma_{i+1}^+)$ such that $\sin \theta$ does not vanish on $[\sigma_i^-, \sigma_{i+1}^+]$, and that it satisfies

$$\begin{aligned} \|\psi_S(s)\|_{L^2(\mathbf{R}_+)} &\leq C (|\Psi_0(s)| + |\Psi_1(s)| + |\Psi_0(\sigma_i^-)| + |\Psi_1(\sigma_i^-)|) \\ &\quad + C \int_{\sigma_i^-}^s (|\Psi_0'(s')| + |\Psi_1'(s')|) ds' \\ &\quad + C \int_{\sigma_i^-}^s (\mu(s') + |\gamma(s')|) (|\Psi_0(s')| + |\Psi_1(s')|) ds', \end{aligned}$$

with $\mu(s) = -1/\sin \theta(s)$, $\gamma(s) = -E^{-1/4} \cos \theta(s)/\sin \theta(s)$.

2.3.3. Boundary conditions for $s \in (s_i, s_{i+1})$. In order to satisfy the boundary conditions

$$\psi|_{\partial\Omega} = 0, \quad \partial_n \psi|_{\partial\Omega} = 0,$$

the South boundary layer term constructed above must be such that for $s \in (s_i, s_{i+1})$,

$$\begin{aligned} \psi_S(s, Z = 0) &= -\psi|_{\partial\Omega}^0(s), \\ E^{-1/4} \partial_Z \psi_S(s, Z = 0) &= -\partial_n \psi|_{\partial\Omega}^0(s). \end{aligned}$$

In equation (2.11), we therefore take, for $s \in (s_i, s_{i+1})$,

$$(2.14) \quad \Psi_0(s) := -\psi|_{\partial\Omega}^0(s),$$

$$(2.15) \quad \Psi_1(s) := E^{1/4} \partial_y \psi|_{\partial\Omega}^0(s).$$

There remains to define Ψ_0 and Ψ_1 for $s \in (\sigma_i^-, s_i] \cup [s_{i+1}, \sigma_{i+1}^+)$.

2.3.4. Connection with East and West boundary layers.

• In the most simple cases, the North or South boundaries are connected on the one hand to the East boundary, and on the other hand to the West boundary. This corresponds to the situation when

- the North boundary $y = y_i$ is a local maximum of the ordinate;
- the South boundary $y = y_i$ is a local minimum of the ordinate.

Without loss of generality, we assume that the corresponding piece of $\partial\Omega$ is a ‘‘South’’ boundary, meaning that $\cos \theta < 0$ in a neighbourhood on the right of s_{i+1} , $\cos \theta > 0$ in a neighbourhood on the right of s_i , and $\sin \theta = -1$ for $s \in [s_i, s_{i+1}]$.

The connection with the East boundary is fairly simple: we will introduce some truncation χ_E of τ close to the East corner (parametrized by s_i). Therefore the solution to the transport equation $\partial_x \psi_t^0 = \tau \chi_E$ is identically zero in a vicinity of (x_i, y_i) , as well as the East boundary layer (which lifts the trace of $\partial_n \psi_t^0$), so that we merely require $\psi_{S|s=\sigma_i^-} = 0$.

Concerning the connection with the West boundary layer, the situation is not as straightforward. In order to satisfy the boundary conditions

$$\psi|_{\partial\Omega} = 0, \quad \partial_n \psi|_{\partial\Omega} = 0,$$

the West and South boundary layer terms must be such that

$$\begin{aligned} \psi_W(s, Z = 0) + \psi_S(s, Z = 0) &= -\psi|_{\partial\Omega}^0(s), \\ \lambda_W \partial_Z \psi_W(s, Z = 0) + \lambda_S \partial_Z \psi_S(s, Z = 0) &= -\partial_n \psi|_{\partial\Omega}^0(s) \end{aligned}$$

in a vicinity of $s = s_{i+1}$. As a consequence, we take

$$\begin{aligned} \psi_W(s, Z = 0) &= -\varphi_{i+1}(s) \psi|_{\partial\Omega}^0(s), \\ \partial_Z \psi_W(s, Z = 0) &= -\lambda_W^{-1} \varphi_{i+1}(s) \partial_n \psi|_{\partial\Omega}^0(s), \end{aligned}$$

and

$$\begin{aligned} \psi_S(s, Z = 0) &= -(1 - \varphi_{i+1}(s)) \psi|_{\partial\Omega}^0(s), \\ \partial_Z \psi_S(s, Z = 0) &= -\lambda_S^{-1} (1 - \varphi_{i+1}(s)) \partial_n \psi|_{\partial\Omega}^0(s), \end{aligned}$$

where φ_{i+1} is a truncation function which we will define precisely in the next chapter. We emphasize that with this definition, the South and West boundary layers are related via their amplitude only: in particular, the sizes of the boundary layers are not related.

- The case when the boundary has non smooth corners between the horizontal part and the meridional boundaries (see paragraph 1.2.4)

$$\cos \theta(s) \equiv 0 \text{ on } \Gamma_N \cup \Gamma_S, \quad \inf_{\Gamma_E \cup \Gamma_W} |\cos \theta| > 0,$$

typically the case of rectangles studied by De Ruijter [20], could seem more singular at first sight but is actually easier to deal with.

Near East corners, there is no need to introduce the truncation χ_E since the solution to the Sverdrup equation is smooth.

Now, on the West boundary, if we lift the function $\psi^0 + \psi_E + \psi_{N,S}$, we obtain local boundary terms which vanish identically when $s \rightarrow s_i$. Hence there is no error due to the trace of West boundary terms on the horizontal part of the boundary. Notice that:

- If the angle between a horizontal and a western boundary is exactly $\pi/2$, our construction works without any adaptation;
- If the angle is obtuse, the North/South boundary layer needs to be extended beyond the corner, in the spirit of the connection between the discontinuity boundary layers and western boundary layers (see Remark 3.4.2).
- If the angle is acute, the correct rescaled boundary layer variable on the western boundary is $(x - x_W(y))/E^{1/3}$ instead of $zE^{-1/3}$, so as not to pollute the trace on the North/South boundary. Notice that this is the boundary layer variable used by Desjardins and Grenier in [6].

We leave the technical details to the reader, and treat in complete detail the case of rectangles in the present article.

• In more complex cases, the line $y = y_i$ intersects the interior of Ω , and we get a singularity which is not only localized in the vicinity of $\partial\Omega$. Techniques of boundary layers allow however to understand the qualitative behaviour of the solution in the vicinity of the line of singularity. This heuristic approach is presented in the next paragraph.

2.4. Discontinuity zones

Discontinuity zones occur when the East boundary $\bar{\Gamma}_E$ is not connected. At leading order, we expect the solution to (1.1) to be approximated by the solution ψ_t^0 to the transport equation with suitable truncations χ_E near East corners

$$\begin{aligned} \partial_x \psi_t^0 &= \tau \chi_E, \\ \psi_t^0|_{\Gamma_E} &= 0. \end{aligned}$$

If the domain Ω has islands, we add to ψ_t^0 the quantity $\sum_{j=2}^K \bar{c}_j \mathbf{1}_{B_j}$. We will give more details on this case in the next section. Note that, because of the truncation χ_E , the main order approximation ψ_t^0 now depends (weakly) on E .

Therefore on every set A_i , ψ_t^0 takes the form

$$\psi_t^0(x, y) = - \int_x^{x_E^i(y)} \tau \chi_E(x', y) dx'.$$

Unlike the main order term in Theorem 1, ψ_t^0 does not belong to $H^2(\Omega)$ in general. It is indeed obvious that ψ_t^0 and $\partial_y \psi_t^0$ may be discontinuous across every (nonempty)

line Σ_{ij} . More precisely, the jump of ψ_t^0 across a given line Σ_{ij} takes the following form (up to an inversion of the indexes i and j)

$$[\psi_t^0]_{\Sigma_{ij}} = - \int_{x_i}^{x_j} (\tau \chi_E)(x', y_{ij}) dx'.$$

The existence of such discontinuities in the main interior term is a serious impediment to energy methods. Indeed, inequality (1.7), for instance, requires the approximate solution to be at least in $H^2(\Omega)$. In fact, we will prove that this discontinuity gives rise to a “**boundary layer singularity**”: the corresponding corrector is a boundary layer term located in the vicinity of Σ_{ij} . Since the normal vector to Σ_{ij} is parallel to e_y , this boundary layer term is apperanted to the North and South boundary terms constructed in Section 2.3, and therefore the size of the boundary layer is $E^{1/4}$.

We now define what we have called in Theorem 2 a “generic forcing τ ”:

DEFINITION 2.4.1. *We say that the forcing τ is **generic** if all the following conditions are satisfied:*

- For all $i \neq j$ such that $\Sigma_{ij} \neq \emptyset$, there exists $c_\Sigma > 0$ such that

$$\text{for all } E \text{ sufficiently small, } \left| \left[\psi_t^0 + \sum_{l=2}^K \bar{c}_l \mathbf{1}_{B_l} \right]_{\Sigma_{ij}} \right| \geq c_\Sigma;$$

- If $\mathring{\Gamma}_{N,S} \neq \emptyset$ (for the induced topology on $\partial\Omega$), then for any $V \subset \Gamma_{N,S}$, there exists $c_V > 0$ such that

$$\text{for all } E \text{ sufficiently small, } \|\psi_t^0|_{\partial\Omega}\|_{L^2(V)} \geq c_V;$$

- For any nonempty $V \subset \Gamma_W$, there exists $c_V > 0$ such that

$$\text{for all } E \text{ sufficiently small, } \left\| \left(\psi_t^0 + \sum_{l=2}^K \bar{c}_l \mathbf{1}_{B_l} \right) \Big|_{\partial\Omega} \right\|_{L^2(V)} \geq c_V.$$

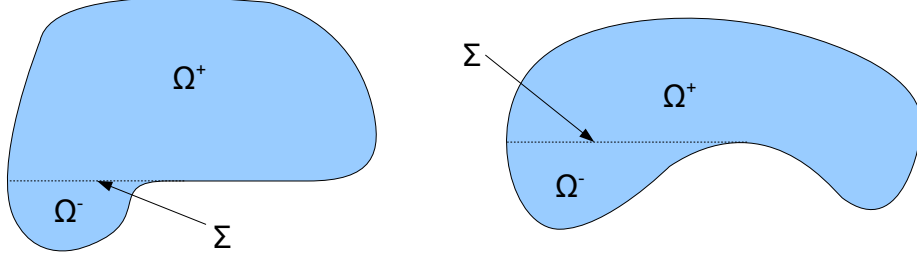
• From now on, for the sake of simplicity, we restrict the presentation to the case when $M = 2$, i.e. $\bar{\Gamma}_E$ **has two connected components**, and ψ_t^0 has exactly one line of discontinuity. Of course, our construction can be immediately generalized to the case when there are more than two connected components, and therefore several lines of discontinuity: we merely add up the local correctors constructed in the vicinity of every Σ_{ij} . In particular, when Ω has islands, there are always at least two discontinuity lines. We will sketch the necessary adaptations in the next paragraph and leave the details to the reader.

Thus we henceforth assume that Ω has the following form (see Figure 2.3):

$$\Omega = \Omega^+ \cup \Omega^- \cup \Sigma,$$

where

- Ω^\pm are non empty, open and convex in x ;
- $\Omega^+ = \{(x, y) \in \Omega, y > y_1 \text{ or } x \geq x_1\}$;
- $\Omega^- = \{(x, y) \in \Omega, y < y_1 \text{ and } x < x_1\}$;
- $\Sigma = \{(x, y_1) \in \Omega, x < x_1\}$

FIGURE 2.3. Two possible configurations for the domain Ω

and $(x_1, y_1) = (x(s_1), y(s_1))$ for some s_1 such that $\cos \theta(s_1) = 0$. Without any loss of generality, we further assume that $\sin \theta(s_1) = -1$. This type of domain can correspond to two different configurations (see Figure 2.3).

We will use the following notations

$$\chi^- = \mathbf{1}_{x < x_1} \mathbf{1}_{y < y_1}, \quad \chi^+ = 1 - \chi^-$$

which somehow stand for $\mathbf{1}_{\Omega^\pm}$: their role is to avoid any artificial singularity on $\partial\Omega$, and $\Gamma_E^\pm = \partial\Omega^\pm \cap \Gamma_E$. We parametrize each set Γ_E^\pm by a graph $(x_E^\pm(y), y)$.

As explained above, the function ψ_t^0 and its y derivative are discontinuous across Σ . More precisely, for $x < x_1$,

$$\begin{aligned} [\psi_t^0]_{|\Sigma}(x, y_1) &= \psi_t^0(x, y_1^+) - \psi_t^0(x, y_1^-) \\ &= - \int_x^{x_E^+(y_1)} (\tau\chi_E)(x', y_1) dx' + \int_x^{x_E^-(y_1)} (\tau\chi_E)(x', y_1) dx' \\ &= - \int_{x_E^-(y_1)}^{x_E^+(y_1)} (\tau\chi_E)(x', y_1) dx'. \end{aligned}$$

Notice in particular that the jump is constant along Σ . In a similar way, since $(\tau\chi_E)(x_E^-(y), y)$ vanishes in a neighbourhood on the left of y_1 ,

$$\begin{aligned} [\partial_y \psi_t^0]_{|\Sigma}(x, y_1) &= -(x_E^+)'(y_1) (\tau\chi_E)(x_E^+(y_1), y_1) \\ &\quad - \int_{x_E^-(y_1)}^{x_E^+(y_1)} \partial_y (\tau\chi_E)(x', y_1) dx'. \end{aligned}$$

We now construct the boundary layer type correctors $\psi^{\text{lift}}, \psi^\Sigma$. The role of ψ^{lift} is to counterbalance the jump of ψ_t^0 , and of its normal derivative $\partial_y \psi_t^0$, across Σ :

$$[\psi^{\text{lift}}]_\Sigma = -[\psi_t^0]_\Sigma, \quad [\partial_y \psi^{\text{lift}}]_\Sigma = -[\partial_y \psi_t^0]_\Sigma.$$

When we lift these boundary conditions, we introduce a source term in the equation. This source term is then handled by a boundary layer type term ψ^Σ , which has no discontinuity across Σ : $\psi^\Sigma \in H^2(\Omega)$.

At first sight, if we consider the whole singular corrector $\psi^{\text{lift}} + \psi^\Sigma$, this problem could seem underdetermined. Indeed there are two jump conditions to be satisfied, and possibly two boundary layers (on Ω^+ and on Ω^-), each of which is the solution

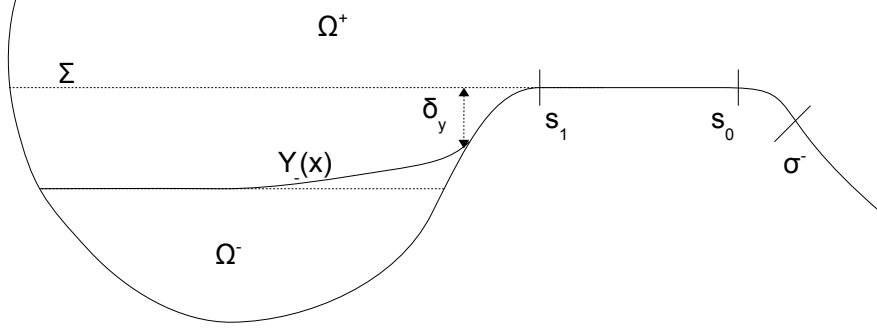


FIGURE 2.4. The interior singular layer

of an equation of the type (2.11), and therefore having each two entries (Ψ_0 and Ψ_1).

However, when looking at the error terms, we see that the traces of $\partial_y^2 \psi|_{\Sigma}^{\text{lift}}$ and $\partial_y^3 \psi|_{\Sigma}^{\text{lift}}$ appear in the energy estimates (see the proof of Lemma 2.4.3 below, and in particular the derivation of equation (2.22)). Therefore we have further to request that

$$[\partial_y^2 \psi|_{\Sigma}^{\text{lift}}]_{\Sigma} = [\partial_y^3 \psi|_{\Sigma}^{\text{lift}}]_{\Sigma} = 0.$$

Because of this constraint, the energy of the boundary layer discontinuity is distributed on both sides of Σ . The boundary term $\psi^{\text{lift}} + \psi^{\Sigma}$ is unequivocally defined, since roughly speaking, there are four jump conditions and four entries for the boundary layer terms.

2.4.1. Lifting the discontinuity. Let us start by introducing some notation. Consider the closed set $\{s \in \partial\Omega, y(s) = y_1\}$. We denote by I_1 its connected component containing s_1 . Then I_1 is a closed interval (possibly reduced to a single point). Without loss of generality, we assume that $s_1 = \sup I_1$, which means (recall that $\sin \theta(s_1) = -1$) that s_1 is the West end of I_1 . Hence we have either $I_1 = \{s_1\}$, or $I_1 = [s_0, s_1]$ with $s_0 < s_1$. As in equation (2.11), we introduce a point σ_{in} which will be the initial point for ψ^{lift} and ψ^{Σ} , and which we will define more precisely in the next chapter (see (3.36) and Figure 2.4).

In order to keep the construction as simple as possible, we use that the cancellation of $\cos \theta$ near $\inf I_1$ is strong (assumption (H4) in section 1.2), which implies in particular that the boundary of Ω^+ near the junction point of arc-length $\inf I_1$ has \mathcal{C}^4 regularity.

The role of ψ^{lift} is to lift both the jumps of ψ_t^0 and $\partial_y \psi_t^0$ across Σ , and the traces of ψ_t^0 and $\partial_n \psi_t^0$ on the portion of $\partial\Omega$ between σ_{in} and s_1 , so that $\psi_t^0 + \psi^{\text{lift}}$ belongs to $H^2(\Omega)$.

We have indeed the following

PROPOSITION 2.4.2. *Let ψ be a function on Ω such that*

$$\psi|_{\Omega^+} \in H^2(\Omega^+), \quad \psi|_{\Omega^-} \in H^2(\Omega^-), \quad \text{and} \quad [\psi]_{\Sigma} = [\partial_n \psi]_{\Sigma} = 0.$$

Then ψ belongs to $H^2(\Omega)$.

PROOF. The squared L^2 norm being additive, the only point to be checked is that

$$(2.16) \quad \Delta\psi = \Delta(\psi\chi^+) + \Delta(\psi\chi^-)$$

in the sense of distributions, which is obtained by a simple duality argument.

For any $\phi \in C_c^\infty(\Omega)$, because of the jump conditions on $\partial_y\psi$ and ψ , we have

$$\begin{aligned} \langle \phi, \Delta\psi \rangle_{\mathcal{D}, \mathcal{D}'} &= \int_{\Omega} \Delta\phi(\psi\chi^+ + \psi\chi^-) \\ &= - \int_{\Sigma} \partial_y\phi\psi|_{\Sigma^+} - \int_{\Omega^+} \nabla\phi \cdot \nabla(\psi\chi^+) \\ &\quad + \int_{\Sigma} \partial_y\phi\psi|_{\Sigma^-} - \int_{\Omega^-} \nabla\phi \cdot \nabla(\psi\chi^-) \\ &= \int_{\Omega^+} \phi\Delta(\psi\chi^+) + \int_{\Sigma} \phi\partial_y\psi|_{\Sigma^+} - \int_{\Omega^-} \phi\Delta(\psi\chi^-) - \int_{\Sigma} \phi\partial_y\psi|_{\Sigma^-} \\ &= \int_{\Omega} \phi(\Delta(\psi\chi^+) + \Delta(\psi\chi^-)) \end{aligned}$$

from which we deduce that $\Delta\psi \in L^2(\Omega)$, and $\Delta\psi = \Delta(\psi\chi^+) + \Delta(\psi\chi^-)$. \square

We therefore seek ψ^{lift} in the form

$$(2.17) \quad \psi^{\text{lift}}(x, y) = \chi^+\chi\left(\frac{y-y_1}{E^{1/4}}\right)[a(x) + b(x)(y-y_1)],$$

where the truncation $\chi \in \mathcal{C}_0^\infty(\mathbf{R})$ is such that

$$(2.18) \quad \text{Supp } \chi \subset [-1, 1], \quad \chi(y) = 1 \text{ for } |y| \leq 1/2,$$

and satisfying the conditions

$$\begin{aligned} (i) \quad &\psi|_{\partial\Omega^+ \cap \partial\Omega}^{\text{lift}} = -(1-\varphi)\psi_{t|\partial\Omega}^0, \text{ for } s \in (\sigma_{in}, s_1), \\ &\partial_n\psi|_{\partial\Omega^+ \cap \partial\Omega}^{\text{lift}} = -(1-\varphi)\partial_n\psi_{t|\partial\Omega}^0 \text{ for } s \in (\sigma_{in}, s_1), \\ (ii) \quad &\psi|_{\Sigma}^{\text{lift}} = -[\psi_t^0]_{\Sigma}, \quad \partial_y\psi|_{\Sigma}^{\text{lift}} = -[\partial_y\psi_t^0]_{\Sigma}. \end{aligned}$$

As in section 2.3, φ is a truncation function in s , whose role is to ensure a smooth transition between the East/West boundary layer on $[\sigma_{in}, \inf I_1]$ and the discontinuity boundary layer.

A straightforward computation then provides the following formulas for the coefficients a and b

$$(2.19) \quad \begin{aligned} b(x) &= (1-\varphi(s)) \left[-\sin\theta(s)\partial_n\psi_{t|\partial\Omega}^0(s) + \cos\theta(s)\partial_s\psi_{t|\partial\Omega}^0(s) \right] \\ &\quad - \cos\theta(s)\varphi'\psi_{t|\partial\Omega}^0(s), \\ a(x) &= -(1-\varphi(s))\psi_{t|\partial\Omega}^0(s) - b(x(s))(y(s) - y_1), \end{aligned}$$

for $x_1 < x = x(s) < x(\sigma_{in})$, and by

$$(2.20) \quad \begin{aligned} a(x) &= -[\psi_t^0]_{\Sigma}, \\ b(x) &= -[\partial_y\psi_t^0]_{\Sigma} \end{aligned} \quad \text{for } x \leq x_1.$$

For $x \geq x(\sigma_{in})$, we merely take $a(x) = b(x) = 0$.

Of course, however, ψ^{lift} is not a solution of (1.1), even in the approximate sense. More precisely, we have the following proposition :

PROPOSITION 2.4.3. *With the previous definition and notations,*

$$(\partial_x - E\Delta^2)(\psi_t^0 + \psi^{\text{lift}}) = \tau + \delta\tau^{\text{lift}} \left(x, \frac{y - y_1}{E^{1/4}} \right) + r_{\text{lift}}^1 + r_{\text{lift}}^2,$$

where

$$\|r_{\text{lift}}^1\|_{L^2(\Omega)} = o(E^{1/8}), \quad \|r_{\text{lift}}^2\|_{H^{-2}(\Omega)} = o(E^{5/8}),$$

and

$$\begin{aligned} \delta\tau^{\text{lift}}(x, Y) &= \chi^+(x, y_1 + E^{1/4}Y)\chi(Y) \left[a'(x) + E^{1/4}b'(x)Y \right] \\ &\quad - \chi^+(x, y_1 + E^{1/4}Y)\chi^{(4)}(Y) \left[a(x) + E^{1/4}b(x)Y \right] \\ &\quad - 4\chi^+(x, y_1 + E^{1/4}Y)E^{1/4}\chi^{(3)}(Y)b(x). \end{aligned}$$

DEFINITION 2.4.4. *In the rest of the paper, we set*

$$(2.21) \quad \psi^{\text{int}} := \psi_t^0 + \psi^{\text{lift}};$$

which is consistent with the statement of Theorem 2, in which ψ^{int} was described as an H^2 regularization of ψ^0 .

SKETCH OF PROOF. By Proposition 2.4.2, $\psi_t^0 + \psi^{\text{lift}}$ belongs to $H^2(\Omega)$ and, using (2.16),

$$\begin{aligned} \partial_x(\psi_t^0 + \psi^{\text{lift}}) &= \tau\chi_E + \chi^+\chi \left(\frac{y - y_1}{E^{1/4}} \right) (a'(x) + b'(x)(y - y_1)), \\ \Delta(\psi_t^0 + \psi^{\text{lift}}) &= \chi^+(\Delta(\psi_t^0 + \psi^{\text{lift}})) + \chi^-(\Delta\psi_t^0). \end{aligned}$$

Moreover, as we have the following trace identities on $\partial\Omega^+$

$$(\Delta\psi^{\text{lift}})|_{\Sigma} = 0, \quad (\partial_y\Delta\psi^{\text{lift}})|_{\Sigma} = 0,$$

using again (2.16), we get

$$\begin{aligned} E\Delta(\chi^+\Delta\psi^{\text{lift}}) &= E\chi^+\chi \left(\frac{y - y_1}{E^{1/4}} \right) (a^{(4)}(x) + b^{(4)}(x)(y - y_1)) \\ &\quad + 4E^{3/4}\chi^+\chi' \left(\frac{y - y_1}{E^{1/4}} \right) b''(x) \\ (2.22) \quad &\quad + 2E^{1/2}\chi^+\chi'' \left(\frac{y - y_1}{E^{1/4}} \right) (a''(x) + b''(x)(y - y_1)) \\ &\quad + \chi^+\chi^{(4)} \left(\frac{y - y_1}{E^{1/4}} \right) [a(x) + b(x)(y - y_1)] \\ &\quad + 4E^{1/4}\chi^+\chi^{(3)} \left(x, \frac{y - y_1}{E^{1/4}} \right) b(x). \end{aligned}$$

We thus define

$$\begin{aligned} r_{\text{lift}}^1 &= 4E^{3/4}\chi^+\chi' \left(\frac{y - y_1}{E^{1/4}} \right) b''(x) + 2E^{1/2}\chi^+\chi'' \left(\frac{y - y_1}{E^{1/4}} \right) (a''(x) + b''(x)(y - y_1)) \\ r_{\text{lift}}^2 &= \tau(\chi_E - 1) + E\chi^+\chi \left(\frac{y - y_1}{E^{1/4}} \right) (a^{(4)}(x) + b^{(4)}(x)(y - y_1)) \\ &\quad - E\Delta^2(\chi^+\psi_t^0) - E\Delta^2(\chi^-\psi_t^0). \end{aligned}$$

so that

$$\begin{aligned} & (\partial_x - E\Delta^2)(\psi_t^0 + \psi^{\text{lift}}) \\ &= \tau + \delta\tau^{\text{lift}} \left(x, \frac{y - y_1}{E^{1/4}} \right) + r_{\text{lift}}^1 + r_{\text{lift}}^2, \end{aligned}$$

- The remainder terms will be dealt with in the last chapter, using some technical estimates on a , b and their derivatives (see Lemmas 3.1.1 and C.1 and paragraph 4.1.2).
- The source term $\delta\tau^{\text{lift}}$ is not an admissible error term in the sense of Definition 1.1.2. Therefore it has to be corrected by yet another boundary layer type term.

□

2.4.2. The interior singular layer. The lifting term ψ^{lift} has introduced a remainder $\delta\tau^{\text{lift}}$ which we now treat as a source term. By analogy with North and South boundary layers, we therefore define ψ^Σ as the solution of the following equation

$$(2.23) \quad \begin{aligned} \partial_x \psi^\Sigma - \partial_Y^4 \psi^\Sigma &= -\delta\tau^{\text{lift}}, \quad x < x(\sigma_{in}), \quad Y > Y_-(x) \\ \psi_{|x=x(\sigma_{in})}^\Sigma &= 0, \quad \psi_{|Y=Y_-(x)}^\Sigma = \partial_n \psi_{|Y=Y_-(x)}^\Sigma = 0, \end{aligned}$$

where the function Y_- parametrizes the lower boundary of the interior singular layer (see Figure 2.4)

- which has typical size δ_y , where δ_y is any parameter such that $\delta_y \gg E^{1/4}$,
- which is of course included in Ω .

Note that we use here cartesian coordinates x and $Y = (y - y_1)/E^{1/4}$, since the discontinuity line Σ is essentially horizontal (which is not the case for North and South boundary layers which are extended on macroscopic parts of the adjacent East and West boundaries). The precise definitions of Y_- and σ_{in} will be given in the next chapter (see page 62).

2.5. The case of islands

When Ω is not simply connected, the solution of (1.14)-(1.15) is given by (see Lemma 1.2.1)

$$\psi = \psi_1 + \sum_{i=2}^K c_i \psi_i,$$

where ψ_i is the solution of a Munk-like equation, and the constants c_i satisfy a linear system whose coefficients depend on the ψ_i .

Therefore, as explained in the introduction (paragraph 1.2.3) the main issue is to compute the asymptotic behaviour of every function ψ_i . Following the steps described in the preceding sections (and which will be developed in the next chapter), we are able to construct a function ψ_i^{app} for every $1 \leq i \leq K$. Of course, there are a few minor changes, due to the non homogeneous boundary condition, but we leave those to the reader, as they do not bear any additional difficulty.

Let us admit for the time being that Theorem 2 holds for the functions ψ_i , namely that we are able to construct ψ_i^{app} such that

$$(2.24) \quad \|\psi_i - \psi_i^{\text{app}}\|_{L^2} = o(E^{1/8}), \quad \|\psi_i - \psi_i^{\text{app}}\|_{H^2} = o(E^{-3/8}).$$

Hence, in order to prove Theorem 2 for the function ψ , it suffices to define

$$\psi_{app} = \psi_1^{app} + \sum_{i=2}^K c_i^{app} \psi_i^{app}$$

where the constants c_i^{app} are such that

$$(2.25) \quad |c_i - c_i^{app}| = o(E^{1/8}), \quad c_i^{app} = O(1).$$

According to (1.21), (1.22), we have $\|\psi_i^{app}\|_{L^2} = O(1)$, $\|\psi_i^{app}\|_{H^2} = O(E^{-1/2})$, so that (2.24) and (2.25) imply

$$\|\psi - \psi_{app}\|_{L^2} = o(E^{1/8}), \quad \|\psi - \psi_{app}\|_{H^2} = o(E^{-3/8}).$$

We now turn to the proof of (2.25), which relies on the following Lemma:

LEMMA 2.5.1. *For all $i \in \{2, \dots, K\}$, let $g_i \in C^\infty(\bar{\Omega})$ such that $g_i \equiv 1$ in a neighbourhood of C_i , and $\text{Supp } g_i \cap C_j = \emptyset$ for $i \neq j$.*

Let us define the matrix

$$M^{app} := \left(- \int_{\Omega} \psi_j^{app} \partial_x g_i \right)_{2 \leq i, j \leq K}$$

and the vector

$$D^{app} := \left(\int_{\Omega} (\tau g_j + \psi_1^{app} \partial_x g_j) + \int_{C_j} \mathcal{T}^\perp \cdot n \right)_{2 \leq j \leq K}.$$

Then the following facts hold (recall that M^E, D^E are defined in Lemma 1.2.1):

- $|M^E - M^{app}| = o(E^{1/8})$, $|D^E - D^{app}| = o(E^{1/8})$;
- M^E, M^{app} are invertible matrices, with $(M^E)^{-1} = O(1)$, $(M^{app})^{-1} = O(1)$;
- $D^{app} = O(1)$.

Before addressing the proof of the lemma, let us explain why (2.25) follows: by definition,

$$M^E \begin{pmatrix} c_2 \\ \vdots \\ c_K \end{pmatrix} = M^E c = D^E,$$

so that

$$M^E(c - c^{app}) = (M^{app} - M^E)c^{app} + D^E - D^{app},$$

where c^{app} is defined by

$$M^{app} c^{app} = D^{app}.$$

Using the last two items of Lemma 2.5.1, we infer that $c^{app} = O(1)$. Therefore

$$M^E(c - c^{app}) = o(E^{1/8}).$$

Since $(M^E)^{-1} = O(1)$, (2.25) holds.

PROOF OF LEMMA 2.5.1. • The first step is to use the equation satisfied by every ψ_i in order to express the coefficients of M^E , D^E as integrals on Ω . More precisely, we have, since $\nabla g_i = 0$ on a neighbourhood of $\partial\Omega$,

$$\begin{aligned} E \int_{C_i} \partial_n \Delta \psi_j &= E \int_{\partial\Omega} g_i \partial_n \Delta \psi_j \\ &= E \int_{\Omega} g_i \Delta^2 \psi_j + E \int_{\Omega} \nabla \Delta \psi_j \cdot \nabla g_i \\ &= \int_{\Omega} g_i \partial_x \psi_j - E \int_{\Omega} \psi_j \Delta^2 g_i \\ &= - \int_{\Omega} \psi_j \partial_x g_i + \delta_{ij} \int_{C_i} e_x \cdot n - E \int_{\Omega} \psi_j \Delta^2 g_i. \end{aligned}$$

Since C_i is a closed curve, we obtain eventually

$$E \int_{C_i} \partial_n \Delta \psi_j = - \int_{\Omega} \psi_j \partial_x g_i - E \int_{\Omega} \psi_j \Delta^2 g_i.$$

In a similar way,

$$-E \int_{C_j} \partial_n \Delta \psi_1 = \int_{\Omega} (\tau g_j + \psi_1 \partial_x g_j) + E \int_{\Omega} \psi_1 \Delta^2 g_i.$$

The energy estimate (1.7) shows that $\|\psi_j\|_{L^2} = O(1)$ for $1 \leq j \leq K$, so that

$$(2.26) \quad \begin{aligned} M^E &= \left(- \int_{\Omega} \psi_j \partial_x g_i \right)_{2 \leq i, j \leq K} + O(E), \\ D^E &= \left(\int_{\Omega} (\tau g_j + \psi_1 \partial_x g_j) + \int_{C_j} \mathcal{T}^\perp \cdot n \right)_{2 \leq j \leq K} + O(E). \end{aligned}$$

Replacing every function ψ_j by ψ_j^{app} and using the error estimate (2.24), we obtain the first point of the lemma.

• To prove the second point of the Lemma, we will use identity (A.2) in the Appendix, from which we will deduce the following coercivity inequality: there exists a constant $C > 0$, independent of E , such that

$$(2.27) \quad \forall a = (a_2, \dots, a_K) \in \mathbf{R}^{K-1}, \quad \sum_{i,j} M_{ij}^E a_i a_j \leq -C|a|^2.$$

Of course this entails easily that $(M^E)^{-1}$ is bounded. Using the first item of the Lemma, we infer that M^{app} satisfies the same type of coercivity property, and therefore $(M^{app})^{-1}$ is bounded as well.

Let us now prove (2.27). Let $a \in \mathbf{R}^{K-1}$ be a fixed vector. Let $V \subset \Omega$ be an open set such that V is a neighbourhood of Γ_W , namely

- $\partial V \cap \partial\Omega \subset \Gamma_W$ (notice that this implies that \bar{V} does not intersect $\Gamma_{N,S}$, nor Γ_E);
- $\bar{V} \cap \Sigma = \emptyset$ (V does not meet the discontinuity boundary layer);
- $V \cap B_i \neq \emptyset$, and $\partial V \cap \bar{B}_i \cap \Gamma_W \neq \emptyset$ for all $2 \leq i \leq K$ (V intersects the West boundary of every set B_i).

Then we may write, with $\psi_a^{app} = \sum a_i \psi_i^{app}$,

$$\begin{aligned} E \int_{\Omega} |\Delta \psi_a|^2 &= E \int_{\Omega} |\Delta \psi_a^{app}|^2 + o(E^{1/8} |a|^2) \\ &\geq E \int_V |\Delta \psi_a^{app}|^2 + o(E^{1/8} |a|^2) \\ &\geq E \sum_{i=2}^K \int_{V \cap B_i} |\Delta \psi_a^{app}|^2 + o(E^{1/8} |a|^2). \end{aligned}$$

Notice that $E \int_{\Omega} |\Delta \psi_a|^2 = O(1)$ (thanks to energy estimates), while ψ_a^{app} is proportional to a_i on $V \cap B_i$. These are the two main ingredients for the proof of the coercivity inequality. More precisely, on every set $V \cap B_i$, by definition¹,

$$\psi_a^{app} = a_i \left(1 + \chi_0(z) \sum_{\pm} \frac{e^{\pm i\pi/6}}{\sqrt{3}} \exp(-\lambda_W e^{\pm i\pi/3} z) \right).$$

Consequently, at the main order,

$$\begin{aligned} E \int_{V \cap B_i} |\Delta \psi_a^{app}|^2 &\simeq E a_i^2 \int_{V \cap B_i} \lambda_W^2 \left| \chi_0(z) \sum_{\pm} \frac{e^{\pm 5i\pi/6}}{\sqrt{3}} \exp(-\lambda_W e^{\pm i\pi/3} z) \right|^2 ds dz \\ &\geq C E a_i^2 \int_{V \cap \bar{B}_i \cap \Gamma_W} \lambda_W^3(s) ds \\ &\geq C a_i^2. \end{aligned}$$

Gathering all the terms, we obtain (2.27).

• The fact that $D^{app} = O(1)$ is obvious and follows from (1.7), which yields $\|\psi_1\|_{L^2} = O(1)$. \square

For the sake of completeness, we also give the system satisfied by $\bar{c} = \lim_{E \rightarrow 0} c = \lim_{E \rightarrow 0} c^{app}$:

COROLLARY 1. *Let \bar{c} be the solution of*

$$\bar{M} \bar{c} = \bar{D},$$

where

$$\begin{aligned} \bar{M} &:= \left(- \int_{\partial B_j \cap C_i} e_x \cdot n \right)_{2 \leq i, j \leq K}, \\ \bar{D} &:= \left(\int_{\Omega} (\tau g_j + \psi^0 \partial_x g_j) + \int_{C_j} \mathcal{T}^\perp \cdot n \right)_{2 \leq j \leq K}. \end{aligned}$$

Then $\bar{c} = \lim_{E \rightarrow 0} c_i$.

PROOF. We pass to the limit in the formulas (2.26). It is proved in [2], and it can be easily deduced from our construction, that

$$\psi_j \rightharpoonup \mathbf{1}_{B_j} \quad \text{in } L^2(\Omega) \text{ for } 2 \leq j \leq K,$$

¹Notice that on B_i , the main order of ψ_i^{app} is $\mathbf{1}_{B_i}$, whose normal derivative on $\partial\Omega$ is zero (away from the end point of Σ). As a consequence, several corrector terms, whose construction is necessary in the general case, are identically zero here.

and therefore

$$-\int_{\Omega} \psi_j \partial_x g_i \rightarrow -\int_{B_j} \partial_x g_i = -\int_{\partial B_j \cap C_i} e_x \cdot n.$$

Hence the identities (2.26) imply easily that

$$M^E \rightarrow \bar{M}, \quad D^E \rightarrow \bar{D}.$$

Passing to the limit in inequality (2.27), we infer that \bar{M} is also coercive, and therefore invertible. Thus \bar{c} is well-defined. The corollary follows. \square

2.6. North and South periodic boundary layers

When the domain Ω is $\Omega := \mathbb{T} \times (y_-, y_+)$, boundary layers occur in the vicinity of y_-, y_+ . Because of the periodicity, they can be easily computed by using Fourier series in the x variable. Therefore this kind of construction is covered by article [9]; we include the computations here for the sake of completeness.

By definition of the Sverdrup term in (1.26), the boundary conditions to be lifted on the north and south boundary have zero average. Let us focus for instance on the south boundary; setting $Z = (y - y_-)/E^{1/4}$, the equation satisfied by the boundary layer term is

$$\begin{aligned} \partial_x \psi_S^{BL} - \partial_Z^4 \psi_S^{BL} &= 0 \quad \text{in } \mathbb{T} \times \mathbf{R}_+, \\ \psi_S^{BL}|_{Z=0} &= f_0, \quad \partial_x \psi_S^{BL}|_{Z=0} = f_1, \end{aligned}$$

with $\langle f_0 \rangle = \langle f_1 \rangle = 0$. Looking for ψ_S^{BL} in the form

$$\psi_S^{BL}(x, Z) = \sum_{k \in 2\pi\mathbf{Z} \setminus \{0\}} \hat{\psi}_k(Z) \exp(ikx),$$

we infer that $\hat{\psi}_k$ is given by

$$\hat{\psi}_k(Z) = \sum_{\pm} A_k^{\pm} \exp(-|k|^{1/4} \lambda_k^{\pm} Z),$$

where

$$\begin{aligned} \lambda_k^+ &= e^{i\pi/8}, \quad \lambda_k^- = e^{-3i\pi/8} \quad \text{if } k > 0, \\ \lambda_k^+ &= e^{3i\pi/8}, \quad \lambda_k^- = e^{-i\pi/8} \quad \text{if } k < 0, \end{aligned}$$

and

$$\begin{aligned} A_k^+ + A_k^- &= \hat{f}_0(k), \\ \lambda_k^+ A_k^+ + \lambda_k^- A_k^- &= -|k|^{-1/4} \hat{f}_1(k). \end{aligned}$$

Hence

$$|A_k^{\pm}| \leq C(|\hat{f}_0(k)| + |k|^{-1/4} |\hat{f}_1(k)|).$$

It follows easily that

$$\begin{aligned} (2.28) \quad \|\psi_S^{BL}\|_{L^2(\mathbb{T}, H^2(\mathbf{R}_+))} &\leq C \left(\sum_{k \neq 0} |k|^{3/4} (|\hat{f}_0(k)| + |k|^{-1/4} |\hat{f}_1(k)|)^2 \right)^{1/2}, \\ \|\partial_x^l \psi_S^{BL}\|_{L^2(\mathbb{T}, H^2(\mathbf{R}_+))} &\leq C \left(\sum_{k \neq 0} |k|^{2l+3/4} (|\hat{f}_0(k)| + |k|^{-1/4} |\hat{f}_1(k)|)^2 \right)^{1/2}. \end{aligned}$$

CHAPTER 3

Construction of the approximate solution

The approximate solutions to the Munk equation (1.1) are obtained by gathering together the different elementary pieces described in the previous chapter, namely

- the interior term which is essentially the solution to the transport equation (1.4), regularized in the vicinity of “East corners” and of the interface Σ ;
- East and West boundary layer terms, which lift locally the boundary conditions but become singular in the vicinity of the points s_i ;
- North and South boundary layer terms, which lift the boundary conditions on the horizontal parts of the boundary but in a non local way;
- singular layer terms, which make up for the source terms introduced by the regularization of the discontinuity at Σ .

We have now to understand the interplay between those different elementary pieces.

Of course, the equation (1.1) being linear, the errors induced by all these terms are simply added, so that the control on the remainders in the approximate equation will be rather simple to obtain. The point to be stressed is that we need each elementary term to be smooth enough (namely H^2), with suitable controls on the corresponding derivatives.

More precisely, we will consider here only one basic problem among (1.16)(1.17), say the case (1.16) with homogeneous boundary conditions $\psi|_{\Omega} = \partial_n \psi|_{\Omega} = 0$ and forcing τ . We will further assume, without loss of generality, that we have the simple geometry described in the previous chapter

$$\Omega = \Omega^+ \cup \Omega^- \cup \Sigma,$$

where

- Ω^{\pm} are non empty, open and convex in x ;
- $\Omega^+ = \{(x, y) \in \Omega, y > y_1 \text{ or } x \geq x_1\}$;
- $\Omega^- = \{(x, y) \in \Omega, y < y_1 \text{ and } x < x_1\}$;
- $\Sigma = \{(x, y_1) \in \Omega, x < x_1\}$

and $(x_1, y_1) = (x(s_1), y(s_1))$ for some s_1 such that $\cos \theta(s_1) = 0$ and $\sin \theta(s_1) = -1$. We emphasize that the only simplification here regards notations, and that more complex domains satisfying assumptions (H1) – (H4) are handled exactly in the same way.

Periodic and rectangle cases will be dealt with separately in Section 3.6.

We will actually focus on the following technical difficulties

- the precise regularization process for the interior term in Ω^{\pm} (section 3.1);

- the fact that the East boundary layer does not lift simultaneously both boundary conditions (correcting the normal derivative introduces indeed an arrow on the trace). This implies that one has to introduce an additional corrector defined on a macroscopic domain in the vicinity of the East boundary but far from corners (x_i, y_i) where it would be singular (section 3.2);
- the connection with North/South boundary layers (even if the corresponding horizontal part of the domain is reduced to one point (x_i, y_i)): we indeed expect these boundary layers to lift the boundary conditions both on the horizontal parts and on the East/West boundaries close to the corners (x_i, y_i) (section 3.3). Note that, since these North/South boundary layers are defined by a non local equation, we need to check that they do not carry any more energy beyond some point, so that they can be truncated and considered as local contributions;
- the truncation of the surface layer near its West end, in order that its trace on the West boundary is not too singular (see 3.4);
- the precise definition of the West boundary layers, which have to lift the traces of all previous contributions (see 3.5);

We will also derive estimates on the terms constructed at every step.

Note that, **for boundary layers, the construction must be performed in a precise order, starting with East boundaries, then defining North/South (and surface) boundary layers, and finally lifting the West boundary conditions.** This dissymmetry between East and West boundaries is similar to what happens at the macroscopic level for the interior term.

3.1. The interior term

As in the previous chapter, we define

$$(3.1) \quad \psi^0(x, y) = - \int_x^{x_E(y)} \tau(x', y) dx',$$

where $x_E(y)$ is the abscissa (or longitude) of some point in Γ_E with ordinate (or latitude) y . We have seen that ψ^0 does not belong to $H^2(\Omega)$ in general: indeed,

- The function x_E has singularities near the points $y_i = y(s_i), i \in I_+$;
- Since x_E takes different values on Ω^+ and Ω^- , the function ψ_t^0 and its y derivative are discontinuous across Σ .

Therefore, ψ^0 cannot be used as such in the definition of the approximate solution ψ^{app} of (1.1).

Let us first consider separately the domains Ω^\pm . We remove the singularities by truncating the function τ near the end points of Γ_E (with abscissa $s_i, i \in I_+$).

The size of the truncation depends on the rate of cancellation of $\cos \theta$ near s_i . Since the rate of cancellation may be different on the left and on the right of s_i , we seek for truncation functions with different behaviours on the left and on the right of every cancellation point. More precisely, let

$$(3.2) \quad \chi_\nu(x, y) := \prod_{i \in I_+} \chi_i(x - x_i, y - y_i)$$

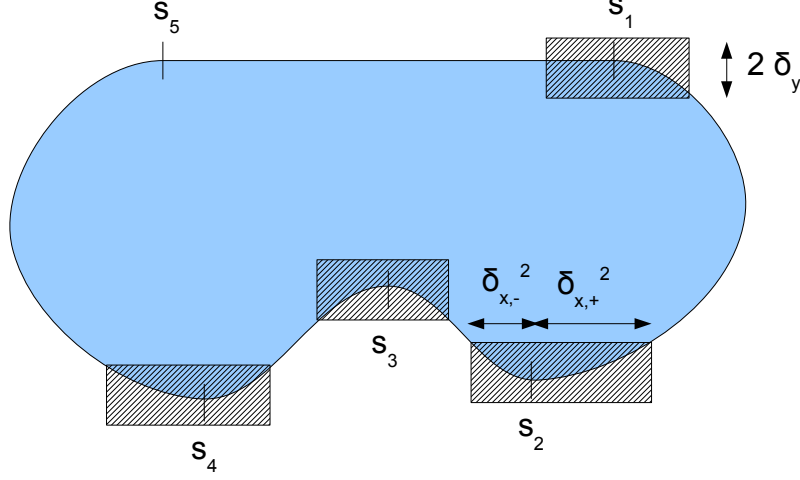


FIGURE 3.1. The function τ is truncated in the hatched zones. In the picture, $I_+ = \{1, 2, 3, 4\}$.

where (x_i, y_i) are the coordinates of the point of $\partial\Omega$ with arc length s_i . Each function χ_i takes the following form:

$$\chi_i(x, y) := \begin{cases} \bar{\chi}\left(\frac{x}{\delta_{x,+}^i}, \frac{y}{\delta_y}\right) & \text{if } x > 0, \\ \bar{\chi}\left(\frac{x}{\delta_{x,-}^i}, \frac{y}{\delta_y}\right) & \text{if } x \leq 0. \end{cases}$$

- The function $\bar{\chi}$ is defined by

$$1 - \bar{\chi}(x, y) := (1 - \bar{\chi}_1(x))(1 - \bar{\chi}_2(y)) \quad (x, y) \in \mathbf{R}^2,$$

with $\bar{\chi}_1, \bar{\chi}_2 \in \mathcal{C}^\infty(\mathbf{R})$, and

$$\begin{aligned} \bar{\chi}_1(x) &= 1 \text{ if } |x| \geq 1 + \frac{1}{|\ln \delta_y|}, \quad \bar{\chi}_1(x) = 0 \text{ if } |x| \leq 1, \\ \bar{\chi}_2(y) &= 1 \text{ if } |y| \geq 1, \quad \bar{\chi}_2(y) = 0 \text{ if } |y| \leq 1/2. \end{aligned}$$

It can be easily checked that all the x derivatives of $\bar{\chi}$ vanish for $x = 0$, so that χ belongs to $\mathcal{C}_0^\infty(\mathbf{R}^2)$. Furthermore, it satisfies obviously

$$\begin{aligned} 0 &\leq \bar{\chi} \leq 1, \\ \bar{\chi}(x, y) &= 0 \text{ if } |y| \leq \frac{1}{2} \text{ and } |x| \leq 1, \\ \bar{\chi}(x, y) &= 1 \text{ if } |y| \geq 1 \text{ or } |x| \geq 1 + \frac{1}{|\ln \delta_y|}. \end{aligned}$$

- Let us now give the definition of the truncation rates. We take

$$(3.3) \quad \delta_y := \nu^{1/4} |\ln \nu|^{1/5} (\ln |\ln \nu|)^{-\beta},$$

for some arbitrary exponent $\beta > 0$. Note that a rate of this type is mandatory in the case of an exponential cancellation (assumption (H2ii)). In the case of an algebraic cancellation (assumption (H2i)), there is more latitude, but the above choice still works.

The definition of δ_x is a little more involved. We define δ_x so that if we start from the point $(x_E(y_i), y_i)$ (i.e. from the end point of Γ_E) and perform a shift of size δ_x in the x direction, $-\sin \theta(s_i)\delta_y$ in the y direction, the end point still belongs to Γ_E . This leads to the following definitions:

- If $x_E(y)$ is defined on both sides of y_i , i.e. if s_i belongs to the interior of $\bar{\Gamma}_E$, then we set

$$(3.4) \quad \begin{aligned} \delta_{x,+}^i &:= |x_E(y_i - \sin \theta(s_i)\delta_y) - x_i|, \\ \delta_{x,-}^i &:= |x_E(y_i + \sin \theta(s_i)\delta_y) - x_i|. \end{aligned}$$

- If $x_E(y)$ is defined only on one side of y_i , say for $y > y_i$, then we take

$$(3.5) \quad \delta_{x,+}^i = \delta_{x,-}^i = |x_E(y_i + \delta_y) - x_i|.$$

In particular, if $(x_E(y), y) \in \text{Supp } \chi_\nu$, then

$$\text{either } \inf_{i \in I_+} \frac{|y - y_i|}{\delta_y} \geq \frac{1}{2} \text{ or } \inf_{i \in I_+} \sup_{\pm} \frac{(x_E(y) - x_i)_\pm}{\delta_{x,\pm}^i} \geq 1.$$

According to (3.4)-(3.5), the above condition becomes

$$\text{either } \inf_{i \in I_+} \frac{|y - y_i|}{\delta_y} \geq \frac{1}{2} \text{ or } \inf_{i \in I_+} \sup_{\pm} \frac{(x_E(y) - x_i)_\pm}{|x_E(y_i \mp \sin \theta(s_i)\delta_y) - x_i|} \geq 1.$$

Since x_E is locally a monotonous function (near $y = y_i$), this amounts to

$$\inf_{i \in I_+} |y - y_i| \geq \delta_y/2.$$

We infer that

$$(3.6) \quad (x_E(y), y) \in \text{Supp } \chi_\nu \Rightarrow \forall i \in I_+, |y - y_i| \geq \frac{\delta_y}{2}.$$

We then set

$$(3.7) \quad \psi_t^0(x, y) := - \int_x^{x_E(y)} \chi_\nu(x', y) \tau(x', y) dx'$$

With this definition, $\psi_t^0 \in H^2(\Omega^\pm)$, but $\psi_t^0 \notin H^2(\Omega)$: indeed ψ_t^0 and $\partial_y \psi_t^0$ are still discontinuous across Σ . The term ψ^{lift} constructed in paragraph 3.4.1 of the present chapter will lift this discontinuity.

Moreover, by definition of ψ_t^0 , we have

$$\partial_x \psi_t^0 - \nu \Delta^2 \psi_t^0 = \tau + \delta\tau \quad \text{on } \Omega^+ \cup \Omega^-$$

with

$$(3.8) \quad \delta\tau = \tau(\chi_\nu - 1) - \nu \Delta^2 \psi_t^0.$$

PROPOSITION 3.1.1. *Assume that Ω satisfies assumptions (H1), (H2). Let $\beta > 0$ be arbitrary, and let δ_y, δ_x as in (3.3), (3.4)-(3.5). Then*

$$\begin{aligned}\|\tau(\chi_\nu - 1)\|_{H^{-2}(\Omega)} &= o(\nu^{5/8}), \\ \|\Delta\psi_t^0\|_{L^2(\Omega^\pm)} &= o(\nu^{-3/8}).\end{aligned}$$

This proposition will be proved in Chapter 4.

REMARK 3.1.2. *Notice that the rate of cancellation in y , namely δ_y , is the same for every function χ_i . It is far from obvious a priori that such a choice can lead to a suitable truncation. However, in order not to further burden the notations, we have chosen to anticipate on this result, which follows from the proof of Lemma 3.1.1 below.*

The x derivative of the function $\bar{\chi}$ is unbounded (it is of order $|\ln \delta_y|$ in L^∞). This choice is mandatory in the case of an exponential cancellation of $\cos \theta$ around s_i (assumption (H2ii)). If the cancellation around s_i is algebraic (assumption (H2i)), the function $\bar{\chi}$ can be any smooth function vanishing near zero and such that $1 - \bar{\chi}$ has compact support.

We conclude this paragraph by giving some estimates on the trace of ψ_t^0 and $\partial_n \psi_t^0$. These estimates will be useful when we construct the boundary layer terms lifting these boundary conditions. For the sake of readability, we introduce the following majorizing functions

$$\begin{aligned}\mathcal{M}(y) &= \prod_{i \in I_+} \frac{1}{\max(|(y - y_i) \ln |y - y_i|, \delta_y| |n \delta_y|)} \\ \mathcal{M}^*(y) &= \mathcal{M}(y) \prod_{i \in I_+} \mathbf{1}_{|y - y_i| \geq \delta_y/2} \dots\end{aligned}$$

LEMMA 3.1.3. *Let ψ_t^0 be defined by*

$$\psi_t^0(x, y) = - \int_x^{x_E(y)} (\tau \chi_\nu)(x', y) dx'.$$

• **Trace estimates on $\partial\Omega$:** *There exists a constant C , depending only on Ω and τ , such that for all $s \in \partial\Omega$,*

$$\begin{aligned}\|\psi_t^0\|_{L^\infty(\partial\Omega)} &\leq C, \\ |\partial_n \psi_t^0|_{\partial\Omega}(s) &\leq C \mathcal{M}(y)\end{aligned}$$

Moreover, for all $s \in \partial\Omega$ such that $(x(s), y(s)) \notin \Sigma$,

$$|\partial_s \psi_t^0|_{\partial\Omega}(s) + \delta_y |\partial_s \partial_n \psi_t^0|_{\partial\Omega}(s) \leq C |\cos \theta(s)| \mathcal{M}(y(s))$$

and

$$\begin{aligned}|\partial_s^2 \psi_t^0|_{\partial\Omega}(s) + \delta_y |\partial_s^2 \partial_n \psi_t^0|_{\partial\Omega}(s) &\leq C \left(\frac{1}{\delta_y} + \frac{|\cos \theta|^2}{\delta_y^2 |\ln \delta_y|} \right), \\ |\partial_s^3 \psi_t^0|_{\partial\Omega}(s) + \delta_y |\partial_s^3 \partial_n \psi_t^0|_{\partial\Omega}(s) &\leq \frac{C}{\delta_y} \left(1 + \frac{|\cos \theta|}{\delta_y} + \frac{|\cos \theta|^3}{\delta_y^2} \right).\end{aligned}$$

Eventually, on the East coast, we have the following more precise estimates:

$$\begin{aligned} |\partial_s^2 \partial_n \psi_t^0|_{\Gamma_E}(s) &\leq C \frac{|\theta'(s)|}{\delta_y |\cos \theta(s)|} \prod_{i \in I_+} \mathbf{1}_{|y(s)-y_i| \geq \delta_y/2}, \\ |\partial_s^3 \partial_n \psi_t^0|_{\Gamma_E}(s) &\leq C \frac{|\theta'(s)|^2}{\delta_y |\cos \theta(s)|^2} \prod_{i \in I_+} \mathbf{1}_{|y(s)-y_i| \geq \delta_y/2}. \end{aligned}$$

• **Jump estimate on Σ :** The jumps $[\psi_t^0]_\Sigma$, $[\partial_y \psi_t^0]_\Sigma$ are constant along Σ . Moreover, there exists a constant C such that

$$|[\psi_t^0]_\Sigma| \leq C, \quad |[\partial_y \psi_t^0]_\Sigma| \leq \frac{C}{\delta_y |\ln \delta_y|}.$$

PROOF. The L^∞ bound on $\psi_t^0|_{\partial\Omega}$ is obvious: we merely observe that

$$\|\psi_t^0|_{\partial\Omega}\|_{L^\infty(\partial\Omega)} \leq \|\tau\|_{L^\infty(\Omega)} \sup_{(x,y),(x',y) \in \Omega} |x - x'|.$$

As for the bound on $\partial_n \psi_t^0$, we have $\partial_n = \cos \theta \partial_x + \sin \theta \partial_y$, so that

$$(3.9) \quad \partial_n \psi_t^0|_{\partial\Omega}(s) = -\sin \theta(s) x'_E(y(s)) (\tau \chi_\nu)(x_E(y(s)), y(s))$$

$$(3.10) \quad -\sin \theta(s) \int_{x(s)}^{x_E(y(s))} \partial_y (\tau \chi_\nu)(x, y(s)) dx$$

$$(3.11) \quad + \cos \theta(s) (\tau \chi_\nu)(x(s), y(s)).$$

The formulas in Appendix B together with (3.6) imply that for all $i \in I_+$, in the vicinity of y_i

$$(3.12) \quad |x'_E(y)| \leq \frac{C}{|y - y_i| |\ln |y - y_i||^2}$$

so that

$$(x_E(y), y) \in \text{Supp } \chi_\nu \Rightarrow |x'_E(y)| \leq C \mathcal{M}(y) \leq \frac{C}{\delta_y |\ln \delta_y|^2}.$$

Recalling eventually that for all $c > 1$, for ν sufficiently small,

$$\|\partial_y \chi_\nu\|_\infty = O(\delta_y^{-1}),$$

$$\text{Supp } \partial_y \chi_\nu \subset \bigcup_{i \in I_+} [x_i - c\delta_{x,-}^i, x_i + c\delta_{x,+}^i] \times [y_i - \delta_y, y_i + \delta_y],$$

with $\delta_{x,\pm}^i = O(|\ln \delta_y|^{-1})$, this leads to the estimate on $\partial_n \psi_t^0|_{\partial\Omega}$. We have indeed

$$(3.13) \quad \begin{aligned} \left| \int_{x(s)}^{x_E(y(s))} \partial_y (\tau \chi_\nu)(x, y(s)) dx \right| &\leq C \sum_{i \in I_+} \|\partial_y \chi_\nu\|_\infty \mathbf{1}_{|y-y_i| \leq \delta_y} \int_{x(s)}^{x_E(y(s))} \mathbf{1}_{|x-x_i| \leq c\delta_x} dx \\ &+ \sum_{i \in I_+} \int_{x(s)}^{x_E(y(s))} \chi_\nu |\partial_y \tau|(x, y(s)) dx \\ &\leq C \left(1 + \frac{\delta_x}{\delta_y} \mathbf{1}_{|y-y_i| \leq \delta_y} \right). \end{aligned}$$

The estimate on $\partial_s \psi_t^0$ is obtained by differentiating the identity

$$\psi_{t|\partial\Omega}^0(s) = - \int_{x(s)}^{x_E(y(s))} (\tau\chi_\nu)(x, y(s)) dx$$

with respect to s . Using (2.2) and therefore the relation

$$\frac{d}{ds}(x(s), y(s)) = (\sin \theta(s), -\cos \theta(s)),$$

we infer that

$$\begin{aligned} \partial_s \psi_{t|\partial\Omega}^0(s) &= \cos \theta(s) x'_E(y(s)) (\tau\chi_\nu)(x_E(y(s)), y(s)) \\ &\quad + \sin \theta(s) (\tau\chi_\nu)(x(s), y(s)) \\ &\quad - \cos \theta(s) \int_{x(s)}^{x_E(y(s))} \partial_y (\tau\chi_\nu)(x, y(s)) dx. \end{aligned}$$

The estimates (3.12) and (3.13) above yield the desired inequality.

The other trace estimates are derived in a similar fashion. Notice that it is hard to derive a sharp global estimate, since the angle $\theta(s)$ for $s \in \Gamma_W$ is in general different from the angle $\theta(s')$, where $s' \in \Gamma_E$ is such that $y(s) = y(s')$. Therefore, there is no simplification for terms of the type $\cos^2(\theta(s)) x'_E(y(s))$ when $s \in \Gamma_W$. On the East coast, however, we can use the formulas in Appendix B, from which we deduce that

$$\cos^2(\theta(s)) x''_E(y(s)) = \frac{\theta'(s)}{\cos \theta(s)} \quad \forall s \in \Gamma_E.$$

Notice also that ψ_t^0 vanishes on Γ_E by definition, that $\partial_n \psi_{t|\Gamma_E}^0$ is supported in $\{|y(s) - y_i| \geq \delta_y/2\}$, and that all terms of the type

$$\int_{x(s)}^{x_E(y(s))} \partial_y^k (\tau\chi_\nu)(x, y(s)) dx$$

are zero for $s \in \Gamma_E$. The upper-bounds for $\partial_s^2 \partial_n \psi_{t|\Gamma_E}^0$, $\partial_s^3 \partial_n \psi_{t|\Gamma_E}^0$ follow.

As for the jump estimates, we recall that with the notations of Chapter 2, Section 2.4,

$$[\psi_t^0]_\Sigma = - \int_{x_1}^{x_E^\pm(y_1)} (\tau\chi_\nu)(x, y_1) dx,$$

and therefore the jump of ψ_t^0 is of order one. Notice that the assumptions of Chapter 2, Section 2.4 imply that $x_1 = x_E^-(y_1)$. The jump of the y derivative is given by

$$\begin{aligned} [\partial_y \psi_t^0]_\Sigma &= -x_E^{+'}(y_1) (\tau\chi_\nu)(x_E^+(y_1), y_1), \\ &\quad + x_E^{-'}(y_1) (\tau\chi_\nu)(x_E^-(y_1), y_1) \\ &\quad - \int_{x_1}^{x_E^+(y_1)} \partial_y (\tau\chi_\nu)(x, y_1) dx. \end{aligned}$$

The points $(x_E^\pm(y_1), y_1)$ do not depend on ν . Therefore, either $(x_E^+(y_1), y_1)$ is an East corner ($1 \in I_+$), and then $(\tau\chi_\nu)(x_E^+(y_1), y_1) = 0$ for all $\nu > 0$, or $(x_E^+(y_1), y_1)$ is not a corner, and then $x_E^{+'}(y_1)$ is bounded by a constant independent of ν . Hence

the first two terms of the right-hand side above are bounded as $\nu \rightarrow 0$. Using inequality (3.13), we infer that

$$\left| \int_{x_1}^{x_E^+(y_1)} \partial_y(\tau\chi_\nu)(x, y_1) dx \right| \leq \frac{C}{\delta_y |\ln \delta_y|}.$$

□

At this stage, we have built on each subdomain Ω^\pm an interior term ψ_t^0

- which vanishes on Γ_E ,
- which belongs to $H^2(\Omega^\pm)$ (typically with derivatives of order $O(\delta_y^{-1})$ in the vicinity of East corners),
- and which approximately satisfies the transport equation $\partial_x \psi_t^0 = \tau$.

Nevertheless ψ_t^0 does not fit the boundary conditions, and therefore boundary layer correctors must be defined. Following the direction of propagation of the equation at main order, we start with the East boundary layers.

3.2. Lifting the East boundary conditions

In this section, we focus on an interval $(s_i, s_{i+1}) \subset \Gamma_E$. We recall that the domain $[s_i^+, s_{i+1}^-] \subset [s_i, s_{i+1}]$ of validity of the East boundary layer is given by

$$s_i^+ := \sup \left\{ s \in \left(s_i, \frac{s_i + s_{i+1}}{2} \right), \nu^{1/3} |\theta'| |\cos \theta|^{-7/3} \geq 1 \right\},$$

$$s_{i+1}^- = \inf \left\{ s \in \left(\frac{s_i + s_{i+1}}{2}, s_{i+1} \right), \nu^{1/3} |\theta'| |\cos \theta|^{-7/3} \geq 1 \right\}.$$

Easy computations based on the explicit rate of cancellation of $\cos \theta$ near s_i provide then the following Lemma, which shows in particular that, because of the truncation, the trace of $\partial_n \psi_t^0$ is zero outside the domain of validity of the East boundary layer.

LEMMA 3.2.1. *Under assumption (H2i),*

$$s_i^+ - s_i \sim C_i \nu^{\frac{1}{4n+3}}.$$

Under assumption (H2ii),

$$s_i^+ - s_i = -\frac{4\alpha}{\ln \nu} \left(1 - 6 \frac{\ln |\ln \nu|}{\ln \nu} + O\left(\frac{1}{\ln \nu}\right) \right).$$

In particular,

$$|y(s_i^+) - y_i| \ll \delta_y$$

and

$$(3.14) \quad \lambda_E(s_i^+) \geq C\nu^{-1/4}.$$

PROOF. Lemma 3.2.1 is obtained by straightforward computations :

- If (H2i) is satisfied, then for $s > s_i$

$$\nu^{1/3} |\theta'(s)| |\cos \theta(s)|^{-7/3} \sim C' \nu^{1/3} |s - s_i|^{-\frac{4n+3}{3}}$$

so that there exists a positive constant C_i such that

$$(3.15) \quad s_i^+ - s_i \sim C_i \nu^{\frac{1}{4n+3}}.$$

- If (H2ii) is satisfied, then for $s > s_i$

$$\nu^{1/3} |\theta'(s)| |\cos \theta(s)|^{-7/3} \sim C' \nu^{1/3} \frac{\exp\left(\frac{4\alpha}{3(s-s_i)}\right)}{(s-s_i)^2}$$

so that

$$-2 \ln(s_i^+ - s_i) + \frac{4\alpha}{3(s_i^+ - s_i)} = -\ln C' - \frac{\ln \nu}{3} + o(1).$$

Computing an asymptotic development of $s_i^+ - s_i$ leads to

$$(3.16) \quad s_i^+ - s_i = -\frac{4\alpha}{\ln \nu} \left(1 - 6 \frac{\ln |\ln \nu|}{\ln \nu} + O\left(\frac{1}{\ln \nu}\right)\right).$$

Using the formulas in Appendix B, we infer in particular that in case (H2i),

$$y(s_i^+) - y_i \sim C \nu^{\frac{n+1}{4n+3}},$$

and $(n+1)/(4n+3) > 1/4$, so that $|y(s_i^+) - y_i| \ll \delta_y$. In a similar way, if (H2ii) is satisfied,

$$\begin{aligned} |y(s_i^+) - y_i| &\sim C |\ln \nu|^{-2} \exp\left(\alpha \frac{\ln \nu}{4\alpha} \left(1 + 6 \frac{\ln |\ln \nu|}{\ln \nu} + O(|\ln \nu|^{-1})\right)\right) \\ &\sim C |\ln \nu|^{-2} \exp\left(\frac{\ln \nu}{4} + \frac{3}{2} \ln |\ln \nu| + O(1)\right) \\ &\sim C |\ln \nu|^{-1/2} \nu^{1/4} \ll \delta_y. \end{aligned}$$

Therefore, in all cases, we have $|y(s_i^+) - y_i| \ll \delta_y$, so that the extremities of the domain of validity of the East boundary layer are in the truncation zone. \square

3.2.1. Traces of the East boundary layers.

By definition of ψ_t^0 , we have

$$(\psi_t^0)|_{\Gamma_E} = 0.$$

We thus define the East boundary layer to lift the trace of $\partial_n \psi_t^0$, i.e.

$$\psi_E(s, Z) = -\lambda_E^{-1}(s) \partial_n \psi_t^0|_{\Gamma_E}(s) \exp(-Z)$$

where

$$\lambda_E(s) = \left(\frac{\cos \theta}{\nu}\right)^{1/3}.$$

But then $\psi_t^0 + \psi_E^{BL}(s, \lambda_E z)$ no longer satisfies the zero trace condition on Γ_E . More precisely, since $\partial_n \psi_t^0|_{\Gamma_E}$ vanishes for $|y(s) - y_i| \leq \delta_y/2$, we have the following trace estimate:

LEMMA 3.2.2. *The trace of ψ_E satisfies the following bound*

$$(3.17) \quad |\psi_E|_{Z=0}(s) \leq C \frac{\nu^{1/3}}{|\cos \theta(s)|^{1/3}} \mathcal{M}^*(y(s)).$$

Moreover, its derivatives with respect to s satisfy

$$\begin{aligned} |\partial_s \psi_E|_{Z=0}(s) &\leq C\nu^{1/3} \delta_y^{-1} |\cos \theta|^{2/3} \mathcal{M}^*(y(s)), \\ |\partial_s^2 \psi_E|_{Z=0}(s) &\leq C\nu^{1/3} \delta_y^{-1} \frac{|\theta'(s)|}{|\cos \theta(s)|^{1/3}} \mathcal{M}^*(y(s)), \\ |\partial_s^3 \psi_E|_{Z=0}(s) &\leq C\nu^{1/3} \delta_y^{-1} \frac{|\theta'(s)|^2}{|\cos \theta(s)|^{4/3}} \mathcal{M}^*(y(s)). \end{aligned}$$

These estimates are a straightforward consequence of Lemma 3.1.3 and of the formula defining ψ_E .

Hence the remaining trace on Γ_E is non-zero, and must be corrected. Note in particular that the bounds above are too singular in the vicinity of $s_i, i \in I_+$ in order that we can lift the remaining trace by a simple macroscopic corrector. We lift the trace in two different ways depending on the value of s :

- If s is far from any point $s_i, i \in I_+$, we lift the trace thanks to a macroscopic corrector ψ_Σ^{corr} , which we construct in the next paragraph;
- If s is in a neighbourhood of size one of any s_i , we lift the trace of ψ_E thanks to the North and South boundary layer terms; we will explain the latter construction in the next section.

REMARK 3.2.3. *Since the East corrector is not captured by energy estimates, as explained in Remark 1.3.1, it would be tempting to construct a boundary layer corrector which is not a solution of the East boundary layer equation, but which lifts the trace of the normal derivative of ψ_t^0 without perturbing the zero order trace on the boundary, namely a corrector of the type*

$$A(s)z \exp(-z/\nu^p),$$

for some $p > 0$ and for an adequate choice of $A(s)$. However, because of the strong singularities of $\partial_n \psi_t^0|_{\partial\Omega}(s)$ near s_i and s_{i+1} , it can be proved that no choice of the parameter p leads to admissible error terms. In other words, close to the singularity zones, the corrector should be an approximate solution of the equation. This justifies the need for an elaborate construction, even though the corresponding terms are negligible in the final energy. This is in fact classical in multi-scale problems: quite often, it is necessary to construct high order correctors, whose sole purpose is to ensure that the remainder terms are admissible, but which are not seen by the total energy.

3.2.2. Definition of the East corrector.

The idea is therefore to split each East boundary component $[s_i, s_{i+1}]$ in three subdomains (independent of ν)

- one which is far from the singularities

$$[\sigma_i^+, \sigma_{i+1}^-] \subset]s_i, s_{i+1}[$$

so that we have uniformly small bounds on the trace of ψ_E^{BL} on $[\sigma_i^+, \sigma_{i-1}^-]$;

- two which are (macroscopic) neighbourhoods of the singularities and such that

$$(3.18) \quad \forall s \in [s_i, \sigma_i^+] \cup [\sigma_{i+1}^-, s_{i+1}], \quad |\sin \theta(s)| \geq \frac{1}{2},$$

so that we can extend the North or South boundary layers on these parts of the boundary.

We therefore define suitable truncation functions in s , namely $\varphi_i^+, \varphi_{i+1}^- \in C^\infty(\mathbf{R}, [0, 1])$

$$(3.19) \quad \begin{aligned} \varphi_i^+(s) &= 1 \text{ if } s \geq \sigma_i^+, \quad \varphi_i^+(s) = 0 \text{ if } s \leq \frac{1}{2}(s_i + \sigma_i^+), \\ \varphi_{i+1}^-(s) &= 1 \text{ if } s \leq \sigma_{i+1}^-, \quad \varphi_{i+1}^-(s) = 0 \text{ if } s \geq \frac{1}{2}(s_{i+1} + \sigma_{i+1}^-). \end{aligned}$$

The East corrector is then expected to lift $\varphi_i^+ \varphi_{i+1}^- \psi_{E|Z=0}$, which is the part of the trace which is known to be small. More precisely, we set, for $s \in [s_{i-1}, s_i]$,

$$(3.20) \quad \psi_E^{corr}(s, z) = -\psi_{E|Z=0}(s) \varphi_i^+(s) \varphi_{i+1}^-(s) \chi_0(z).$$

Using Lemmas 3.2.1 and 3.2.2, we infer that

$$(3.21) \quad \|\psi_E^{corr}\|_{H^m(\Omega)} = O(\nu^{1/3}) \quad \forall m \leq 3.$$

Moreover, $\psi_E^{corr}|_{\partial\Omega \setminus \Gamma_E} = 0$, and $\partial_n \psi_E^{corr} = 0$ by definition of χ_0 . Notice also that

$$\begin{aligned} (\psi_t^0 + \psi_E(s, \lambda_E z) + \psi_E^{corr})|_{[\sigma_i^+, \sigma_{i+1}^-]} &= 0, \\ \partial_n(\psi_t^0 + \psi_E(s, \lambda_E z) + \psi_E^{corr})|_{\Gamma_E} &= 0. \end{aligned}$$

Therefore, at this point, we have restored both boundary conditions on the subdomain $[\sigma_i^+, \sigma_{i+1}^-]$ of $[s_i, s_{i+1}]$, but not the condition on the trace in the neighborhoods of the singularities. This is handled by the North and South boundary layers, which we now address.

In the next section, we set

$$(3.22) \quad \bar{\psi} := \psi_t^0 + \chi_0(\psi_E^{corr} + \psi_E(s, \lambda_E(s)z)),$$

where χ_0 is the macroscopic truncation defined in (1.10), so that on the boundary

$$\begin{aligned} \bar{\psi}|_{\partial\Omega \setminus \Gamma_E} &= \psi_t^0|_{\partial\Omega \setminus \Gamma_E}, \\ \partial_n \bar{\psi}|_{\partial\Omega \setminus \Gamma_E} &= \partial_n \psi_t^0|_{\partial\Omega \setminus \Gamma_E}, \\ \bar{\psi}|_{(s_i, s_{i+1})} &= (1 - \varphi_i^+ \varphi_{i+1}^-) \psi_{E|Z=0}(s), \\ \partial_n \bar{\psi}|_{\Gamma_E} &= 0, \end{aligned}$$

where (s_i, s_{i+1}) is a connected component of Γ_E .

3.3. North and South boundary layers

In order to lift the boundary conditions both on the horizontal parts and on the East/West boundaries close to the points (x_i, y_i) for $1 \leq i \leq k$, we then define North and South boundary layers. Without loss of generality, we focus on the case of South boundaries (the case of North boundaries can be deduced by a simple symmetry).

Denote by s_i and s_{i-1} the curvilinear abscissa of the endpoints of the horizontal part to be considered, with $s_i = s_{i-1}$ if it is just an isolated point of the boundary with horizontal tangent.

There are several point to be discussed in the present section:

- In the first paragraph, we give a precise definition of the interval on which energy is injected in the South boundary layer (i.e. we define the support of the functions Ψ_0 and Ψ_1 appearing in (2.11));

- The second paragraph is devoted to the derivation of regularity and moment estimates on ψ_S ;
- Eventually, we explain how we truncate ψ_S beyond the support of Ψ_0 and Ψ_1 .

3.3.1. Definition of the initial boundary value problem.

We first define the extremal points of the **interval on which some energy is injected in the boundary layer** :

- If $\cos \theta(s) > 0$ for $s > s_i$ (East boundary), we need to lift the East boundary condition on a macroscopic neighborhood of s_i , namely $[s_i, \sigma_i^+]$ where σ_i^+ is defined by (3.18).
Similarly, if $\cos \theta(s) > 0$ for $s < s_{i-1}$, we define σ_{i-1}^- by (3.18). Truncation functions $\varphi_i^+, \varphi_{i-1}^-$ are then defined by (3.19).
- If $\cos \theta(s) < 0$ for $s > s_i$ (West boundary), we only need to lift the West boundary condition on a small neighbourhood of s_i , precisely when the West boundary layer become too singular (Recall that the trace of ψ_i^0 is not zero in general in the vicinity of s_i on West boundaries). We therefore denote by σ_i^+ the arc-length beyond which no energy is injected in the boundary layer, and we expect that $\sigma_i^+ > s_i^+ > s_i$, and $|\sigma_i^+ - s_i| \ll 1$.

Note that, in order that the transition between the two types of boundary layers is not too singular, we have to choose $|\sigma_i^+ - s_i^+|$ as large as possible. But in order that the transport coefficient b in (2.11) remains bounded in L^1 , we need that $\cos \theta(s)$ is small for $s \leq \sigma_i^+$. We thus choose σ_i^+ satisfying the condition

$$(3.23) \quad \int_{s_i}^{\sigma_i^+} |\cos \theta(s)| ds = \nu^{1/4}.$$

We further define $\varphi_i^+ \in C^\infty(\mathbf{R}, [0, 1])$ such that

$$(3.24) \quad \varphi_i^+(s) = 0 \text{ if } s \leq s_i^+ \text{ and } \text{Supp}(1 - \varphi_i^+) \subset (-\infty, \sigma_i^+].$$

Similarly, if $\cos \theta(s) < 0$ for $s < s_{i-1}$, we define σ_{i-1}^- by

$$\int_{\sigma_{i-1}^-}^{s_{i-1}} |\cos \theta(s)| ds = \nu^{1/4},$$

and $\varphi_{i-1}^- \in C^\infty(\mathbf{R}, [0, 1])$ such that

$$\text{Supp}(1 - \varphi_{i-1}^-) \subset]\sigma_{i-1}^-, \infty) \text{ and } \varphi_{i-1}^-(s) = 0 \text{ if } s \geq s_{i-1}^-.$$

We will take

$$\varphi_i^+(s) = \Phi \left(s_i^+ + \frac{s - s_i^+}{\sigma_i^+ - s_i^+} \right)$$

where $\Phi \in C^\infty(\mathbf{R})$ is such that $\Phi(s) = 0$ for $s \leq 0$, $\Phi(s) = 1$ for $s \geq \frac{1}{2}$, $0 \leq \Phi \leq 1$ and all the derivatives of Φ are bounded.

REMARK 3.3.1. *There are two cases of South boundary layer terms which will not be considered in this section, since they both give rise to interface boundary layer terms:*

- when $\cos \theta(s) < 0$ for $s < s_{i-1}$ (West boundary on the right) and $\cos \theta(s) > 0$ for $s > s_i$ (East boundary on the left);

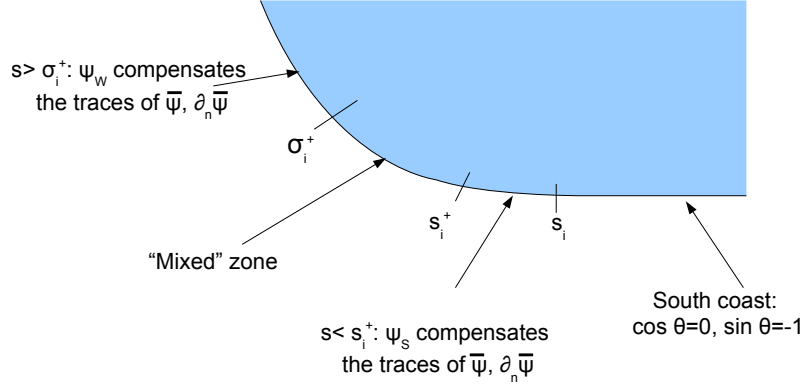


FIGURE 3.2. The interplay between the North/South and East/West boundary layers

- when $\cos \theta(s) > 0$ for $s < s_{i-1}$ and for $s > s_i$ (East boundary on both sides), with $s_{i-1} < s_i$;

We will indeed consider the corresponding South boundary together with the singular interface Σ in the next section.

As in Lemma 3.2.1, we can compute asymptotic developments for σ_i^+ and σ_{i-1}^- .

LEMMA 3.3.2. *The transition functions φ_i^\pm have controlled variations:*

- If $\cos \theta > 0$ for $s \in (s_i, s_{i+1})$ (East coast), then there exists a constant $c > 0$ such that

$$|s_i - \sigma_i^+| \geq C,$$

and therefore

$$\|\partial_s^k \varphi_i^+\|_\infty = O(1) \quad \forall k.$$

- If $\cos \theta < 0$ for $s \in (s_i, s_{i+1})$ (West coast), then

$$\sigma_i^+ - s_i \sim C'_i \nu^{\frac{1}{4n+4}}$$

if (H2i) is satisfied in a neighbourhood on the right of s_i , and

$$\sigma_i^+ - s_i = -\frac{4\alpha}{\ln \nu} \left(1 - \frac{8 \ln |\ln \nu|}{\ln \nu} + O(|\ln \nu|^{-1}) \right)$$

if (H2ii) is satisfied in a neighbourhood on the right of s_i .

In this case,

$$\begin{aligned} \|\partial_s^k \varphi_i^+\|_\infty &\leq C \nu^{-k/7}, \\ \|\partial_s^k \varphi_i^+\|_1 &\leq C \nu^{-(k-1)_+/7}. \end{aligned}$$

PROOF. The proof follows from the definition of σ_i^+ and φ_i^+ on East and West coasts.

On the West coast, if $\cos \theta$ vanishes algebraically near s_i , then

$$s_i^+ - s_i \sim \nu^{\frac{1}{4n+3}} \text{ and } \sigma_i^+ - s_i \sim \nu^{\frac{1}{4n+4}}.$$

On the other hand, if $\cos \theta$ vanishes exponentially near s_i , then

$$s_i^+ - s_i \sim \sigma_i^+ - s_i \sim -4\alpha / \ln \nu \text{ and } \sigma_i^+ - s_i^+ = O(\ln(|\ln \nu|)/(\ln \nu)^2) \ll |s_i^+ - s_i|.$$

Therefore we always have $|s_i^+ - s_i| \ll |\sigma_i^+ - s_i|$.

Furthermore, by construction, we have

$$(3.25) \quad \begin{aligned} \|\partial_s^k \varphi_i^\pm\|_\infty &= O(|s_i^\pm - \sigma_i^\pm|^{-k} + |s_i^\pm - s_i|^{-k}) \text{ if } (s_i, s_{i\pm 1}) \subset \Gamma_W, \\ \|\partial_s^k \varphi_i^\pm\|_\infty &= O(|s_i^\pm - \sigma_i^\pm|^{-k}) \text{ if } (s_i, s_{i\pm 1}) \subset \Gamma_E. \end{aligned}$$

so that

$$\|\partial_s^k \varphi_i^+\|_\infty = O(\nu^{-k/(4n+3)}) \quad (\text{resp. } \|\partial_s^k \varphi_i^+\|_\infty = O\left(\left(\frac{|\ln \nu|^2}{\ln |\ln \nu|}\right)^k\right).$$

To derive L^1 estimates, we merely multiply the L^∞ estimate by the size of $\text{Supp } \partial_s^k \varphi_i^+$. \square

Let us then define the **initial boundary value problem on $s \geq \sigma_{i-1}^-$, $Z \geq 0$** . Denote

$$(3.26) \quad \gamma_{i-1,i} = (1 - \varphi_i^+)(1 - \varphi_{i-1}^-).$$

The South boundary layer is therefore described by

$$(3.27) \quad \begin{aligned} \partial_s \psi_S + \gamma(s) \partial_Z \psi_S + \mu(s) \partial_Z^4 \psi_S &= 0, \quad s \geq \sigma_{i-1}^-, \quad Z > 0, \\ \psi_S|_{Z=0} &= \Psi_0, \quad \partial_Z \psi_S|_{Z=0} = \Psi_1 \end{aligned}$$

where $\gamma(s) = -\nu^{-1/4} \cos \theta(s) / \sin \theta(s)$, $\mu(s) = -1 / \sin \theta(s)$, with boundary condition

$$(3.28) \quad \begin{aligned} \Psi_0(s) &:= -\gamma_{i-1,i} \bar{\psi}|_{\partial\Omega}(s), \\ \Psi_1(s) &:= -\nu^{1/4} \gamma_{i-1,i} \partial_n \bar{\psi}|_{\partial\Omega}(s), \end{aligned}$$

and zero initial data prescribed at $s = \sigma_{i-1}^-$. We recall that $\bar{\psi}$ is defined by (3.22).

Note that, when the two types of boundary layers meet (that is on the support of $(\gamma_{i-1,i})'$), the width of the North/South boundary layer is much larger than the one of the East/West boundary layer. Indeed, the width of the North/South boundary layer is always $\nu^{1/4}$, while that of the East/West boundary layer is at most, using Lemma 3.2.1 and hypothesis (H2),

$$\lambda_{E,W}^{-1}(s_i^+) = \left(\frac{\nu}{\cos \theta(s_i^+)}\right)^{1/3} \leq C \begin{cases} \nu^{\frac{n+1}{4n+3}} & \text{in case (H2i),} \\ \nu^{1/4} |\ln \nu|^{-3/2} & \text{in case (H2ii).} \end{cases}$$

Therefore, it seems more accurate to talk about a superposition of the boundary layers, rather than a connection.

3.3.2. Estimates for $\psi_{N,S}$.

LEMMA 3.3.3 (Trace estimates). *The functions Ψ_0, Ψ_1 defined by (3.28) for $s \in (\sigma_{i-1}^-, s_{i+1})$ are such that*

$$\begin{aligned} \|\partial_s^k \Psi_0\|_{L^\infty} + \|\partial_s^k \Psi_1\|_{L^\infty} &\leq C\nu^{-k/7} \quad \text{for } k \in \{0, \dots, 3\}, \\ \|\partial_s \Psi_0\|_{L^1} + \|\partial_s \Psi_1\|_{L^1} &\leq C, \\ \|\partial_s^2 \Psi_0\|_{L^1} + \|\partial_s^2 \Psi_1\|_{L^1} &\leq C\delta_y^{-1}, \quad \|\partial_s^3 \Psi_0\|_{L^1} + \|\partial_s^3 \Psi_1\|_{L^1} \leq C\delta_y^{-1}\nu^{-1/7}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\gamma \Psi_0\|_{L^1} + \|\gamma \Psi_1\|_{L^1} &= O(1), \\ \|\partial_s(\gamma \Psi_0)\|_{L^1} + \|\partial_s(\gamma \Psi_1)\|_{L^1} &= o(\nu^{-1/6}), \\ \|\partial_s^2(\gamma \Psi_0)\|_{L^1} + \|\partial_s^2(\gamma \Psi_1)\|_{L^1} &= O(\nu^{-2/7}). \end{aligned}$$

PROOF. We recall that $\bar{\psi} = \psi_t^0 + \chi_0(\psi_E^{corr} + \psi_E(s, \lambda_E(s)z))$. Notice that thanks to assumption (H3), ψ_t^0 is continuous (and even \mathcal{C}^2) on the interval $(\sigma_{i-1}^-, \sigma_i^+)$. The estimates are slightly different depending on whether (s_{i-2}, s_{i-1}) and (s_i, s_{i+1}) are portions of Γ_E or Γ_W . We focus for instance on the portion $(\sigma_{i-1}^-, s_{i-1})$, keeping in mind that the portion (s_i, σ_i^+) is analogous.

Connection with West boundaries.

If $(\sigma_{i-1}^-, s_{i-1}) \subset \Gamma_W$, then by definition (3.23) of σ_{i-1}^- ,

$$(3.29) \quad \forall s \in (\sigma_{i-1}^-, s_{i-1}), \quad |y(s) - y_i| \leq \nu^{1/4} \ll \delta_y.$$

Moreover, $\bar{\psi} = \psi_t^0$ on (σ_{i-1}^-, s_i) in this case. Therefore Lemma 3.3.3 is a consequence of the trace estimates on ψ_t^0 stated in Lemma 3.1.3.

First, the L^∞ bounds together with the definition of φ_{i-1}^- imply that $\|\Psi_0\|_{L^\infty} = O(1)$. Moreover, for $s \in (\sigma_{i-1}^-, s_i)$, using Lemma 3.1.3 and inequality (3.29), we have

$$|\partial_s \psi_t^0|_{\partial\Omega}(s)| \leq C \left(1 + \frac{|\cos \theta(s)|}{\delta_y |\ln \delta_y|} \right),$$

so that, using the definition (3.18) of σ_i^\pm together with the definition (3.3) of δ_y ,

$$\begin{aligned} \|\partial_s \psi_t^0|_{\partial\Omega}\|_{L^1(\sigma_{i-1}^-, s_i)} &\leq C \left(1 + \nu^{-1/4} |\ln \nu|^{-6/5} (\ln |\ln \nu|)^\beta \int_{\sigma_{i-1}^-}^{\sigma_i^+} |\cos \theta(s)| ds \right) \\ &\leq C(1 + |\ln \nu|^{-6/5} (\ln |\ln \nu|)^\beta). \end{aligned}$$

We infer that

$$\|\Psi_0'\|_{L^1(\sigma_{i-1}^-, s_i)} \leq \|\psi_t^0|_{\partial\Omega}\|_\infty \|\partial_s \gamma_{i-1, i}\|_{L^1(\sigma_{i-1}^-, s_i)} + \|\partial_s \psi_t^0|_{\partial\Omega}\|_{L^1(\sigma_{i-1}^-, s_i)} \leq C.$$

In a similar fashion, Lemma 3.3.2 and assumption (H2) yield

$$\begin{aligned} \|\Psi_1\|_{L^\infty(\sigma_{i-1}^-, s_i)} &\leq \nu^{1/4} \|\partial_n \psi_t^0|_{\partial\Omega}\|_{L^\infty(\sigma_{i-1}^-, s_i)} \\ &\leq C\nu^{1/4} \left(1 + \frac{1}{\delta_y |\ln \delta_y|} \right) \leq C, \end{aligned}$$

and eventually

$$\begin{aligned}
\|\Psi_1'\|_{L^1} &\leq \nu^{1/4} \|\partial_s \gamma_{i-1,i}\|_{L^1(\sigma_{i-1}^-, s_i)} \|\partial_n \psi_t^0|_{\partial\Omega}\|_{L^\infty(\sigma_{i-1}^-, s_i)} \\
&\quad + C\nu^{1/4} \|\partial_s \partial_n \psi_t^0|_{\partial\Omega}\|_{L^1(\sigma_{i-1}^-, s_i)} \\
&\leq C \left(1 + \nu^{1/4} \int_{\sigma_{i-1}^-}^{s_i} \left(\frac{1}{\delta_y} + \frac{|\cos \theta|}{\delta_y^2 |\ln \delta_y|} \right) \right) \\
&\leq C \left(1 + \frac{\nu^{1/4}}{\delta_y} + \frac{\nu^{1/2}}{\delta_y^2 |\ln \delta_y|} \right) \leq C.
\end{aligned}$$

The higher order estimates are obtained in a similar way. We use the estimates of Lemma 3.3.2, which lead to

$$\|\partial_s^k \gamma_{i-1,i}\|_\infty = O(\nu^{-k/7}), \quad \|\partial_s^k \gamma_{i-1,i}\|_1 = O(\nu^{-(k-1)/7});$$

In a similar fashion,

$$\|\cos^k \theta\|_{L^\infty(\sigma_{i-1}^-, s_i)} = O(\nu^{k/8}), \quad \|\cos^k \theta\|_{L^1(\sigma_{i-1}^-, s_i)} = O(\nu^{(k+1)/8}).$$

Lemma 3.1.3 implies that for $s \in (\sigma_{i-1}^-, s_i)$

$$\begin{aligned}
|\Psi_0''(s)| + |\Psi_1''(s)| &\leq C \left(\delta_y^{-1} \left(1 + \frac{\cos^2 \theta(s)}{\delta_y} \right) + \nu^{-1/7} \left(1 + \frac{|\cos \theta(s)|}{\delta_y} \right) \right) \\
&\quad + C (|\partial_s^2 \varphi_i^+| + |\partial_s^2 \varphi_{i-1}^-|), \\
|\partial_s^3 \Psi_0(s)| + |\partial_s^3 \Psi_1(s)| &\leq C \delta_y^{-1} \left(1 + \frac{|\cos \theta|}{\delta_y} + \frac{|\cos \theta|^3}{\delta_y^2} \right) \\
&\quad + C \nu^{-1/7} \delta_y^{-1} \left(1 + \frac{\cos^2 \theta(s)}{\delta_y} \right) \\
&\quad + C \nu^{-2/7} \left(1 + \frac{|\cos \theta(s)|}{\delta_y} \right) + C (|\partial_s^3 \varphi_i^+| + |\partial_s^3 \varphi_{i-1}^-|),
\end{aligned}$$

from which we easily infer the estimates of the Lemma.

The estimates on $\gamma\Psi_0$ and $\gamma\Psi_1$ are a consequence of the following estimates on γ :

$$\begin{aligned}
\gamma &\equiv 0 \text{ on } (s_{i-1}, s_i), \\
\|\gamma\|_{L^1(\sigma_{i-1}^-, s_{i-1})} &= 1
\end{aligned}$$

by definition of σ_{i-1}^- on West coasts, and

$$\begin{aligned}
\|\gamma\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})}, \|\partial_s \gamma\|_{L^1(\sigma_{i-1}^-, s_{i-1})} &= O(\nu^{-1/8}), \\
\|\partial_s^2 \gamma\|_{L^1(\sigma_{i-1}^-, s_{i-1})} &= O(\nu^{-1/6}).
\end{aligned}$$

Indeed

$$\|\partial_s^k \gamma\|_{L^1(\sigma_{i-1}^-, s_{i-1})} \leq C(1 + \nu^{-1/4} \|\partial_s^k \theta\|_{L^1(\sigma_{i-1}^-, s_{i-1})}).$$

Using assumption (H2) together with the definition of σ_{i-1}^- , we infer the desired result; notice that the most singular case corresponds to $n = 1$ in (H2i) for the estimates on γ and $\partial_s \gamma$, and to $n = 2$ for the estimate on $\partial_s^2 \gamma$.

Connection with East boundaries

If $(\sigma_{i-1}^-, s_{i-1}) \subset \Gamma_E$, the estimates are different in several regards:

- The function $\gamma_{i-1,i}$ has bounded derivatives;

- Because of the truncation χ_ν , the traces of ψ_t^0 and $\partial_n \psi_t^0$ are identically zero on the vicinity of s_{i-1} ;
- The trace of ψ_t^0 is zero on $(\sigma_{i-1}^-, s_{i-1})$.
- The normal derivative of $\bar{\psi} = \psi_t^0 + \psi_E + \psi_E^{corr}$ is identically zero on $(\sigma_{i-1}^-, s_{i-1})$ by definition of ψ_E , so that $\Psi_1 = 0$ on $(\sigma_{i-1}^-, s_{i-1})$;

Therefore it suffices to prove that $\gamma_{i-1,i} \psi_E|_{Z=0}$ satisfies the desired estimates on the interval $(\sigma_{i-1}^-, s_{i-1})$. Notice that the estimates on (s_{i-1}, s_i) can be treated with the same arguments as in the first case.

We use the estimates of Lemma 3.2.2 and assumption (H2), together with the formulas of Appendix B, in order to compute $y(s)$, $\cos \theta$ and θ' in terms of s . We infer that on $(\sigma_{i-1}^-, s_{i-1})$, Ψ_0 satisfies the following bounds:

- If (H2i) is satisfied in a neighbourhood on the left of s_{i-1} , then for $k = 1, 2, 3$, $s \in (\sigma_{i-1}^-, s_{i-1})$,

$$\begin{aligned} |\Psi_0| &\leq C\nu^{1/3} \left(|s - s_{i-1}|^{-\frac{4n+3}{3}} |\ln(s_{i-1} - s)|^{-1} \mathbf{1}_{C\delta_y^{1/(n+1)} \leq |s - s_{i-1}| \leq 1/2} + 1 \right), \\ |\partial_s^k \Psi_0| &\leq \nu^{1/3} \delta_y^{-1} \left(|s - s_{i-1}|^{-\frac{n}{3} - k} |\ln(s_{i-1} - s)|^{-1} \mathbf{1}_{C\delta_y^{1/(n+1)} \leq |s - s_{i-1}| \leq 1/2} + 1 \right). \end{aligned}$$

We infer in particular that

$$\|\Psi_0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})} \leq C\nu^{1/3} \delta_y^{-\frac{4n+3}{3(n+1)}} |\ln \delta_y|^{-1}.$$

Since $\frac{4n+3}{3(n+1)} < 4/3$ for all $n \geq 1$, we infer that

$$\|\Psi_0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})} \leq C\nu^{1/3} \delta_y^{-4/3} |\ln \nu|^{-1} \leq C |\ln \nu|^{-1}.$$

As for the other estimates, we have, for $k = 1, 2, 3$,

$$\|\partial_s^k \Psi_0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})} \leq C\nu^{1/3} \delta_y^{-1} \delta_y^{-\frac{n+3k}{3(n+1)}} |\ln \nu|^{-1},$$

$$\|\partial_s^k \Psi_0\|_{L^1(\sigma_{i-1}^-, s_{i-1})} \leq C\nu^{1/3} \delta_y^{-1} \delta_y^{-\frac{n+3(k-1)}{3(n+1)}} |\ln \nu|^{-1}.$$

It can be checked that the most singular estimates on the L^∞ norm, and on the L^1 norm as soon as $k \geq 2$, correspond to $n = 1$. We then obtain the following (non optimal) upper bounds

$$\begin{aligned} \|\partial_s \Psi^0\|_{L^1(\sigma_{i-1}^-, s_{i-1})} &\leq C |\ln \nu|^{-1}, \\ \|\partial_s \Psi^0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})}, \|\partial_s^2 \Psi^0\|_{L^1(\sigma_{i-1}^-, s_{i-1})} &\leq C\nu^{-1/12} |\ln \nu|^{-1}, \\ \|\partial_s^2 \Psi^0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})}, \|\partial_s^3 \Psi^0\|_{L^1(\sigma_{i-1}^-, s_{i-1})} &\leq C\nu^{-5/24} |\ln \nu|^{-1}, \\ \|\partial_s^3 \Psi^0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})} &\leq C\nu^{-1/3} |\ln \nu|^{-1}. \end{aligned}$$

- If (H2ii) is satisfied in a neighbourhood of the left of s_{i-1} , then for $s \in (\sigma_{i-1}^-, s_{i-1})$ and $k = 1, 2, 3$,

$$\begin{aligned} |\Psi_0(s)| &\leq C\nu^{1/3} |s - s_{i-1}|^{-1} \exp\left(\frac{4\alpha}{3|s - s_{i-1}|}\right) \mathbf{1}_{|y(s) - y_{i-1}| \geq \delta_y/2}, \\ |\partial_s^k \Psi_0(s)| &\leq C\nu^{1/3} \delta_y^{-1} |s - s_{i-1}|^{-1-2k} \exp\left(\frac{\alpha}{3|s - s_{i-1}|}\right) \mathbf{1}_{|y(s) - y_{i-1}| \geq \delta_y/2}. \end{aligned}$$

Moreover, in case (H2ii), we recall that

$$y(s) - y_{i-1} \sim C(s - s_{i-1})^2 \exp\left(-\frac{\alpha}{|s - s_{i-1}|}\right),$$

so that for $k = 0, 1, 2$

$$\begin{aligned} & \|\partial_s^k \Psi^0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})}, \|\partial_s^{k+1} \Psi^0\|_{L^1(\sigma_{i-1}^-, s_{i-1})} \\ & \leq C\nu^{1/3} \delta_y^{-4/3} |\ln \nu|^{2k - \frac{5}{3}} \leq C |\ln \nu|^{2k - \frac{5}{3}} \end{aligned}$$

and similarly

$$\|\partial_s^3 \Psi^0\|_{L^\infty(\sigma_{i-1}^-, s_{i-1})} \leq C |\ln \nu|^{13/3}.$$

Therefore we retrieve the desired on the norm of Ψ_0 . Concerning the estimates on $b\Psi_0$, we use the following inequalities on $(\sigma_{i-1}^-, s_{i-1})$

$$\begin{aligned} |\gamma\Psi_0(s)| & \leq C\nu^{1/12} |\cos \theta|^{2/3} \mathcal{M}(y(s)), \\ |\partial_s(\gamma\Psi_0(s))| & \leq C\nu^{1/12} \left(\delta_y^{-1} |\cos \theta|^{5/3} + |\theta'(s)| |\cos \theta|^{-1/3} \right) \mathcal{M}(y(s)) \\ |\partial_s^2(\gamma\Psi_0(s))| & \leq C\nu^{1/12} \left(\delta_y^{-1} |\cos \theta|^{2/3} |\theta'(s)| + \frac{|\theta''(s)|}{|\cos \theta|^{-1/3}} \right) \mathcal{M}(y(s)). \end{aligned}$$

Once again, we distinguish between (H2i) and (H2ii), and we obtain in the worst situation

$$\begin{aligned} \|\gamma\Psi_0\|_{L^1} & \leq C |\ln \nu|^{-1}, \\ \|\partial_s(\gamma\Psi_0)\|_{L^1} & \leq C\nu^{1/12} \delta_y^{-1} \leq C\nu^{-1/6} |\ln \nu|^{-1/5} |\ln |\ln \nu||^\beta, \\ \|\partial_s^2(\gamma\Psi_0)\|_{L^1} & \leq C\nu^{1/12} \delta_y^{-7/6} \leq C\nu^{-5/24}. \end{aligned}$$

□

By Lemma 2.3.1, we then have the well-posedness of equation (3.27), as well as suitable a priori estimates for the solution ψ_S

$$\begin{aligned} \|\psi_S(s)\|_{L^2(\mathbf{R}_+)} & \leq C(|\Psi_0(s)| + |\Psi_1(s)|) \\ & + C \int_{\sigma_{i-1}^-}^s (|\Psi_0'(s')| + |\Psi_1'(s')|) ds' \\ & + C \int_{\sigma_{i-1}^-}^s (\mu(s') + |b(s')|)(|\Psi_0(s')| + |\Psi_1(s')|) ds'. \end{aligned} \tag{3.30}$$

In order to show that the South boundary layer is stable, we will need additional estimates on the boundary layer terms, namely regularity and moment estimates.

LEMMA 3.3.4 (Regularity and moment estimates). *Let ψ_S be the solution to (3.27) with boundary conditions (3.28), defined on*

$$I_S := \{s \geq \sigma_{i-1}^- / \forall s' \in [\sigma_{i-1}^-, s], |\sin \theta(s)| \geq \frac{1}{4}\}.$$

Then

$$\begin{aligned} & \|(1 + Z^k)\psi_S(s, Z)\|_{L^\infty(I_S, L_Z^2)} + \|(1 + Z^k)\partial_Z^2 \psi_S\|_{L^2(I_S \times \mathbf{R}_+)} = O(1), \\ & \|(1 + Z^k)\partial_s \psi_S\|_{L^\infty(I_S, L_Z^2)} + \|(1 + Z^k)\partial_Z^2 \partial_s \psi_S\|_{L^2(I_S \times \mathbf{R}_+)} = O(\delta_y^{-1}), \\ & \|(1 + Z^k)\partial_s^2 \psi_S\|_{L^\infty(I_S, L_Z^2)} + \|(1 + Z^k)\partial_Z^2 \partial_s^2 \psi_S\|_{L^2(I_S \times \mathbf{R}_+)} = O(\nu^{-1/2} |\ln \nu|^{-1}). \end{aligned} \tag{3.31}$$

REMARK 3.3.5. *The condition $|\sin \theta| \geq 1/4$ in the definition of I_S is somewhat arbitrary: we just want to prevent $\sin \theta$ from touching zero, but obviously any fixed positive number will do.*

PROOF. Such a priori estimates are obtained by combining precised energy estimates for the parabolic equation (3.38), together with high order estimates on the traces.

- Define as in the previous chapter

$$\begin{aligned} S(s, Z) := & -\Psi'_0(s)(Z+1)\exp(-Z) - \Psi'_1(s)Z\exp(-Z) \\ & + \gamma(s)\Psi_0(s)Z\exp(-Z) - \gamma(s)\Psi_1(s)(1-Z)\exp(-Z) \\ & - \mu(s)\Psi_0(s)(Z-3)\exp(-Z) - \mu(s)\Psi_1(s)(Z-4)\exp(-Z). \end{aligned}$$

From the trace estimates in Lemma 3.3.3, we deduce that, for all $k \geq 0$,

$$\begin{aligned} \|Z^k S\|_{L^1} &\leq C, \\ \|Z^k \partial_s S\|_{L^1} &\leq C\delta_y^{-1}, \quad \|Z^k \partial_s^2 S\|_{L^1} \leq C\delta_y^{-1}\nu^{-1/7}. \end{aligned}$$

Notice also that by definition (3.19), (3.24) of φ_i^\pm , we have $\text{Supp}(1 - \varphi_{i-1}^-) \subset]\sigma_{i-1}^-, \infty[$, so that Ψ_0 and Ψ_1 are identically zero on a neighbourhood of σ_{i-1}^- . Therefore $\partial_s^k \psi_S|_{s=\sigma_{i-1}^-} = 0$.

- In order to derive estimates on $\partial_s^k \psi_S$ for $k \geq 1$, we lift the boundary conditions to get

$$(3.32) \quad \begin{aligned} \partial_s g + \gamma(s)\partial_Z g + \mu(s)\partial_Z^4 g &= S(s, Z), \quad s \in (0, T), \quad Z > 0, \\ g|_{s=0} &= 0, \\ g|_{Z=0} &= 0, \quad \partial_Z g|_{Z=0} = 0, \end{aligned}$$

then differentiate the equation and proceed by induction on k . We start with

$$\partial_s(\partial_s^k g) + \partial_s^k(\gamma(s)\partial_Z g) + \partial_s^k(\mu(s)\partial_Z^4 g) = \partial_s^k S.$$

Integrating against $\partial_s^k g$, we get

$$\begin{aligned} & \frac{1}{2} \|\partial_s^k g(s)\|_{L_Z^2}^2 + \int_{\sigma_{i-1}^-}^s \mu(s') \|\partial_Z^2 \partial_s^k g(s')\|_{L_Z^2}^2 ds' \\ & \leq \int_{\sigma_{i-1}^-}^s \|\partial_s^k S(s')\|_{L_Z^2} \|\partial_s^k g(s')\|_{L_Z^2} ds' \\ & + \sum_{j=0}^{k-1} C_k^j \int_{\sigma_{i-1}^-}^s |\partial_s^{k-j} \gamma(s')| \|\partial_Z \partial_s^j g(s')\|_{L_Z^2} \|\partial_s^k g(s')\|_{L_Z^2} ds' \\ & + \sum_{j=0}^{k-1} C_k^j \int_{\sigma_{i-1}^-}^s \|\partial_Z^2 \partial_s^k g(s')\|_{L_Z^2} |\partial_s^{k-j} \mu(s')| \|\partial_Z^2 \partial_s^j g(s')\|_{L_Z^2} ds', \end{aligned}$$

and therefore, since μ is bounded from below,

$$\begin{aligned} & \|\partial_s^k g\|_{L^\infty((\sigma_{i-1}^-, s), L_Z^2)}^2 + \|\partial_s^k \partial_Z^2 g\|_{L^2((\sigma_{i-1}^-, s), L_Z^2)}^2 \\ & \leq C \|\partial_s^k S\|_{L^1((\sigma_{i-1}^-, s), L_Z^2)} \|\partial_s^k g\|_{L^\infty((\sigma_{i-1}^-, s), L_Z^2)} \\ & + C \sum_{j=0}^{k-1} \|\partial_s^{k-j} \gamma\|_{L^{4/3}(\sigma_{i-1}^-, s)} \|\partial_s^j g\|_{L_s^\infty(L_Z^2)}^{1/2} \|\partial_s^j \partial_Z^2 g\|_{L_s^2(L_Z^2)}^{1/2} \|\partial_s^k g\|_{L_s^\infty(L_Z^2)} \\ & + C \sum_{j=0}^{k-1} \|\partial_s^{k-j} \mu\|_{L^\infty(\sigma_{i-1}^-, s)} \|\partial_s^j \partial_Z^2 g\|_{L_s^2(L_Z^2)} \|\partial_s^k \partial_Z^2 g\|_{L_s^2(L_Z^2)}. \end{aligned}$$

Eventually, using the Cauchy-Schwarz inequality, we obtain for $k \leq 2$

$$\begin{aligned} (3.33) \quad & \|\partial_s^k g\|_{L^\infty((\sigma_{i-1}^-, s), L_Z^2)}^2 + \|\partial_s^k \partial_Z^2 g\|_{L^2((\sigma_{i-1}^-, s), L_Z^2)}^2 \\ & \leq C \|\partial_s^k S\|_{L^1((\sigma_{i-1}^-, s), L_Z^2)}^2 \\ & + C \sum_{j=0}^{k-1} \|\partial_s^{k-j} \gamma\|_{L^{4/3}(\sigma_{i-1}^-, s)}^2 \|\partial_s^j g\|_{L^\infty((\sigma_{i-1}^-, s), L_Z^2)} \|\partial_s^j \partial_Z^2 g\|_{L^2((\sigma_{i-1}^-, s) \times \mathbf{R}_+)} \\ & + C \sum_{j=0}^{k-1} \|\partial_s^{k-j} \mu\|_{L^\infty(\sigma_{i-1}^-, s)}^2 \|\partial_s^j \partial_Z^2 g\|_{L^2((\sigma_{i-1}^-, s) \times \mathbf{R}_+)}^2. \end{aligned}$$

Next recall that μ is bounded, and that all its derivatives are of order $O(1)$

$$\mu \geq \mu_0 > 0, \quad \|\partial_s^j \mu\|_\infty \leq C.$$

In order to bound the term involving γ , we have to distinguish between the cases when $(s_{i-2}, s_{i-1}) \subset \Gamma_E$ or Γ_W :

- If $(s_{i-2}, s_{i-1}) \subset \Gamma_W$, then $\gamma = \nu^{-1/4} \cos \theta \mu$ satisfies

$$\begin{aligned} \|\partial_s \gamma\|_{L^{4/3}(\sigma_{i-1}^-, \sigma_i^+)} &= O(\nu^{-\frac{1}{4} + \frac{3}{4} \frac{1}{8}}) = O(\nu^{-5/32}), \\ \|\partial_s^2 \gamma\|_{L^{4/3}(\sigma_{i-1}^-, \sigma_i^+)} &= O(\nu^{-\frac{1}{4} + \frac{3}{4} \frac{1}{12}}) = O(\nu^{-3/16}). \end{aligned}$$

Notice that in this case, we also have $(s_i, s_{i+1}) \subset \Gamma_W$ since we have excluded for the moment the case leading to a discontinuity line Σ .

Inequality (3.33) therefore yields, for $k = 0$,

$$\|g\|_{L_s^\infty(L_Z^2)} + \|\partial_Z^2 g\|_{L_{s,Z}^2} \leq C \|S\|_{L_s^1(L_Z^2)} \leq C,$$

then, for $k = 1$,

$$\|\partial_s g\|_{L_s^\infty(L_Z^2)} + \|\partial_Z^2 \partial_s g\|_{L_{s,Z}^2} \leq C \delta_y^{-1},$$

and, for $k = 2$,

$$\|\partial_s^2 g\|_{L_s^\infty(L_Z^2)} + \|\partial_Z^2 \partial_s^2 g\|_{L_{s,Z}^2} \leq C \delta_y^{-1} \nu^{-3/16} = o(\nu^{-7/16}).$$

- If $(s_{i-2}, s_{i-1}) \subset \Gamma_E$, then for $s \in (\sigma_{i-1}^-, s_{i-1})$

$$\|\partial_s^j \gamma\|_{L^{4/3}(\sigma_{i-1}^-, s)} = O(\nu^{-1/4})$$

for all j and for all $s > \sigma_{i-1}^-$. In this case, we refer to the proof of Lemma 3.3.3 and we state that

$$\begin{aligned} \|S\|_{L^1((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} &= O(|\ln \nu|^{-1}), \\ \|\partial_s S\|_{L^1((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} &= o(\nu^{-1/6}), \\ \|\partial_s^2 S\|_{L^1((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} &= o(\nu^{-5/24}). \end{aligned}$$

Inequality (3.33) therefore yields, for $k = 0$,

$$\|g\|_{L^\infty((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} + \|\partial_Z^2 g\|_{L^2((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} \leq C |\ln \nu|^{-1},$$

then, for $k = 1$,

$$\|\partial_s g\|_{L^\infty((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} + \|\partial_Z^2 \partial_s g\|_{L^2((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} \leq C \nu^{-1/4} |\ln \nu|^{-1},$$

and, for $k = 2$,

$$\|\partial_s^2 g\|_{L^\infty((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} + \|\partial_Z^2 \partial_s^2 g\|_{L^2((\sigma_{i-1}^-, s_{i-1}), L_Z^2)} \leq C \nu^{-1/2} |\ln \nu|^{-1}.$$

There remains to derive bounds for $s > s_{i-1}$. There are essentially two cases:

- either $(s_i, s_{i+1}) \subset \Gamma_W$, and in this case the estimates on (s_{i-1}, σ_i^+) are similar to the ones of the case $(s_{i-2}, s_{i-1}) \subset \Gamma_W$. The only change lies in the initial data: we merely replace

$$\partial_s^k g|_{s=\sigma_{i-1}^-} = 0$$

by

$$\|\partial_s^k g|_{s=s_{i-1}}\|_{L_Z^2} = O(\nu^{-k/4} |\ln \nu|^{-1}) \quad \text{for } k = 0, 1, 2.$$

We infer that

$$\begin{aligned} \|g\|_{L_s^\infty(L_Z^2)} + \|g\|_{L_{s,Z}^2} &= O(1), \\ \|\partial_s g\|_{L_s^\infty(L_Z^2)} + \|\partial_s g\|_{L_{s,Z}^2} &= O(\delta_y^{-1}), \\ \|\partial_s^2 g\|_{L_s^\infty(L_Z^2)} + \|\partial_s^2 g\|_{L_{s,Z}^2} &= O(\nu^{-1/2} |\ln \nu|^{-1}). \end{aligned}$$

- or $(s_i, s_{i+1}) \subset \Gamma_E$. Notice that in this case, $s_i = s_{i-1}$ (i.e. there is no flat horizontal boundary), otherwise the corresponding piece of flat boundary would be the beginning of a Σ layer, which will be treated in the next section. Whence we can treat the interval (s_i, σ_i^+) exactly as $(\sigma_{i-1}^-, s_{i-1})$, and we obtain

$$\begin{aligned} \|g\|_{L_s^\infty(L_Z^2)} + \|g\|_{L_{s,Z}^2} &= O(|\ln \nu|^{-1}), \\ \|\partial_s^k g\|_{L_s^\infty(L_Z^2)} + \|\partial_s^k g\|_{L_{s,Z}^2} &= O(\nu^{-k/4} |\ln \nu|^{-1}) \text{ for } k = 1, 2. \end{aligned}$$

- We get moment estimates in a similar fashion

$$\partial_s(Z^k g) + Z^k(\gamma(s)\partial_Z g) + Z^k(\mu(s)\partial_Z^4 g) = Z^k S.$$

Integrating against $Z^k g$, we get

$$\begin{aligned}
& \frac{1}{2} \|Z^k g(s)\|_{L_Z^2}^2 + \int_{\sigma_{i-1}^-}^s \mu(s') \|Z^k \partial_Z^2 g(s')\|_{L_Z^2}^2 ds' \\
& \leq \int_{\sigma_{i-1}^-}^s \|Z^k S(s')\|_{L_Z^2} \|Z^k g(s')\|_{L_Z^2} ds' \\
& \quad + 2k \int_{\sigma_{i-1}^-}^s |\gamma(s')| \|Z^k g(s')\|_{L_Z^2} \|Z^{k-1} g(s')\|_{L_Z^2} ds' \\
& \quad + 2k \int_{\sigma_{i-1}^-}^s |\mu(s')| \|Z^k \partial_Z^2 g(s')\|_{L_Z^2} \|Z^{k-1} \partial_Z g(s')\|_{L_Z^2} ds' \\
& \quad + k(k-1) \int_{\sigma_{i-1}^-}^s |\mu(s')| \|Z^k \partial_Z^2 g(s')\|_{L_Z^2} \|Z^{k-2} g(s')\|_{L_Z^2} ds',
\end{aligned}$$

A simple recursion then leads to the expected uniform bound for the moments of ψ_S and $\partial_Z^2 \psi_S$.

• Combining both arguments we obtain the moment estimates for higher order derivatives. □

3.3.3. Extinction and truncation.

We emphasize that $\psi_S|_{s=\sigma_i^+}$ is not small in general: there is still some energy in the South boundary layer, even though the boundary conditions are absorbed by the West boundary layer only for $s \geq \sigma_i^+$. We deal with this somewhat unexpected phenomenon by propagating the boundary layer term ψ_S beyond σ_i^+ . We prove that when $\cos \theta$ becomes bounded away from zero, ψ_S vanishes. As before, there are two cases, depending on whether (s_i, s_{i+1}) is a part of Γ_W or Γ_E :

• First case: $(s_i, s_{i+1}) \subset \Gamma_E$: looking closely at the proof of Lemma 3.3.4, we see that in this case,

$$(3.34) \quad \|\psi_S\|_{L^\infty((s_i, \bar{s}), L_Z^2)} = O(|\ln \nu|^{-1})$$

for all fixed $\bar{s} > \sigma_i^+$ (independent of ν) such that $|\sin \theta| \geq 1/2$ on (s_i, \bar{s}) . Therefore ψ_S is small on the West boundary layer, on intervals of size one.

• Second case: $(s_i, s_{i+1}) \subset \Gamma_W$: we use the following result:

LEMMA 3.3.6 (Extinction of the South boundary layer). *Let ψ_S be the solution to (3.27) with boundary conditions (3.28), defined for $s \geq s_i$ as long as $\sin \theta(s) < 0$. Then*

$$\int_{s_i}^s \nu^{-1/4} |\cos \theta(s')| \|\psi_S(s')\|_{L_Z^2}^2 ds' \leq C.$$

Before proving Lemma 3.3.6, we now explain how we truncate the South boundary layer on (s_i, s_{i+1}) . We introduce a function $\tilde{\gamma}_{i-1,i} \in \mathcal{C}^\infty(\partial\Omega)$ by

$$(3.35) \quad \begin{aligned} & \text{Supp } \tilde{\gamma}_{i-1,i} \subset \{s \in]s_{i-2}, s_{i+1}[\mid \sin \theta(s) < 0\}, \\ & \tilde{\gamma}_{i-1,i} \equiv 1 \text{ on a neighbourhood of } \text{Supp}(1 - \varphi_i^+) \cap \text{Supp}(1 - \varphi_{i-1}^-). \end{aligned}$$

Notice that we can always assume that $\tilde{\gamma}_{i-1,i}$ has bounded derivatives.

The South boundary layer on (s_{i-2}, s_{i+1}) is then defined by

$$\tilde{\gamma}_{i-1,i}(s) \psi_S(s, z\nu^{-1/4}).$$

Using either Lemma 3.3.6 or estimation (3.34), we infer that $\|\psi_S\|_{L^2_{s,Z}} = O(|\ln \nu|^{-1})$ on the support of $\tilde{\gamma}'_{i-1,i}$. We will prove in the next chapter that this is enough to ensure that the error terms generated by $\tilde{\gamma}_{i-1,i}$ are admissible in the sense of Definition 1.1.2.

PROOF OF LEMMA 3.3.6. By Lemma 3.3.4, for every $s \in (s_i, s_{i+1})$ such that $\sin \theta(s) < 0$,

$$\|(1+Z)\psi_S\|_{L^2_Z} = O(1),$$

and

$$\int_{\sigma_{i-1}^-}^s (|\Psi'_0(s)| + |\Psi'_1(s)| + (\mu(s) + |\gamma(s)|)(|\Psi_0(s)| + |\Psi_1(s)|)) ds = O(1).$$

Moreover, for $s \geq \sigma_i^+$, $\Psi_0(s) = \Psi_1(s) = 0$. Integrating (3.27) against $Z\psi_S$, we infer that there exists a constant $C > 0$, which does not depend on ν , such that for all $s \geq s_i$ such that $\sin \theta(s) < 0$

$$\int_0^\infty Z\psi_S^2(s) dZ + \int_{s_i}^s \nu^{-1/4} |(\tan \theta(s'))^{-1}| \|\psi_S(s')\|_{L^2}^2 ds' \leq C.$$

which gives the desired estimate. \square

3.4. The interface layer

Let us now focus on the interface $\Sigma = \partial\Omega^+ \cap \partial\Omega^-$. We recall that, by construction, ψ_t^0 is discontinuous across Σ so that

- we need to lift this discontinuity to get a smooth (H^2) approximate solution,
- we then have to define a corrector (in the form of a singular layer term) to remove the energy created by the lifting term.

3.4.1. The lifting term ψ^{lift} .

We first construct the term ψ^{lift} which lifts the discontinuity across Σ . The construction has already been explained in Chapter 2. Since σ_i^\pm is now defined precisely by (3.23) for West boundaries and (3.18) for East boundaries, we can define σ_{in} in the following way

$$(3.36) \quad \sigma_{in} := \begin{cases} \sigma_1^- & \text{if } I_1 = \{s_1\}, \\ \sigma_0^- & \text{if } I_1 = [s_0, s_1] \text{ and } (\sigma_0^-, s_0) \subset \Gamma_W, \\ s_0 & \text{if } I_1 = [s_0, s_1] \text{ and } (\sigma_0^-, s_0) \subset \Gamma_E. \end{cases}$$

Notice that if $I_1 = \{s_1\}$, then we always have $(\sigma_1^-, s_1) \subset \Gamma_W$ (an isolated point of cancellation of $\cos \theta$ on the East boundary does not create a discontinuity line). The reason why we choose to take s_0 instead of σ_0^- as an initial point in the last line is essentially technical: we have chosen to write the equation on the discontinuity type boundary layer in cartesian coordinates, rather than curvilinear ones (see equation (2.23)). This is legitimate as long as $\cos \theta \ll 1$, so that the normal vector coincides almost with $-e_y$, while the tangent vector coincides with $-e_x$. However, by definition of σ_i^\pm on East boundaries, $\cos \theta$ takes "large" values on (σ_0^-, s_0) , and therefore the present construction cannot be used. We by-pass this problem by considering separately the East boundary for $s < s_0$ and the discontinuity boundary layer (which will be justified in Remark 3.4.4).

We thus define a truncation φ by

$$\varphi := \begin{cases} \varphi_1^- & \text{if } I_1 = \{s_1\}, \\ \varphi_0^- & \text{if } I_1 = [s_0, s_1] \text{ and } (\sigma_0^-, s_0) \subset \Gamma_W, \\ 0 & \text{if } I_1 = [s_0, s_1] \text{ and } (\sigma_0^-, s_0) \subset \Gamma_E. \end{cases}$$

Notice that in the last case, ψ_t^0 vanishes identically in a neighbourhood of (x_0, y_0) , and thus there is no need for a truncation in s .

We then define as in (2.17)

$$\psi^{\text{lift}}(x, y) = \chi^+ \chi \left(\frac{y - y_1}{\nu^{1/4}} \right) [a(x) + b(x)(y - y_1)],$$

where χ is a truncation function as in (2.18) and a and b are defined by (2.19) and (2.20).

Eventually, in order to simplify the present analysis, we truncate the function τ in the vicinity of $\inf I_1$ and $\sup I_1$, even in the case when these points do not belong to I_+ . Note that this does not change anything to the estimates of Lemma 3.1.1.

3.4.2. The interior singular layer ψ^Σ . The lifting term ψ^{lift} has introduced a remainder $\delta\tau^{\text{lift}}$ in the approximate Munk equation (see Lemma 2.4.3), which we have to treat as a source term. As suggested in the previous chapter, we therefore define ψ^Σ as the solution of equation (2.23)

$$\begin{aligned} \partial_x \psi^\Sigma - \partial_Y^4 \psi^\Sigma &= -\delta\tau^{\text{lift}}, \quad x < x(\sigma_{in}), \quad Y > Y_-(x) \\ \psi^\Sigma|_{x=x(\sigma_{in})} &= 0, \quad \psi^\Sigma|_{Y=Y_-(x)} = \partial_n \psi^\Sigma|_{Y=Y_-(x)} = 0. \end{aligned}$$

The function Y_- is any C^4 function satisfying the following conditions:

$$\begin{aligned} Y_-(x) &= \frac{y(x) - y_1}{\nu^{1/4}} \quad \text{for } x \in (x(t_1^+), x(\sigma_{in})), \\ Y_-(x) &\equiv \bar{Y}_- > \delta_y \nu^{-1/4} \quad \text{for } x < x(t_1^+), \\ \forall k \leq 4, \forall p \geq 1, \exists C > 0, \quad \|Y_-\|_{W^{k,p}(\bar{x}, x(\sigma_{in}))} &\leq C \|Y_-\|_{W^{k,p}(x(t_1^+), x(\sigma_{in}))}, \end{aligned}$$

where

$$(3.37) \quad t_1^+ = \inf\{s > s_1, \quad |y(s) - y(s_1)| = \delta_y\},$$

$y(x)$ is the ordinate of the point of $\partial\Omega$ with abscissa x in the vicinity of (x_1, y_1) and of $(x(s_0), y(s_0))$, and \bar{x} is an arbitrary fixed point such that the vertical line $x = \bar{x}$ does not intersect $\bar{\Omega}$.

Then the solution ψ^Σ of equation (2.23) can be expressed as

$$\psi^\Sigma(x, Y) = \phi^\Sigma(x, Y - Y_-(x)),$$

where ϕ^Σ is the solution of the backward equation

$$(3.38) \quad \begin{cases} \partial_x \phi^\Sigma - Y'_-(x) \partial_Z \phi^\Sigma - \partial_Z^4 \phi^\Sigma = \\ = -\delta\tau^{\text{lift}}(x, Y_-(x) + Z), \quad x < x(\sigma_{in}), \quad Z > 0, \\ \phi^\Sigma|_{x=x(\sigma_{in})} = 0, \quad \phi^\Sigma|_{Z=0} = \partial_Z \phi^\Sigma|_{Z=0} = 0. \end{cases}$$

which is precisely an equation of the type (2.13). In the equation above, we still denote the rescaled boundary layer coordinate by Z , in order to be consistent with (2.13). However, in the present context, Z is just a scaled cartesian coordinate.

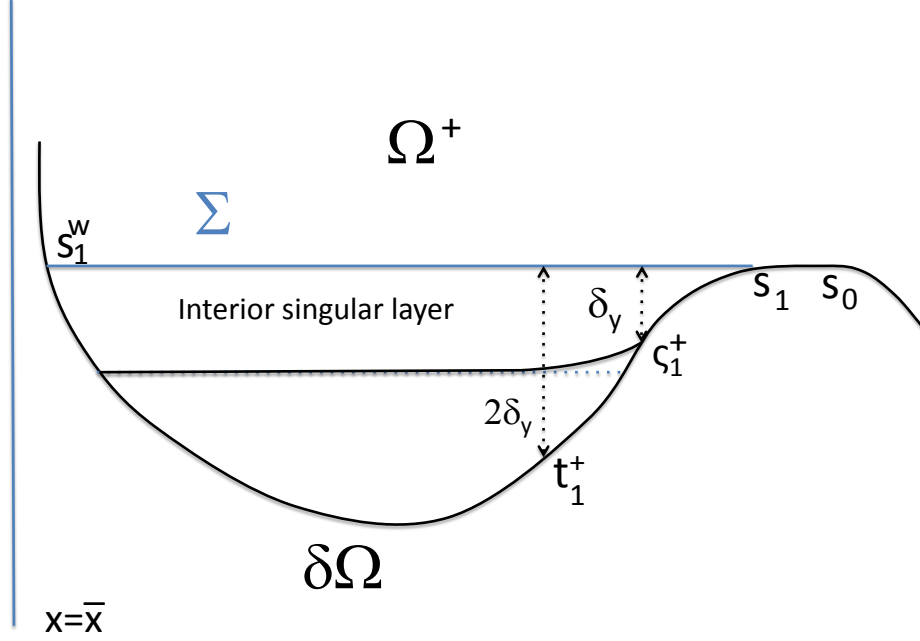


FIGURE 3.3. The interior singular layer

Notice that ψ^Σ and ϕ^Σ depend in fact on ν through the function Y_- and the coefficients a and b . The asymptotic value of ψ^Σ as ν vanishes will be given later on.

This indeed requires a refined version of Lemma 2.3.1 giving high order *a priori* estimates on the solutions to the singular layer equation (2.23), as well as Lemma C.1 controlling the sizes of a , b and their derivatives.

LEMMA 3.4.1. *Let ϕ^Σ be the solution to (3.38), and $\psi^\Sigma(x, Y) = \phi^\Sigma(x, Y - Y_-(x))$.*

Denote by $(x_1^W, y_1) = (x(s_1^W), y(s_1^W))$ the projection of (x_1, y_1) on the West boundary.

Then, for all $\bar{x} < x_1^W$,

$$\begin{aligned} \|\psi^\Sigma\|_{L^\infty((\bar{x}, x(\sigma_{in})), L^2(Y_-, \infty))} &= O(1), \\ \|\partial_x \psi^\Sigma\|_{L^4((\bar{x}, x(\sigma_{in})), L^2(Y_-, \infty))} &= O(\nu^{-1/19}), \\ \|\partial_x^2 \psi^\Sigma\|_{L^2((\bar{x}, x(\sigma_{in})), L^2(Y_-, \infty))} &= O(\nu^{-2/19}). \end{aligned}$$

Furthermore, for all integers k, l with $k + l \leq 3$,

$$\|\partial_x^k \partial_Z^l \psi^\Sigma\|_{L^\infty(\bar{x}, x(t_1^+))} \leq C\nu^{-\frac{1+2l+8k}{8 \times 19}},$$

where t_1^+ is defined by (3.37).

REMARK 3.4.2. *The function ψ^Σ is defined beyond the West end of $\{(x, y) \in \Omega\} / y = \bar{Y}_-$: indeed, for some values of s in a neighbourhood of s_1^W , $x(s)$ may*

be smaller than x_1^W (notice that this is the case as soon as $\sin \theta(s_1^W) \neq 0$, i.e. as soon as the tangent to $\partial\Omega$ at $s = s_1^W$ is not vertical). Thus the trace $\psi|_{\Gamma_W}^\Sigma$ involves values of ψ^Σ at points $x < x_1^W$.

PROOF. Such a priori estimates are obtained by combining precised energy estimates for the parabolic equation (3.38), together with a refined description of the geometry Y_- , and high order estimates on the source terms.

• **Estimates on ϕ^Σ .**

Denoting by $S(x, Z) = -\delta\tau^{\text{lift}}(x, Y_-(x) + Z)$ the right-hand side of (3.38), we recall that

$$\begin{aligned} S(x, Z) &= -\left(1 - \mathbf{1}_{x < x_1, Z + Y_-(x) < 0}\right) \chi(Z + Y_-(x)) \left[a'(x) + \nu^{1/4} b'(x)(Z + Y_-(x)) \right] \\ &\quad + \left(1 - \mathbf{1}_{x < x_1, Z + Y_-(x) < 0}\right) \chi^{(4)}(Z + Y_-(x)) \left[a(x) + \nu^{1/4} b(x)(Z + Y_-(x)) \right] \\ &\quad + 4\left(1 - \mathbf{1}_{x < x_1, Z + Y_-(x) < 0}\right) \chi^{(3)}(Z + Y_-(x)) \nu^{1/4} b(x). \end{aligned}$$

Notice that the indicator function does not create any singularity on the domain of integration $x \in (\bar{x}, x(\sigma_{in})) \times (0, \infty)$: indeed, $\chi^{(3)}$ and $\chi^{(4)}$ are identically zero in a neighbourhood of zero, so that $\mathbf{1}_{Y < 0} \chi^{(k)}(Y)$ is in fact a C^∞ function for $k = 3, 4$. Moreover, for $x < x_1$, a and b are constant, so that

$$\mathbf{1}_{x < x_1} \mathbf{1}_{Z + Y_-(x) < 0} \chi(Z + Y_-(x)) \left(a'(x) + \nu^{1/4} b'(x)(Z + Y_-(x)) \right) \equiv 0.$$

Eventually, for $x = x_1$, $Y_- = 0$, so that $\mathbf{1}_{Z > 0} \mathbf{1}_{Z + Y_-(x_1) < 0} = 0$.

It is easily checked that

$$\|S(x, \cdot)\|_{L^2(\mathbf{R}_+)} \leq C(|a| + |a'| + \nu^{1/4}|b| + \nu^{1/4}|b'|),$$

so that, using Lemma C.1 in the Appendix,

$$\|S\|_{L^1((\bar{x}, x(\sigma_{in})), L^2(\mathbf{R}_+))} \leq C.$$

According to Lemma 2.3.1, we then have

$$\|\phi^\Sigma\|_{L^\infty(\bar{x}, x(\sigma_{in})), L^2(\mathbf{R}_+)} + \|\partial_Z^2 \phi^\Sigma\|_{L^2((\bar{x}, x(\sigma_{in})) \times \mathbf{R}_+)} = O(1).$$

In order to derive estimates on $\partial_x^k \phi^\Sigma$ for $k \geq 1$, we differentiate equation (3.38) and we proceed by induction on k . Notice that because of the various truncations and of the choice of σ_{in} , ϕ^Σ is identically zero in a neighbourhood of $x = x(\sigma_{in})$, so that the initial data is always zero, together with the boundary conditions. Moreover, as explained before, the indicator function in the right-hand side does not raise any singularity. Using the same kind of estimates as in Lemma 3.3.4, we infer that

$$\begin{aligned} (3.39) \quad & \|\partial_x^k \phi^\Sigma\|_{L_x^\infty(L_Z^2)} + \|\partial_Z^2 \partial_x^k \phi^\Sigma\|_{L_{x,Z}^2} \\ & \leq C \|\partial_x^k S\|_{L^1((\bar{x}, x(\sigma_{in})), L^2(\mathbf{R}_+))} \\ & \quad + C \sum_{l=0}^{k-1} \|Y_-^{(k-l)}\|_{L^{4/3}(\bar{x}, x(\sigma_{in}))} \left(\|\partial_x^l \phi^\Sigma\|_{L_x^\infty(L_Z^2)} + \|\partial_Z^2 \partial_x^l \phi^\Sigma\|_{L_{x,Z}^2} \right). \end{aligned}$$

• **Estimates on $\partial_x^k S$.**

As in the proof of Lemma 3.3.2, the most singular estimates for $Y_-^{(k)}$ are obtained when $\cos \theta$ vanishes algebraically near $s = s_1$, with the lowest possible exponent n . Moreover, by assumption (H4), the exponent n above satisfies $n \geq 4$. Hence we have

$$(3.40) \quad \begin{aligned} \|Y_-^{(k)}\|_\infty &\leq C |\ln \nu|^{1/5} \nu^{-k/20}, \\ \|Y_-^{(k)}\|_{L^{4/3}} &\leq C |\ln \nu|^{1/5} \nu^{(\frac{3}{4}-k)\frac{1}{20}}. \end{aligned}$$

We now use the estimates of Lemma C.1 in Appendix C together with (3.40). Denoting f

$$A_k := \|a^{(k)}\|_{L^1(x_{\min}, x_{\max})} + \nu^{1/4} \|b^{(k)}\|_{L^1(x_{\min}, x_{\max})} = O(\nu^{\frac{1-k}{19}}).$$

or all $k \in \{1, \dots, 4\}$, we have

$$\begin{aligned} \|\partial_x S\|_{L^1 L^2} &\leq C((A_1 + A_2) + \|Y'_-\|_\infty(A_0 + A_1)), \\ \|\partial_x^2 S\|_{L^1 L^2} &\leq C(A_2 + A_3) + \|Y'_-\|_\infty(A_1 + A_2) \\ &\quad + C(\|Y''_-\|_\infty + \|Y'_-\|_\infty^2)(A_0 + A_1), \\ \|\partial_x^3 S\|_{L^1 L^2} &\leq C(A_3 + A_4) + \|Y'_-\|_\infty(A_2 + A_3) \\ &\quad + C(\|Y''_-\|_\infty + \|Y'_-\|_\infty^2)(A_1 + A_2) \\ &\quad + C(\|Y_-^{(3)}\|_\infty + \|Y'_-\|_\infty \|Y''_-\|_\infty + \|Y'_-\|_\infty^3)(A_0 + A_1), \end{aligned}$$

so that

$$(3.41) \quad \|\partial_x^k S\|_{L_x^1(L_x^2)} \leq C \nu^{-k/19}.$$

• $W^{k,\infty}$ estimates on ψ^Σ :

• Gathering (3.39), (3.40) and (3.41), we prove easily by induction that for $k \in \{0, \dots, 3\}$,

$$\|\partial_x^k \phi^\Sigma\|_{L^\infty((\bar{x}, x(\sigma_0^-), L^2(\mathbf{R}_+))} + \|\partial_Z^2 \partial_x^k \phi^\Sigma\|_{L^2((\bar{x}, x(\sigma_0^-), L^2(\mathbf{R}_+))} = O(\nu^{-k/19}).$$

The first estimates on ψ^Σ follow from the identities

$$\begin{aligned} \partial_x \psi^\Sigma(x, Y) &= \partial_x \phi^\Sigma(x, Y - Y_-) - Y'_- \partial_Z \phi^\Sigma(x, Y - Y_-), \\ \partial_x^2 \psi^\Sigma(x, Y) &= \partial_x^2 \phi^\Sigma(x, Y - Y_-) - 2Y'_- \partial_x \partial_Z \phi^\Sigma(x, Y - Y_-) \\ &\quad - Y''_- \partial_Z \phi^\Sigma(x, Y - Y_-) - Y_-'^2 \partial_Z^2 \phi^\Sigma(x, Y - Y_-). \end{aligned}$$

Then, notice that for $x < x(t_1^+)$, $Y_- \equiv \bar{Y}_-$, so that

$$\|\psi^\Sigma\|_{W^{k,\infty}((\bar{x}, x(t_1^+)) \times (\bar{Y}_-, \infty))} = \|\phi^\Sigma\|_{W^{k,\infty}((\bar{x}, x(t_1^+)) \times (0, \infty))}.$$

We then use Sobolev inequalities in order to bound the L^∞ norms. More precisely, we use repeatedly the following Lemma :

LEMMA 3.4.3. *There exists a constant $C > 0$ such that for any $f \in H^4(\mathbf{R}_+)$ satisfying $f|_{Z=0} = \partial_Z f|_{Z=0} = 0$, we have $f \in W^{3,\infty}(\mathbf{R}_+)$ and for $k = 0, 1, 2, 3$,*

$$\|\partial_Z^k f\|_{L^\infty(\mathbf{R}_+)} \leq C \|f\|_{L^2(\mathbf{R}_+)}^{\frac{7-2k}{8}} \|\partial_Z^4 f\|_{L^2(\mathbf{R}_+)}^{\frac{1+2k}{8}}.$$

Lemma 3.4.3 is proved in Appendix D. Notice that for $x < x(t_1^+)$, ϕ^Σ satisfies

$$\begin{aligned} \partial_x \phi^\Sigma - \partial_Z^4 \phi^\Sigma &= (\mathbf{1}_{Z+\bar{Y}^- < 0} - 1) \chi^{(4)}(Z + \bar{Y}^-) (\bar{a} + \nu^{1/4} \bar{b}(Z + \bar{Y}^-)) \\ &\quad + 4(1 - \mathbf{1}_{x < x_1, Z+Y_-(x) < 0}) \chi^{(3)}(Z + Y_-(x)) \nu^{1/4} b(x), \end{aligned}$$

so that for $k \geq 2$,

$$\partial_x^k \phi^\Sigma = \partial_Z^4 \partial_x^{k-1} \phi^\Sigma, \quad x < x(t_1^+), Z > \bar{Y}^-.$$

Lemma 3.4.3 and equation (3.38) imply that for $k \geq 1$,

$$\begin{aligned} \|\partial_x^k \phi^\Sigma\|_{L^\infty((\bar{x}, x(t_1^+)) \times (0, \infty))} &\leq \|\partial_x^k \phi^\Sigma\|_{L_x^\infty(L_Z^2)}^{7/8} \|\partial_Z^4 \partial_x^k \phi^\Sigma\|_{L_x^\infty(L_Z^2)}^{1/8} \\ &\leq \|\partial_x^k \phi^\Sigma\|_{L_x^\infty(L_Z^2)}^{7/8} \|\partial_x^{k+1} \phi^\Sigma\|_{L_x^\infty(L_Z^2)}^{1/8} \\ &\leq C \nu^{-\frac{7k}{8 \times 19}} \nu^{-\frac{k+1}{8 \times 19}} \\ &\leq C \nu^{-\frac{k}{19} - \frac{1}{8 \times 19}}, \end{aligned}$$

while

$$\begin{aligned} \|\partial_Z^l \phi^\Sigma\|_{L^\infty((\bar{x}, x(t_1^+)) \times (0, \infty))} &\leq C \|\phi^\Sigma\|_{L_x^\infty(L_Z^2)}^{(7-2l)/8} \|\partial_Z^4 \phi^\Sigma\|_{L_x^\infty(L_Z^2)}^{(1+2l)/8} \\ &\leq C \left(\|\partial_x \phi^\Sigma\|_{L_x^\infty(L_Z^2)} + \|S\|_{L_x^\infty(L_Z^2)} \right)^{(1+2l)/8} \\ &\leq C \nu^{-\frac{1+2l}{8 \times 19}}. \end{aligned}$$

Similar estimates can be derived for terms of the type $\partial_x^k \partial_Z^l \phi^\Sigma$, with $k+l \leq 3$. This concludes the proof of Lemma 3.4.1. \square

REMARK 3.4.4. • *Because of the function Y_- and the coefficients a and b , the profile ψ^Σ depends on ν in general, which is not entirely satisfactory for an approximate solution. However, the estimates of Lemma 3.4.1 show that ψ^Σ is uniformly bounded in $L^\infty(L^2) \cap L^2(H^2)$.*

Let $\underline{x} \in (\bar{x}, x_1)$ arbitrary. Then $x(t_1^+) > \underline{x}$ for ν small enough, so that $Y_-(x) = \bar{Y}_-$ for $x \in (\bar{x}, \underline{x})$. Extending ψ^Σ by zero below \bar{Y}_- , we infer that ψ^Σ is uniformly bounded in $L^2((\bar{x}, \underline{x}), H^2(\mathbf{R}))$. Up to the extraction of a sub-sequence, ψ^Σ converges towards a function $\bar{\psi}^\Sigma$ weakly in $L^2((\bar{x}, \underline{x}), H^2(\mathbf{R}))$. Passing to the limit in the equation satisfied by ψ^Σ , we obtain

$$\partial_x \bar{\psi}^\Sigma - \partial_Y^4 \bar{\psi}^\Sigma = \bar{S}(Y), \quad \text{in } \mathcal{D}'((\bar{x}, \underline{x}) \times \mathbf{R}),$$

where

$$\bar{S}(Y) = -\bar{a} \chi^{(4)}(Y), \quad \text{with } \bar{a} = - \int_{x_E^-(y_1)}^{x_E^+(y_1)} \tau(x, y_1) dx.$$

• *Notice that once ψ^{lift} and ψ^Σ are constructed, there remains a non zero trace on the East part of the boundary (s_1, s_2) , and also on (s_{k-1}, s_0) if $(s_{k-1}, s_0) \subset \Gamma_E$ (notice that $\sigma_{in} = s_0$ in this case according to (3.36), and therefore the trace of ψ_E on (σ_0^-, s_0) is not lifted by ψ^{lift}).*

These traces correspond to $\psi_E|_{Z=0}(1 - \varphi_1^+ \varphi_2^-)$ and $\psi_E|_{Z=0}(1 - \varphi_0^- \varphi_{k-1}^+)$. They can be lifted thanks to South type boundary layer terms, exactly as explained in the previous section.

3.4.3. Connection with the West boundary. The question is then to describe the West junction, that is the connection with the West boundary layer. The traces of the discontinuity terms $\psi^{\text{lift}} + \psi^\Sigma \chi\left(\frac{y-y_1}{\delta_y}\right)$ on the West boundary create yet another singularity in the West boundary layer term: indeed, the coefficient $A(s)$ in (2.7) now lifts the traces of $\psi^{\text{int}} + \psi^\Sigma$, and thus changes abruptly near the point $s_1^W \in \Gamma_W$. We recall that $\psi^{\text{int}} = \psi_t^0 + \psi^{\text{lift}}$.

At this stage we encounter another difficulty: the function $\partial_s A$ can be written as

$$\begin{aligned} \partial_s A_\pm(s) &= \pm \left| \frac{E}{\cos \theta} \right|^{1/3} \cos \theta(s) \sin \theta(s) \partial_y^2 \psi^{\text{int}}(x(s), y(s)) \\ &\quad + \text{differentiable terms.} \end{aligned}$$

The function $\partial_y^2(\psi_t^0 + \psi^{\text{lift}})$ is discontinuous across Σ : more precisely,

$$[\partial_y^2(\psi_t^0 + \psi^{\text{lift}})]_\Sigma = [\partial_y^2 \psi_t^0]_\Sigma,$$

and the jump of $\partial_y^2 \psi_t^0$ across Σ is constant along Σ and of order δ_y^{-2} . As a consequence, A' is not continuous, which is problematic, because the error estimates for the West boundary layer involve L^p norms of A'' , as we will see in the next chapter (see the proof of Lemma 4.2.6).

In order to avoid the apparition of a Dirac mass in A'' , we add yet another corrector term which has the following form

$$(3.42) \quad \psi_\Sigma^{\text{corr}} = -[\partial_y^2 \psi_t^0]_\Sigma \mathbf{1}_{y > y_1} (y - y_1)^2 \chi\left(\frac{y - y_1}{E^{1/4}}\right) \chi\left(\frac{x - x(s_1^W)}{|\log E|^{-1/5}}\right).$$

It is obvious that $\psi_\Sigma^{\text{corr}} \in H^2(\Omega)$. Moreover, $\psi_\Sigma^{\text{corr}}$ satisfies the following estimates:

LEMMA 3.4.5. *For all $k \in \{0, 1, 2\}$, we have*

$$\|\psi_\Sigma^{\text{corr}}\|_{W^{k, \infty}} = O(\delta_y^{-2} E^{-\frac{k-2}{4}}) = o(E^{-k/4}).$$

Moreover,

$$\|\partial_x \psi_\Sigma^{\text{corr}}\|_{L^2(\Omega)} = o(E^{1/8}), \quad \|E \Delta^2 \psi_\Sigma^{\text{corr}}\|_{H^{-2}(\Omega)} = o(E^{5/8}).$$

PROOF. The estimates are fairly straightforward, noticing that for $l \in \{0, 1, 2\}$, $k \in \mathbf{N}$,

$$\left\| (y - y_1)^l \chi^{(k)}\left(\frac{y - y_1}{E^{1/4}}\right) \right\|_{L^\infty} = O(E^{l/4}).$$

Since $[\partial_y^2 \psi_t^0]_\Sigma$ does not depend on x nor y , we have

$$\partial_x^k \psi_\Sigma^{\text{corr}} = -[\partial_y^2 \psi_t^0]_\Sigma \mathbf{1}_{y > y_1} (y - y_1)^2 \chi\left(\frac{y - y_1}{E^{1/4}}\right) |\ln E|^{k/5} \chi^{(k)}\left(\frac{x - x(s_1^W)}{|\log E|^{-1/5}}\right),$$

so that

$$\|\partial_x^k \psi_\Sigma^{\text{corr}}\|_\infty = O(\delta_y^{-2} E^{1/2} |\ln E|^{k/5}) = O(|\ln E|^{\frac{k-2}{5}} |\ln |\ln E||^{2\beta}).$$

Let us now estimate $\partial_y \psi_\Sigma^{corr}$; the rest of the estimates are left to the reader. We have

$$\begin{aligned} \partial_y \psi_\Sigma^{corr}(x, y) &= -[\partial_y^2 \psi_t^0]_\Sigma \chi \left(\frac{x - x(s_1^W)}{|\log E|^{-1/5}} \right) \mathbf{1}_{y > y_1} (y - y_1) \\ &\quad \times \left[2\chi \left(\frac{y - y_1}{E^{1/4}} \right) + E^{-1/4} (y - y_1) \chi' \left(\frac{y - y_1}{E^{1/4}} \right) \right], \end{aligned}$$

so that

$$\|\partial_y \psi_\Sigma^{corr}\|_\infty \leq C |[\partial_y^2 \psi_t^0]_\Sigma| E^{1/4} \leq C \delta_y^{-2} E^{1/4}.$$

More generally, for $k = 0, 1, 2$

$$\|\partial_y^k \psi_\Sigma^{corr}\|_\infty = O(\delta_y^{-2} E^{\frac{2-k}{4}}) = O(E^{-k/4} |\ln E|^{-2/5} |\ln |\ln E||^{2\beta}).$$

The $W^{k,\infty}$ estimates follow easily. Moreover, since the support of ψ_Σ^{corr} is included in the rectangle

$$[x_1^W - 2|\ln E|^{-1/5}, x_1^W + 2|\ln E|^{-1/5}] \times [y_1 - 2E^{1/4}, y_1 + 2E^{1/4}],$$

we infer that

$$\begin{aligned} \|\partial_x \psi_\Sigma^{corr}\|_{L^2(\Omega)} &= O(|\ln E|^{-1/5} |\ln |\ln E||^{2\beta} |\ln E|^{-1/10} E^{1/8}) = o(E^{1/8}), \\ \|\Delta \psi_\Sigma^{corr}\|_{L^2(\Omega)} &= O(E^{-1/2} |\ln E|^{-2/5} |\ln |\ln E||^{2\beta} |\ln E|^{-1/10} E^{1/8}) = o(E^{-3/8}). \end{aligned}$$

□

In the rest of the paper, we set

$$\bar{\psi}_\Sigma := \psi^{\text{lift}} + \psi^\Sigma \chi \left(\frac{y - y_1}{\delta_y} \right) + \psi_\Sigma^{corr}.$$

With this definition, $\psi_t^0 + \bar{\psi}_\Sigma \in H^2(\Omega)$.

Moreover, we have

$$\partial_x(\psi_t^0 + \bar{\psi}_\Sigma) - E\Delta^2(\psi_t^0 + \bar{\psi}_\Sigma) = \tau + \delta\tau + r_{\text{lift}}^1 + r_{\text{lift}}^2 + \delta\tau_\sigma \quad \text{on } \Omega$$

where $\delta\tau$ has been estimated in Proposition 3.1.1, $r_{\text{lift}}^1 + r_{\text{lift}}^2$ is controlled by Proposition 2.4.3 and

$$(3.43) \quad \delta\tau_\Sigma = \partial_x \psi_\Sigma^{corr} - E\Delta^2 \psi^{corr} + E(\partial_y^4 - \Delta^2) \psi^\Sigma,$$

PROPOSITION 3.4.6. *Assume that Ω satisfies assumptions (H1)-(H4). Then $\delta\tau_\Sigma$ is an admissible remainder in the sense of Definition 1.1.2.*

This proposition will be proved in Chapter 4.

Note further that, by assumption on Σ , the trace of $\bar{\psi}_\Sigma + \psi_t^0 + \bar{\psi}_{N,S}$ on the boundary Γ_W vanishes in the vicinity the points $s_i \in \Gamma_W$.

REMARK 3.4.7. *If one looks carefully at the construction above, there are several reasons why the assumption (H4) is necessary: some of them are technical, and others are probably more fundamental. We first notice that if $I_1 = [s_0, s_1]$ with $s_0 < s_1$, assumption (H4) is always satisfied, as $\partial\Omega$ is C^4 near s_0 . Therefore we focus on the case $I_1 = \{s_1\}$.*

- First, assumption (H4) is required so that the coefficients a and b have derivatives of order four; if $\cos \theta$ vanishes only at order n near s_1 , with $n \leq 3$ in (H2i), then such a regularity is not achieved in general. Therefore, if (H4) is not satisfied, an additional regularization of a and b must be performed.
- Moreover, the estimates on ψ^Σ are more singular if the cancellation of $\cos \theta$ near $s = s_1$ has a lower order. This could have two different consequences: on the one hand, it could be possible that the estimates on ψ^Σ and its derivatives in the interior of Ω are not sufficient to prove that the error terms are admissible. This, however, is unlikely: indeed, the estimates are exactly the same as the ones for North and South boundary layer terms, for which we are able to prove that the error terms are admissible even when $\cos \theta$ vanishes at the lowest possible order.

On the other hand, if $\cos \theta$ vanishes at a low order near $s = s_1$, the estimations on $\|\psi^\Sigma\|_{W^{k,\infty}}$ become much more singular, and therefore the trace of ψ^Σ and its derivatives on the West boundary are much larger. In this regard, assumption (H4) does not seem to be merely technical: it is possible that the discontinuity boundary layer may destabilize the West boundary layer if ψ^Σ becomes very large. This could in fact be achieved with a low order of cancellation of $\cos \theta$ near s_1 . However, such considerations go beyond the scope of this paper.

3.5. Lifting the West boundary conditions

We recall that the boundary layer term on West boundaries takes the form $\psi_W(s, Z) = A(s)^t f(Z)$, where $A(s)$ is defined by

$$(3.44) \quad A(s) = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\pi/6} & -i\lambda_W^{-1} \\ e^{-i\pi/6} & i\lambda_W^{-1} \end{pmatrix} \begin{pmatrix} -(\psi_t^0 + \bar{\psi}_\Sigma)|_{\partial\Omega}(s) \\ \partial_n(\psi_t^0 + \bar{\psi}_\Sigma)|_{\partial\Omega}(s) \end{pmatrix}$$

for $s \in \Gamma_W$ and

$$f(Z) = \begin{pmatrix} \exp(-e^{i\pi/3}Z) \\ \exp(-e^{-i\pi/3}Z) \end{pmatrix}.$$

LEMMA 3.5.1. *Define the amplitude of the West boundary layer by (3.44). Let (s_i, s_{i+1}) be a portion of Γ_W .*

Then there exists a constant c_0 such that the following estimates hold:

- *Far from the West end of Σ , i.e. for $|s - s_1^W| \geq c_0\delta_y$, and for all $s \in \text{Supp}(\varphi_i^+ \varphi_{i-1}^-)$,*

$$(3.45) \quad \begin{aligned} |A(s)| &\leq C, \\ |A'(s)| &\leq C(1 + |\cos \theta| \mathcal{M}(y(s))) \end{aligned}$$

$$(3.46) \quad |A''(s)| \leq C \left(\frac{1}{\delta_y} + \frac{(\cos \theta)^2}{\delta_y^2 |\ln \delta_y|} \right).$$

- *If $|s - s_1^W| \leq c_0\delta_y$,*

$$(3.47) \quad |\partial_s^k A(s)| \leq CE^{-\frac{k}{4} - \frac{1+2k}{8 \times 19}} \quad \text{for } k \in \{0, 1, 2\}.$$

PROOF. Consider a component $[s_{i-1}, s_i]$ of the West boundary. There are two zones on which the s derivatives of A become singular:

- near s_{i-1} and s_i , or, more generally, near points of $\partial\Omega$ such that $|y(s) - y_j| = 0$ ($y_j \neq y_1$); on such zones, the derivatives of the function ψ_t^0 become unbounded, while $\bar{\psi}_\Sigma$ is identically zero.
- near s_1^W , the end point of Σ ; we recall that the trace of ψ_t^0 is discontinuous at this point, and that this discontinuity is lifted by the term ψ^{lift} . Additionally, the s derivative of $\partial_n \psi_t^0$ is also discontinuous, and this discontinuity is corrected by the term $\psi_\Sigma^{\text{corr}}$ introduced in equation (3.42). Notice however that on this zone, $\cos \theta$ is bounded away from zero.

Away from these zones, A and its derivatives are bounded, and $\cos \theta$ is bounded away from zero. Whence we now focus on the two pathological zones. Since the proof of the estimation on each zone is rather different, we separate the two.

- **Estimate near points such that $|y(s) - y_j| \ll 1$ ($y_j \neq y_1$):**

In the vicinity of such points, thanks to assumption (H3), ψ_t^0 is C^∞ (with derivatives which are not uniformly bounded). We also use the following bound:

$$\sup_{s \in \text{Supp}(\varphi_i^- \varphi_{i-1}^+)} |\lambda_W^{-1}(s)| \leq C \begin{cases} E^{\frac{n+1}{4n+3}} & \text{in case (H2i),} \\ E^{1/4} |\ln E|^{-1/2} & \text{in case (H2ii),} \end{cases}$$

so that $\sup_{s \in \text{Supp}(\varphi_i^- \varphi_{i-1}^+)} |\lambda_W^{-1}(s)| \ll E^{1/4} \ll \delta_y$. In the rest of the proof, we write λ for λ_W . Using Lemma 3.1.3, we infer that for all $s \in \text{Supp}(\varphi_i^- \varphi_{i-1}^+)$,

$$(3.48) \quad |A(s)| \leq C (1 + |\lambda(s)|^{-1} \mathcal{M}(y(s))) \leq C.$$

The estimates of the s derivatives are a little more involved. Using (3.44), we have

$$|A'(s)| \leq C \left(|\partial_s \psi_t^0|_{\partial\Omega}(s) + |\lambda|^{-1} |\partial_s \partial_n \psi_t^0|_{\partial\Omega}(s) + \left| \frac{\lambda'}{\lambda^2} \right| |\partial_n \psi_t^0|_{\partial\Omega}(s) \right).$$

Notice that on the support of $\varphi_i^- \varphi_{i-1}^+$,

$$\left| \frac{\lambda'}{\lambda^2} \right| \leq \frac{E^{1/3} |\theta'|}{|\cos \theta|^{4/3}} \leq |\cos \theta|$$

thanks to the validity condition (2.8). Lemma 3.1.3 then implies, for $s \in \text{Supp}(\varphi_i^- \varphi_{i-1}^+)$,

$$|A'(s)| \leq C \left(1 + \frac{|\cos \theta|}{\delta_y |\ln \delta_y|} \sum_{j \in I_+} \mathbf{1}_{|y(s) - y_j| \leq \delta_y} + |\cos \theta| \mathcal{M}(y(s)) \right).$$

In a similar fashion,

$$\begin{aligned} |A''(s)| &\leq C \left(|(\partial_s^2 \psi_t^0)|_{\partial\Omega}(s) + \left| \frac{\lambda'}{\lambda^2} \right| |(\partial_s \partial_n \psi_t^0)|_{\partial\Omega}(s) \right) \\ &\quad + C \left(|\lambda|^{-1} |(\partial_s^2 \partial_n \psi_t^0)|_{\partial\Omega}(s) + \left| \left(\frac{\lambda'}{\lambda^2} \right)' \right| |(\partial_n \psi_t^0)|_{\partial\Omega}(s) \right). \end{aligned}$$

We have

$$\left| \left(\frac{\lambda'}{\lambda^2} \right)' \right| \leq C \frac{E^{1/3}}{|\cos \theta|^{7/3}} (|\theta'|^2 + |\theta''| |\cos \theta|).$$

Since $\cos \theta$ vanishes always at a higher order than θ' , we infer that on $\text{Supp}(\varphi_i^- \varphi_{i-1}^+)$,

$$\left| \left(\frac{\lambda'}{\lambda^2} \right)' \right| \leq C \frac{E^{1/3} |\theta'|^2}{|\cos \theta|^{7/3}} \leq C.$$

Finally, we obtain, for $s \in \text{Supp}(\varphi_i^- \varphi_{i-1}^+)$,

$$|A''(s)| \leq C \left(\frac{1}{\delta_y} + \frac{(\cos \theta)^2}{\delta_y^2 |\ln \delta_y|} \right).$$

• **Estimates near the end point of Σ :**

As explained above, since the interior term is now $\psi_t^0 + \bar{\psi}_\Sigma$, the coefficient $A(s)$ changes abruptly near s_1^W , on an interval whose length is of order δ_y . Moreover, since (H3) is satisfied, we have

$$\cos \theta(s_1^W) \neq 0,$$

so that $\varphi_i^- \varphi_{i-1}^+$ and its derivatives are $O(1)$ in a neighbourhood V of s_1^W independent of E . As before, we use (3.44) and we obtain

$$\begin{aligned} \|A\|_{L^\infty(V)} &\leq C(\|\psi_t^0 + \bar{\psi}_\Sigma\|_{L^\infty} + E^{1/3}\|\psi_t^0 + \bar{\psi}_\Sigma\|_{W^{1,\infty}}), \\ \|A'\|_{L^\infty(V)} &\leq C(\|\psi_t^0 + \bar{\psi}_\Sigma\|_{W^{1,\infty}} + E^{1/3}\|\psi_t^0 + \bar{\psi}_\Sigma\|_{W^{2,\infty}}). \end{aligned}$$

The $W^{k,\infty}$ norms of the right-hand side are to be understood in a neighbourhood of size $O(1)$ in Ω of $(x(s_1^W), y(s_1^W))$. We now recall that thanks to the addition of the corrector term ψ_Σ^{corr} , the amplitude A' is differentiable at $s = s_1^W$, and

$$\|A''\|_{L^\infty(V)} \leq C(\|\psi_t^0 + \bar{\psi}_\Sigma\|_{W^{2,\infty}} + E^{1/3}\|\psi_t^0 + \bar{\psi}_\Sigma\|_{W^{3,\infty}}).$$

We now estimate $\|\psi_t^0 + \bar{\psi}_\Sigma\|_{W^{k,\infty}}$ using Lemmas 3.1.3, 3.4.1, 3.4.5, together with the explicit definition of ψ^{lift} . We infer that

$$\|\psi_t^0 + \bar{\psi}_\Sigma\|_{W^{k,\infty}} = O(E^{-\frac{k}{4} - \frac{1+2k}{8 \times 19}}) \quad \text{for } k \in \{0, \dots, 3\},$$

which leads to the desired estimates. □

3.6. Approximate solution in the periodic and rectangle case

• We start with the periodic case, for which the solution has already been completely defined (see (1.26) and section 2.6). We therefore only give here a few estimates on that solution.

We recall that

$$\psi_{app} = \psi^{circ} + \psi_{per}^0 + \psi_{N,S}^{BL}$$

where ψ_{per}^0 and $\psi_{N,S}^{BL}$ are defined as before and $\psi^{circ} = \psi^{circ}(y)$ is the circumpolar current, due to the average forcing by the wind, namely

$$\begin{aligned} -E\partial_y^4 \psi^{circ} &= \langle \tau \rangle (y), \quad y \in (y_-, y_+), \\ \psi_{|y=y_\pm}^{circ} &= 0, \\ \partial_y \psi_{|y=y_\pm}^{circ} &= 0. \end{aligned}$$

Provided τ is sufficiently smooth (i.e. belongs to H^s for s large enough), we have

$$(3.49) \quad \begin{aligned} \|\psi_{circ}\|_{H^s} &\leq CE^{-1}\|\tau\|_{H^{(s-4)_+}}, \\ \|\psi^0\|_{H^s} &\leq C\|\tau\|_{H^s}, \end{aligned}$$

and consequently by (2.28)

$$(3.50) \quad \begin{aligned} \|\partial_x^k \partial_Z^l \psi_{N,S}^{BL}\|_{L^2(\mathbb{T} \times \mathbf{R}_+)} &\leq C(\|\psi_{y=y_\pm}^0\|_{H^{k+\frac{1}{4}-\frac{1}{8}}(\mathbb{T})} + E^{1/4}\|\partial_y \psi_{y=y_\pm}^0\|_{H^{k+\frac{1}{4}-\frac{3}{8}}(\mathbb{T})}) \\ &\leq C(\|\tau\|_{H^{k+\frac{1}{4}+\frac{3}{8}}} + E^{1/4}\|\tau\|_{H^{k+\frac{1}{4}+\frac{3}{8}}}). \end{aligned}$$

• In the rectangle case, i.e. when $\Omega = (x_-, x_+) \times (y_-, y_+)$, the construction explained in the preceding sections for a smooth domain $\Omega \subset \mathbf{R}^2$ still works:

- Since $\cos \theta$ does not vanish at the north and south ends of the east boundary, there is no need for a truncation. We therefore simply define

$$\begin{aligned} \partial_x \psi^0 &= \tau \quad \text{in } \Omega, \\ \psi_{|x=x_+}^0 &= 0. \end{aligned}$$

- We also construct an East boundary layer corrector and a macroscopic corrector (see section 3.2) on the whole East boundary; again, this is possible because there is no singularity in ψ^0 . We denote by $\bar{\psi}_E(x, y)$ the sum of these two terms.
- We then construct north and south boundary layers $\psi_{N,S}^{BL}$, lifting the traces of $(\psi^0 + \bar{\psi}_E)|_{y=y_\pm}$, $\partial_y(\psi^0 + \bar{\psi}_E)|_{y=y_\pm}$, with zero initial condition at $x = x_+$. When τ is sufficiently smooth, $\Psi_0 = -(\psi^0 + \bar{\psi}_E)|_{y=y_\pm}$ and $\Psi_1 = -E^{1/4}\partial_y(\psi^0 + \bar{\psi}_E)|_{y=y_\pm}$ are smooth functions of x with large derivatives of order 2 or higher (because of the east boundary layer), which satisfy better estimates than in the case of a regular domain. Typically

$$\begin{aligned} \|\Psi_j\|_{L^\infty} + \|\partial_x \Psi_j\|_{L^\infty} &\leq C, \\ \partial_x^2 \Psi_j &= O(1)_{L^\infty} + O(E^{-1/3} \mathbf{1}_{|x-x_+| \leq CE^{1/3}}). \end{aligned}$$

Furthermore, as $\psi_{N,S}^{BL}$ is a smooth function of x and Z ,

$$\partial_x \psi_{N,S}^{BL}|_{x=x_+} = \partial_Z^4 \psi_{N,S}^{BL}|_{x=x_+} = 0.$$

As a consequence, the trace of $\psi_{N,S}^{BL}$ on the east boundary and the trace of its normal derivative (i.e. its x derivative) is identically zero: **the North and South boundary layer correctors do not pollute the East boundary.**

- The next and last step is the construction of the western boundary layer. The latter lifts the trace of $\psi_t^0 + \psi_{N,S}^{BL}$ and its normal derivative on $x = x_-$. Let

$$\begin{pmatrix} A_+(y) \\ A_-(y) \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\pi/6} & -iE^{1/3} \\ e^{-i\pi/6} & iE^{1/3} \end{pmatrix} \begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix},$$

where

$$\begin{aligned} f_0(y) &= - \left(\psi_{t|x=x_-}^0 + \psi_N^{BL} \left(x_-, \frac{y_+ - y}{E^{1/4}} \right) \chi_0(y) + \psi_S^{BL} \left(x_-, \frac{y - y_-}{E^{1/4}} \right) \chi_0(y) \right), \\ f_1(y) &= \partial_x \psi_{t|x=x_-}^0 + \partial_x \left(x_-, \frac{y_+ - y}{E^{1/4}} \right) \chi_0(y) + \partial_x \psi_S^{BL} \left(x_-, \frac{y - y_-}{E^{1/4}} \right) \chi_0(y). \end{aligned}$$

as the macroscopic truncation χ_0 can be chosen as a function of y only. Notice that by definition of $\psi_{N,S}^{BL}$ $A_+(y)$ and $A_-(y)$ both vanish at y_- and y_+ . Therefore the amplitude of the western boundary layer is zero at $y = y_\pm$: **its trace vanishes on the northern and southern boundaries.** Thus there is no need for additional south and north boundary layers.

Using the energy estimates of Lemma 3.3.4, we obtain

$$\|\partial_x^k \psi_{N,S}^{BL}\|_{L_x^\infty(L_Z^2)} + \|\partial_x^k \partial_Z^2 \psi_{N,S}^{BL}\|_{L_x^2(L_Z^2)} \leq CE^{-(k-1)_+/3}$$

from which we infer, using Lemma 3.4.3, for $k \geq 0$ and $l \in \{0, 1, 2, 3\}$,

$$\begin{aligned} \|\partial_x^k \partial_Z^l \psi_{N,S}^{BL}\|_{L_{x,Z}^\infty} &\leq C \|\partial_x^k \psi_{N,S}^{BL}\|_{L_x^\infty(L_Z^2)}^{\frac{7-2l}{8}} \|\partial_x^k \partial_Z^4 \psi_{N,S}^{BL}\|_{L_x^\infty(L_Z^2)}^{\frac{1+2l}{8}} \\ &\quad + CE^{-(k-1)_+/3} \\ &\leq C \|\partial_x^k \psi_{N,S}^{BL}\|_{L_x^\infty(L_Z^2)}^{\frac{7-2l}{8}} \|\partial_x^{k+1} \psi_{N,S}^{BL}\|_{L_x^\infty(L_Z^2)}^{\frac{1+2l}{8}} \\ &\quad + CE^{-(k-1)_+/3} \\ &\leq CE^{-\frac{(7-2l)(k-1)_+ - (1+2l)k}{24}}. \end{aligned}$$

This implies that for $l \in \{0, 1, 2\}$

$$(3.51) \quad \|\partial_y^l A_\pm\|_{L^\infty(y_-, y_+)} \leq CE^{-l/4}.$$

The construction of the approximate solution in all the cases considered in the introduction is now complete. There remains to check that the function which we built is indeed an approximate solution in the sense of Definitions 1.1.1 and 1.3.4. In other words, we still need to prove that the remainder terms created by the various approximations are admissible. This is the goal of the next chapter.

Proof of convergence

The approximate solution ψ_{app} we have built in the previous chapter

- (i) satisfies exactly the boundary conditions

$$\psi_{app}|_{\partial\Omega} = (\partial_n \psi_{app})|_{\partial\Omega} = 0$$

by definition of the boundary layer terms;

- (ii) has H^2 regularity, but with non uniform bounds on the derivatives (see for instance the definition of the interior term near East corners and surface discontinuities);
- (iii) but does not satisfy the Munk equation, only the approximate version

$$\partial_x \psi_{app} - E\Delta^2 \psi_{app} = \tau + \delta\tau,$$

where $\delta\tau$ comes from both the interior and boundary terms.

The goal of this final chapter is to estimate the different contributions to this error term $\delta\tau$, and to check that it is admissible in the sense of Definition 1.1.2 (or satisfies (1.25) in the periodic case). At this stage, there are no more conceptual difficulties: all the proofs are based on technical computations. Exactly as in the previous chapters, we focus mainly on the case when Ω is a smooth domain in \mathbf{R}^2 , and we have gathered in a paragraph at the end of the chapter the estimates corresponding to the periodic and rectangular cases.

4.1. Remainders stemming from the interior term $\psi^{int} = \psi_t^0 + \psi^{lift}$

Going back to the construction of the interior term, we see that its contribution $\delta\tau_{int}$ to the remainder is made of two kinds of terms.

- Because of the truncation, the solution ψ_t^0 to the transport equation satisfies the Munk equation with source terms $\tau(1 - \chi_E)$ and $-E\Delta^2 \psi_t^0$, which are admissible remainders according to Proposition 3.1.1.
- The lifting term which restores the H^2 regularity near discontinuity surfaces also introduces correctors. Part of them, namely $\delta\tau^{lift}$, are considered as source terms for the singular surface boundary layers. The other ones will be proved to be admissible remainders as stated in Proposition 2.4.3.

4.1.1. Error terms due to the truncation χ_E .

Let us first prove Proposition 3.1.1, i.e. the following estimates :

$$\begin{aligned} \|\tau(\chi_E - 1)\|_{H^{-2}(\Omega)} &= o(E^{5/8}), \\ \|\Delta \psi_t^0\|_{L^2(\Omega^\pm)} &= o(E^{-3/8}). \end{aligned}$$

• We start by localizing the problem around (x_i, y_i) thanks to a partition of unity: for $i \in I_+$, let $f_i \in \mathcal{C}_0^\infty(\mathbf{R}^2)$ such that

$$\sum_{i \in I_+} f_i = 1 \quad \text{on } \bar{\Omega},$$

and

- for all $i \in I_+$, $f_i \equiv 1$ in a neighbourhood of (x_i, y_i) ,
- for all $i \neq j$, $d((x_i, y_i), \text{Supp } f_j) > 0$.

Then, according to the definition of χ_E , for E small enough,

$$\begin{aligned} \tau \chi_E &= \sum_{i \in I_+} f_i \tau \chi_E \\ &= \sum_{i \in I_+} f_i \tau \chi_i(x - x_i, y - y_i). \end{aligned}$$

Thus, defining ψ_i^0 by

$$\begin{aligned} \partial_x \psi_i^0 &= f_i \tau \chi_i(x - x_i, y - y_i), \\ \psi_i^0|_{\Gamma_E} &= 0, \end{aligned}$$

we have $\psi_i^0 = \sum \psi_i^0$, and it suffices to prove that under the assumptions of Proposition 3.1.1,

$$\begin{aligned} \left\| f_i \tau [\chi_i(x - x_i, y - y_i) - 1] \right\|_{H^{-2}(\Omega)} &= o(E^{5/8}), \\ \|E \Delta^2 \psi_i^0\|_{H^{-2}(\Omega^\pm)} &= E \|\Delta \psi_i^0\|_{L^2(\Omega^\pm)} = o(E^{5/8}). \end{aligned}$$

In order not to burden the notation, we drop the indices i and we shift the origin of the arc-length parametrization so that $s_i = 0$. We also shift the origin of the axes so that $(x_i, y_i) = (0, 0)$, and we rename respectively τ and χ_E the functions $f_i \tau$ and $\chi_i(x - x_i, y - y_i)$ (which have the same regularity as the original functions).

Note that, by definition of χ_E , we have the obvious estimates

$$(4.1) \quad \begin{aligned} \|\partial_x^{s_1} \partial_y^{s_2} \chi_E\|_{L^\infty} &\leq C \max_{\pm} \left(\frac{|\ln \delta_y|}{\delta_x^\pm} \right)^{s_1} \left(\frac{1}{\delta_y} \right)^{s_2}, \\ \|\partial_x^{s_1} \partial_y^{s_2} (1 - \chi_E)\|_{L^2} &\leq C \max_{\pm} \left(\left(\frac{|\ln \delta_y|}{\delta_x^\pm} \right)^{s_1} \left(\frac{1}{\delta_y} \right)^{s_2} (\delta_x^\pm \delta_y)^{1/2} \right). \end{aligned}$$

• We now give an upper-bound for the source term $\tau(1 - \chi_E)$ in terms of δ_x^\pm, δ_y . Without loss of generality, we can assume that $s = 0$ is a North boundary point of Γ_E . Define, for $(x, y) \in \Omega$,

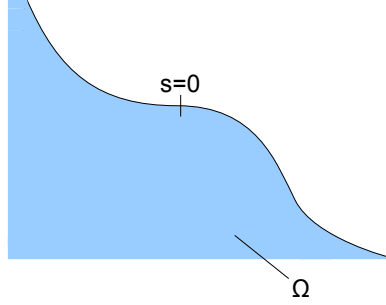
$$\phi_E(x, y) = \int_{-\infty}^y dy' \int_{-\infty}^{y'} dz (\chi_E - 1)(z).$$

so that

$$\partial_y^2 \phi_E = \chi_E - 1, \quad \phi_E(x, y) = 0 \quad \text{for } (x, y) \in \Omega, \quad d((x, y), \partial\Omega) \gg \delta_x, \delta_y.$$

Since the support of $1 - \chi_E$ is included in the rectangle

$$[-\delta_x^-(1 + |\ln \delta_y|^{-1}), \delta_x^+(1 + |\ln \delta_y|^{-1})] \times [-\delta_y, \delta_y],$$


 FIGURE 4.1. $\cos \theta$ has a local minimum at $s = 0$

we have

$$\begin{aligned} \phi_E(x, y) = 0 \text{ if } x \geq \delta_x^+(1 + |\ln \delta_y|^{-1}) \text{ or } x \leq -\delta_x^-(1 + |\ln \delta_y|^{-1}), \\ \text{and } \phi_E(x, y) = 0 \text{ if } y < -\delta_y. \end{aligned}$$

Notice also that

$$|\phi_E(x, y)| \leq C(\delta_y^2 + |y|\delta_y) \quad \forall (x, y) \in \Omega.$$

Therefore, parametrizing locally $\partial\Omega$ by a graph $(x, y_N(x))$, we obtain

$$\|\phi_E\|_{L^2(\Omega)}^2 \leq C \int_{-\delta_x^-(1+|\ln \delta_y|^{-1})}^{\delta_x^+(1+|\ln \delta_y|^{-1})} \int_{-\delta_y}^{y_N(x)} (\delta_y^4 + \delta_y^2 y^2) dy dx.$$

In order to bound the right-hand side of the above inequality, we have to distinguish between two cases, depending on whether $\cos \theta$ has a local extremum at $s = 0$.

- **No local extremum at $s = 0$:** either $\cos \theta$ changes sign across $s = 0$, or $\cos \theta$ is identically zero in a neighbourhood on the left or on the right of $s = 0$. Notice that in this case, $x_E(y)$ is only defined on a neighbourhood of the left of $y = 0$, and thus we have $\delta_x^- = \delta_x^+ =: \delta_x$. Moreover the function $y_N(x)$ remains non-positive for $|x| \leq \delta_x(1 + |\ln \delta_y|^{-1})$. In this case, we simply obtain

$$\|\phi_E\|_{L^2(\Omega)}^2 \leq C\delta_y^5\delta_x.$$

- **Local extremum at $s = 0$** (see Figure 4.1) : we have

$$\|\phi_E\|_{L^2(\Omega)}^2 \leq C\delta_y^5\delta_x^+ + C \int_{-\delta_x^-(1+|\ln \delta_y|^{-1})}^0 (\delta_y^4 |y_N(x)| + \delta_y^2 |y_N(x)|^3) dx.$$

We now estimate the right-hand side using the formulas in Appendix B:

$$\begin{aligned} y_N(x) &\sim Cx^{n+1} \text{ in case (i),} \\ y_N(x) &\sim Cx^2 \exp\left(-\frac{\alpha}{|x|}\right) \text{ in case (ii).} \end{aligned}$$

As a consequence,

$$\int_{-\delta_x^-(1+|\ln \delta_y|^{-1})}^0 |y_N(x)| dx \leq C \begin{cases} (\delta_x^-)^{n+2} & \text{in case (i),} \\ (\delta_x^-)^4 \exp\left(-\frac{\alpha}{\delta_x^-(1+|\ln \delta_y|^{-1})}\right) & \text{in case (ii)} \end{cases}$$

and

$$\int_{-\delta_x^-(1+|\ln \delta_y|^{-1})}^0 |y_N(x)|^3 dx \leq C \begin{cases} (\delta_x^-)^{3n+4} & \text{in case (i),} \\ (\delta_x^-)^8 \exp\left(-\frac{3\alpha}{\delta_x^-(1+|\ln \delta_y|^{-1})}\right) & \text{in case (ii)} \end{cases}$$

We infer that in all cases,

$$\|\phi_E\|_{L^2(\Omega)}^2 \leq C \delta_y^5 (\delta_x^+ + \delta_x^-).$$

An integration by part shows that

$$\|\tau(\chi_E - 1)\|_{H^{-2}(\Omega)} \leq \|\phi_E \tau\|_{L^2(\Omega)} + 2\|\phi_E \partial_y \tau\|_{L^2(\Omega)} + \|\phi_E \partial_y^2 \tau\|_{L^2(\Omega)}$$

Then, by definition of δ_x^\pm and δ_y , we get

(4.2)

$$\|\tau(\chi_E - 1)\|_{H^{-2}(\Omega)} \leq C \|\tau\|_{W^{2,\infty}} E^{5/8} |\ln E|^{1/2} (\ln |\ln E|)^{-5\beta/2} \delta_x^{1/2} = o(E^{5/8}),$$

since in the worse case

$$\delta_x = \max(\delta_x^+, \delta_x^-) \sim \frac{\alpha}{|\ln \delta_y|} \sim \frac{4\alpha}{|\ln E|}.$$

Note that the choice of δ_y is dictated by this estimate (which fixes both the power of E and the power of $|\ln E|$).

• It remains then to estimate the viscous term $E\Delta^2 \psi_t^0$ in $H^{-2}(\Omega^\pm)$, that is to estimate $E\Delta \psi_t^0$ in $L^2(\Omega^\pm)$. We indeed recall that, by Proposition 2.4.2,

$$\Delta \psi^{int} = \chi^+ \Delta \psi^{int} + \chi^- \Delta \psi^{int}.$$

We then start from the identity

$$\begin{aligned} \Delta \psi_t^0 &= \partial_x(\tau \chi_E) - \partial_y^2 \int_x^{x_E(y)} \tau \chi_E \\ (4.3) \quad &= \chi_E \partial_x \tau + \tau \partial_x \chi_E - \int_x^{x_E(y)} (\chi_E \partial_y^2 \tau + 2\partial_y \tau \partial_y \chi_E + \tau \partial_y^2 \chi_E) \\ (4.4) \quad &\quad - 2x'_E(y) (\chi_E \partial_y \tau + \tau \partial_y \chi_E)|_{x=x_E(y)} - x''_E(y) (\tau \chi_E)(x_E(y), y) \\ (4.5) \quad &\quad - |x'_E(y)|^2 (\chi_E \partial_x \tau + \tau \partial_x \chi_E)|_{x=x_E(y)}. \end{aligned}$$

Using (4.1), we can easily check that

$$\|(4.3)\|_{L^2} \leq \max_{\pm} \left(1 + \delta_y^{1/2} (\delta_x^\pm)^{-1/2} |\ln \delta_y| + \delta_x^\pm \delta_y^{-3/2}\right).$$

Note that we use here in a crucial way the fact that there is no discontinuity line in the domains Ω^\pm : the singularity is localized in a rectangle of width δ_x .

As for (4.4), (4.5), we use (3.6) together with the formulas of Appendix B:

• **Case (i):** we deduce that

$$\begin{aligned} \|(4.4)\|_2 &\leq \frac{C}{\delta_y} \left(\int_{|y| \geq \delta_y/2} |x'_E(y)|^2 dy \right)^{1/2} + C \left(\int_{|y| \geq \delta_y/2} |x''_E(y)|^2 dy \right)^{1/2} \\ &\leq C \delta_y^{-\frac{n}{n+1} - \frac{1}{2}} = C \delta_y^{-\frac{3n+1}{2(n+1)}}, \\ \|(4.5)\|_2 &\leq \max_{\pm} \frac{C}{\delta_x^{\pm}} \left(\int_{|y| \geq \delta_y/2} |x'_E(y)|^4 dy \right)^{1/2} \\ &\leq C \delta_y^{-\frac{1}{n+1}} \delta_y^{\frac{1}{2} - \frac{2n}{n+1}} = C \delta_y^{-\frac{3n+1}{2(n+1)}}. \end{aligned}$$

Hence the choice $\delta_y = E^{1/4} |\ln(E)|^{1/5} (\ln |\ln(E)|)^{-\beta}$, with $\beta > 0$ arbitrary, ensures that $\delta\tau$ is an admissible remainder. Notice that any choice of the type $\delta_y = E^\gamma$, with $\gamma \in \left(\frac{5(n+1)}{4(5n+6)}, \frac{3(n+1)}{4(3n+1)} \right)$, also works in this case.

• **Case (ii):** similarly, we have

$$\begin{aligned} \|(4.4)\|_2 &\leq \frac{C}{\delta_y} \left(\int_{|y| \geq c\delta_y} \frac{1}{y^2 (\ln |y|)^4} dy \right)^{1/2} + C \left(\int_{|y| \geq c\delta_y} \frac{1}{y^4 (\ln |y|)^4} dy \right)^{1/2} \\ &\leq \delta_y^{-3/2} |\ln |\delta_y||^{-2} \\ \|(4.5)\|_2 &\leq \max_{\pm} \frac{C}{\delta_x^{\pm}} \left(\int_{|y| \geq c\delta_y} \frac{1}{y^4 (\ln |y|)^8} dy \right)^{1/2} \\ &\leq C \delta_y^{-3/2} |\ln |\delta_y||^{-3}. \end{aligned}$$

Thus, with the choice $\delta_y = E^{1/4} |\ln(E)|^{1/5} (\ln |\ln(E)|)^{-\beta}$, we infer that $\delta\tau$ is an admissible remainder.

This completes the proof of Proposition 3.1.1.

4.1.2. Error terms due to the lifting term ψ^{lift} . We give the rest of the proof of Proposition 2.4.3. Since we have already checked that all the terms are well-defined, there only remains to estimate the size of the remainders coming from ψ^{lift} . More precisely, we prove that

$$(4.6) \quad \|r_{lift}^1\|_{L^2(\Omega)} = o(E^{1/8}), \quad \|r_{lift}^2\|_{H^{-2}(\Omega)} = o(E^{5/8}),$$

where

$$\begin{aligned} r_{lift}^1 &= 4E^{3/4} \chi^+ \chi' \left(\frac{y-y_1}{E^{1/4}} \right) b''(x) + 2E^{1/2} \chi^+ \chi'' \left(\frac{y-y_1}{E^{1/4}} \right) (a''(x) + b''(x)(y-y_1)) \\ r_{lift}^2 &= \tau(\chi_E - 1) + E \chi^+ \chi \left(\frac{y-y_1}{E^{1/4}} \right) (a^{(4)}(x) + b^{(4)}(x)(y-y_1)) \end{aligned}$$

We rely on Lemma C.1 in Appendix C, which yields

$$\begin{aligned} &\left\| 2E^{1/2} \chi'' \left(\frac{y-y_1}{E^{1/4}} \right) (a'' + b''(y-y_1)) + 4E^{3/4} \chi' \left(\frac{y-y_1}{E^{1/4}} \right) b'' \right\|_{L^2(\Omega)} \\ &= O(E^{5/8} (\|a''\|_{L^2} + E^{1/4} \|b''\|_{L^2})) = o(E^{1/8}). \end{aligned}$$

As for r_{lift}^2 , we have already proved (see Proposition 3.1.1) that $\tau(\chi_E - 1) = o(E^{5/8})$ in $H^{-2}(\Omega)$. And we have clearly

$$\begin{aligned} & E\chi^+\chi\left(\frac{y-y_1}{E^{1/4}}\right)(a^{(4)}(x)+b^{(4)}(x)(y-y_1)) \\ &= \partial_x^2\left(E\chi^+\chi\left(\frac{y-y_1}{E^{1/4}}\right)(a''(x)+b''(x)(y-y_1))\right), \end{aligned}$$

so that

$$\|r_{\text{lift}}^2\|_{H^{-2}(\Omega)} = o(E^{5/8}) + O\left(E^{9/8}(\|a''\|_{L^2} + E^{1/4}\|b''\|_{L^2})\right) = o(E^{5/8}).$$

Gathering all the terms, we obtain (4.6).

4.2. Remainders coming from the boundary terms

Following the construction of the approximate solution, we now turn to the remainders coming from boundary layer terms. This step is more technical since boundary terms are defined through rescaled curvilinear coordinates, so that estimating the bilaplacian requires tedious computations.

Actually most of the terms will be estimated in H^{-2} norm, so that we will need to compute only one iteration of the laplacian.

4.2.1. Laplacian in curvilinear coordinates. In order to get the boundary layer equation (2.3), we have kept only the leading order term of the bilaplacian, i.e. the fourth derivative with respect to the scaled normal variable Z . Rewriting this boundary layer equation as a Munk equation with remainder terms, we have to estimate the lower order terms of the bilaplacian, which can be done thanks to the following

PROPOSITION 4.2.1. *Denote as previously by (z, s) the local coordinates on a tubular neighbourhood of the boundary. Let $f \equiv F(s, \lambda(s)z)$ be a smooth function with $\text{Supp } f \subset \text{Supp } \chi_0$.*

Then, we have the following error estimate

$$\begin{aligned} |\Delta f - \partial_{zz}f| &\leq C(|\partial_{ss}F| + (\lambda'/\lambda)^2|\partial_{ZZ}F| + |\lambda'/\lambda||Z\partial_sZ F|) \\ &\quad + C\left(\left|\frac{\theta''}{\lambda}\right||Z\partial_sF| + \left|\frac{\lambda'\theta''}{\lambda^2}\right||Z^2\partial_ZF| + \left|\frac{\lambda''}{\lambda}\right||Z\partial_ZF| + |\lambda\theta'|\partial_ZF\right). \end{aligned}$$

In particular,

- if λ does not depend on s

$$(4.7) \quad |\Delta f - \partial_{zz}f| \leq C\left(|\partial_{ss}F| + \left|\frac{\theta''}{\lambda}\right||Z\partial_sF| + |\lambda\theta'|\partial_ZF\right);$$

- if F is an exponential profile with respect to Z , namely if $F(s, Z) = A(s)\exp(-\mu Z)$, with $\mu \in \mathbf{C}$ a constant of order one with positive real part,

$$(4.8) \quad \begin{aligned} |\Delta f - \partial_{zz}f| &\leq C\left(|\partial_{ss}A| + \left(\left|\frac{\theta''}{\lambda}\right| + \left|\frac{\lambda'}{\lambda}\right|\right)|\partial_sA|\right)e^{-cZ} \\ &\quad + C\left(\left|\frac{\lambda'\theta''}{\lambda^2}\right| + \left|\frac{\lambda''}{\lambda}\right| + |\lambda\theta'| + \left(\frac{\lambda'}{\lambda}\right)^2\right)|A|e^{-cZ} \end{aligned}$$

where $c = \Re(\mu)/2$.

PROOF. We start from the expression of the Laplacian in curvilinear coordinates

$$\Delta = \frac{1}{1+z\theta'} \left[\partial_z((1+z\theta')\partial_z) + \partial_s \left(\frac{1}{1+z\theta'} \partial_s \right) \right].$$

Given the specific form of the profile $f \equiv F(s, \lambda(s)z)$, we thus have

$$(4.9) \quad \begin{aligned} \Delta f &= \lambda^2 \partial_{ZZ} F + \frac{\lambda\theta'}{1+z\theta'} \partial_Z F - \frac{z\theta''}{(1+z\theta')^3} (\partial_s F + \lambda' z \partial_Z F) \\ &+ \frac{1}{(1+z\theta')^2} [\partial_{ss} F + 2\lambda' z \partial_{sZ} F + \lambda'' z \partial_Z F + (\lambda' z)^2 \partial_{ZZ} F] \end{aligned}$$

Then, using the fact that on the support of χ_0 the jacobian of the change of variables is uniformly bounded from above and from below

$$0 < \frac{1}{C} \leq 1+z\theta' \leq C,$$

we immediately get the first, general, estimate.

For North, South and discontinuity layers, we always choose

$$\lambda = E^{-1/4}$$

so that $\lambda' = \lambda'' = 0$. The remainder consists then only on three terms.

For East and West boundary layers, f is a combination of exponential profiles. Therefore, multiplying by Z or differentiating with respect to Z does not change the size of the error term. □

In order to get the expected estimates on the remainders, we first note that

$$\begin{aligned} \Delta^2 &= (\lambda^2 \partial_{ZZ})^2 + \Delta(\Delta - \lambda^2 \partial_{ZZ}) + (\Delta - \lambda^2 \partial_{ZZ})(\lambda^2 \partial_{ZZ}) \\ &= (\lambda^2 \partial_{ZZ})^2 + (\Delta + \lambda^2 \partial_{ZZ})(\Delta - \lambda^2 \partial_{ZZ}) + [\Delta - \lambda^2 \partial_{ZZ}, \lambda^2 \partial_{ZZ}] \end{aligned}$$

- The first term of the right-hand side has been considered in the equation for the boundary layers, so it does not introduce any remainder.
- For the second term, we expect that it can be estimated in H^{-2} using the previous proposition together with the controls on the boundary layer profiles and their derivatives up to second order.
- The difficulty comes then from the third term, which is a commutator between two second-order differential operators. Roughly speaking, it should be a third-order differential operator, so that we could estimate the corresponding remainder in L^2 in terms of the third-order derivative of the boundary layer profiles.

However we will not proceed exactly in that way, first of all because the commutator $[\Delta - \lambda^2 \partial_{ZZ}^2, \lambda^2 \partial_{ZZ}^2]$ involves too many terms so that the computations would be very tedious, and overall because some terms are not small in L^2 and need to be estimated in H^{-2} . The proof will actually be adapted to the case to be considered: recall indeed that for North, South and discontinuity layers, λ does not depend on s , whereas for East and West boundary layers, F is an exponential profile with respect to Z .

A useful tool in the case when λ does not depend on s is the following commutator estimate :

LEMMA 4.2.2. *Let $\rho \in H^2(\Omega)$, with support in $\{d(x, \partial\Omega) \leq \delta\}$, and let $f \in L^2(\partial\Omega \times \mathbf{R}_+)$. We set $\lambda := E^{-1/4}$. Then*

$$\begin{aligned} \|\rho(s, z)(\lambda^2 \partial_Z^2) f(s, \lambda z)\|_{H^{-2}(\Omega)} &\leq C |\lambda|^{-1/2} \|f\|_{L^2(\partial\Omega \times \mathbf{R}_+)} \sum_{k=0}^2 \|\partial_z^k \rho\|_{L^\infty(\Omega)}, \\ \|\rho(s, z) \Delta(f(s, \lambda z))\|_{H^{-2}(\Omega)} &\leq C |\lambda|^{-1/2} \|f\|_{L^2(\partial\Omega \times \mathbf{R}_+)} \|\rho\|_{W^{2, \infty}(\Omega)}. \end{aligned}$$

PROOF. • Let us first recall that

$$\lambda \partial_Z : f(s, \lambda Z) \in H^s(\mathbf{R}_+) \rightarrow \lambda \partial_Z f(s, \lambda Z) \in H^{s-1}(\mathbf{R}_+)$$

is a bounded operator (uniformly with respect to E) for all s .

Let $\varphi \in H^2(\Omega)$ be an arbitrary test function such that $\varphi|_{\partial\Omega} = \partial_n \varphi|_{\partial\Omega} = 0$. Then, using the Cauchy-Schwarz inequality,

$$\begin{aligned} &\int_{\Omega} \rho(s, z)(\lambda^2 \partial_Z^2) f(s, \lambda z) \varphi(s, z) (1 + z\theta') \, ds \, dz \\ &= \int_{\Omega} f(s, \lambda z) \partial_z^2 (\rho(s, z) \varphi(s, z) (1 + z\theta')) \, ds \, dz \\ &\leq C |\lambda|^{-1/2} \|f\|_{L^2(\partial\Omega \times \mathbf{R}_+)} \sum_{k=0}^2 \|\partial_z^k (\rho \varphi)\|_{L^2(\Omega)} \\ &\leq C |\lambda|^{-1/2} \|f\|_{L^2(\partial\Omega \times \mathbf{R}_+)} \|\varphi\|_{H^2(\Omega)} \sum_{k=0}^2 \|\partial_z^k \rho\|_{L^\infty(\Omega)}. \end{aligned}$$

By definition of $H^{-2}(\Omega)$,

$$\begin{aligned} &\|\rho(s, z)(\lambda^2 \partial_Z^2) f(s, \lambda z)\|_{H^{-2}(\Omega)} \\ &= \sup_{\substack{\varphi \in H_0^2(\Omega), \\ \|\varphi\|_{H^2(\Omega)} \leq 1}} \int_{\Omega} \rho(s, z)(\lambda^2 \partial_Z^2) f(s, \lambda z) \varphi(s, z) (1 + z\theta') \, ds \, dz, \end{aligned}$$

from which we deduce the first commutator estimate.

• Concerning the second inequality, we merely develop the commutator $[\Delta, \rho]$. We have

$$\begin{aligned} \rho(s, z) \Delta f(s, \lambda z) - \Delta(\rho(s, z) f(s, \lambda z)) &= -2 \nabla \rho \nabla(f(s, \lambda z)) - \Delta \rho f(s, \lambda z) \\ &= -2 \operatorname{div}(f(s, \lambda z) \nabla \rho) + \Delta \rho f(s, \lambda z). \end{aligned}$$

The estimate follows. □

When the profile is exponential, we will rather use the following Lemma :

LEMMA 4.2.3. *Let $I \subset \Gamma_E \cup \Gamma_W$ be a closed interval, and let $\lambda \in L^\infty(I, \mathbf{C})$ such that $\inf_I \Re(\lambda) > 0$.*

Let $\phi \in L^\infty(\Omega)$ such that $\operatorname{Supp} \phi \subset I$. Then

$$\|\phi(s) \chi_0(z) \exp(-\lambda z)\|_{L^2(\Omega)} \leq C \left(\int_I \phi(s)^2 (\Re \lambda(s))^{-1} \, ds \right)^{1/2}$$

and

$$\begin{aligned} & \|\phi(s)\chi_0(z)\exp(-\lambda z)\|_{H^{-2}(\Omega)} \\ & \leq C \left(\int_I \phi(s)^2 (\Re\lambda(s))^{-5} ds \right)^{1/2} + C\|\phi\|_\infty \exp(-C \inf_I \Re\lambda). \end{aligned}$$

PROOF. We begin with the L^2 estimate. We recall that the jacobian of the change of variables $(x, y) \rightarrow (s, z)$ is equal to $(1 + z\theta')^{-1}$, and is thus bounded in L^∞ . As a consequence,

$$\begin{aligned} & \|\phi(s)\chi_0(z)\exp(-\lambda z)\|_{L^2(\Omega)}^2 \\ & = \int_{\partial\Omega} \int_0^\delta \phi(s)^2 \chi_0(z)^2 \exp(-2\Re(\lambda)z) (1 + z\theta')^{-1} ds dz \\ & \leq C \int_I \phi^2 \left(\int_0^\infty \exp(-2\Re(\lambda)z) \right) ds dz \\ & \leq C \int_I \phi^2 (\Re\lambda)^{-1}. \end{aligned}$$

As for the H^{-2} estimate, we have

$$\begin{aligned} \phi(s)\chi_0(z)\exp(-\lambda z) & = \frac{\partial^2}{\partial z^2} (\phi(s)\chi_0(z)(\lambda(s))^{-2} \exp(-\lambda z)) \\ & \quad + 2\phi(s)\chi_0'(z)(\lambda)^{-1} \exp(-\lambda z) \\ & \quad - \phi(s)\chi_0''(z) \exp(-\lambda z). \end{aligned}$$

The last two terms are supported in $\{\delta/2 \leq z \leq \delta\}$, hence they are exponentially small in L^2 . More precisely, they are bounded in $L^\infty(\Omega)$ by

$$C\|\phi\|_\infty \exp(-\delta \inf_I \Re(\lambda)/2).$$

As for the first term, if $\zeta \in H_0^2(\Omega)$ is an arbitrary test function, then

$$\begin{aligned} & \int_\Omega \frac{\partial^2}{\partial z^2} (\phi(s)\chi_0(z)(\lambda(s))^{-2} \exp(-\lambda z)) \zeta(s, z) \frac{1}{1 + z\theta'} ds dz \\ & = \int_{\partial\Omega} \int_0^\delta \frac{\partial^2}{\partial z^2} (\phi(s)\chi_0(z)(\lambda(s))^{-2} \exp(-\lambda z)) \zeta(s, z) \frac{1}{1 + z\theta'} ds dz \\ & = \int_{\partial\Omega} \int_0^\delta \phi(s)\chi_0(z)(\lambda)^{-2} \exp(-\lambda z) \frac{\partial^2}{\partial z^2} \left(\zeta(s, z) \frac{1}{1 + z\theta'} \right) ds dz \\ & \leq C\|\zeta\|_{H^2(\Omega)} \left(\int_{\partial\Omega} \int_0^\delta \phi(s)^2 |\lambda|^{-4} \exp(-2\Re(\lambda)z) ds dz \right)^{1/2} \\ & \leq C\|\zeta\|_{H^2(\Omega)} \left(\int_{\partial\Omega} I\phi^2 (\Re\lambda)^{-5} \right)^{1/2}. \end{aligned}$$

By definition of the H^{-2} norm, we infer the estimate of Lemma 4.2.3. \square

4.2.2. Error terms associated with North and South layers.

In order to simplify the presentation, we focus on a South boundary layer, as we did in the preceding chapter. Of course, the case of North boundary layers is strictly identical. We denote by s_{i-1}, s_i the end points of the South boundary under consideration, and we introduce a macroscopic truncation function $\tilde{\gamma}_{i-1,i}$ (with bounded derivatives) as in (3.35).

LEMMA 4.2.4. *Let ψ_S be the solution to (3.27).*

Then

$$[\partial_x - E\Delta^2] \left(\tilde{\gamma}_{i-1,i}(s) \psi_S(s, E^{-1/4}z) \chi_0(z) \right) = r_i^1 + r_i^2,$$

with

$$\|r_i^1\|_{L^2(\Omega)} = o(E^{1/8}), \quad \|r_i^2\|_{H^{-2}(\Omega)} = o(E^{5/8}).$$

PROOF OF LEMMA 4.2.4. Since ψ_S satisfies (3.27), there are several kinds of terms in

$$[\partial_x - E\Delta^2] \left(\psi_S(s, E^{-1/4}z) \tilde{\gamma}_{i-1,i}(s) \chi_0(z) \right) :$$

- All terms in which at least one derivative of χ_0 or $\tilde{\gamma}_{i-1,i}$ occurs are small: indeed, we have one derivative with respect to z less acting on ψ_S , so that we gain a power $E^{1/4}$. By Lemma 4.2.2, we indeed have

$$\begin{aligned} E\partial_z^4(\psi_S\chi_0) - E\chi_0\partial_z^4\psi_S &= E \sum_{j=0}^3 C_4^j \lambda^j \partial_Z^j \psi_S(s, \lambda z) \partial_z^{4-j} \chi_0 \\ (4.10) \qquad \qquad \qquad &= O(E^{5/8} \|\psi_S\|_{L_s^2 H_z^2}) \text{ in } L^2(\Omega) \\ &\quad + O(E^{7/8} \|\partial_Z \psi_S\|_{L_{s,z}^2}) \text{ in } H^{-2}(\Omega). \end{aligned}$$

In the same way,

$$\begin{aligned} &\psi_S(s, E^{-1/4}z) \partial_x(\chi_0(z) \tilde{\gamma}_{i-1,i}(s)) \\ &= \psi_S(s, E^{-1/4}z) \left[\frac{\sin \theta(s)}{1+z\theta'} \partial_s \tilde{\gamma}_{i-1,i}(s) \chi_0(z) + \cos \theta(s) \tilde{\gamma}_{i-1,i}(s) \partial_z \chi_0(z) \right]. \end{aligned}$$

Using the estimates of paragraph 3.3.3, we infer that

$$\left\| \psi_S(s, E^{-1/4}z) \cos \theta(s) \tilde{\gamma}_{i-1,i}(s) \partial_z \chi_0(z) \right\|_{L^2(\Omega)} = O(E^{1/8} |\ln E|^{-1}).$$

Moreover, on the support of $\partial_s \tilde{\gamma}_{i-1,i}$, either ψ_S is zero or $\|\psi_S\|_{L_s^\infty(L_z^2)} = O(|\ln E|^{-1})$. We deduce that

$$\left\| \psi_S(s, E^{-1/4}z) \sin \theta(s) \partial_s \tilde{\gamma}_{i-1,i}(s) \chi_0(z) \right\|_{L^2(\Omega)} = O(E^{1/8} |\ln E|^{-1}),$$

so that eventually

$$(4.11) \quad \left\| \psi_S(s, E^{-1/4}z) \partial_x(\chi_0(z) \tilde{\gamma}_{i-1,i}(s)) \right\|_{L^2(\Omega)} = O(E^{1/8} |\ln E|^{-1}).$$

- The terms stemming from the ∂_x derivative applied to ψ_S can be split into a remainder term coming from the jacobian

$$\frac{z\theta' \sin \theta}{1+z\theta'} \chi_0(z) \tilde{\gamma}_{i-1,i}(s) \partial_s \psi_S(s, E^{-1/4}z),$$

and

$$\chi_0(z) \tilde{\gamma}_{i-1,i}(s) \left(E^{-1/4} \cos \theta \partial_Z \psi_S(s, E^{-1/4}z) + \sin \theta \partial_s \psi_S(s, E^{-1/4}z) \right),$$

which will simplify with the term $E\lambda^4\chi_0(z)\tilde{\gamma}_{i-1,i}(s)(\partial_Z^4)\psi_S(s, E^{-1/4}z)$ coming from the bilaplacian.

From the moment estimate in (3.31), we deduce that

$$(4.12) \quad \left\| \frac{z\theta' \sin \theta}{1+z\theta'} \tilde{\gamma}_{i-1,i}(s)\chi_0(z)\partial_s\psi_S(s, E^{-1/4}z) \right\|_{L^2} = O(E^{1/8}E^{1/4}\delta_y^{-1}) = o(E^2).$$

• The most technical part is the control of the remainder terms coming from the bilaplacian. For the sake of simplicity, instead of commuting the whole laplacian with $\lambda^2\partial_Z^2$, we will first get rid of the corrections coming from the jacobian, then commute the remaining part of the laplacian. More precisely, we start from (4.9) with a constant λ

$$\begin{aligned} & \Delta^2 - (\lambda^2\partial_Z^2)^2 \\ &= \Delta(\Delta - \lambda^2\partial_Z^2) + (\Delta - \lambda^2\partial_Z^2)\lambda^2\partial_Z^2 \\ &= \Delta(\Delta - \lambda^2\partial_{ZZ}) + \left(\frac{\lambda\theta'}{1+z\theta'}\partial_Z - \frac{z\theta''}{(1+z\theta')^3}\partial_s + \frac{1}{(1+z\theta')^2}\partial_{ss} \right) \lambda^2\partial_{ZZ} \\ &= \Delta(\Delta - \lambda^2\partial_{ZZ}) + \lambda^2\partial_{ZZ} \left(\lambda\theta'\partial_Z - z\theta''\partial_s + \partial_{ss} \right) + 2\lambda\theta''\partial_{Zs} \\ &\quad - \left(\frac{\lambda z(\theta')^2}{1+z\theta'}\partial_Z - \frac{z\theta''((1+z\theta')^3-1)}{(1+z\theta')^3}\partial_s + \frac{(1+z\theta')^2-1}{(1+z\theta')^2}\partial_{ss} \right) \lambda^2\partial_{ZZ}. \end{aligned}$$

We have, using the regularity and moment estimates (3.31)

$$\begin{aligned} & E(\Delta - \lambda^2\partial_{ZZ})\psi_S \\ &= E \frac{\lambda\theta'}{1+z\theta'}\partial_Z\psi_S - E \frac{\lambda^{-1}Z\theta''}{(1+z\theta')^3}\partial_s\psi_S + \frac{E}{(1+z\theta')^2}\partial_{ss}\psi_S \\ &= O(EE^{-1/4}E^{1/8})_{L^2} + O(EE^{1/4}\delta_y^{-1}E^{1/8})_{L^2} + O(EE^{-1/2}|\ln E|^{-1}E^{1/8})_{L^2}. \end{aligned}$$

We now use the second commutator estimates stated in Lemma 4.2.2 to get

$$(4.13) \quad \|\chi_0(z)\tilde{\gamma}_{i-1,i}(s)E\Delta(\Delta - \lambda^2\partial_{ZZ})\psi_S(s, \lambda z)\|_{H^{-2}(\Omega)} = o(E^{5/8}).$$

In the same way, we have

$$\begin{aligned} & E\lambda\theta'\partial_Z\psi_S - Ez\theta''\partial_s\psi_S + E\partial_{ss}\psi_S \\ &= O(EE^{-1/4}E^{1/8})_{L^2} + O(EE^{1/4}\delta_y^{-1}E^{1/8})_{L^2} + O(EE^{-1/2}|\ln E|^{-1}E^{1/8})_{L^2} \end{aligned}$$

so that, using some integration by parts together with the first estimate of Lemma 4.2.2,

$$(4.14) \quad E\chi_0(z)\tilde{\gamma}_{i-1,i}(s)\lambda^2\partial_{ZZ}(\lambda\theta'\partial_Z - z\theta''\partial_s + \partial_{ss})\psi_S = o(E^{5/8})_{H^{-2}}.$$

For the corrections coming from the jacobian, we use the first estimate of the commutator lemma 4.2.2:

$$\begin{aligned}
& \left\| \chi_0 \tilde{\gamma}_{i-1,i} \left(\frac{\lambda z (\theta')^2}{1+z\theta'} \partial_Z - \frac{z\theta''((1+z\theta')^3-1)}{(1+z\theta')^3} \partial_s + \frac{(1+z\theta')^2-1}{(1+z\theta')^2} \partial_{ss} \right) \lambda^2 \partial_{ZZ} \psi_S \right\|_{H^{-2}} \\
&= O(E^{1/8}) \|Z \partial_Z \psi_S\|_{L_{s,Z}^2} \left\| \frac{(\theta')^2}{1+z\theta'} \chi_0 \tilde{\gamma}_{i-1,i} \right\|_{W^{2,\infty}(\Omega)} \\
&+ O(E^{1/8} E^{1/2}) \|Z^2 \partial_s \psi_S\|_{L_{s,Z}^2} \left\| \frac{\theta''((1+z\theta')^3-1)}{z(1+z\theta')^3} \chi_0 \tilde{\gamma}_{i-1,i} \right\|_{W^{2,\infty}(\Omega)} \\
&+ O(E^{1/8} E^{1/4}) \|Z \partial_s^2 \psi_S\|_{L_{s,Z}^2} \left\| \frac{\theta''((1+z\theta')^2-1)}{z(1+z\theta')^2} \chi_0 \tilde{\gamma}_{i-1,i} \right\|_{W^{2,\infty}(\Omega)}
\end{aligned}$$

which implies that

$$\begin{aligned}
(4.15) \quad & \left\| E \chi_0 \tilde{\gamma}_{i-1,i} \left(\frac{\lambda z (\theta')^2}{1+z\theta'} \partial_Z - \frac{z\theta''((1+z\theta')^3-1)}{(1+z\theta')^3} \partial_s + \frac{(1+z\theta')^2-1}{(1+z\theta')^2} \partial_{ss} \right) \lambda^2 \partial_{ZZ} \psi_S \right\|_{H^{-2}} \\
&= o(E^{5/8}).
\end{aligned}$$

The only remaining term is

$$(4.16) \quad 2E\lambda\theta'' \chi_0(z) \tilde{\gamma}_{i-1,i}(s) \partial_{Zs} \psi_S = O(EE^{-1/4} \delta_y^{-1} E^{1/8})_{L^2}.$$

Combining estimates (4.10)-(4.16) shows that the remainders coming from the South and North boundary layers are admissible in the sense of Definition 1.1.2. \square

4.2.3. Error terms associated with East and West boundary layers.

We recall that the boundary layer term on the East or West boundary $[s_i, s_{i+1}]$ takes the form

$$\psi_{E,W}(s, Z) = (\varphi_i^\pm \varphi_{i+1}^\mp)(s) A(s)^t f(Z),$$

where

- $A(s) = -\lambda_E^{-1}(s) \partial_n \psi_{t|\Gamma_E}^0(s)$ and $f(Z) = \exp(-Z)$ on East coasts,
- A is the matrix defined by (3.44) and

$$f(Z) = \begin{pmatrix} \exp(-e^{i\pi/3} Z) \\ \exp(-e^{-i\pi/3} Z) \end{pmatrix}$$

on West coasts,

- φ_i^\pm are defined by (3.19) on the East coast, and by (3.23)-(3.24) on the West coast (see also Lemma 3.3.2).

Combining the definitions of s_i^\pm , σ_i^\pm and the equivalents given in Lemma 3.2.1 and Appendix B, we can check that the eigenvalues $\lambda_{E,W}$ satisfy the following estimates on the domain of validity of the East and West boundary layers:

LEMMA 4.2.5. *Under the compatibility condition (2.8), for $s \in \text{Supp } \varphi_i$,*

$$\left| \frac{\lambda'}{\lambda^3} \right| \ll 1, \quad \left| \frac{\lambda''}{\lambda^2} \right| \ll 1, \quad \left| \frac{(\lambda')^2}{\lambda^3} \right| \ll 1,$$

and

$$\left| E\lambda'\lambda^{1/2} \right| \leq C|\theta'| \left| \frac{E}{\cos \theta} \right|^?.$$

We can then prove that the error terms associated with East and West boundary layers are admissible.

PROPOSITION 4.2.6. *Let A, f be defined by the expressions above and (3.44).*

Then

$$[\partial_x - E\Delta^2] \left(\psi_{E,W}(s, \lambda_{E,W} z) \chi_0(z) \right) = r_{E,W}^1 + r_{E,W}^2,$$

with

$$\|r_{E,W}^1\|_{L^2(\Omega)} = o(E^{1/8}), \quad \|r_{E,W}^2\|_{H^{-2}(\Omega)} = o(E^{5/8}).$$

PROOF. Throughout the proof, we write λ instead of $\lambda_{E,W}$, and we set $\varphi_i = \varphi_i^+ \varphi_{i+1}^-$. Since f is an exponential profile, we will use the second part of Proposition 4.2.1 to estimate the remainder terms in the bilaplacian. In a first step, we bound the error terms $r_{E,W}^1$ and $r_{E,W}^2$ by expressions involving E, θ and A , and in a second step, we prove that these bounds yield the desired estimates.

First step: estimates in terms of the amplitude A

We claim that for all i such that $(s_i, s_{i+1}) \subset \Gamma_W \cap \Gamma_E$,

$$(\partial_x - E\Delta^2)(\varphi_i(s)A(s)f(\lambda z)) = r_i^1 + r_i^2,$$

where

$$(4.17) \quad \begin{aligned} \|r_i^1\|_{L^2(\Omega)} &\leq CE^{5/6} \|(\varphi_i A)''\|_{L^2(\partial\Omega)} \\ &\quad + CE^{1/2} |\ln E|^{1/2} \|(\varphi_i A)'\|_{L^\infty(\partial\Omega)} \\ &\quad + CE^{1/6} \|A\|_{L^\infty(\partial\Omega)} + CE^{-K} \exp(-C/E^{1/4}), \end{aligned}$$

and

$$(4.18) \quad \begin{aligned} \|r_i^2\|_{H^{-2}(\Omega)} &\leq CE \left(\int (\partial_s^2(\varphi_i A))^2 \left| \frac{E}{\cos(\theta(s))} \right|^{1/3} ds \right)^{1/2} \\ &\quad + C \left(\int (\partial_s(\varphi_i A))^2 \left| \frac{E}{\cos(\theta(s))} \right|^{5/3} ds \right)^{1/2} \\ &\quad + C \left(\int_{\text{Supp } \varphi_i} A(s)^2 (\theta')^2 E^{5/3} |\cos \theta|^{-11/3} ds \right)^{1/2} \\ &\quad + C \|A\|_\infty E^{5/6} + CE^{-K} \exp(-C/E^{1/4}). \end{aligned}$$

We recall the expression of ∂_x in curvilinear coordinates:

$$\frac{\partial}{\partial x} = -\cos \theta \frac{\partial}{\partial z} + \frac{\sin \theta}{1+z\theta'} \frac{\partial}{\partial s}$$

We now estimate

$$[\partial_x - E\Delta^2] (\varphi_i(s)A(s) \exp(-\lambda(s)z) \chi_0(z)).$$

• We start by commuting the differential operator $\partial_x - E\Delta^2$ with the multiplication by χ_0 . Since $\chi_0(z) \equiv 1$ for z in a neighbourhood of zero, all terms in which at least one derivative of χ_0 appears are exponentially small: indeed, for all $k \geq 1$, and for $\mu \in \{1, e^{\pm i\pi/3}\}$,

$$|\partial_z^k \chi_0(z) \exp(-\mu\lambda(s)z)| \leq C_k \exp(-CE^{-1/4}) \quad \forall (s, z) \in \text{Supp } \varphi_i \times \text{Supp } \chi_0.$$

We also have (see Lemmas 3.1.3, 3.4.1, 3.4.5)

$$|\partial_s^k, \partial_z^k(\varphi_i(s)A(s) \exp(-\mu\lambda(s)z))| \leq CE^{-K} \exp(-c\lambda(s)z),$$

for some positive constants c, C, K . Hence we infer that

$$\begin{aligned} & [\partial_x - E\Delta^2](\varphi_i(s)A(s) \exp(-\lambda(s)z)\chi_0(z)) \\ &= \chi_0(z)[\partial_x - E\Delta^2](\varphi_i(s)A(s) \exp(-\lambda(s)z)) \\ & \quad + O(E^{-K} \exp(-CE^{-1/4})) \text{ in } L^2(\Omega). \end{aligned}$$

- The terms stemming from $\chi_0 \partial_x(\varphi_i(s)A(s) \exp(-\lambda(s)z))$ can be split into

$$\lambda \cos \theta \varphi_i(s)A(s)(\partial_Z f)(\lambda z)\chi_0,$$

which will simplify with the term $E\lambda^4 \varphi_i(s)A(s)(\partial_Z^4 f)(\lambda z)\chi_0$ coming from the bi-laplacian, and a remainder term

$$(4.19) \quad \frac{\sin \theta}{1+z\theta'} [\partial_s(\varphi_i A) f(\lambda z) + A(s)\varphi_i(s)z\lambda'(\partial_Z f)(\lambda z)] \chi_0.$$

By Lemma 4.2.3, replacing $(1+z\theta')^{-1}$ by 1 in (4.19) yields an error term whose square L^2 norm is bounded by

$$\begin{aligned} & C \int_{\text{Supp } \varphi_i} \theta'^2 \frac{E}{|\cos \theta|} \left(|\partial_s(A\varphi_i)|^2 + \frac{\theta'^2}{\cos^2 \theta} |A(s)|^2 \right) ds \\ & \leq CE |\ln E| \|\partial_s(\varphi_i A)\|_{L^\infty(\partial\Omega)}^2 + CE^{5/7} \|A\|_{L^\infty(\partial\Omega)}^2. \end{aligned}$$

Indeed, by definition of φ_i ,

$$\text{Supp } \varphi_i \subset (s_i^+, s_{i+1}^-),$$

and using assumption (H2), it can be easily proved that for all i such that (s_i, s_{i+1}) is a western or eastern boundary,

$$\begin{aligned} & \int_{s_i^+}^{s_{i+1}^-} \theta'^2 |\cos \theta|^{-1} = O(|\ln E|), \\ & \int_{s_i^+}^{s_{i+1}^-} \theta'^4 |\cos \theta|^{-3} = O(E^{-2/7}). \end{aligned}$$

Then, using Lemma 4.2.3, we infer that

$$\begin{aligned} & \|\sin \theta \chi_0(z) \partial_s(\varphi_i A) f(\lambda z)\|_{H^{-2}(\Omega)}^2 \\ & \leq C \int_{\partial\Omega} (\partial_s(\varphi_i A))^2 \left| \frac{E}{\cos(\theta(s))} \right|^{5/3} ds + O(E^{-K} \exp(-CE^{-1/4})) \end{aligned}$$

and

$$\begin{aligned} & \|\sin \theta \chi_0(z) A(s)\varphi_i(s)z\lambda' f(\lambda z)\|_{H^{-2}(\Omega)}^2 \\ & \leq C \int_{\text{Supp } \varphi_i} A(s)^2 \theta'^2 E^{5/3} |\cos \theta|^{-11/3} ds + O(E^{-K} \exp(-CE^{-1/4})). \end{aligned}$$

- We now develop the bilaplacian of $(\varphi_i A)(s)f(\lambda z)$, using Proposition 4.2.1. We bound all the derivatives of θ by a positive constant. Moreover, by Lemma

4.2.5,

$$\begin{aligned} & |(\Delta - \lambda^2 \partial_{ZZ})(\varphi_i A)(s) f(\lambda z)| \\ & \leq C \left(|(A\varphi_i)''| + \left| \frac{\lambda'}{\lambda} \right| |(A\varphi_i)'| + |\lambda A\varphi_i| \right) \exp(-c|\lambda|z). \end{aligned}$$

We now evaluate the right-hand side of the above inequality in $L^2(\Omega)$ (after multiplication by χ_0). By Lemma 4.2.3, we have

$$\|(\lambda A\varphi_i)(s) \exp(-c|\lambda|z) \chi_0(z)\|_{L^2(\Omega)} \leq C \|A\varphi_i \lambda^{1/2}\|_{L^\infty} \leq C \|A\|_\infty E^{-1/6}.$$

Therefore

$$\begin{aligned} \Delta((A\varphi_i)(s) f(\lambda z)) &= (A\varphi_i)(s) \lambda^2 f''(\lambda z) \\ &+ O\left(\|A\|_{L^\infty} E^{-1/6} + \|(A\varphi_i)' \lambda' \lambda^{-3/2}\|_{L^2} + \|(A\varphi_i)'' \lambda^{-1/2}\|_{L^2}\right)_{L^2} \end{aligned}$$

Commuting once again χ_0 and Δ , we infer that

$$\begin{aligned} E\chi_0(z) \Delta^2((A\varphi_i)(s) f(\lambda z)) &= E\chi_0(z) \Delta((A\varphi_i)(s) \lambda^2 f''(\lambda z)) + O(\exp(-CE^{1/4}))_{L^2} \\ &+ O(\|A\|_\infty E^{5/6} + E\|(A\varphi_i)' \lambda' \lambda^{-3/2}\|_{L^2} + E\|(A\varphi_i)'' \lambda^{-1/2}\|_{L^2})_{H^{-2}} \end{aligned}$$

Lemma 4.2.5 then implies

$$E\|(A\varphi_i)' \lambda' \lambda^{-3/2}\|_{L^2} = o\left(\left(\int_{\partial\Omega} (\partial_s(\varphi_i A))^2 \left|\frac{E}{\cos(\theta(s))}\right|^{5/3} ds\right)^{1/2}\right).$$

It remains then to deal with the first term $E\chi_0(z) \Delta((A\varphi_i)(s) \lambda^2 f''(\lambda z))$. We develop the laplacian one more time. Since f is an exponential, we can use the computations above and simply replace A by $A\lambda^2$. Then, by Lemma 4.2.5,

$$\begin{aligned} & E\chi_0(z) \Delta((A\varphi_i)(s) \lambda^2 f''(\lambda z)) \\ &= E\chi_0(z) (A\varphi_i)(s) \lambda^4 f^{(4)}(\lambda z) + \frac{E\chi_0(z)}{(1+z\theta')^2} (A\varphi_i)'' \lambda^2 f''(\lambda z) \\ &+ EO(\|A\varphi_i \lambda^{5/2}\|_{L^2} + \|(A\varphi_i)' \lambda' \lambda^{1/2}\|_{L^2}) \text{ in } L^2(\Omega). \end{aligned}$$

We estimate the second term of the right-hand side in $H^{-2}(\Omega)$. This is the only time where we really need to commute the jacobian terms with the z derivatives, as explained at the end of paragraph 4.2.1. Using Lemma 4.2.3, we have

$$\begin{aligned} & \frac{\chi_0(z)}{(1+z\theta')^2} (A\varphi_i)'' \lambda^2 f''(\lambda z) \\ &= \frac{\partial^2}{\partial z^2} \left(\frac{\chi_0(z)}{(1+z\theta')^2} (A\varphi_i)'' f(\lambda z) \right) \\ &+ 4 \frac{\chi_0(z)\theta'}{(1+z\theta')^3} \lambda f'(\lambda z) (A\varphi_i)'' - 6 \frac{\chi_0(z)\theta'^2}{(1+z\theta')^4} f(\lambda z) (A\varphi_i)'' \\ &+ O\left(\|(A\varphi_i)'' \lambda^{1/2}\|_{L^2} \exp(-C/E^{1/4})\right)_{L^2(\Omega)} \\ &= O\left(\|(A\varphi_i)'' \lambda^{-1/2}\|_{L^2}\right)_{H^{-2}(\Omega)} + O\left(\|(A\varphi_i)'' \lambda^{1/2}\|_{L^2}\right)_{L^2(\Omega)}. \end{aligned}$$

The second statement in Lemma 4.2.5 leads then to

$$\|(A\varphi_i)' \lambda' \lambda^{1/2}\|_{L^2} \leq C \|(A\varphi_i)'\|_{\infty} E^{1/2} |\ln E|^{1/2}.$$

Gathering all the terms, we obtain (4.17), (4.18).

Second step: quantitative bounds for r_i^1, r_i^2 :

We now use the definition of A together with Lemma 3.1.3 and Lemma 3.5.1 in order to estimate the right-hand sides of (4.17), (4.18). Notice that the estimates on West coasts are always more singular than the ones on East coasts: indeed, φ_i has bounded derivatives on East coasts, and unbounded on West coasts. Moreover, the estimates of Lemma 3.1.3 show that A is always larger on West coasts, together with its derivatives. Therefore, we focus on a West part of the boundary in the following, i.e. we assume that $(s_i, s_{i+1}) \subset \Gamma_W$.

Following Lemma 3.5.1, we treat separately the estimates in the vicinity of s_1^W and in the vicinity of points s such that $\cos \theta = 0$ or $y(s) = y_j$ for some $j \in I_+$. In the rest of the domain, the estimates on r_i^1 and r_i^2 are easily proved.

- **Estimate near points such that $|y(s) - y_j| \ll 1$ ($y_j \neq y_1$):**

Let us first recall (see Lemma 3.3.2)

$$\|\partial_s^k \varphi_i\|_{\infty} \leq C E^{-k/7} \quad k \in \{0, 1, 2\}.$$

Therefore

$$\|r_i^1\|_{L^2} \leq C \left(E^{1/6} \|A\|_{\infty} + E^{1/2} |\ln E|^{1/2} \|A'\|_{\infty} + E^{5/6} \|A''\|_2 + E^{-K} \exp(-C/E^{1/4}) \right).$$

Inequalities (3.48), (3.45), (3.46) yield respectively

$$(4.20) \quad \|A\|_{\infty} = O(1), \quad \|A'\|_{\infty} = O(\delta_y^{-1} |\ln \delta_y|^{-1}), \quad \|A''\|_{\infty} = O(\delta_y^{-2} |\ln \delta_y|^{-1}),$$

so that $\|r_i^1\|_{L^2} = o(E^{1/8})$. We now address the bound of r_i^2 . Using once again (4.20), we have

$$\begin{aligned} & E \left(\int_{\partial\Omega} (\partial_s^2(\varphi_i A))^2 \left| \frac{E}{\cos(\theta(s))} \right|^{1/3} ds \right)^{1/2} \\ & \leq C E^{7/6} (\|\varphi''\|_{\infty} + \|\varphi'\|_{\infty} \|A'\|_{\infty} + \|A''\|_{\infty}) \left(\int \mathbf{1}_{\text{Supp } \varphi_i} |\cos \theta|^{-1/3} \right)^{1/2} \\ & \leq C E^{2/3} |\ln E|^{-1} \left(\int \mathbf{1}_{\text{Supp } \varphi_i} |\cos \theta|^{-1/3} \right)^{1/2}. \end{aligned}$$

Using the formulas in Appendix B together with the definition of s_i^+, s_{i+1}^- , it can be proved (treating cases (i) and (ii) in assumption (H2) separately) that

$$(4.21) \quad \int_{\text{Supp } \varphi_i} |\cos \theta(s)|^{-\gamma} ds = O(E^{-\gamma/4} |\ln E|^{-(2+3\gamma/2)}).$$

We infer that

$$E \left(\int_{\partial\Omega} (\partial_s^2(\varphi_i A))^2 \left| \frac{E}{\cos(\theta(s))} \right|^{1/3} ds \right)^{1/2} = o(E^{5/8}).$$

In a similar way, we have

$$\int_{\text{Supp } \varphi_i} (\theta')^2 |\cos \theta|^{-11/3} = O\left(E^{-5/12} |\ln E|^{-1/2}\right),$$

so that

$$\left(\int_{\text{Supp } \varphi_i} A(s)^2 (\theta')^2 E^{5/3} |\cos \theta|^{-11/3} ds \right)^{1/2} = O(E^{\frac{5}{8} - \frac{5}{24}} |\ln E|^{-1/4}) = o(E^{5/8}).$$

We now tackle the term

$$(4.22) \quad \int_{\partial\Omega} A^2(s) (\partial_s \varphi_i(s))^2 \left| \frac{E}{\cos(\theta(s))} \right|^{5/3} ds.$$

By definition, the support of $\partial_s \varphi_i$ is located in neighbourhoods of s_i and s_{i+1} , so that we can use assumption (H2). We distinguish between the cases (i) and (ii). By definition of φ_i^+ ,

$$\text{Supp } \partial_s \varphi_i^+ \subset \left(s_i^+, s_i^+ + \frac{|s_i - s_i^+|}{2} \right).$$

In a neighbourhood of $s = s_i$, in case (i), we have

$$|\partial_s \varphi_i^+(s)| \leq C E^{-\frac{1}{4n+3}} \mathbf{1}_{C_0 E^{\frac{1}{4n+3}} \leq s - s_i \leq 2C_0 E^{\frac{1}{4n+3}}},$$

so that (4.22) is bounded by

$$C E^{\frac{5}{3} - \frac{1}{2(4n+3)} + \frac{1}{4n+3} (1 - \frac{5n}{3})} = O\left(E^{\frac{10n+11}{8n+6}}\right) = o(E^{5/4}).$$

In case (ii), using Lemmas 3.2.1 and 3.3.2, we infer that

$$|s_i^\pm - \sigma_i^\pm| \sim C \frac{\ln |\ln E|}{(\ln E)^2},$$

while

$$\int_{s_i^\pm}^{\sigma_i^\pm} |\cos \theta|^{-5/3} \leq C E^{-5/12} |\ln E|^{-9/2}.$$

We deduce that the contribution of (4.22) is bounded by

$$|s_i^\pm - \sigma_i^\pm|^{-2} E^{5/3} \int_{s_i^\pm}^{\sigma_i^\pm} |\cos \theta|^{-5/3} = O(E^{5/4} |\ln E|^{-1/2} (\ln |\ln E|)^{-2}) = o(E^{5/4}).$$

There only remains to check that

$$\int_{\text{Supp } \varphi_i} (A'(s))^2 \left| \frac{E}{\cos(\theta(s))} \right|^{5/3} ds = o(E^{5/4}).$$

We use inequality (3.45). We have

$$\begin{aligned} & \int_{\text{Supp } \varphi_i} (A'(s))^2 |\cos \theta|^{-5/3} ds \\ & \leq C \int_{\text{Supp } \varphi_i} |\cos \theta|^{-5/3} ds + C \int_{\text{Supp } \varphi_i} |\cos \theta|^{1/3} (\mathcal{M}(y(s)))^2 ds. \end{aligned}$$

Using (4.21), it can be checked that the first term is $o(E^{-5/12})$. There remains to estimate the second and third term of the right-hand side. We have to distinguish between two cases:

- If $|\cos \theta| \ll 1$ in a neighbourhood of the point where $|y(s) - y_j| \ll 1$, then we can use assumption (H2) and the estimates of Appendix B. We infer that

$$\int_{\text{Supp } \varphi_i} |\cos \theta|^{1/3} (\mathcal{M}(y(s)))^2 \leq C \delta_y^{-5/3} |\ln \delta_y|^{-10/3}.$$

- If $\cos \theta$ is bounded away from zero at the point where $y(s) = y_j$ for some $s \in \text{Supp } \varphi_i$, $j \in I_+$, then we can use the change of variables $s \rightarrow y$ in the integrals. Since $dy(s)/ds = -\cos \theta(s)$, the jacobian is not singular, and we have

$$\begin{aligned} \int |\cos \theta|^{1/3} (\mathcal{M}(y(s)))^2 &\leq C \int_{\delta_y/2 \leq |y| \leq 1} |y|^{-2} (\ln |y|)^{-2} \\ &\leq C \delta_y^{-1} |\ln \delta_y|^{-2}. \end{aligned}$$

In both cases, we infer that

$$E^{5/3} \int_{\text{Supp } \varphi_i} (A'(s))^2 |\cos \theta|^{-5/3} ds = o(E^{5/4}).$$

This concludes the estimates in the neighbourhood of a point $s \in \partial\Omega$ such that $\cos \theta$ vanishes or $y(s) = y_j$ for some $j \in I_+$.

- **Estimates near the end point of Σ :**

Using once again Lemma 3.5.1, we infer that the contribution of this zone to $\|r_i^1\|_{L^2}$ is bounded by

$$C(E^{\frac{1}{6} - \frac{1}{8 \times 19}} + E^{1/2} |\ln E|^{1/2} E^{-\frac{1}{4} - \frac{3}{8 \times 19}} + E^{\frac{5}{6} - \frac{1}{2} - \frac{5}{8 \times 19}}) = o(E^{1/8}),$$

while its contribution to $\|r_i^2\|_{H^{-2}}$ is bounded by

$$C(E^{\frac{5}{6} - \frac{1}{8 \times 19}} + E^{\frac{5}{6} - \frac{1}{4} - \frac{3}{8 \times 19}} \delta_y^{1/2} + E^{\frac{7}{6} - \frac{1}{2} - \frac{5}{8 \times 19}} \delta_y^{1/2}) = o(E^{5/8}).$$

This concludes the proof. \square

4.2.4. Error terms associated with discontinuity layers.

Since we have already proved that the error terms associated with ψ_Σ^{corr} are admissible in the sense of Definition 1.1.2, in order to establish Proposition 3.4.6, there only remains to prove that

$$(\partial_x - E\Delta^2) \left(\psi^\Sigma \left(x, \frac{y - y_1}{E^{1/4}} \right) \chi \left(\frac{y - y_1}{\delta_y} \right) \right) = \delta\tau^{\text{lift}} \left(x, \frac{y - y_1}{E^{1/4}} \right) + r_\Sigma^1 + r_\Sigma^2,$$

where

$$\|r_\Sigma^1\|_{L^2(\Omega)} = o(E^{1/8}), \quad \|r_\Sigma^2\|_{H^{-2}(\Omega)} = o(E^{5/8}).$$

Note that the computations here are much simpler since all the formulas are given in cartesian coordinates.

By definition of ψ^Σ , we have

$$(4.23) \quad (\partial_x - E\Delta^2) \left(\psi^\Sigma \left(x, \frac{y-y_1}{E^{1/4}} \right) \chi \left(\frac{y-y_1}{\delta_y} \right) \right) - \delta\tau^{\text{lift}} \left(x, \frac{y-y_1}{E^{1/4}} \right) \\ = (\chi - 1) \left(\frac{y-y_1}{\delta_y} \right) \delta\tau^{\text{lift}} \left(x, \frac{y-y_1}{E^{1/4}} \right)$$

$$(4.24) \quad -E(2\partial_y^2 + \partial_x^2)\partial_x^2 \left(\psi^\Sigma \left(x, \frac{y-y_1}{E^{1/4}} \right) \chi \left(\frac{y-y_1}{\delta_y} \right) \right)$$

$$(4.25) \quad -E \sum_{j=0}^3 C_4^j E^{-j/4} \delta_y^{j-4} \partial_Y^j \psi^\Sigma \left(x, \frac{y-y_1}{E^{1/4}} \right) \chi^{(4-j)} \left(\frac{y-y_1}{\delta_y} \right).$$

The term (4.23) is supported in

$$\{|y - y_1| \leq E^{1/4}\} \cap \{|y - y_1| \geq \delta_y\}.$$

Since $\delta_y \gg E^{1/4}$, (4.23) is zero for E small enough. Moreover, using Lemma 3.4.1, we infer that (4.24) is $O(EE^{-2/19}E^{1/8})$ in $H^{-2}(\Omega)$. Eventually, using the same type of commutations as in the proof of Lemma 4.2.4, we prove that (4.25) is $o(E^{1/8})_{L^2} + o(E^{5/8})_{H^{-2}}$.

4.3. Remainders in the periodic and rectangular cases

• We begin with the rectangular case, which is closer to the case of a smooth domain in \mathbf{R}^2 . The remainder term coming from the interior is $-E\Delta^2\psi^0$, which is $O(E)$ in L^2 if τ is smooth (say, $\tau \in H^4$), and is therefore admissible. The north and south boundary layers give rise to the same type of error terms as in the case of a smooth domains; hence these error terms are admissible as well. Eventually, the error terms coming from the western boundary layer are

$$(E\partial_y^4 + 2E\partial_y^2\partial_x^2) \left(\sum_{\pm} A_{\pm}(y) \exp \left(-e^{\pm i\pi/3} \frac{x-x_-}{E^{1/3}} \right) \chi_0(x) \right),$$

which are bounded in H^{-2} by

$$E \left\| \partial_y^2 \sum_{\pm} A_{\pm}(y) \exp \left(-e^{\pm i\pi/3} \frac{x-x_-}{E^{1/3}} \right) \chi_0(x) \right\|_{L^2}.$$

Using estimates (3.51), we infer that the above quantity is bounded by

$$CE^{7/6} \sum_{\pm} \|\partial_y^2 A_{\pm}\|_{L^2(y_-, y_+)} \\ \leq CE^{7/6} E^{-1/2} = o(E^{5/8}),$$

hence the remainder term is admissible.

• In the periodic case, there are very few remainder terms. Concerning the interior part, we have

$$(\partial_x - E\Delta^2)(\psi^0 + \psi_{\text{circ}}) = \tau - E\Delta^2\psi^0,$$

and

$$\|E\Delta^2\psi^0\|_{H^{-2}(\mathbb{T} \times (y_-, y_+))} \leq CE\|\tau\|_{H^2(\mathbb{T} \times (y_-, y_+))}.$$

As for the boundary layer terms, we have

$$\begin{aligned} & (\partial_x - E\Delta^2)(\psi_N^{BL}(x, (y_+ - y)/E^{1/4})\chi_0(y)) \\ = & -E(\partial_x^4 + 2\partial_x^2\partial_y^2)(\psi_N^{BL}(x, (y_+ - y)/E^{1/4})\chi_0(y)) \\ & + \text{exponentially small terms in } L^2 \end{aligned}$$

and using (3.49), the first term in the right-hand side is bounded in H^{-2} by

$$\begin{aligned} & CE \left\| \partial_x^2 \psi_N^{BL}(x, (y_+ - y)/E^{1/4})\chi_0(y) \right\|_{L^2(\Omega)} \\ \leq & CE^{7/8} (\|\psi_{y=y_\pm}^0\|_{H^{2-\frac{1}{8}}(\mathbb{T})} + E^{1/4}\|\partial_y\psi_{y=y_\pm}^0\|_{H^{2-\frac{3}{8}}(\mathbb{T})}) \\ \leq & CE^{7/8} (\|\tau\|_{H^{2+\frac{3}{8}}} + E^{1/4}\|\tau\|_{H^{2+\frac{9}{8}}}). \end{aligned}$$

Hence for $\tau \in H^{25/8}$, the remainder terms are all $O(E)$ in H^{-2} , which completes the proof of Corollary 1.3.5.

Discussion: Physical relevance of the model

Of course it is not completely clear whether our simplified model is really relevant from the physical point of view. Let us therefore explain briefly the derivation of this model (following Desjardins and Grenier [6]), and track the different simplifications that should be compared with experimental data. For further discussions regarding the physical relevance of such models, we refer to [16],[17] or [10].

- As usual in large-scale oceanography, we start with the 3D incompressible Navier-Stokes equations in a rotating frame: the velocity field u is assumed to be divergence-free

$$\nabla \cdot u = 0,$$

and to satisfy the dynamical equation

$$\partial_t u + (u \cdot \nabla)u + 2\omega \wedge u = \frac{\nabla p}{\rho} + g + \frac{1}{\rho} \mathcal{F}u$$

expressing the fact that the fluid evolves under the combined effects of the Coriolis force, the pressure, the gravity and some turbulent dissipation mechanism. The precise formulation of this last contribution (involving a turbulent viscosity)

$$\frac{1}{\rho} \mathcal{F}u = \nu_h \Delta_h u + \nu_3 \partial_{33} u$$

even commonly used by physicists, has no real justification, which is a first limitation of our study.

These equations are set in a bounded domain

$$\mathcal{D} = \{x \in \Omega \times \mathbf{R} / h_B(x_h) \leq x_3 \leq 0\}$$

and supplemented by boundary conditions. On the bottom (which is described by the topography h_B) and on the lateral boundaries, we assume that the fluid-solid interaction can be caught through a no slip condition (Dirichlet condition) :

$$u|_{x_3=h_B} = 0, \quad u|_{\partial\Omega_h} = 0.$$

For the sake of simplicity, the free surface is replaced by a prescribed spherical boundary corresponding to the depth $x_3 = 0$ (rigid lid approximation), and the effect of the wind is modeled by some non homogeneous Navier condition

$$u_n|_{x_3=0} = 0, \quad \nu_h(\nabla u + (\nabla u)^T)|_{x_3=0} = \mathcal{T}.$$

This drastic but standard simplification is investigated for instance in [15, 7], but has no rigorous justification, which is the second weakness of the model.

- Far from the poles and the equator, i.e. around a latitude $\vartheta_0 \in (0, \pi/2)$, these equations can be rewritten in scaled cartesian-like coordinates. This involves many non dimensional parameters characterizing the physical properties of the flow, especially

- the Rossby number which measures the size of the Coriolis force

$$\epsilon = \frac{U}{2\omega \sin \vartheta_0 L}$$

where U and L are the typical velocity of the fluid and (horizontal) length of observation;

- the aspect ratio and the curvature parameter which characterize the geometry of the domain, defined respectively by

$$\rho = \frac{\bar{h}}{L}, \quad r_* = \frac{R_0}{L}$$

\bar{h} being the typical height of the ocean and R_0 the radius of the Earth;

- the horizontal and vertical Ekman numbers which account for the viscous effects

$$E_3 = \frac{\nu_3}{2\omega \sin \vartheta_0 \bar{h}^2}, \quad E_h = \frac{\nu_h}{2\omega \sin \vartheta_0 L^2}.$$

The Munk equation is obtained in the fast rotating limit with thin layer approximation, that is when $\epsilon \rightarrow 0$, with the following choice of scaling

$$\rho, r_*^{-1} = O(\epsilon), \quad E_3 = O(\epsilon)$$

$$h_B = -\bar{h}(1 + \epsilon \eta_B),$$

$$\text{and } \sin \vartheta = \sin \vartheta_0(1 + \epsilon \beta y).$$

where

$$\beta = \frac{r_* \cos \vartheta_0}{\epsilon \sin \vartheta_0}.$$

- A formal asymptotic analysis (based for instance on asymptotic expansions) shows that the limiting flow is purely two-dimensional: we indeed obtain at leading order

$$(5.1) \quad u_3 = 0, \quad \partial_3 u_h = 0, \quad \nabla_h \cdot u_h = 0,$$

or equivalently

$$u = \nabla_h^\perp \psi$$

for some scalar stream function ψ depending only on the horizontal variables. Since these conditions are not compatible with the bottom and surface boundary conditions, one has to introduce boundary layer corrections, referred to as Ekman layers, which contribute to the global energy balance via Ekman pumping.

The dynamical equation is then obtained at next order (more or less as a solvability condition): since u_h is divergence-free, it is indeed completely determined by its vorticity:

$$(5.2) \quad \nabla^\perp \cdot (\partial_t u + u \cdot \nabla_h u - \beta y u^\perp) = \nabla^\perp \cdot (E_h \Delta_h u - \beta \mathcal{T} + \eta_B u^\perp - ru)$$

where $-ru$ is the Ekman pumping associated to the energy dissipation by friction on the bottom, $\eta_B u^\perp$ accounts for the effects of the topography, and $-\beta \mathcal{T}$ is the source term resulting from the wind forcing. When the domain Ω is not simply connected, (5.2) must be supplemented with further compatibility conditions (see paragraph 1.2.3).

Note that this formal derivation can be justified by classical energy methods [6], at least for well-prepared initial data (i.e. for initial data satisfying the constraint equations (5.1)). Starting from the limiting system (5.1)(5.2), one can indeed build

a smooth approximate solution, then control its L^2 distance to the solution u_ϵ of the Navier-Stokes equations using some strong-weak stability principle. Note that one even obtains a rate of convergence.

- The apparition of Munk boundary layers and the intensification of oceanic currents on western coasts we would like to describe with such a model are physical phenomena which are typically linear and which result from the fact that the no-slip Dirichlet condition on the coasts of the basin becomes a non admissible boundary condition if the viscous dissipation is not a leading order term in the equation (5.2).

More precisely, we expect to exhibit such a behaviour when the remaining rotating term $\beta \nabla^\perp \cdot (yu^\perp) = \beta u_y$ (due to the inhomogeneities of the local rotation vector) is large compared to $E_h \Delta \nabla^\perp \cdot u$. In particular, we expect that boundary layers

- should not depend on the topographical effects and Ekman pumping (if β is large enough)
- decouple from the convection (at least if boundary layers are stable in the sense that their kinetic energy remain small)
- are quasi-stationary, meaning that the equation governing boundary layers does not involve time.

We emphasize that our study remains entirely valid if in equation (1.1), β is not a parameter, but a smooth function which remains bounded away from zero.

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Appendix

Appendix A: The case of islands: derivation of the compatibility condition (1.15) and proof of Lemma 1.2.1

• The compatibility condition (1.15) is inherited from the Navier-Stokes system satisfied by $u = \nabla^\perp \psi$ (see [14] for a similar argument in the inviscid case). Indeed, we start from the stationary Stokes-Coriolis system in dimension 2, with β -plane approximation, namely

$$(A.1) \quad \begin{aligned} \frac{1}{\epsilon}(1 + \beta \epsilon y)u^\perp - E\Delta u + \nabla p &= \mathcal{T}, \text{ in } \Omega, \\ \operatorname{div} u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

The idea is to take the curl of the first equation in order to get rid of the pressure term. However, for $\Phi \in L^2(\Omega)^2$, the identity $\operatorname{curl} \Phi = 0$ does not imply the existence of $q \in H^1(\Omega)$ satisfying $\Phi = \nabla q$. Indeed, Φ is a gradient if and only if its circulation around any closed contour \mathcal{C} in Ω vanishes:

$$\oint_{\mathcal{C}} \Phi \cdot t = 0,$$

where t is the tangent vector to the curve \mathcal{C} . If $\operatorname{curl} \Phi = 0$, this condition becomes $\int_{C_i} \Phi \cdot t = 0$ for all $i \geq 2$. Hence we obtain

$$\exists q \in H^1(\Omega), \quad \Phi = \nabla q \iff \begin{cases} \operatorname{curl} \Phi = 0 \\ \text{and } \int_{C_i} \Phi \cdot t = 0 \text{ for all } i \geq 2. \end{cases}$$

Therefore (A.1) is equivalent to

$$\begin{aligned} \beta u_2 - E\Delta \operatorname{curl} u &= \operatorname{curl} \mathcal{T} \text{ in } \Omega, \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= 0, \\ E \int_{C_i} \Delta u \cdot t + \int_{C_i} \mathcal{T} \cdot t &= 0 \quad \forall i \geq 2. \end{aligned}$$

This amounts to (1.14)-(1.15) with $u = \nabla^\perp \psi$ (notice that, as above, the existence of ψ is ensured by the divergence free condition on u and by the Dirichlet boundary conditions).

• Proof of Lemma 1.2.1: the existence and uniqueness of the functions $\psi_i \in H^2(\Omega)$ follow for instance from the Lax-Milgram Lemma. Identity (1.18) is a consequence of the linearity of the equation (1.14) and of uniqueness, and the equality $M^E \mathcal{C} = D^E$ follows from the compatibility condition (1.15) and the decomposition (1.18).

Concerning the invertibility of M^E , we prove in fact that for all $(a_2, \dots, a_K) \in \mathbf{R}^{K-1}$

$$(A.2) \quad \sum_{i,j} a_i a_j E \int_{C_j} \partial_n \Delta \psi_i = -E \int_{\Omega} |\Delta \psi_a|^2 \leq 0,$$

where $\psi_a = \sum_{i=2}^K a_i \psi_i$.

We have

$$\begin{aligned} \partial_x \psi_a - E \Delta^2 \psi_a &= 0, \\ \partial_n \psi_a|_{\partial\Omega} &= 0, \\ \psi_a|_{C_1} = 0, \quad \psi_a|_{C_i} &= a_i, \quad i \geq 2. \end{aligned}$$

Therefore, since C_i is a closed contour,

$$\int_{\Omega} \partial_x \psi_a \psi_a = \frac{1}{2} \sum_{i=2}^K a_i^2 \int_{C_i} e_x \cdot n = 0,$$

and

$$\begin{aligned} E \int_{\Omega} \Delta^2 \psi_a \psi_a &= E \int_{\Omega} |\Delta \psi_a|^2 + \sum_{i=2}^K a_i E \int_{C_i} \partial_n \Delta \psi_a \\ &= E \int_{\Omega} |\Delta \psi_a|^2 + E \sum_{i,j} a_i a_j \int_{C_i} \partial_n \Delta \psi_j. \end{aligned}$$

The identity (A.2) follows.

Appendix B: Equivalents for the coordinates of boundary points near horizontal parts

In the vicinity of a cancellation point $s_0 \in \partial\Gamma_E$, the East part of the boundary can be described as the graph of some function $y \mapsto x_E(y)$. Assumption (H2) provides then asymptotic expansions of the function x_E .

Let $(x(s), y(s))$ be the coordinates of the point with arc-length s on $\partial\Omega$. By definition, if $\cos(\theta(s)) > 0$,

$$x(s) = x_E(y(s)).$$

Moreover,

$$(B.1) \quad \frac{d}{ds} \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} \sin(\theta(s)) \\ -\cos(\theta(s)) \end{pmatrix}$$

and setting the origin of the axes so that $(x(s_0), y(s_0)) = (0, 0)$, we get

$$(B.2) \quad \begin{aligned} x'_E(y(s)) &= -\tan(\theta(s)), \\ x''_E(y(s)) &= \frac{\theta'(s)}{\cos^3(\theta(s))}. \end{aligned}$$

Similar formulas hold for $x_E^{(3)}$ and $x_E^{(4)}$.

In case (H2i), we have

$$y(s) \sim C(s - s_0)^{n+1} \sim Cx(s)^{n+1},$$

and thus

$$(B.3) \quad \begin{aligned} x_E(y) &\sim C|y|^{\frac{1}{n+1}}, \\ x_E^{(k)}(y) &\sim C_k |y|^{\frac{1}{n+1} - k} \quad \text{for } k \in \{1, 2, 3, 4\}. \end{aligned}$$

In case (H2ii), using the same kind of calculations as above, we infer that as y vanishes

$$(B.4) \quad \begin{aligned} x_E(y) &\sim \frac{C}{\ln|y|}, \\ x_E^{(k)}(y) &\sim \frac{C_k}{y^k(\ln|y|)^2} \quad \text{for } k \in \{1, 2, 3, 4\}. \end{aligned}$$

We have used the following fact:

$$\int_{s_0}^s \exp\left(-\frac{\alpha}{s' - s_0}\right) ds' \sim \frac{(s - s_0)^2}{\alpha} \exp\left(-\frac{\alpha}{s - s_0}\right) \quad \text{as } s \rightarrow s_0.$$

Appendix C: Estimates on the coefficients a and b .

We recall that (see (2.17), (2.19), (2.20))

$$\psi^{\text{lift}}(x, y) = \chi^+ \chi \left(\frac{y - y_1}{E^{1/4}} \right) [a(x) + b(x)(y - y_1)],$$

where a and b are defined by

$$\begin{aligned} b(x) &= (1 - \varphi(s)) \left[-\sin \theta(s) \partial_n \psi_{t|\partial\Omega}^0(s) + \cos \theta(s) \partial_s \psi_{t|\partial\Omega}^0(s) \right] \\ &\quad - \cos \theta(s) \varphi' \psi_{t|\partial\Omega}^0(s), \\ a(x) &= -(1 - \varphi(s)) \psi_{t|\partial\Omega}^0(s) - b(x(s))(y(s) - y_1), \end{aligned}$$

for $x_1 < x = x(s) < x(\sigma_{in})$, and by

$$\begin{aligned} a(x) &= -[\psi_t^0]_{\Sigma}, \\ b(x) &= -[\partial_y \psi_t^0]_{\Sigma} \quad \text{for } x \leq x_1. \end{aligned}$$

For $x \geq x(\sigma_{in})$, we merely take $a(x) = b(x) = 0$.

LEMMA C.1. *Let*

$$\begin{aligned} x_{min} &:= \inf\{x \in \mathbf{R}, \exists y \in \mathbf{R}, (x, y) \in \Omega\}, \\ x_{max} &:= \sup\{x \in \mathbf{R}, \exists y \in \mathbf{R}, (x, y) \in \Omega\} \end{aligned}$$

Then for all $k \in \{1, \dots, 4\}$,

$$A_k := \|a^{(k)}\|_{L^1(x_{min}, x_{max})} + E^{1/4} \|b^{(k)}\|_{L^1(x_{min}, x_{max})} = O(E^{\frac{1-k}{19}}).$$

Additionally,

$$\begin{aligned} \|a\|_{L^\infty(x_{min}, x_{max})} + E^{1/4} \|b\|_{L^\infty(x_{min}, x_{max})} &= O(1), \\ \|a''\|_{L^2(x_{min}, x_{max})} + E^{1/4} \|b''\|_{L^2(x_{min}, x_{max})} &= o(E^{1/2}). \end{aligned}$$

PROOF. We recall that a and b were defined in paragraph 3.4.1. Since a and b are constant for $x < x_1$ and for $x > x(\sigma^-)$, it suffice to prove the estimates on $(x_1, x(\sigma^-))$. On this interval, a and b are defined through formulas of the type

$$a(x(s)) = \mathbf{a}(s), \quad b(x(s)) = \mathbf{b}(s).$$

Differentiating these inequalities with respect to s , we obtain

$$x'(s)a'(x(s)) = \mathbf{a}'(s),$$

and thus, using the formula (B.1),

$$a'(x(s)) = \frac{\mathbf{a}'(s)}{\sin \theta(s)}.$$

Iterating this process and using the fact that $\sin \theta$ remains bounded away from zero in the interval under consideration, we infer that for all $k \in \{0, \dots, 4\}$,

$$\left| a^{(k)}(x(s)) \right| \leq C \sum_{l=0}^k |\mathbf{a}^{(l)}(s)|,$$

and eventually, for all $p \in [1, \infty]$,

$$\|a^{(k)}\|_{L^p(x_i 1x(\sigma^-))} \leq C \sum_{l=0}^k \|\mathbf{a}^{(l)}\|_{L^p(\sigma^-, s_1)}.$$

Of course the same type of inequality holds for b as well. Therefore we now compute the derivatives of \mathbf{a} and \mathbf{b} with respect to s up to order 4. Since

$$\mathbf{a}(s) = -(1 - \varphi(s))\psi_{t|\partial\Omega}^0(s) - \mathbf{b}(s)(y(s) - y_1).$$

we start with the derivatives of \mathbf{b} . Recall that

$$\begin{aligned} \mathbf{b}(s) &= (1 - \varphi(s)) \left[-\sin \theta(s) \partial_n \psi_{t|\partial\Omega}^0(s) + \cos \theta(s) \partial_s \psi_{t|\partial\Omega}^0(s) \right] \\ &\quad - \cos \theta(s) \varphi'(s) \psi_{t|\partial\Omega}^0(s). \end{aligned}$$

The computations are lengthy but do not raise any difficulty. It can be easily checked that the most singular estimates correspond to the case when $\cos \theta$ vanishes algebraically near $s = \inf I_1$ (assumption (H2i)), with the lowest possible exponent n . Indeed, if $\cos \theta$ vanishes exponentially near $s = \inf I_1$, then for all $k \geq 0$

$$\|\partial_s^k \varphi\|_{L^\infty} = O(|\ln E|^{2k}),$$

while in case (i)

$$\|\partial_s^k \varphi\|_{L^\infty} = O(E^{-\frac{k}{4n+3}}).$$

Similar formulas hold for $\partial_s^k \psi_{t|\partial\Omega}^0$, $\partial_s^k \partial_n \psi_{t|\partial\Omega}^0$.

Therefore in the rest of the proof, we only treat the case (H2i) with $n = 4$ (Remember that because of assumption (H4), $\cos \theta$ vanishes at least like $(s - \inf I_1)^4$ near $\inf I_1$). We explain with some detail the estimates of \mathbf{b} and \mathbf{b}' , and we leave the rest of the derivatives to the reader.

First, we recall that by definition of σ^- (see (3.36)),

$$|y(s) - y_1| \leq E^{1/4} \ll \delta_y$$

for all $s \in (\sigma^-, s_1)$. Therefore, using the estimates of Lemma 3.1.3, we have

$$|\mathbf{b}(s)| \leq C \left(\frac{1}{\delta_y |\ln \delta_y|} + E^{-1/19} \cos \theta(s) \right),$$

so that

$$E^{1/4} \|\mathbf{b}\|_\infty = O(|\ln E|^{-1}) = o(1).$$

Differentiating \mathbf{b} with respect to s , we obtain, for $s \in (\sigma^-, s_1)$,

$$\begin{aligned} \mathbf{b}'(s) &= (1 - \varphi) \left[-\theta' \cos \theta \partial_n \psi_{t|\partial\Omega}^0 - \sin \theta \partial_s \partial_n \psi_{t|\partial\Omega}^0 \right. \\ &\quad \left. - \theta' \sin \theta \partial_s \psi_{t|\partial\Omega}^0 + \cos \theta \partial_s^2 \psi_{t|\partial\Omega}^0 \right] \\ &\quad + \varphi' \left[\sin \theta \partial_n \psi_{t|\partial\Omega}^0 - 2 \cos \theta \partial_s \psi_{t|\partial\Omega}^0 + \theta' \sin \theta \psi_{t|\partial\Omega}^0 \right] \\ &\quad - \varphi'' \cos \theta \psi_{t|\partial\Omega}^0, \end{aligned}$$

so that

$$\begin{aligned} |\mathbf{b}'(s)| &\leq C(1 - \varphi) \left[1 + \frac{|\cos \theta|}{\delta_y} + \frac{|\cos \theta|^3}{\delta_y^2 |\ln \delta_y|} \right] \\ &\quad + C|\varphi'| \frac{1}{\delta_y |\ln \delta_y|} + C|\varphi''| |\cos \theta|. \end{aligned}$$

Eventually, we retrieve

$$\|\mathbf{b}'\|_{L^1} = O(E^{-1/4} |\ln E|^{-1}).$$

The same kind of estimate yields, for $k = 2, 3, 4$,

$$\|\mathbf{b}^{(k)}\|_{L^1} = O(E^{-\frac{1}{4} + \frac{1-k}{19}} |\ln E|^{-1}).$$

Now, since

$$\mathbf{a}^{(k)} = \sum_{l=0}^k C_k^l \left(\varphi^{(l)} \partial_s^{k-l} \bar{\psi}_{|\partial\Omega} - \mathbf{b}^{(l)} \frac{d^{k-l}(y(s) - y_1)}{ds^{k-l}} \right),$$

we infer, for $k \geq 1$,

$$\begin{aligned} \|\mathbf{a}^{(k)}\|_{L^1} &\leq C \|\varphi^{(k)}\|_{L^1} \\ &\quad + C \sum_{l=0}^{k-1} E^{-l/19} \|\partial_s^{k-l} \bar{\psi}_{|\partial\Omega}\|_{L^1(\sigma^-, s_1)} \\ &\quad + C \sum_{l=0}^k E^{-\frac{1}{4} + \frac{1-l}{19}} |\ln E|^{-1} \left\| \frac{d^{k-l}(y(s) - y_1)}{ds^{k-l}} \right\|_{L^\infty(\sigma^-, s_1)}. \end{aligned}$$

It can be checked that for $l \in \{1, \dots, 4\}$,

$$\begin{aligned} \|\partial_s^l \bar{\psi}_{|\partial\Omega}\|_{L^1(s_-^N, s_+)} &\leq C |\ln E|^{-1} E^{\frac{1-l}{24}}, \\ \left\| \frac{d^l(y(s) - y_1)}{ds^l} \right\|_{L^\infty(s_-^N, s_+)} &\leq C E^{\frac{6-l}{24}}. \end{aligned}$$

Gathering all the terms, we obtain, for $k \geq 1$,

$$\|\mathbf{a}^{(k)}\|_{L^1} = O(E^{\frac{1-k}{19}}).$$

The estimates of \mathbf{a}'' and \mathbf{b}'' in L^2 go along the same lines.

□

Appendix D: Proof of Lemma 3.4.3

The estimates on f and f' are derived in a classical fashion: for instance, write

$$f(Z)^2 = -2 \int_Z^\infty f(Z') \partial_Z f(Z') dZ',$$

so that

$$\|f\|_\infty^2 \leq 2\|f\|_2 \|\partial_Z f\|_2.$$

Integrating by parts and using the fact that $f|_{Z=0} = \partial_Z f|_{Z=0} = 0$, we infer that

$$\|\partial_Z f\|_2^2 = - \int_{\mathbf{R}_+} f \partial_Z^2 f$$

and similarly

$$(D.1) \quad \|\partial_Z^2 f\|_2^2 = \int_{\mathbf{R}_+} f \partial_Z^4 f.$$

Using several times the Cauchy-Schwarz inequality leads to

$$\|f\|_\infty \leq \sqrt{2} \|f\|_2^{7/8} \|\partial_Z^4 f\|_2^{1/8}.$$

The estimate on $\partial_Z f$ goes along the same lines. The estimate on $\partial_Z^2 f$ is a little more tricky: we write

$$(D.2) \quad \begin{aligned} \|\partial_Z^2 f\|_\infty^2 &\leq 2\|\partial_Z^2 f\|_2 \|\partial_Z^3 f\|_2 \\ &\leq 2\|f\|_2^{1/2} \|\partial_Z^3 f\|_2 \|\partial_Z^4 f\|_2^{1/2}. \end{aligned}$$

On the other hand, an integration by parts yields

$$\|\partial_Z^3 f\|_2^2 = - \int_{\mathbf{R}_+} \partial_Z^2 f \partial_Z^4 f - \partial_Z^2 f|_{Z=0} \partial_Z^3 f|_{Z=0}$$

and

$$\partial_Z^3 f|_{Z=0}^2 = -2 \int_{\mathbf{R}_+} \partial_Z^3 f \partial_Z^4 f.$$

Therefore, using (D.1)

$$\|\partial_Z^3 f\|_2^2 \leq C\|f\|_2^{1/2} \|\partial_Z^4 f\|_2^{3/2} + C\|\partial_Z^2 f\|_\infty \|\partial_Z^4 f\|_2^{1/2} \|\partial_Z^3 f\|_2^{1/2}.$$

We deduce that

$$\|\partial_Z^3 f\|_2^2 \leq C\|f\|_2^{1/2} \|\partial_Z^4 f\|_2^{3/2} + C\|\partial_Z^2 f\|_\infty^{4/3} \|\partial_Z^4 f\|_2^{2/3}.$$

Inserting the above inequality into (D.2) and using Young's inequality leads eventually to

$$\|\partial_Z^2 f\|_\infty^2 \leq C\|f\|_2^{3/4} \|\partial_Z^4 f\|_2^{5/4},$$

so that

$$\|\partial_Z^3 f\|_2 \leq C\|f\|_2^{1/4} \|\partial_Z^4 f\|_2^{3/4}.$$

The last estimate on $\|\partial_Z^3 f\|_\infty$ follows easily. □

Notations

A_i, B_i subdomains of Ω where the solution to the Sverdrup equation is continuous, p. 8, 10

Γ_E, Γ_W East and West parts of the boundary, p. 6

Γ_N, Γ_S North and South parts of the boundary, p. 6

$\gamma_{i,i-1}$ truncation of the boundary conditions to be lifted by the North/South boundary layers, p. 52

$\tilde{\gamma}_{i,i-1} \geq \gamma_{i,i-1}$ truncation of the North/South boundary layers, p. 60

$\delta_y = E^{1/4} |\ln E|^{1/5} (\ln |\ln E|)^{-\beta}$ truncation parameter with respect to y , p. 41

$\delta_x^{i,\pm} = x_E(y_i \pm \delta_y) - x_E(y_i)$ local truncation parameter with respect to x , p. 42

$\theta(s)$ angle between the horizontal vector e_x and the exterior normal to the boundary, p. 6

I_+ set of indices corresponding to East corners ($s_i \in \partial\Gamma_E$), p. 6

λ_E, λ_W inverse sizes of the East and West boundary layers, p. 21

$\lambda_N = \lambda_S = \lambda_\Sigma = E^{-1/4}$ inverse size of the horizontal boundary layers, p. 23

\mathcal{M} majorizing function, p. 43

ρ_i partition of unity, p. 6

s curvilinear abscissa, p. 5

s_i curvilinear abscissa of the singular points ($s_i \in \partial\Gamma_E \cup \partial\Gamma_W$), p. 6

s_i^\pm extremal points of the domain of validity of East and West boundary layers (if relevant), p. 23

t_1^\dagger curvilinear abscissa of the East end of Σ , p. 62

σ_i^\pm extremal points where energy is injected in North and South boundary layers (if relevant), p. 48, 50

σ_{in} initial point for the discontinuity boundary layer $\psi^{\text{lift}}, \psi^\Sigma$, p. 61

Σ surfaces of discontinuity (ordinates of critical points), p. 29

τ forcing term (coming from Ekman pumping), p. 1

φ_i^\pm truncation dealing with the transition between East/West and North/South boundary layer terms, p. 49, 50

χ_0 localization near the boundary, p. 5

χ_\pm localization on Ω^\pm , p. 30

χ_E truncation of τ near the singular point s_i , p. 41

$x_E^\pm \equiv x_E^\pm(y)$ graphs of the East boundaries of Ω^\pm , p. 30

x_1^W abscissa of the West end of Σ , p. 63

(x_i, y_i) coordinates of the singular points $((x_i, y_i) = (x(s_i), y(s_i)))$, p. 6

ψ^0 solution to the transport equation, p. 40

ψ_t^0 solution to the transport equation with truncated source term, p. 42

$\psi_{N,S}$ North and South boundary layer terms, p. 24

$\psi_{E,W}$ East and West boundary layer terms, p. 22

ψ^{lift} interior singular terms lifting the discontinuities of ψ^0 , p. 62

ψ^Σ interior singular layer terms, p. 34

$\psi_{int} = \psi_t^0 + \psi^{\text{lift}}$ regularization of the solution to the Sverdrup equation, p. 33

$\bar{\psi} = \psi_t^0 + \psi_E + \psi_E^{\text{cor}}$ sum of all terms whose trace is lifted by the North/South boundary layers, p. 49

$\bar{\psi}_\Sigma = \psi^{\text{lift}} + \psi^\Sigma \chi\left(\frac{y-y_1}{\delta_y}\right) + \psi_W^{\text{cor}}$ sum of all singular layer terms whose trace is lifted on the West boundary, p. 68

Y^- parametrization of the interior singular layer, p. 62

$\Omega = \Omega_1 \setminus \cup_{i=2}^K \Omega_i$ domain with islands Ω_i , p.9

Ω^\pm subdomains of Ω where the transport equation has a continuous solution, p. 29

z distance to the boundary, p. 5

$Z = \lambda(s)z$ scaled distance to the boundary, p. 20

Sizes of parameters and terms

Parameter	Size/Definition	
	Case (H2i)	Case (H2ii)
$s_i^\pm - s_i$	$C_i E^{\frac{1}{4n+3}}$	$\frac{\mp 4\alpha}{\ln E} \left(1 - 6 \frac{\ln \ln E }{\ln E} + O\left(\frac{1}{\ln E}\right) \right)$
$\sigma_i^\pm - s_i$ (East coast)	C_i''	
$\sigma_i^\pm - s_i$ (West coast)	$C_i' E^{\frac{1}{4n+4}}$	$\frac{\mp 4\alpha}{\ln E} \left(1 - 8 \frac{\ln \ln E }{\ln E} + O\left(\frac{1}{\ln E}\right) \right)$
σ_{in}	$\begin{cases} \sigma_1^- & \text{if } I_1 = \{s_1\}, \\ \sigma_0^- & \text{if } I_1 = [s_0, s_1] \text{ and } (\sigma_0^-, s_0) \subset \Gamma_W, \\ s_0 & \text{if } I_1 = [s_0, s_1] \text{ and } (\sigma_0^-, s_0) \subset \Gamma_E. \end{cases}$	
$\lambda_{N,S}$	$E^{-1/4}$	
$\lambda_{E,W}(s)$	$\left(\frac{ \cos \theta(s) }{E} \right)^{1/3}$	
$\lambda'_{E,W}(s)$	$\frac{E^{1/3} \text{sign}(\cos \theta(s)) \theta'(s) \sin \theta(s)}{3 \cos \theta(s) ^{4/3}}$	
$\lambda_{E,W}(s_i^\pm)$	$E^{-\frac{n+1}{4n+3}}$	$E^{-1/4} \ln E ^{1/2}$
$\lambda_{E,W}(\sigma_i^\pm)$	$E^{-\frac{3n+4}{12(n+1)}}$	$E^{-1/4} \ln E ^{2/3}$

Term	Typical size in $L^2(\Omega)$	Typical size in $H^2(\Omega^\pm)$
ψ_i^0	1	$o(E^{-3/8})$
$\psi_{N,S}$	$E^{1/8}$	$E^{-3/8}$
ψ_W	$E^{1/6}$	$E^{-1/2}$
ψ_E	$E^{1/2}$	$o(E^{-3/8})$
ψ^{lift}	$E^{1/8}$	$E^{-3/8}$
ψ^Σ	$E^{1/8}$	$E^{-3/8}$
$\psi_\Sigma^{\text{corr}}$	$o(E^{1/8})$	$o(E^{-3/8})$
	$\ \psi_\Sigma^{\text{corr}}\ _{W^{k,\infty}} = O(\delta_y^{-2} E^{-\frac{k-2}{4}})$	

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