LONG TIME BEHAVIOR OF PARABOLIC SCALAR CONSERVATION LAWS WITH SPACE PERIODIC FLUX

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Abstract. This paper is concerned with the stability of stationary solutions of the conservation law $\partial_t u + \mathrm{div}_y A(y,u) - \Delta_y u = 0$, where the flux A is periodic with respect to its first variable. Essentially two kinds of asymptotic behaviors are studied here: the case when the equation is set on $\mathbb{R}$, and the case when it is endowed with periodic boundary conditions. In the whole space case, we first prove the existence of standing shocks which connect two different periodic stationary solutions to one another. We prove that standing shocks are stable in L^1, provided the initial disturbance satisfies some appropriate boundedness conditions. Furthermore, a recent result enables us to extend this stability property to arbitrary initial data. In the periodic case, we prove that periodic stationary solutions are always stable. The proof of this result relies on the derivation of uniform L^∞ bounds on the solution of the conservation law, and on sub- and super-solution techniques.

Keywords. Viscous shocks; shock stability; viscous scalar conservation laws; periodic flux.

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1. Introduction

This paper is devoted to the analysis of the long-time behavior of the solution $u \in \mathcal{C}([0,\infty), L^1_{\mathrm{loc}}(Q)) \cap L^\infty_{\mathrm{loc}}([0,\infty), L^\infty(Q))$ of the equation

$$(1)\qquad \begin{aligned} &\partial_t u + \mathrm{div}_y A(y,u) - \Delta_y u = 0, \quad t>0,\ y\in Q,\\ &u_{|t=0} = u_0 \in L^\infty(Q). \end{aligned}$$

Above, Q denotes either $\mathbb{R}$ or $\mathbb{T}^N$, the N-dimensional torus ($\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$), and $A \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{T}^N \times \mathbb{R})^N$ is an N-dimensional flux (with $N=1$ when $Q=\mathbb{R}$).

Heuristically, it can be expected that the parabolicity of equation (1) may yield some compactness on the trajectory $\{u(t)\}_{t\geq 0}$. Hence, it is legitimate to conjecture that the family $u(t)$ will converge as $t\to\infty$ towards a stationary solution of (1). Such a result was proved when $Q=\mathbb{T}^N$ by the author in [7] for a certain class of initial conditions, namely when u_0 is bounded from above and below by two stationary solutions of (1). This kind of assumption is in fact classical in the framework of conservation laws which admit a comparison principle: we refer for instance to [3], where the authors study the long time behavior of the fast diffusion equation, and assume that the initial data is bounded by two Barenblatt profiles. The same kind of assumption was made in the context of traveling waves by Stanley Osher and James Ralston in [19]; let us also mention the review paper by Denis Serre [22], which is devoted to the stability of standing shocks of scalar conservation laws, and in which the author assumes at first that the initial data is bounded from above and below by shifted standing shocks. Nonetheless, in [12] (see also [21, 22]), Heinrich Freistühler and Denis Serre remove this hypothesis, and prove that shock stability holds under a mere L^1 assumption on the initial data.

The goal of this paper is to extend the result of [7] to arbitrary initial data, that is, to prove that solutions of (1) converge towards a stationary solution for any initial data $u_0 \in L^\infty(\mathbb{T}^N)$. We also tackle similar issues on the stability of standing shocks in dimension one, when the equation is set on the whole space case ($Q = \mathbb{R}$). Thus, a large part of the paper is devoted to the proof of the existence of standing shocks, and to the analysis of their properties. We will see that the question of shock stability reduces in fact to the stability of periodic stationary solutions of (1) in $L^1(\mathbb{R})$, an issue which is treated in the companion paper [8].

The proof of stability in the periodic setting relies strongly on the derivation of uniform L^∞ bounds on the family $\{u(t)\}_{t\geq 0}$. In the whole space case, the first step of the analysis is to prove the property for initial data which are bounded from above and below by standing shocks; in fact, this result is similar to the one proved in [7], and uses arguments from dynamical systems theory, following an idea by S. Osher and J. Ralston [19] (see also [22, 2]). But the derivation of uniform L^∞ bounds is not sufficient to obtain a general stability result in the whole space case. Thus the idea is to use existing results on the stability of stationary periodic solutions of (1) in $L^1(\mathbb{R})$. This question was addressed, when the flux A is linear, by Adrien Blanchet, Jean Dolbeault, and Michal Kowalczyk in [4]. Their techniques were then extended to arbitrary fluxes by the author in [8].

Throughout the paper, we use the following notation: if $v \in L^1(\mathbb{T}^N)$,

$$\langle v \rangle = \int_{\mathbb{T}^N} v.$$

We denote by $L^1_0(Q)$ the set of integrable functions with zero mass:

$$L^1_0(Q) := \{u \in L^1(Q),\ \int_Q u = 0\}.$$

Following [16], for $\alpha \in (0,1)$, we define, if I is an interval in $(0,\infty)$ and Ω is a domain in $\mathbb{R}^N$,

$$H^{\frac{\alpha}{2},\alpha}(I\times\Omega) = \{u \in \mathcal{C}(\bar{I}\times\bar{\Omega}),\ \ \|u\|_{H^{\alpha/2,\alpha}(I\times\Omega)} < \infty\},$$

where

$$\begin{aligned}\|u\|_{H^{\frac{\alpha}{2},\alpha}(I\times\Omega)} \quad &:= \quad \max_{(t,x)\in\bar{I}\times\bar{\Omega}} |u(t,x)| \\ &+ \sup_{\substack{(x,t)\in I\times\Omega,\\ (x',t')\in I\times\Omega,\\ |t-t'|\leq\rho}} \frac{|u(t,x)-u(t',x)|}{|t-t'|^{\alpha/2}} + \sup_{\substack{(x,t)\in I\times\Omega,\\ (x',t')\in I\times\Omega,\\ |x-x'|\leq\rho}} \frac{|u(t,x)-u(t,x')|}{|x-x'|^{\alpha}};\end{aligned}$$

above, ρ is any positive number. We also set

$$\mathcal{C}^\alpha(\Omega) := \left\{ u \in \mathcal{C}(\bar{\Omega}),\ \sup_{(x,x')\in\Omega^2} \frac{|u(x)-u(x')|}{|x-x'|^\alpha} < +\infty \right\}.$$

Eventually, for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, $h \in \mathbb{R}$, we set $\tau_h f = f(\cdot + h)$.

2. Main results

Before stating our main results, we recall general features of equation (1), together with some facts related to the stationary solutions of this equation.

In the rest of this paper, we denote by S_t the semi-group associated with equation (1). Notice that S_t is always well-defined on $L^\infty(Q)$, thanks to the papers by Kružkov [14, 15]. Moreover, we recall that the following properties hold true (these are called the **Co-properties** in [22]):

- **Comparison:** if $a, b \in L^\infty(Q)$ are such that $a \le b$, then $S_t a \le S_t b$ for all $t \ge 0$.
- **Contraction:** if $a, b \in L^\infty(Q)$ are such that $a-b \in L^1(Q)$, then $S_t a - S_t b \in L^1(Q)$ for all $t \ge 0$ and
$$\|S_t a - S_t b\|_{L^1} \le \|a-b\|_{L^1} \quad \forall t \ge 0.$$
- **Conservation:** if $a, b \in L^\infty(Q)$ are such that $a - b \in L^1(Q)$, then $S_t a - S_t b \in L^1(Q)$ for all $t \ge 0$ and
$$\int_Q (S_t a - S_t b) = \int_Q (a-b) \quad \forall t \ge 0.$$

Thanks to the Contraction property, the semi-group S_t can be extended on $L^\infty(Q) + L^1(Q)$. The so-called "Constant property" in [22] is not true in the present setting, since the flux A does not commute with translations. In other words, constants are not stationary solutions of equation (1) in general. The existence of space-periodic stationary solutions of (1) was proved by the author in [6], and we recall the corresponding result below:

Proposition 2.1. *Let $A \in W^{1,\infty}_{loc}(\mathbb{T}^N \times \mathbb{R})^N$. Assume that there exist $C_0 > 0$, $m \in [0, \infty)$, $n \in [0, \frac{N+2}{N-2})$ when $N \ge 3$, such that for all $(y,p) \in \mathbb{T}^N \times \mathbb{R}$*

$$|\partial_p A_i(y,p)| \le C_0 \left(1 + |p|^m\right) \quad \forall\ 1 \le i \le N, \tag{2}$$
$$|\mathrm{div}_y A(y,p)| \le C_0 \left(1 + |p|^n\right). \tag{3}$$

Assume as well that one of the following conditions holds:

(4)
$$m = 0 \text{ or } 0 \le n < 1 \text{ or } \left(n < \min\left(\frac{N+2}{N}, 2\right) \text{ and } \exists p_0 \in \mathbb{R},\ \mathrm{div}_y A(\cdot, p_0) = 0\right).$$

Then for all $p \in \mathbb{R}$, there exists a unique solution $v(\cdot, p) \in H^1_{per}(\mathbb{T}^N)$ of the equation

$$-\Delta_y v(y,p) + \mathrm{div}_y A(y, v(y,p)) = 0, \quad \langle v(\cdot,p)\rangle = p. \tag{5}$$

The family $(v(\cdot,p))_{p\in\mathbb{R}}$ satisfies the following properties:

(i) *Regularity estimates: For all $p \in \mathbb{R}$, $v(\cdot,p)$ belongs to $W^{2,q}_{per}(\mathbb{T}^N)$ for all $1 < q < +\infty$ and additionally*

$$\forall R > 0 \quad \exists C_R > 0 \quad \forall p \in [-R, R] \quad ||v(\cdot,p)||_{W^{2,q}(\mathbb{T}^N)} \le C_R. \tag{6}$$

(ii) *Growth property: if $p > p'$, then*

$$v(y,p) > v(y,p') \quad \forall y \in \mathbb{T}^N.$$

(iii) *Behavior at infinity: assume that*

$$\sup_{v\in\mathbb{R}} \|\partial_v A(\cdot, v)\|_{L^\infty(\mathbb{T}^N)} < +\infty. \tag{7}$$

Then

$$\lim_{p\to-\infty} \sup_{y\in\mathbb{T}^N} v(y,p) = -\infty, \quad \lim_{p\to+\infty} \inf_{y\in\mathbb{T}^N} v(y,p) = +\infty.$$

Remark 1. Assumption (4) is not completely optimal, as some examples in [6] show; in particular, it would be interesting to have an existence theory wich unites the three regimes in (4). However, it is proved in [6] that the exponent $(N+2)/N$ in assumption (4) is optimal in the following sense: consider a flux A of the type

$$A(y,u) = \nabla_y \phi(y)(1+|u|^2)^{n/2}, \quad y\in\mathbb{T}^N,\ u\in\mathbb{R}.$$

Then if $n > (N+2)/N$, there exists $\phi \in \mathcal{C}^2(\mathbb{T}^N)$ and $p_- < p_+$ in $\mathbb{R}$ such that equation (5) has no solutions for $p \notin [p_-, p_+]$.

2.1. ***A priori* bounds for solutions of scalar conservation laws.** Our first result is concerned with the derivation of *a priori* bounds in L^∞ which are uniform in time. Notice that such a result is not trivial in general: in the homogeneous case, that is, when the flux A does not depend on the space variable x, this result follows from the comparison principle stated earlier. However, in the present case, this argument does not hold, since constants are not stationary solutions of (1). Of course, if there exists a constant C such that $u_0 \le v(\cdot, C)$, then the comparison principle entails that $S_t u_0 \le v(\cdot, C)$. Hence, the derivation of *a priori* bounds is easy when the initial data is bounded from above and below by solutions of equation (5). Consequently, the goal of this paragraph is to present similar results when the initial data does not satisfy such an assumption.

Proposition 2.2. *Assume that the flux A satisfies the assumptions of Proposition 2.1. Assume also that for all $K>0$, there exists a positive constant C_K, such that for all $v\in[-K,K]$, for all $w\in\mathbb{R}$,*

$$\begin{aligned} |\mathrm{div}_y A(y, v+w) - \mathrm{div}_y A(y,v)| &\le C_K(|w| + |w|^n), \\ |\partial_v A(y, v+w) - \partial_v A(y,v)| &\le C_K(|w| + |w|^n), \end{aligned} \tag{8}$$

where $n < (N+2)/N$.

Let $u_0 \in L^\infty(Q)$, and assume that there exists a stationary solution $U_0 \in W^{1,\infty}(Q)$ of (1) such that $u_0 \in U_0 + L^1(Q)$.

Then

$$\sup_{t\ge 0} \|S_t u_0\|_{L^\infty(Q)} < +\infty.$$

Notice that in the above proposition, we do not assume that the stationary solution U_0 is periodic. Thus U_0 is not necessarily a solution of equation (5), and may be, for instance, a standing shock (see Proposition 2.4 below). In the periodic case, any function $u_0 \in L^\infty$ is such that $u_0 - v(\cdot, 0) \in L^1(\mathbb{T}^N)$, and thus the result holds for all functions in L^∞.

Remark 2. First, let us emphasize that assuming polynomial growth for the flux A in (2), (3), (8) is classical in the framework of boundedness theorems for parabolic equations, see for instance the book [20] by P. Quittner and P. Souplet. Moreover, the result in Proposition 2.2 may be compared with blow-up theorems for superlinear parabolic equations with gradient terms. For instance, the article [1] by J. Aguirre and M. Escobedo (see also Chapter 4 in [20]) is dedicated to the study of equations of the type

$$\partial_t u - \Delta_y u = u|u|^{p-1} + a \cdot \nabla_y (u|u|^{q-1}), \quad t > 0, y \in \mathbb{R}^N, \tag{9}$$

with $p > 1$, $q \geq 1$ and $a \in \mathbb{R}^N$. Blow-up and global existence results are given, depending on the value of the parameters p and q. Notice that equation (1) falls more or less into the regime $q > p$: indeed, in (1), there is no term $u|u|^{p-1}$ in the right-hand side. Thus the (conservative) gradient term "dominates" $u|u|^{p-1}$.

In the regime $q > p$, it is proved in [1, 20] that solutions of (9) are global, and that

$$\sup_{t>0} \|u(t)\|_{L^\infty(\mathbb{R}^N)} < +\infty,$$

which is coherent with Proposition 2.2. However, it is also proved in [1, 20] that in the regime

$$q \leq p \leq \min\left(\frac{N+2}{N}, 1 + \frac{2q}{N+1}\right)$$

there is blow-up in finite time. Thus Proposition 2.2 is suprising in this regard.

Remark 3. Proposition 2.2 remains true when $Q = \mathbb{R}^N$ with $N \geq 2$. The proof is essentially the same as the one in Section 5 in the case $Q = \mathbb{R}$, with minor changes due to the dependance of Sobolev embeddings with respect to the space dimension. The details are given in the Appendix of [8].

2.2. **Stability of stationary periodic solutions in the periodic case.** The derivation of uniform *a priori* bounds for the solutions of equation (1) allows us to extend the stability results proved in [7] to general classes of initial data. Let us first recall the stability result of [7]:

Proposition 2.3. *Assume that the flux A satisfies the assumptions of Proposition 2.1. Let $u_0 \in L^\infty(\mathbb{T}^N)$ such that there exists $\beta_1, \beta_2 \in \mathbb{R}$ such that*

$$v(\cdot, \beta_1) \leq u_0 \leq v(\cdot, \beta_2). \tag{10}$$

Then as $t \to \infty$

$$S_t u_0 \to v(\cdot, \langle u_0 \rangle) \quad \text{in } L^\infty(\mathbb{T}^N).$$

It was also proved in [7] that under additional regularity assumptions on the flux A, the speed of convergence is exponential, due to a spectral gap result.

We now remove assumption (10) thanks to Proposition 2.2:

Theorem 2.1. *Assume that the flux A satisfies the assumptions of Proposition 2.1, together with* (8). *Then for all $u_0 \in L^\infty(\mathbb{T}^N)$, as $t \to \infty$,*

$$S_t u_0 \to v(\cdot, \langle u_0 \rangle) \quad \text{in } L^\infty(\mathbb{T}^N).$$

The proof of this result relies mainly on Proposition 2.2 and on sub- and super-solution methods based on the Comparison principle. Once again, it can be proved that the speed of convergence is exponential, provided the flux A is sufficiently smooth. For more details regarding that point, we refer to [7].

2.3. **Existence of standing shocks.** We now consider equation (1) set in $Q = \mathbb{R}$. Our goal here is to prove the stability of a special class of stationary solutions, called "standing shocks". By analogy with the definition in [22] of shocks in homogeneous conservation laws, a *standing shock* is a stationary solution U of equation (1) which is asymptotic to solutions of equation (5) at infinity, namely

$$\exists (p_l, p_r) \in \mathbb{R}^2, \quad \lim_{y\to-\infty} (U(y) - v(y, p_l)) = 0, \ \lim_{y\to+\infty} (U(y) - v(y, p_r)) = 0.$$

Because of the spatial dependence of the flux A, it does not seem to be possible to restrict the study of general shocks to standing shocks. For that matter, we wish to emphasize that the definition of a viscous shock with non-zero speed should not be exactly the same as in [22]; indeed, it can be easily checked that if

$$u(t, x) = U(x - st)$$

is a solution of (1), then $s = 0$ necessarily. Thus, for $s \neq 0$, a standing shock is a solution of (1) of the form

$$u(t, x) = U(t, x - st),$$

where for all t, $U(t)$ is asymptotic to solutions of equation (5) at infinity. This is related (although not equivalent to) the definition of traveling pulsating fronts, see for instance the paper of Xue Xin [23]. The existence of non-stationary shocks and their stability is beyond the scope of this paper, and thus, we will focus on standing shocks from now on.

Our first result is concerned with the existence of standing shocks. We define the averaged (or homogenized) flux $\bar{A}$ by

$$\bar{A}(p) := \int_{\mathbb{T}} A(y, v(y, p))\, dy,$$

where $v(\cdot, p)$ is the solution of the cell problem (5) with mean value p.

Proposition 2.4 (Existence of standing shocks). *Assume that there exists $p^-, p^+ \in \mathbb{R}$ such that $p^- < p^+$ and*

$$\bar{A}(p^+) = \bar{A}(p^-) =: \alpha, \tag{11}$$

and define $v_\pm := v(\cdot, p^\pm)$.

Let $U_0 \in \mathbb{R}$ such that

$$v_-(0) < U_0 < v_+(0),$$

and let $U : I \to \mathbb{R}$ be the maximal solution of the differential equation

$$\frac{dU(x)}{dx} = A(x, U(x)) - \alpha, \tag{12}$$

$$U_{|x=0} = U_0. \tag{13}$$

Then U satisfies the following properties:

- **(i)** *The function U is a global solution of (12); in other words, $I = \mathbb{R}$.*
- **(ii)** *For all $x \in \mathbb{R}$,*
$$v_-(x) \le U(x) \le v_+(x);$$
- **(iii)** *There exist $q_l, q_r \in [p^-, p^+]$ such that $\bar{A}(q_l) = \bar{A}(q_r) = \alpha$ and*
$$\lim_{x\to-\infty} (U(x) - v(x, q_l)) = 0, \quad \lim_{x\to+\infty} (U(x) - v(x, q_r)) = 0.$$

As a consequence, the solution U of (12)-(13) is a standing shock.

Remark 4. **(i)** Assumption (11) is the analogue of the Rankine-Hugoniot condition for homogeneous conservation laws. It is in fact a necessary condition, as demonstrated in Lemma 3.1 below.

(ii) The solution of (12)-(13) may in fact be a periodic stationary solution of (1) (notice that our definition allows periodic solutions to be standing shocks; in this case, the asymptotic states are identical). This occurs if and only if there exists $p \in (p_-, p_+)$ such that

$$\bar{A}(p) = \alpha \text{ and } v(0,p) = u_0.$$

We refer to Corollary 2.1 below for more details.

(iii) In general, the asymptotic states $v(\cdot, q_l)$, $v(\cdot, q_r)$ are different from $v(\cdot, p^+)$, $v(\cdot, p^-)$. Proposition 2.4 only ensures that

$$\bar{A}(q_l) = \bar{A}(q_r) = \bar{A}(p^\pm).$$

However, the asymptotic states can be identified:

Corollary 2.1. *Assume that the hypotheses of Proposition 2.4 are satisfied. Then the following properties hold:*

(i) *There exists a standing shock connecting $v(\cdot, p^-)$ and $v(\cdot, p^+)$ if and only if*

$$\forall p \in (p^-, p^+), \ \bar{A}(p) \neq \alpha. \tag{14}$$

In this case, there exists an infinity of such shocks.

(ii) *With the notation of Proposition 2.4, let*

$$q_+ := \inf \left\{ p \in [p^+, p^-], U_0 \leq v(0,p) \text{ and } \bar{A}(p) = \alpha \right\},$$
$$q_- := \sup \left\{ p \in [p^+, p^-], U_0 \geq v(0,p) \text{ and } \bar{A}(p) = \alpha \right\}.$$

Then

$$\{q_l, q_r\} = \{q_+, q_-\}.$$

In particular, if $q_+ = q_-$, then U is a periodic stationary solution of (1).

2.4. **Stability of standing shocks in the whole space case.** We are now ready to state results on shock stability for equation (1). Our first result is the analogue of Proposition 2.3: indeed, Theorem 2.2 below states that $S_t u_0$ converges towards a standing shock, provided u_0 is bounded from above and below by the asymptotic states of the shock. In view of Theorem 2.1, it is natural to expect that this result remains true for arbitrary initial data. Unfortunately, we have not been able to provide a simple proof of this result in complete generality: we merely prove that stationary shocks are stable in L^1 provided stability holds (in $L^1(\mathbb{R})$) for solutions of equation (5).

Theorem 2.2. *Assume that the flux A satisfies the assumptions of Proposition 2.1. Let $p_l, p_r \in \mathbb{R}$ such that $p_l \neq p_r$ and $\bar{A}(p_r) = \bar{A}(p_l) =: \alpha$, and assume that $\bar{A}, p_l, p_r$ satisfy Oleinik's condition* (14).

Let U be a standing shock connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$. Let $u_0 \in U + L^1(\mathbb{R})$ such that for almost every $x \in \mathbb{R}$,

$$v(x, \min(p_l, p_r)) \leq u_0(x) \leq v(x, \max(p_l, p_r)). \tag{15}$$

Then there exists a standing shock V connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$ and such that $u \in V + L^1_0(\mathbb{R})$. Moreover,

$$\lim_{t\to\infty} \|S_t u_0 - V\|_{L^1(\mathbb{R})} = 0.$$

As outlined before, hypothesis (15) should be compared with assumption (10). Thus, the next step would be to prove that stability holds even when (15) is false. In fact, we are able to prove the following:

Proposition 2.5. *Assume that the flux A satisfies the assumptions of Proposition 2.2. Let $p_l, p_r \in \mathbb{R}$ such that $p_l \neq p_r$, $\bar{A}(p_r) = \bar{A}(p_l)$, and such that* (14) *is satisfied. Assume that the following assertion is true:*

(H) *For $p \in \{p_l, p_r\}$, there exists $\delta > 0$ such that for all $u_0 \in v(\cdot, p) + L^1_0(\mathbb{R})$,*

$$\|u_0 - v(p)\|_1 \leq \delta \Rightarrow \lim_{t\to\infty} \|S_t u_0 - v(\cdot, p)\|_{L^1(\mathbb{R})} = 0.$$

Let U be a standing shock connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$, and let $u_0 \in U + L^1(\mathbb{R})$. Then there exists a standing shock V connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$ and such that $u \in V + L^1_0(\mathbb{R})$. Moreover,

$$\lim_{t\to\infty} \|S_t u_0 - V\|_{L^1(\mathbb{R})} = 0.$$

Let us now discuss the validity of assumption **(H)**. If the flux A is linear, namely if

$$A(y, u) = \alpha(y)u \quad y \in \mathbb{T}^N,\ u \in \mathbb{R},$$

then the analysis performed by A. Blanchet, J. Dolbeault and M. Kowalczyk (see [4]) shows that **(H)** holds, under some technical assumptions on the fourth order moments of the solutions of (1). This result was then extended to nonlinear fluxes by the author. More precisely, it is proved in [8] that **(H)** holds for all fluxes $A \in W^{5,\infty}(\mathbb{T} \times \mathbb{R})$ satisfying (8). Hence the following result follows:

Corollary 2.2. *Assume that the flux $A \in W^{5,\infty}(\mathbb{T} \times \mathbb{R})$ satisfies the assumptions of Proposition 2.2. Let $p_l, p_r \in \mathbb{R}$ such that $p_l \neq p_r$, $\bar{A}(p_r) = \bar{A}(p_l)$, and such that* (14) *is satisfied.*

Let U be a standing shock connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$, and let $u_0 \in U + L^1(\mathbb{R})$. Then there exists a standing shock V connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$ and such that $u \in V + L^1_0(\mathbb{R})$. Moreover,

$$\lim_{t\to\infty} \|S_t u_0 - V\|_{L^1(\mathbb{R})} = 0.$$

The plan of the paper is the following: given the similarity between the statements for periodic solutions when $Q = \mathbb{T}^N$, and stationary shocks when $Q = \mathbb{R}$, we first prove the existence of standing shocks (i.e. Proposition 2.4) and the shock stability result under boundedness conditions on the initial data (i.e. Theorem 2.2) in sections 3 and 4 respectively. At this stage, we are able to treat both models simultaneously, and thus we prove Proposition 2.2 in Section 5. Section 6 is devoted to the proof of Theorem 2.1, and at last, we show Proposition 2.5 in Section 7.

Throughout the paper, we will often denote by $v(p)$ the function $v(\cdot, p)$ (i.e. the solution of (5)), for the sake of brevity.

3. Existence of one dimensional stationary standing shocks

This section is devoted to the proof of Proposition 2.4 and Corollary 2.1, together with a number of results related to standing shocks which will be useful in the proof of Theorem 2.2. These auxiliary results (monotonicity, integrability of the difference between two standing shocks, etc.) can be found in paragraph 3.4.

We begin with some comments on assumption (11).

3.1. **Analysis of necessary conditions.**

Lemma 3.1. *Let $q_l, q_r \in \mathbb{R}$, and let $U \in W^{1,\infty}(\mathbb{R})$ be such that*

$$\begin{aligned} &U(x) - v(x, q_l) \to 0 \quad \text{as } x \to -\infty, \\ &U(x) - v(x, q_r) \to 0 \quad \text{as } x \to +\infty, \\ &-\frac{d^2U(x)}{dx^2} + \frac{d}{dx}(A(x, U(x))) = 0. \end{aligned}$$

Then $\bar{A}(q_r) = \bar{A}(q_l) =: \alpha$, and u satisfies

$$-\frac{dU(x)}{dx} + A(x, U(x)) = \alpha.$$

Proof. We deduce from the differential equation that there exists a constant C such that

$$-\frac{dU(x)}{dx} + A(x, U) = C,$$

and the goal is to prove that $\bar{A}(q_r) = C = \bar{A}(q_l)$. We recall first that for all $p \in \mathbb{R}$, $v(\cdot, p)$ is a solution of

$$-\frac{\partial v(x,p)}{\partial x} + A(x, v(x,p)) = \bar{A}(p).$$

Indeed, integrating (5) on $\mathbb{R}$, we infer that for all $p \in \mathbb{R}$ there exists a constant C_p such that

$$\forall x \in \mathbb{R}, \quad -\frac{\partial v(x,p)}{\partial x} + A(x, v(x,p)) = C_p.$$

Taking the average of the above equality over $\mathbb{T}$, we deduce that $C_p = \bar{A}(p)$.

As a consequence, we have

$$-\frac{d}{dx}\left(U(x) - v(x, q_r)\right) + \left[A(x, U(x)) - A(x, v(x, q_r))\right] = C - \bar{A}(q_r) \tag{16}$$

Now, let $\delta > 0$ arbitrary. There exists $x_r > 0$ such that

$$x \geq x_r \Rightarrow \left(|U(x) - v(x, q_r)| \leq \delta, \quad |A(x, U(x)) - A(x, v(x, q_r))| \leq \delta\right).$$

Integrating (16) on the interval $[x_r, x_r + 1]$, we deduce that

$$|C - \bar{A}(q_r)| \leq 3\delta.$$

Since the above inequality is true for all $\delta > 0$, we infer that $C = \bar{A}(q_r)$. The other equality is treated similarly.

□

Remark 5. Notice that couples (p_l, p_r) such that $p_l \neq p_r$ and $\bar{A}(p_l) = \bar{A}(p_r)$ do not always exist. Indeed, consider the case of a linear flux $A(x, v) = a(x)v$, with $a \in \mathcal{C}^1(\mathbb{T})$. Then, for all $p \in \mathbb{R}$, we have $v(x, p) = pm(x)$, where m is the unique probability measure on $\mathbb{T}$ satisfying

$$-\frac{d^2m(x)}{dx^2} + \frac{d}{dx}(a(x)m(x)) = 0, \quad x \in \mathbb{T}.$$

The positivity of m is a consequence of the Krein-Rutman Theorem; we refer to [6] for more details.

Therefore, for all $p \in \mathbb{R}$,

$$\bar{A}(p) = \langle av(\cdot, p)\rangle = p\,\langle am\rangle\,.$$

Hence, as long as $\langle am\rangle \neq 0$, $\bar{A}(p) \neq \bar{A}(q)$ for all $p, q \in \mathbb{R}$ such that $p \neq q$. In particular, if a is a non-zero constant, assumption (11) is never satisfied.

3.2. **Proof of Proposition 2.4.** We begin with the *a priori* bound **(ii)**, from which we deduce that u is a global solution.

The inequality **(ii)** follows directly from classical results in differential equations; indeed, assume that there exists $x_0 \in I$ such that

$$U(x_0) \geq v_+(x_0);$$

since $U(0) < v_+(0)$, there exists $x_1 \in [0, x_0]$ such that $U(x_1) = v_+(x_1)$. But U and v_+ are solutions of the same first-order differential equation, whence the Cauchy-Lipschitz Theorem implies that $U = v_+$, which is false. Thus

$$U(x) < v_+(x) \quad \forall x \in I. \tag{17}$$

The lower bound is proved in the same way.

As a consequence, we deduce that U remains bounded on its (maximal) interval of existence I. Using once again the Cauchy-Lipschitz Theorem, we infer that $I = \mathbb{R}$, and thus U is a global solution.

We now tackle the core of Proposition 2.4. First, since the flux A is $\mathbb{T}$-periodic, the function $U(\cdot + 1)$ is also a solution of equation (12). Hence the function $x \mapsto U(x+1) - U(x)$ keeps a constant sign on $\mathbb{R}$, which entails in particular that for all $x \in \mathbb{R}$, the sequences $(U(x \pm k))_{k \in \mathbb{N}}$ are monotonous. Consider for instance the sequence of functions

$$U_k : x \in [0,1] \mapsto U(x+k).$$

According to the above remarks, the sequence (U_k) is monotonous and bounded in L^∞; hence for all $x \in [0,1]$, $U_k(x)$ has a finite limit, which we denote by $U_\infty(x)$. Moreover, thanks to the uniform bound **(ii)** and the differential equation (12), U belongs to $W^{1,\infty}(\mathbb{R})$, and thus the sequence U_k is uniformly bounded (with respect to k) in $W^{1,\infty}([0,1])$. Thus $U_\infty \in W^{1,\infty}([0,1])$, and U_∞ is a continuous function. According to Dini's Theorem, we eventually deduce that U_k converges towards U_∞ in $L^\infty([0,1])$. Notice that U_∞ is periodic by definition, and passing to the limit in equation (12), we deduce that U_∞ is a solution of (12). Hence U_∞ belongs to $W^{1,\infty}(\mathbb{T})$ and satisfies

$$-\frac{d^2 U_\infty(x)}{dx^2} + \frac{d}{dx}(A(x, U_\infty(x))) = 0,$$

which means exactly that U_∞ is a periodic solution of equation (5); according to Proposition 2.1, there exists $q_r \in \mathbb{R}$ such that $U_\infty = v(\cdot, q_r)$. Inequality (17) ensures that

$$v(x, q_r) \leq v_+(x) \quad \forall x \in \mathbb{R},$$

and thus $q_r \leq p^+$ according to Proposition 2.1. In a similar way, $q_r \geq p^-$. Eventually, since U_∞ is a solution of (12), we infer that $\alpha = \bar{A}(q_r)$. To sum up, we have proved that there exists $q_r \in [p^-, p^+]$, such that $\bar{A}(q_r) = \bar{A}(p^\pm)$, and such that

$$\lim_{k \to +\infty} \sup_{x \in [0,1]} |U(x+k) - v(x, q_r)| = 0.$$

The above convergence is strictly equivalent to $U(x) - v(x, q_r) \to 0$ as $x \to \infty$, and thus the third point of the Proposition is proved. The limit as $x \to -\infty$ is treated similarly.

3.3. **Proof of Corollary 2.1.** We begin with the proof of (i), frow which (ii) follows easily. First, assume that p^+, p^- satisfy (14). We construct a standing shock U with asymptotic states q_l, q_r as in Proposition 2.4. Point (ii) in Proposition 2.4 entails that

$$\{q_l, q_r\} \subset \{p \in [p^-, p^+],\ \bar{A}(p) = \alpha\} = \{p^-, p^+\}.$$

Hence we only have to prove that $q_l \neq q_r$: if $q_l = q_r$, then the proof of Proposition 2.4 shows that $U(\cdot + k)$, $k \in \mathbb{Z}$ is a monotonous sequence of functions, with the same asymptotic states as $k \to \pm\infty$. Consequently, $U(\cdot + k)$ is a constant sequence, which means that the function U is periodic: we have

$$U(x) = v(x, q_l) \quad \forall x \in \mathbb{R}.$$

In particular, $U_0 = v(0, q_l)$; since $q_l \in \{p^-, p^+\}$ and $U_0 \in (v(0, p^-), v(0, p^+))$, there is a contradiction. Thus $q_l \neq q_r$, and we infer that

$$\{q_l, q_r\} = \{p^-, p^+\}.$$

Conversely, assume that there exists a standing shock U connecting $v(\cdot, p^-)$ and $v(\cdot, p^+)$, and let us prove (14). Without loss of generality, assume that $q_l = p^+$, $q_r = p^-$. Assume by contradiction that there exists $p \in (p^-, p^+)$ such that $\bar{A}(p) = \alpha$. Then

$$U(k) - v(k, q_r) = U(k) - v(0, q_r) \underset{\substack{k\to+\infty,\\ k\in\mathbb{N}}}{\longrightarrow} 0,$$
$$U(-k) - v(-k, q_r) = U(-k) - v(0, q_r) \underset{\substack{k\to+\infty,\\ k\in\mathbb{N}}}{\longrightarrow} 0.$$

Hence there exists $k_+, k_- \in \mathbb{N}$ such that

$$v(0, p) > U(k_+), \quad v(0, p) < U(-k_-).$$

Moreover, $\tau_{k_+}U$, $\tau_{-k_-}U$ and $v(\cdot, p)$ are all solutions of the differential equation (12). The Cauchy-Lipschitz uniqueness Theorem entails that $\tau_{k_+}U - v(\cdot, p)$, $\tau_{-k_-}U - v(\cdot, p)$ keep a constant sign on $\mathbb{R}$. Thus for all $x \in \mathbb{R}$, we have

$$U(x + k_+) = \tau_{k_+}U(x) < v(x, p), \quad U(x - k_-) = \tau_{-k_-}U(x) > v(x, p).$$

Since the function v is periodic, we infer that $v(x, p) < U(x) < v(x, p)$ for all $x \in \mathbb{R}$, which is absurd. Hence (14) holds.

Additionally, if (14) holds, than using the construction of Proposition 2.4, any real number $U_0 \in (v(0, p^-), v(0, p^+))$ gives rise to a shock $U[U_0]$ connecting $v(\cdot, p^-)$ to $v(\cdot, p^+)$. The Cauchy-Lipschitz Theorem entails that

$$\forall U_0, U_0', \quad U_0 < U_0' \Rightarrow U[U_0](x) < U[U_0'](x) \text{ for all } x \in \mathbb{R}.$$

In particular, $U[U_0] \neq U[U_0']$ for $U_0 \neq U_0'$, which means that there is an infinite number of shocks connecting $v(\cdot, p^-)$ to $v(\cdot, p^+)$. This completes the proof of (i).

We now tackle the proof of (ii). First, the continuity of $v(y, p)$ with respect to p and that of the flux $\bar{A}$ entail that

$$v(0, q_-) \leq U_0 \leq v(0, q_+), \quad \bar{A}(q_-) = \bar{A}(q_+) = \alpha.$$

As a consequence, if $q_+ = q_- =: q$, then $U_0 = v(0, q)$ with $\bar{A}(q)$. Then $v(\cdot, q)$ is a solution of (12)-(13), and thus the solution U of (12)-(13) is a periodic stationary solution of (1).

We now assume that $q_+ \neq q_-$. The same arguments as before show that

$$v(0, q_-) < U_0 < v(0, q_+).$$

Moreover, by construction, q_-, q_+ satisfy assumption (14). According to point (i), the solution U of (12)-(13) is a standing shock connecting $v(\cdot, q_-)$ to $v(\cdot, q_+)$. Thus Corollary 2.1 is proved.

3.4. **Further results on standing shocks.** We have gathered in this paragraph some results which will be important in the proof of Theorem 2.2. The first lemma gives a criterion allowing us to distinguish the asymptotic states at $\pm\infty$.

Lemma 3.2. *Let $p_l, p_r \in \mathbb{R}$ such that $\bar{A}(p_l) = \bar{A}(p_r)$, and let U be a standing shock such that*

$$\lim_{x\to-\infty} [U(x) - v(x, p_l)] = \lim_{x\to+\infty} [U(x) - v(x, p_r)] = 0.$$

Then

$$\langle \partial_v A(\cdot, v(\cdot, p_l))\rangle \geq 0, \quad \langle \partial_v A(\cdot, v(\cdot, p_r))\rangle \leq 0.$$

Moreover, if one of the above inequalities is strict, then U converges exponentially fast toward the corresponding solution of equation (5)*; for instance, if*

$$\bar{a}_r := \int_{\mathbb{T}} \partial_v A(y, v(y, p_r))\, dy < 0,$$

then for all $a \in (0, -\bar{a}_r)$, there exists a constant C_a such that for all $y \in [0, \infty)$,

$$|U(y) - v(y, p_r)| \leq C_a \exp(-ay).$$

Proof. Throughout the proof, we use the notation $v(p) = v(\cdot, p)$.

Since U is a standing shock and $v(p_l)$, $v(p_r)$ are solutions of equation (5), we have

$$U'(x) = A(x, U(x)) - \alpha,$$
$$\partial_x v(x, p_l) = A(x, v(x, p_l)) - \alpha,$$
$$\partial_x v(x, p_r) = A(x, v(x, p_r)) - \alpha,$$

where α denotes the common value of $\bar{A}(p_l)$ and $\bar{A}(p_r)$.

Consequently, the function $U - v(p_r)$, for instance, satisfies the linear equation

$$\partial_x(U(x) - v(x, p_r)) = b(x)(U(x) - v(x, p_r)), \tag{18}$$

where

$$b(x) = \int_0^1 \partial_v A(x, \tau U(x) + (1-\tau) v(x, p_r))\, d\tau.$$

Notice that since U converges towards $v(p_r)$ as $x \to +\infty$, we obtain

$$\lim_{x\to+\infty} [b(x) - \partial_v A(x, v(x, p_r))] = 0. \tag{19}$$

On the other hand, equation (18) implies that

$$U(x) - v(x, p_r) = [U(0) - v(0, p_r)] \exp\left(\int_0^x b(y)\, dy\right).$$

Once again, since $U - v(p_r)$ converges towards zero, we infer that

$$\lim_{x\to+\infty} \int_0^x b(y)\, dy = -\infty. \tag{20}$$

The first statement of the proposition follows easily from (19), (20); indeed, assume that $\bar{a}_r > 0$. Then there exists a positive number K such that

$$x \geq K \Rightarrow b(x) - \partial_v A(x, v(x, p_r)) \geq -\frac{\bar{a}_r}{2},$$

and consequently, using the fact that $x \mapsto \partial_v A(x, v(x, p_r))$ is a periodic function, we obtain for $x \geq K$

$$\begin{aligned}\int_K^x b(y)\, dy &\geq \int_K^x \partial_v A(y, v(y, p_r))\, dy - (x-K)\frac{\bar{a}_r}{2} \\ &\geq \lfloor x-K \rfloor \bar{a}_r - x\frac{\bar{a}_r}{2} - C \\ &\geq x\frac{\bar{a}_r}{2} - C.\end{aligned}$$

The above inequality is obviously in contradiction with (20). Hence $\bar{a}_r \leq 0$, which proves the first statement in the proposition.

Now, assume that $\bar{a}_r < 0$, and choose $a \in (0, -\bar{a}_r)$ arbitrary. As before, we pick $K > 0$ such that

$$x \geq K \Rightarrow b(x) - \partial_v A(x, v(x, p_r)) \leq -\bar{a}_r - a.$$

We then obtain an inequality of the type

$$\begin{aligned}\int_K^x b(y)\, dy &\leq (-\bar{a}_r - a)(x-K) + \lfloor x-K \rfloor \bar{a}_r + C \\ &\leq -ax + C.\end{aligned}$$

Inserting this inequality back into the formula for $U - v(p_r)$ yields the exponential convergence result.

□

The next result is concerned with the integrability of the difference between two standing shocks.

Lemma 3.3. *Let $p_l, p_r \in \mathbb{R}$ such that $p_l \neq p_r$ and $\bar{A}(p_l) = \bar{A}(p_r)$, and let U, V be two standing shocks with asymptotic states $v(\cdot, p_l)$, $v(\cdot, p_r)$.*

Then $U - V \in L^1(\mathbb{R})$.

Proof. Set

$$U_0 := U(0),\ V_0 := V(0),$$

and assume for instance that $U_0 \leq V_0$. If $U_0 = V_0$, then $U = V$ according to the Cauchy-Lipschitz Theorem (see the proof of Proposition 2.4), and the result is obvious. Thus we assume from now on that $U_0 < V_0$. As a consequence, we have

$$\forall y \in \mathbb{R}, \quad v(y, \min(p_l, p_r)) < U(y) < V(y) < v(y, \max(p_l, p_r)).$$

We recall that the sequence $(U(k))_{k\in\mathbb{Z}}$ is monotonous, and converges towards $v(0, p_l)$ (resp. $v(0, p_r)$) as $k \to -\infty$ (resp. $k \to +\infty$). Hence, there exists an integer $k_0 \in \mathbb{Z}$ such that

$$U_0 < V_0 < U(k_0), \tag{21}$$

from which we infer that $U \leq V \leq \tau_{k_0} U$.

As a consequence, it is sufficient to prove that $\tau_k U - U$ is integrable, for all $k \in \mathbb{Z}$.

First, remember that $\tau_k U - U$ has a constant sign, since $\tau_k U$ and U are both standing shocks. Thus we only have to prove that the family

$$\int_{-A}^{A} (\tau_k U - U)$$

remains bounded as $A \to \infty$. A simple calculation leads to

$$\begin{aligned}\int_{-A}^{A} (\tau_k U - U) &= \int_{-A}^{A} U(y+k)\,dy - \int_{-A}^{A} U(y)\,dy \\ &= \int_{k-A}^{k+A} U(y)\,dy - \int_{-A}^{A} U(y)\,dy \\ &= \int_{A}^{k+A} U(y)\,dy - \int_{-A}^{k-A} U(y)\,dy.\end{aligned}$$

Thus, recalling that U is a bounded function, we obtain

$$\sup_{A>0} \left| \int_{-A}^{A} (\tau_k U - U) \right| \leq 2k \|U\|_{L^\infty(\mathbb{R})}.$$

We deduce that $\tau_k U - U \in L^1(\mathbb{R})$ for all $k \in \mathbb{Z}$, and eventually that $U - V \in L^1(\mathbb{R})$ according to (21).

□

The next result is in fact the first part of the statement of Theorem 2.2:

Lemma 3.4. *Let $p_l, p_r \in \mathbb{R}$ such that the assumptions of Theorem 2.2 are satisfied, and let U be a standing shock connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$.*

Let $u \in U + L^1$. Then there exists a unique standing shock V, with asymptotic states $v(\cdot, p_l)$ and $v(\cdot, p_r)$, and such that $u \in V + L^1_0(\mathbb{R})$.

Proof. According to Lemma 3.3, we already know that for every standing shock V, we have $u - V \in L^1$. Hence, the question is to find a standing shock V such that

$$\int_{\mathbb{R}} (u - V) = 0. \tag{22}$$

Notice that such a standing shock is necessarily unique: indeed, the Cauchy-Lipschitz uniqueness principle entails that the difference of two standing shocks is a function which keeps a constant sign. Hence, if V_1, V_2 are standing shocks satisfying $\int_{\mathbb{R}} (V_1 - V_2) = 0$, then $V_1 = V_2$.

We now prove that there exists a standing shock V such that $u - V \in L^1_0(\mathbb{R})$. As before, we set $p^- = \min(p_l, p_r)$, $p^+ = \max(p_l, p_r)$. For all $\xi \in (v(0, p^-), v(0, p^+))$, we denote by V_ξ the solution of

$$\begin{aligned}\frac{dV(x)}{dx} &= A(x, V(x)) - \bar{A}(p_l), \\ V_{|x=0} &= \xi.\end{aligned}$$

Then, according to Proposition 2.4 and Lemma 3.3, for all ξ, V_ξ is a standing shock connecting $v(p_l)$ to $v(p_r)$, and additionally $u - V_\xi \in L^1(\mathbb{R})$. Moreover, if $\xi > \xi'$, then $V_\xi(x) > V_{\xi'}(x)$ for all x; hence the function

$$F : \xi \in (v(0, p^-), v(0, p^+)) \mapsto \int_{\mathbb{R}} (u(x) - V_\xi(x))\,dx$$

is well-defined and decreasing with respect to ξ; using classical results on differential equations, it can easily be proved that F is continuous. We wish to find ξ_0 such that $F(\xi_0) = 0$; thus it suffices to show that

$$\lim_{\xi \to v(0,p^-)_+} F(\xi) > 0 \quad \text{and} \quad \lim_{\xi \to v(0,p^+)_-} F(\xi) < 0.$$

The above result is a direct consequence of Lebesgue's monotone convergence Theorem and of the fact that

$$\forall x \in \mathbb{R}, \quad \lim_{\xi \to v(0,p^-)_+} V_\xi(x) = v(x, p^-). \tag{23}$$

The same kind of result holds with $v(p^+)$. Let us now prove (23). Let $R > 0$ be arbitrary, and let $\varepsilon > 0$. Without loss of generality, assume that $p_r = p^-$. Then there exists $K \in \mathbb{N}$ such that

$$x \geq K \Rightarrow v(x, p_r) \leq U(x) \leq v(x, p_r) + \varepsilon.$$

In particular, $\tau_{K+\lfloor R \rfloor+1} U$ is a standing shock which satisfies

$$\tau_{K+\lfloor R \rfloor+1} U(x) \leq v(x, p_r) + \varepsilon \quad \forall x \in [-R, R].$$

Let $\bar{\xi} := \tau_{K+\lfloor R \rfloor+1} U(0) = U(K + \lfloor R \rfloor + 1)$. The Cauchy-Lipschitz Theorem entails that $V_{\bar{\xi}} = \tau_{K+\lfloor R \rfloor+1} U$. As a consequence, for all $\xi < \bar{\xi}$, for all $x \in [-R, R]$, we have

$$v(x, p_r) \leq V_\xi(x) \leq V_{\bar{\xi}}(x) \leq v(x, p_r) + \varepsilon.$$

The convergence result (23) follows, and thus there exists a standing shock V such that $u_0 \in V + L^1_0(\mathbb{R})$.

□

The next lemma allows us to replace inequality (15) by an inequality in which the upper and lower bounds are standing shocks, which will be useful in the proof of Theorem 2.2 in Section 4.

Lemma 3.5. *Let p_l, p_r such that the hypotheses of Theorem 2.2 are satisfied. Let U be a standing shock connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$. Let $u \in L^\infty(\mathbb{R})$ such that $u \in U + L^1_0(\mathbb{R})$ and assume that for almost every $y \in \mathbb{R}$,*

$$v(y, \min(p_r, p_l)) \leq u(y) \leq v(y, \max(p_l, p_r)).$$

Let $\varepsilon > 0$ be arbitrary. Then there exists a function $u^\varepsilon \in U + L^1_0(R)$, together with standing shocks $U^\varepsilon_\pm$ connecting $v(p_l)$ to $v(p_r)$, such that

$$\|u - u^\varepsilon\|_{L^1} \leq \varepsilon, \quad U^\varepsilon_- \leq u^\varepsilon \leq U^\varepsilon_+.$$

Proof. First, since $u - U \in L^1(\mathbb{R})$, there exists a positive number A^ε such that

$$\int_{|x| \geq A^\varepsilon} |u - U| \leq \varepsilon.$$

Hence, for $|x| \geq A^\varepsilon$, we take $u^\varepsilon(x) = U(x)$.

The definition of u^ε on the interval $[-A^\varepsilon, A^\varepsilon]$ is slighlty more technical, because of the various constraints bearing on u^ε. Once again, we assume that $p_l > p_r$ in order to lighten the notation. We first consider a function $v^\varepsilon \in \mathcal{C}([-A^\varepsilon, A^\varepsilon])$ which satisfies

$$\int_{|x| \leq A^\varepsilon} |u(x) - v^\varepsilon(x)| \, dx \leq \varepsilon$$

and such that

$$v(x,p_r) < v^\varepsilon(x) < v(x,p_l) \quad \forall x \in [-A^\varepsilon, A^\varepsilon].$$

We denote by α^ε a positive number such that

$$v(x,p_r) + \alpha^\varepsilon \le v^\varepsilon(x) \le v(x,p_l) - \alpha^\varepsilon \quad \forall x \in [-A^\varepsilon, A^\varepsilon].$$

Notice that α^ε can be chosen as small as desired. For further purposes, we choose α^ε so that

$$\alpha^\varepsilon A^\varepsilon \le 2 \int_{|x|\le A^\varepsilon} (U - v(p_r)).$$

The constraint $u^\varepsilon \in U + L^1_0(\mathbb{R})$ entails that the function u^ε should satisfy

$$\int_{|x|\le A^\varepsilon} (u^\varepsilon - U) = 0.$$

However, the function v^ε does not satisfy the above constraint in general: we merely have

$$\begin{aligned} \left|\int_{|x|\le A^\varepsilon} (v^\varepsilon - U)\right| &\le \left|\int_{|x|\le A^\varepsilon} (v^\varepsilon - u)\right| + \left|\int_{|x|\le A^\varepsilon} (u - U)\right| \\ &\le \int_{|x|\le A^\varepsilon} |v^\varepsilon - u| + \int_{|x|\ge A^\varepsilon} |u - U| \\ &\le 2\varepsilon. \end{aligned}$$

Assume for instance that $\int_{|x|\le A^\varepsilon}(v^\varepsilon - U) > 0$. We then define a non-negative function $\rho^\varepsilon \in L^\infty([-A^\varepsilon, A^\varepsilon])$ such that

$$\begin{aligned} v^\varepsilon(x) - \rho^\varepsilon(x) \ge v(x,p_r) + \frac{\alpha^\varepsilon}{2} \quad &\text{a.e. on } [-A^\varepsilon, A^\varepsilon] \\ \text{and} \quad \int_{|x|\le A^\varepsilon} (v^\varepsilon - \rho^\varepsilon - U) = 0.& \end{aligned} \tag{24}$$

Such a function ρ^ε exists provided

$$\int_{|x|\le A^\varepsilon} (v^\varepsilon - U) \le \int_{|x|\le A^\varepsilon} \left(v^\varepsilon - v(p_r) - \frac{\alpha^\varepsilon}{2}\right),$$

and the above inequality is equivalent to

$$\int_{|x|\le A^\varepsilon} (U - v(p_r)) \ge \frac{\alpha^\varepsilon A^\varepsilon}{2}.$$

The previous condition is satisfied by definition of α^ε. Thus there exists a function ρ^ε which satisfies conditions (24).

We then set

$$u^\varepsilon(x) = v^\varepsilon(x) - \rho^\varepsilon(x) \quad \text{for } x \in [-A^\varepsilon, A^\varepsilon].$$

The construction is similar when $\int_{|x|\le A^\varepsilon}(v^\varepsilon - U) < 0$.

At this stage, we have defined a function $u^\varepsilon \in U + L^1_0$ which satisfies

$$\begin{aligned} v(x,p_r) + \frac{\alpha^\varepsilon}{2} \le u^\varepsilon(x) \le v(x,p_l) - \frac{\alpha^\varepsilon}{2} \quad &\forall x \in [-A^\varepsilon, A^\varepsilon], \\ u^\varepsilon(x) = U(x) \quad &\forall x \in \mathbb{R} \setminus [-A^\varepsilon, A^\varepsilon], \\ \text{and} \quad \int_{\mathbb{R}} |u - u^\varepsilon| \le 4\varepsilon.& \end{aligned}$$

Now, by definition of the standing shock U, there exists a positive constant R^ε such that

$$x \geq R^\varepsilon \Rightarrow |U(x) - v(x, p_r)| \leq \frac{\alpha^\varepsilon}{2}.$$

Let k^+ be a positive integer such that $k^+ > R^\varepsilon + A^\varepsilon$. Then for all $x \in [-A^\varepsilon, A^\varepsilon]$, we have

$$v(x, p_r) \leq \tau_{k^+} U(x) \leq v(x, p_r) + \frac{\alpha^\varepsilon}{2} \leq u^\varepsilon(x).$$

Similarly, there exists a negative integer k^- such that for all $x \in [-A^\varepsilon, A^\varepsilon]$,

$$u^\varepsilon(x) \leq v(x, p_l) - \frac{\alpha^\varepsilon}{2} \leq \tau_{k^-} U(x).$$

Notice that $\tau_{k^\pm} U$ are also standing shocks. We now set

$$U_+^\varepsilon := \sup(\tau_{k^+} U, U), \quad U_-^\varepsilon := \inf(\tau_{k^-} U, U).$$

Since standing shocks are ordered, the functions $U_\pm^\varepsilon$ are standing shocks, and

$$U_-^\varepsilon \leq u^\varepsilon \leq U_+^\varepsilon \quad \text{a.e.}$$

Hence the lemma is proved. □

Let us now provide an explicit example for which the existence of standing shocks can be proved.

Lemma 3.6. **(i)** *Assume that for all $y \in \mathbb{T}$, $A(y, \cdot)$ is a convex function. Then the homogenized flux $\bar{A}$ is convex.*

Furthermore, if $A(y, \cdot)$ is strictly convex for all y, then $\bar{A}$ is also strictly convex, and thus satisfies the Oleinik condition of Corollary 2.1.

(ii) *Assume that*

$$\inf_{y \in \mathbb{T}} A(y, p) \underset{|p| \to \infty}{\longrightarrow} +\infty.$$

Then $\lim_{|p| \to \infty} \bar{A}(p) = +\infty$.

These properties are proved in [18]. In the case of equation (5), the strict convexity of $\bar{A}$ comes from the elliptic nature of the equation; if the viscosity term is removed from (5), then examples in [18] show that the homogenized flux $\bar{A}$ may not be striclty convex, even if the flux A is. For the reader's convenience, we have reproduced the proof of Lemma 3.6 in Appendix B.

Example. Assume that

$$A(y, p) = V(y) + |p|^2,$$

for some function $V \in \mathcal{C}^2(\mathbb{T})$. Then according to Lemma 3.6, $\bar{A}$ is strictly convex and $\lim_{|p| \to \infty} \bar{A}(p) = +\infty$. As a consequence, there exists an infinite number of couples $(p^-, p^+) \in \mathbb{R}^2$ satisfying (11). Moreover, the strict convexity of $\bar{A}$ implies that any such couple satisfies Oleinik's condition (14). Hence there exist couples $(p^-, p^+) \in \mathbb{R}^2$ which satisfy the assumptions of Theorem 2.2. Additionally, with the same notation as in Proposition 2.4, we have

$$q_l = p^+ \quad \text{and} \quad q_r = p^-.$$

Indeed, according to Corollary 2.1, we have $\{q_l, q_r\} = \{p^+, p^-\}$. Since the flux A is strictly convex, $\partial_v A(y, \cdot)$ is strictly increasing, and

$$\left\langle \partial_v A(\cdot, v(\cdot, p^-)) \right\rangle < \left\langle \partial_v A(\cdot, v(\cdot, p^+)) \right\rangle.$$

Proposition 3.2 then allows us to conclude that $p^- = q_r$, $p^+ = q_l$.

4. Stability of standing shocks in one space dimension - Part I

This section is devoted to the proof of Theorem 2.2. Hence, throughout this section, we consider an initial data u_0 which satisfies (15), and such that $u_0 \in U + L^1$, where U is a standing shock. Using Lemma 3.4, we deduce that there exists another shock V such that $u \in V + L^1_0(\mathbb{R})$. Then, using Lemma 3.5 together with the Contraction principle, we can restrict the analysis to the class of initial data u_0 such that

$$\exists (U_-, U_+) \text{ standing shocks}, \quad U_- \leq u_0 \leq U_+. \tag{25}$$

Indeed, assume that Theorem 2.2 holds for all $v_0 \in V + L^1_0$ such that (25) is satisfied. Consider now a function $u_0 \in V + L^1_0$ satisfying (15), and let $\varepsilon > 0$ be arbitrary. According to Lemma 3.5, there exists $u_0^\varepsilon \in V + L^1_0$ satisfying (25) and such that $\|u_0 - u_0^\varepsilon\|_1 \leq \varepsilon$. The L^1 contraction principle entails that for all $t \geq 0$,

$$\|S_t u_0 - V\|_1 \leq \|S_t u_0 - S_t u_0^\varepsilon\|_1 + \|S_t u_0^\varepsilon - V\|_1 \leq \varepsilon + \|S_t u_0^\varepsilon - V\|_1.$$

Notice also that by the Contraction principle, the function $t \mapsto \|S_t u_0 - V\|_1$ is non-increasing, and thus has a finite limit as $t \to \infty$. We infer that

$$\forall \varepsilon > 0, \ \lim_{t\to\infty} \|S_t u_0 - V\|_1 \leq \varepsilon,$$

and thus $S_t u_0$ converges toward V as $t \to \infty$.

There remains to prove Theorem 2.2 for initial data which satisfy (25). As emphasized in Section 2, inequalities (15) or (25) should be seen as the analogues of (10) in the context of shock stability. The proof of Theorem 2.2 in this case relies on arguments from dynamical systems theory, which are due to S. Osher and J. Ralston (see [19]; similar ideas are developed by D. Amadori and D. Serre in [2]). The aim is to prove that the ω-limit set of the trajectory $S_t u_0$ is reduced to $\{V\}$, by using a suitable Lyapunov function. Hence, we first prove that the ω-limit set, denoted by Ω, is non-empty, then we state some properties of the ω-limit set, and eventually we prove that $\Omega = \{V\}$.

First step. Compactness in L^1 of the trajectories.

Throughout this section, we set $w(t) := S_t u_0 - V$. Notice first that by the comparison principle for equation (1), inequality (25) is preserved by the semi-group S_t: for all $t \geq 0$, we have

$$U_- \leq S_t u_0 \leq U_+.$$

Hence, for all $t \geq 0$,

$$U_- - U \leq w(t) \leq U_+ - U.$$

Since $U_+ - U$ and $U_- - U$ are integrable functions, the family $\{w(t)\}_{t\geq 0}$ is equi-integrable in L^1. Moreover, since $U_+ - U$ and $U - U_-$ are bounded, it follows that w is uniformly bounded in L^∞. The function w satisfies a linear parabolic equation of the type

$$\partial_t w + \partial_y (b(t, y) w) - \partial_{yy} w = 0, \quad t > 0, \ y \in \mathbb{R},$$

with $b \in L^\infty([0, \infty) \times \mathbb{R})$. Theorem 10.1 in Chapter III of [16] then implies that there exists $\alpha > 0$ such that for all $t_0 \geq 1$, for all $R > 0$,

$$\|u(t)\|_{H^{\alpha/2,\alpha}((t_0,t_0+1)\times(-R,R))} < \infty.$$

Thus the family $\{w(t)\}_{t\geq 0}$ is also equi-continuous in L^1.

Whence it follows from the Riesz-Fréchet-Kolmogorov Theorem that the family $\{w(t)\}_{t\geq 0}$ is relatively compact in $L^1(\mathbb{R})$. Thus the ω-limit set

$$\Omega := \left\{ W \in V + L^1(\mathbb{R}), \exists (t_n)_{n\in\mathbb{N}}, t_n \underset{n\to\infty}{\longrightarrow} \infty,\ S_{t_n} u_0 \to W \text{ in } L^1(\mathbb{R}) \right\}$$

is non-empty.

Second step. Properties of the ω-limit set Ω.

First, Ω is forward and backward invariant by the semi-group S_t, meaning that for all $t \geq 0$,

$$S_t \Omega = \Omega.$$

This important property is a generic one for ω-limit sets. It follows immediately, thanks to parabolic regularity, that all functions in Ω are smooth: $\Omega \subset H^1_{\text{loc}}(\mathbb{R})$, for instance. As a consequence, if $W \in \Omega$ and $w_1(t) := S_t W$, Theorem 6.1 in Chapter III of [16] entails that $w_1 \in L^2([0,T], H^2(B_R)) \cap H^1([0,T], L^2(B_R))$ for all $T, R > 0$.

The second property which is important for our analysis is the LaSalle invariance principle (see [17]), which requires the existence of a Lyapunov function. In the case of scalar conservation laws, a classical choice for a Lyapunov function is $F[u] = \|u - V\|_1$. The Contraction principle entails that $t \mapsto F[S_t u_0]$ is non-increasing. Thus F takes a constant value on Ω, which we denote by C_0.

Eventually, using the conservation of mass, we deduce that Ω is a subset of $V + L^1_0$.

Third step. Conclusion.

We now prove, using the parabolic structure of equation (26), that $\Omega = \{V\}$.

Let $W_0 \in \Omega$ be arbitrary, and let $W(t) = S_t(W_0)$. Notice that $W(t) \in \Omega$ for all $t \geq 0$, according to the previous step. Moreover, $W - V$ satisfies

$$\partial_t (W - V) + \partial_y \left(A(y, W) - A(y, V) \right) - \partial_{yy} (W - V) = 0.$$

Multiplying the above equation by $\operatorname{sgn}(W - V)$, we obtain

$$\partial_t |W - V| + \partial_y \left[\operatorname{sgn}(W - V) \left(A(y, W(t)) - A(y, V) \right) \right] - \operatorname{sgn}(W - V) \partial_{yy} (W - V) = 0.$$

Let ϕ be a cut-off function, i.e. $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\phi \geq 0$ and $\phi \equiv 1$ in a neighbourhood of zero. For $R > 0$, we set $\phi_R := \phi(\cdot/R)$. We now multiply the above equality by ϕ_R and integrate on $[t, t'] \times \mathbb{R}$. Recalling that $\int_{\mathbb{R}} |W(t) - V| = C_0$ for all t, we infer that for all $t' > t \geq 0$, there exists a function $\varepsilon_{t,t'} : [0, \infty) \to [0, \infty)$ such that $\lim_{R\to\infty} \varepsilon_{t,t'}(R) = 0$ and

$$\left| \int_t^{t'} \int_{\mathbb{R}} \operatorname{sgn}(W(s) - V) \partial_{yy} (W(s, y) - V(y)) \phi_R(y) \, ds \, dy \right| \leq \varepsilon_{t,t'}(R).$$

Thus, using a slightly modified version of Lemma 1 in the Appendix, we infer that

$$\operatorname{sgn}(W(s) - V) \partial_{yy} (W(s) - V) = \partial_{yy} |(W(s) - V)|$$

almost everywhere and in the sense of distributions. Consequently, the function $|W - V|$ is a non-negative solution of a parabolic equation of the type

$$\partial_t |W - v| + \partial_y (b(t, y) |W - V|) - \partial_{yy} |W - V| = 0,$$

with $b \in L^\infty([0, \infty) \times \mathbb{R})$. We now conclude thanks to Harnack's inequality (see [11]): let $x_0 \in \mathbb{R}$ be arbitrary, and let K be any compact set in $\mathbb{R}$ such that $x_0 \in K$. Then there exists a constant C_K such that

$$|(W_0 - V)(x_0)| \leq \sup_{x\in K} |(W_0 - V)(x)| \leq C_K \inf_{x\in K} |(W_{|s=1} - V)(x)|.$$

Now, $(W_{|s=1} - V) \in L^1_0 \cap H^1_{\text{loc}}(\mathbb{R})$, and thus there exists $x_1 \in \mathbb{R}$ such that

$$W(1, x_1) - V(x_1) = 0.$$

Choose K such that $x_1 \in K$. Then $W_0 - V$ vanishes uniformly on K, and in particular, $(W_0 - V)(x_0) = 0$. Since x_0 was chosen arbitrarily, we deduce that $W_0 = V$. Hence $\Omega = \{V\}$, and Theorem 2.2 is proved.

5. Uniform in time *a priori* bounds for viscous scalar conservation laws

This section is devoted to the proof of Proposition 2.2. As far as possible, we will treat both models simultaneously. We set

$$w(t) := S_t u_0 - U_0, \quad t \geq 0.$$

The function w satisfies the following equation

$$\partial_t w(t, y) + \operatorname{div}_y B(y, w(t, y)) - \Delta_y w(t, y) = 0, \quad t > 0, \ y \in Q, \tag{26}$$

where

$$B(y, w) = A(y, U_0(y) + w) - A(y, U_0(y)), \quad y \in Q, \ w \in \mathbb{R}.$$

Due to the Contraction principle recalled in Section 2, it is known that w is bounded in $L^\infty([0, \infty), L^1(Q))$, and

$$\forall t \in \mathbb{R}_+, \quad \|w(t)\|_{L^1} \leq \|u_0 - U_0\|_{L^1}. \tag{27}$$

The idea of this section is to use this uniform L^1 bound in order to derive uniform L^p bounds on w for all $p \in [1, \infty]$. To that end, we proceed by induction on the exponent p. The first step is dedicated to the derivation of a differential inequality relating the derivative of the L^p norm to a viscous dissipation term. The calculations are very similar to those developed in [6] to derive *a priori* bounds for solutions of equation (5). Then, we use Poincaré inequalities to control the L^p norm by the dissipation. Eventually, we conclude thanks to a Gronwall type argument.

Preliminary for the whole space case.

We begin by recalling some regularity results about the solutions of equation (1) in the case $Q = \mathbb{R}$. According to the papers by Kružkov [14, 15], it is known that $w \in L^\infty_{\text{loc}}([0, \infty), L^\infty(Q))$. As a consequence, $w \in L^\infty_{\text{loc}}([0, \infty), L^p(Q))$ for all p.

Then, multiplying (26) by $w\chi$ where $\chi \in \mathcal{C}^\infty_0(\mathbb{R})$ is an arbitrary non-negative cut-off function, and integrating in space and time, it is easily proved that for all $T > 0$, w satisfies an inequality of the type

$$\int_0^T \int_{\mathbb{R}} |\partial_y w(s, y)|^2 \chi(y) \, dy \, ds \leq C_T,$$

where the constant C_T depends on T, $\|w\|_{L^\infty([0,T]\times\mathbb{R})}$ and $\|w_{t=0}\|_1$, but not on χ. We deduce that $\partial_y w \in L^2_{\text{loc}}([0, \infty), L^2(\mathbb{R}))$.

First step. A differential inequality.

In this step, we treat the periodic and the full space models simultaneously; our goal is to prove an inequality of the type

$$\frac{d}{dt} \int |w|^{q+1} + c_q \int \left| \nabla |w|^{\frac{q+1}{2}} \right|^2 \leq C_q \left(\int |w|^{q+n} + \int |w|^{q+1} \right),$$

where $q \geq 1$ is arbitrary, n is the exponent appearing in (8), and the constants c_q and C_q depend on q, n, N, and $\|U_0\|_{W^{1,\infty}}$.

To that end, we take $q \geq 1$, multiply (26) by $w|w|^{q-1}$ and integrate on Q; we obtain

$$\begin{aligned} &\frac{1}{q+1}\frac{d}{dt}\int_Q |w|^{q+1} + q\int_Q |\nabla w|^2|w|^{q-1} \\ = \quad & q\int_Q \nabla_y w(t,y)\cdot B(y,w(t,y))|w(t,y)|^{q-1}\,dy. \end{aligned}$$

Notice that all terms are well-defined thanks to the preliminary step.

For $(y,w)\in Q\times\mathbb{R}$, set

$$b_q(y,w) = q\int_0^w B(y,w')|w'|^{q-1}dw';$$

Since $w\in L^\infty_{\text{loc}}([0,\infty),L^\infty(Q))$, there exists a constant C_T such that

$$|b_q(y,w(t,y))| \leq C_T|w(t,y)|^q \quad \forall t\in[0,T],\ \forall y\in Q.$$

As a consequence, $b_q(y,w(t,y))\in L^\infty_{\text{loc}}([0,\infty),L^1(Q))$, and

$$\begin{aligned} & -q\int_Q \nabla_y w(t,y)\cdot B(y,w(t,y))|w(t,y)|^{q-1}\,dy \\ = \quad & \int_Q \left[-\text{div}_y\left(b_q(y,w(t,y))\right) + (\text{div}_y b_q)(y,w(t,y))\right]\,dy \\ = \quad & q\int_Q\int_0^{w(t,y)} (\text{div}_y B)(y,w')|w'|^{q-1}\,dw'. \end{aligned}$$

Thus, we now compute, for $(y,w')\in Q\times\mathbb{R}$,

$$\begin{aligned} \text{div}_y B(y,w') \quad &= \quad \text{div}_y\left[A(y,U_0(y)+w') - A(y,U_0(y))\right] \\ &= \quad (\text{div}_y A)(y,U_0(y)+w') - (\text{div}_y A)(y,U_0(y)) \\ &+ \quad \nabla_y U_0\cdot\left[(\partial_v A)(y,U_0(y)+w') - (\partial_v A)(y,U_0(y))\right]. \end{aligned}$$

Consequently, according to hypothesis (8), we deduce that there exists a positive constant C depending only on $\|U_0\|_{W^{1,\infty}}$ and q such that

$$\left|q\int \nabla_y w(t,y)\cdot B(y,w(t,y))|w(t,y)|^{q-1}\,dy\right| \leq C\left(\int |w(t)|^{q+1} + \int |w(t)|^{q+n}\right).$$

Eventually, we infer that for all $q\geq 1$, there exist positive constants c_q, C_q such that for all $t>0$,

$$\frac{d}{dt}\int |w(t)|^{q+1} + c_q\int\left|\nabla|w(t)|^{\frac{q+1}{2}}\right|^2 \leq C_q\left(\int |w(t)|^{q+1} + \int |w(t)|^{q+n}\right). \tag{28}$$

Second step. Control of L^p norms by the dissipation term (Poincaré inequalities).

In this step, we treat the periodic case and the whole space case separately, and we begin with the periodic case.

First, remember that for all $p\in(1,\infty)$ such that $\frac{1}{p}\geq\frac{1}{2}-\frac{1}{N}$, there exists a positive constant C_p such that for all $\phi\in H^1_{\text{per}}(\mathbb{T}^N)$,

$$\|\phi - \langle\phi\rangle\|_p \leq C_p\|\nabla\phi\|_2. \tag{29}$$

Taking $\phi = |w|^{\frac{q+1}{2}}$, we deduce that

$$\|w\|_r \leq C_r\left(\left\|\nabla|w|^{\frac{q+1}{2}}\right\|_2^{\frac{2}{q+1}} + \|w\|_{\frac{q+1}{2}}\right),$$

where $r \in (1,\infty)$ is such that

$$\frac{1}{r} \geq \frac{1}{q+1} - \frac{2}{N(q+1)}. \tag{30}$$

Now, the idea is to interpolate the L^{n+q} and the L^{q+1} norms in the right-hand side of inequality (28) between L^1 and L^r, where r satisfies the constraint above. It can be easily checked that when $n < (N+2)/N$, we have

$$\frac{1}{n+q} > \frac{1}{q+1} - \frac{2}{N(q+1)};$$

hence the interpolation is always possible, and we have

$$\|w\|_{q+1} \leq \|w\|_1^{1-\alpha}\|w\|_r^{\alpha},$$
$$\|w\|_{q+n} \leq \|w\|_1^{1-\beta}\|w\|_r^{\beta},$$

where

$$\frac{1}{q+1} = 1-\alpha+\frac{\alpha}{r}, \quad \frac{1}{q+n} = 1-\beta+\frac{\beta}{r}.$$

Gathering all inequalities, we infer that

$$\begin{aligned}
& \frac{d}{dt}\|w\|_{q+1}^{q+1} + C_q\left\|\nabla|w|^{\frac{q+1}{2}}\right\|_2^2 \\
\leq\ & C\|w\|_1^{(q+1)(1-\alpha)}\left\|\nabla|w|^{\frac{q+1}{2}}\right\|_2^{2\alpha} + C\|w\|_1^{(q+1)(1-\alpha)}\|w\|_{\frac{q+1}{2}}^{\alpha(q+1)} \\
+\ & C\|w\|_1^{(q+n)(1-\beta)}\left\|\nabla|w|^{\frac{q+1}{2}}\right\|_2^{\frac{2\beta(q+n)}{q+1}} + C\|w\|_1^{(q+n)(1-\beta)}\|w\|_{\frac{q+1}{2}}^{\beta(q+n)}.
\end{aligned}$$

Remember that the L^1 norm is bounded. For the time being, we leave aside the $L^{\frac{q+1}{2}}$ norms of the right-hand side: those will be treated in the very last step. In order to control the right-hand side by the dissipation term in the left-hand side, it suffices to find r (and thus α and β) such that

$$2\alpha < 2, \quad \frac{2\beta(q+n)}{q+1} < 2. \tag{31}$$

Remembering the definition of β, we deduce that we have to find $r \in (q+1,\infty)$ satisfying the two inequalities

$$1-\frac{1}{r} > \frac{q+n-1}{q+1},$$
$$\frac{1}{r} \geq \frac{1}{q+1} - \frac{2}{N(q+1)}.$$

This is possible if and only if the couple (n,q) satisfies

$$\begin{cases} \dfrac{q+n-1}{q+1} < 1, \\[2ex] \dfrac{1}{q+1} - \dfrac{2}{N(q+1)} < 1 - \dfrac{q+n-1}{q+1} \end{cases}$$

which amounts to the condition $n < \min(2,(N+2)/N)$. In the case when $N=1$, this yields $n < 2$, which is more restrictive than the assumption of Proposition 2.2 ($n < 3$). However, when $N=1$, the same arguments as in the whole space case can be used (see below), and lead to $n < 3$. Thus, under the hypotheses of Theorem

2.1, for all $q\geq 1$, we may find $r>\max(q+1,q+n)$ such that conditions (30), (31) are fulfilled. Young's inequality then implies that

$$\frac{d}{dt}\int|w(t)|^{q+1}+C_1\int\left|\nabla|w(t)|^{\frac{q+1}{2}}\right|^2\leq C_2\left(\|w(t)\|^{p_1}_{\frac{q+1}{2}}+\|w(t)\|^{p_2}_{\frac{q+1}{2}}\right),\tag{32}$$

where the constant C_2 depends on $\|u_0-U_0\|_1$, and the exponents p_1,p_2 on n, q and N. According to the Poincaré-Wirtinger inequality, we have

$$\begin{aligned}\left\|\nabla|w|^{\frac{q+1}{2}}\right\|_2^2 &\geq c\left\||w|^{\frac{q+1}{2}}-\left\langle|w|^{\frac{q+1}{2}}\right\rangle\right\|_2^2\\ &= c\left(\left\||w|^{\frac{q+1}{2}}\right\|_2^2-\left\langle|w|^{\frac{q+1}{2}}\right\rangle^2\right)\\ &= c\left(\int|w|^{q+1}-\|w\|^{q+1}_{\frac{q+1}{2}}\right).\end{aligned}$$

Eventually, we deduce that for all $q\geq 1$, there exists constants C_1,C_2,p_1,p_2 such that

$$\frac{d}{dt}\int|w(t)|^{q+1}+C_1\int|w(t)|^{q+1}\leq C_2\left(\|w(t)\|^{p_1}_{\frac{q+1}{2}}+\|w(t)\|^{p_2}_{\frac{q+1}{2}}\right).\tag{33}$$

Let us now treat the one-dimensional model set in the whole space. In dimension one, the H^1 and L^1 norms control the L^∞ norm. Hence we now interpolate the two integrals in the right-hand side of (28) between $L^{\frac{q+1}{2}}$ and L^∞:

$$\int|w|^{q+1}\leq\|w\|^{\frac{q+1}{2}}_{\frac{q+1}{2}}\|w\|_\infty^{\frac{q+1}{2}},$$

$$\int|w|^{q+n}\leq\|w\|^{\frac{q+1}{2}}_{\frac{q+1}{2}}\|w\|_\infty^{\frac{q+2n-1}{2}}.$$

We use the following Poincaré inequality, which involves the dissipation term in the right-hand side of (28) (the proof of this inequality is classical and left to the reader: we refer to [13] for the proof of similar inequalities): there exists a constant C_q, depending only on q, such that for all $w\in L^1\cap L^\infty\cap H^1(\mathbb{R})$[1]

$$\|w\|_\infty\leq C_q\|w\|^{1/3}_{\frac{q+1}{2}}\left(\int\left|\partial_y|w|^{\frac{q+1}{2}}\right|^2\right)^{\frac{2}{3(q+1)}}.$$

Consequently, there exist positive constants C,p such that for all $w\in L^1\cap L^\infty\cap H^1(\mathbb{R})$,

$$\int|w|^{q+n}\leq C\|w\|^p_{\frac{q+1}{2}}\left(\int\left|\partial_y|w|^{\frac{q+1}{2}}\right|^2\right)^{\frac{q+2n-1}{3(q+1)}}.$$

Hence, in order that the dissipation term controls the right-hand side of (28), the exponent n should satisfy

$$\frac{q+2n-1}{3(q+1)}<1\quad\forall q\geq 1,$$

[1]If $w\in L^\infty\cap H^1(\mathbb{T})$, the corresponding inequality is

$$\|w\|_{L^\infty(\mathbb{T})}\leq C_q\|w\|^{1/3}_{\frac{q+1}{2}}\left(\left(\int\left|\partial_y|w|^{\frac{q+1}{2}}\right|^2\right)^{\frac{2}{3(q+1)}}+\|w\|^{2/3}_{\frac{q+1}{2}}\right).$$

which leads to the condition $n < 3$. Using Young's inequality, we conclude that (32) is satisfied. Moreover, the Poincaré inequality used above entails that for all $\lambda > 0$,

$$\begin{aligned}\int |w|^{q+1} &\leq C\|w\|_{\frac{q+1}{2}}^{\frac{2(q+1)}{3}} \left(\int \left|\partial_y |w|^{\frac{q+1}{2}}\right|^2\right)^{1/3} \\ &\leq \frac{C}{\sqrt{\lambda}}\|w\|_{\frac{q+1}{2}}^{q+1} + \lambda \int \left|\partial_y |w|^{\frac{q+1}{2}}\right|^2.\end{aligned}$$

Eventually, we deduce that inequality (33) is also satisfied in the whole space case.

Third step. Uniform bounds in L^q for all $q < \infty$.

We now conclude thanks to Gronwall's lemma, using an inductive argument. Notice indeed that inequality (33) implies that for all $q \geq 1$,

$$w \in L^\infty([0,\infty), L^q(Q)) \Rightarrow w \in L^\infty([0,\infty), L^{2q}(Q)). \tag{34}$$

Indeed, assume that $w \in L^\infty([0,\infty), L^q)$ for some $q \geq 1$. According to (33), we have

$$\frac{d}{dt}\int |w(t)|^{2q} + C_1 \int |w(t)|^{2q} \leq C_2,$$

where the constant C_2 depends on $\|U_0\|_{W^{1,\infty}}$ and on $\|w\|_{L^\infty([0,\infty),L^1\cap L^q)}$, so that, using Gronwall's lemma,

$$\int |w(t)|^{2q} \leq e^{-C_1 t}\int |w_{|t=0}|^{2q} + \frac{C_2}{C_1}(1 - e^{-C_1 t}) \leq C.$$

Thus $w \in L^\infty([0,\infty), L^{2q})$ and (34) is proved. Since $w \in L^\infty([0,\infty), L^1)$, we deduce that $w \in L^\infty([0,\infty), L^q)$ for all $q \in [1,\infty)$.

Fourth step. Uniform bounds in L^∞ and $W^{1,p}$.

We now derive some L^∞ bounds thanks to parabolic regularity results. First, notice that in equation (26), the flux B can be written as

$$B(y, w(t,y)) = b(t,y)w,$$

where

$$b(t,y) = \int_0^1 a(y, v_0(y) + \tau w(t,y))\, d\tau.$$

According to the previous steps, $b(t,y) \in L^\infty([0,\infty), L^q_{\text{loc}}(Q))$ for all $q > 0$; in particular, in the whole space case, for all $q > 1$ there exists a constant C_q such that for all $y_0 \in \mathbb{R}$,

$$\sup_{t\geq 0} \|b(t)\|_{L^q(y_0-2,y_0+2)} \leq C_q.$$

We now use Theorem 8.1 in Chapter III of [16]: we have, for all $y_0 \in Q$, for all $t_0 \geq 1$,

$$|w(t_0,y_0)| \leq C\left(\|w\|_{L^2(Q_{t_0,y_0})}, \|b\|_{L^q(Q_{t_0,y_0})}\right),$$

where $Q_{t_0,y_0} := (t_0 - 1, t_0 + 1) \times (y_0 - 1, y_0 + 1)$ and q is some parameter chosen sufficiently large. The right-hand side is bounded uniformly in y_0 and t_0 by a positive constant C, and we infer that for all $y_0 \in Q, t_0 \geq 1$,

$$|w(t_0,y_0)| \leq C.$$

Thus $w \in L^\infty([0,\infty) \times Q)$. Using Theorem 10.1 in Chapter III of [16], we also deduce that there exists $\alpha > 0$ and a constant $C > 0$ such that for all $t_0 \geq 1$, for all $x_0 \in Q$,

$$\|w\|_{H^{\alpha/2,\alpha}((t_0,t_0+1)\times(x_0-1,x_0+1))} \leq C.$$

As a consequence, we obtain

$$\|w\|_{L^\infty([1,\infty),\mathcal{C}^\alpha(Q))} \leq C.$$

6. Long time behavior of solutions for the periodic model

Throughout this section, we assume that $Q = \mathbb{T}^N$, and we consider a solution $u(t) = S_t u_0$ of equation (1) ($t \geq 0$). Our goal is to prove, under the assumptions of Theorem 2.1, that $u(t) - v(\cdot, \langle u_0 \rangle)$ vanishes in L^∞ as $t \to \infty$. The idea is to prove in a first step the convergence for initial data which are bounded from above or from below by a solution of equation (5), and then to extend this result to arbitrary initial data thanks to the L^∞ bounds proved in the previous section (see Proposition 2.2). We thus begin with the following Proposition:

Proposition 6.1. *Let $u_0 \in L^\infty(\mathbb{T}^N)$ such that*

$$\exists p_0 \in \mathbb{R}, \quad u_0(y) \leq v(y, p_0) \quad \text{for a.e. } y \in \mathbb{T}^N. \tag{35}$$

Let $u(t) = S_t u_0$ for $t \geq 0$. Then, as $t \to \infty$,

$$u(t) \to v(\cdot, \langle u_0 \rangle) \quad \text{in } L^\infty(\mathbb{T}^N).$$

Of course, the same result holds when the upper-bound is replaced by a lower-bound:

Corollary 6.1. *Let $u_0 \in L^\infty(\mathbb{T}^N)$ such that*

$$\exists p_0 \in \mathbb{R}, \quad u_0(y) \geq v(y, p_0) \quad \text{for a.e. } y \in \mathbb{T}^N. \tag{36}$$

Let $u(t) = S_t u_0$ for $t \geq 0$. Then, as $t \to \infty$,

$$u(t) \to v(\cdot, \langle u_0 \rangle) \quad \text{in } L^\infty(\mathbb{T}^N).$$

Proof of Proposition 6.1. According to the previous section (see Proposition 2.2),

$$\sup_{t \geq 0} \|u(t)\|_{L^\infty(\mathbb{T}^N)} < +\infty.$$

Additionally, the Comparison principle yields

$$u(t, y) \leq v(y, p_0) \quad \forall t > 0, \ \forall y \in \mathbb{T}^N.$$

From now on, the proof is very close to that in [7], Section 2: we recall the main steps for the reader's convenience. Set

$$U(t, y) := \sup_{t' \geq t} u(t', y), \quad t \geq 0, y \in \mathbb{T}^N,$$

$$p^*(t) := \inf \left\{ p \in \mathbb{R}, v(y, p) \geq U(t, y) \text{ for a.e. } y \in \mathbb{T}^N \right\}, t \geq 0.$$

Then U belongs to $L^\infty([0,\infty) \times \mathbb{T}^N)$ (since u is uniformly bounded in time), and U is clearly a non-increasing function with respect to t. Moreover, U satisfies

$$U(t, y) \leq v(y, p_0) \quad \forall t > 0, \ \forall y \in \mathbb{T}^N.$$

As a consequence, $p^*(t)$ is bounded from above by p_0, and p^* is a non-increasing function. Moreover, p^* is bounded from below, since for almost every $y \in \mathbb{T}^N$,

$$v(y, p^*(t)) \geq U(t, y) \geq -\|u\|_{L^\infty([0,\infty)\times\mathbb{T}^N)},$$

and thus

$$\forall t \geq 0, \quad p^*(t) = \langle v(\cdot, p^*(t)) \rangle \geq -\|u\|_{L^\infty([0,\infty)\times\mathbb{T}^N)}.$$

Hence p^* is a bounded non-increasing function, and thus $p^*(t)$ has a finite limit, which we denote by $\bar{p}^*$, as $t \to \infty$.

The idea is to prove that $u(t) - v(\cdot, \bar{p}^*)$ converges towards zero as $t \to \infty$. Let $\varepsilon > 0$ be arbitrary. We first choose $t_0 > 0$ such that

$$\|v(p^*(t)) - v(\bar{p}^*)\|_\infty \leq \varepsilon \quad \forall t \geq t_0,$$

and then we pick $p < \bar{p}^*$ and $y_0 \in \mathbb{T}^N$ such that

$$v(y_0, \bar{p}^*) - \varepsilon \leq v(y_0, p) \leq U(t_0 + 1, y_0) \leq v(y_0, p^*(t_0 + 1)) \leq v(y_0, \bar{p}^*) + \varepsilon.$$

Now, choose $t_1 \geq t_0 + 1$ such that

$$U(t_0 + 1, y_0) - \varepsilon \leq u(t_1, y_0) \leq U(t_0 + 1, y_0).$$

By construction, the function

$$V : (s, y) \in (-1, 1) \times \mathbb{T}^N \mapsto v(y, p^*(t_0)) - u(t_1 + s, y)$$

is a non-negative solution of a linear diffusion equation of the type

$$\partial_t V + \mathrm{div}_y(bV) - \Delta_y V = 0$$

for some vector field $b \in L^\infty([-1, 1] \times \mathbb{T}^N)^N$. Hence by Harnack's inequality, there exists a constant C such that

$$\sup_{y \in \mathbb{T}^N} V\left(-\frac{1}{2}, y\right) \leq C \inf_{y \in \mathbb{T}^N} V(0, y) \leq C\varepsilon.$$

Thus, there exists a sequence of positive numbers (t_n) such that $\lim_{n\to\infty} t_n = +\infty$ and such that $u(t_n)$ converges towards $v(\bar{p}^*)$ in L^∞. The L^1 contraction principle, together with parabolic regularity results, entails that the whole family $u(t)$ converges. Eventually, we obtain that $\bar{p}^* = \langle u_0 \rangle$ by conservation of mass.

□

The core of the proof of Theorem 2.1 then lies in the following argument: if $u_0 \in L^\infty$ is arbitrary, we set

$$\tilde{u}_0 := \inf(u_0, v(\cdot, p)),$$
$$\tilde{u} := S_t \tilde{u}_0.$$

The value of parameter p above is irrelevant. One can choose for instance $p = 0$, or $p = \langle w_0 \rangle$.

The function $\tilde{u}_0$ obviously satisfies the assumptions of Proposition 6.1. Hence as $t \to \infty$,

$$\tilde{u}(t) \to v(\cdot, \langle \tilde{u}_0 \rangle) \quad \text{in } L^\infty,$$

and thus there exists a positive time t_0 such that for $t \geq t_0$, for all $y \in \mathbb{T}^N$,

$$\tilde{u}(t, y) \geq v(y, \langle \tilde{u}_0 \rangle - 1).$$

On the other hand, notice that $\tilde{u}_0 \leq u_0$ by definition, and thus by the comparison principle,

$$\tilde{u}(t) \leq u(t) \quad \forall t.$$

Hence, for $t \geq t_0$,

$$u(t) \geq v\left(\cdot, \langle \tilde{u}_0 \rangle - 1\right).$$

In particular, $u(t_0)$ satisfies the assumptions of Corollary 6.1, and thus, as $t \to \infty$,

$$S_t u(t_0) \to v\left(\langle u(t_0) \rangle\right).$$

Since

$$u(t) = S_{t-t_0} u(t_0)$$

and $\langle u(t_0) \rangle = \langle u_0 \rangle$ by the Conservation property, we deduce eventually that

$$u(t) \to v\left(\langle u_0 \rangle\right) \quad \text{as } t \to \infty.$$

Thus Theorem 2.1 is proved.

7. Stability of standing shocks in one space dimension - Part II

This section is devoted to the proof of shock stability in the whole space case, for general initial data. We first prove Proposition 2.5, and then we discuss recent results around the validity of assumption **(H)**.

We start by introducing some notation. Following [22], we denote by $\mathcal{G}$ the set of standing shocks connecting $v(\cdot, p_l)$ to $v(\cdot, p_r)$, and we set

$$\mathcal{A} := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}),\ \exists U \in \mathcal{G},\ u \in U + L^1(\mathbb{R}) \right\},$$
$$\mathcal{A}_0 := \left\{ u \in \mathcal{A},\ v(\cdot, \min(p_l, p_r)) \leq u \leq v(\cdot, \max(p_l, p_r)) \right\}.$$

Our goal is to prove that for all $u_0 \in \mathcal{A}$,

$$\lim_{t \to \infty} d(S_t u_0, \mathcal{G}) = 0,$$

where $d(u, A)$ denotes the L^1 distance from u to a set A. Notice that the Contraction principle easily entails that the function $t \mapsto d(S_t u_0, \mathcal{G})$ is decreasing. Hence, its limit as $t \to \infty$ exists; for all $u_0 \in \mathcal{A}$, set

$$\ell_0(u_0) := \lim_{t \to \infty} d(S_t u_0, \mathcal{G}).$$

Theorem 2.2 states that $\ell_0(u) = 0$ for all $u \in \mathcal{A}_0$. Moreover, it follows from the Contraction principle that $\ell_0(u_0)$ is a contraction, i.e.

$$|\ell_0(u) - \ell_0(v)| \leq \|u - v\|_{L^1} \quad \forall u, v \in \mathcal{A}.$$

Additionally, for all $t \geq 0$ and for all $u \in \mathcal{A}$,

$$\ell_0(u) = \ell_0(S_t u).$$

Similarly, we define, for all $u_0 \in \mathcal{A}$,

$$\ell_1(u_0) := \lim_{t \to \infty} d(S_t u_0, \mathcal{A}_0).$$

The function ℓ_1 is well-defined: indeed, the Comparison property entails that $\mathcal{A}_0$ is stable by the semi-group S_t. Consequently, by the Contraction principle, the function $t \mapsto d(S_t u_0, \mathcal{A}_0)$ is decreasing and non-negative, and thus has a finite limit as $t \to \infty$. Moreover, the functional ℓ_1 enjoys the same properties as ℓ_0: ℓ_1 is a contraction on $\mathcal{A}$ and $\ell_1(u) = \ell_1(S_t u)$ for all $t \geq 0$. Eventually, since $\mathcal{G} \subset \mathcal{A}_0$, we deduce that

$$\ell_1(u) \leq \ell_0(u) \quad \forall u \in \mathcal{A}.$$

In the rest of this section, we denote by $v(p)$ the function $v(\cdot,p)$. We now tackle the proof of Proposition 2.5, which is very similar to [22], paragraph 3.5. Let $u_0\in\mathcal{A}$ be arbitrary. For all $v\in\mathcal{A}_0$, we have

$$\ell_0(u_0)\le \ell_0(v)+\|u_0-v\|_1\le\|u_0-v\|_1.$$

Thus for all $u_0\in\mathcal{A}$,

$$\ell_0(u_0)\le d(u_0,\mathcal{A}_0).$$

Replacing u_0 by S_tu_0 in the previous inequality, we infer that for all $u_0\in\mathcal{A}$,

$$\ell_0(u_0)\le\lim_{t\to\infty}d(S_tu_0,\mathcal{A}_0)=\ell_1(u_0).$$

Thus ℓ_0 and ℓ_1 take the same values on $\mathcal{A}$, and it suffices to prove that

$$\ell_1(u_0)=\lim_{t\to\infty}d(S_tu_0,\mathcal{A}_0)=0. \tag{37}$$

Notice that if $u\in\mathcal{A}$, then, with $p^+=\max(p_l,p_r)$, $p^-=\min(p_l,p_r)$,

$$d(u,\mathcal{A}_0)=\left\|\left(u-v(p^+)\right)_+\right\|_1+\left\|\left(u-v(p^-)\right)_-\right\|_1.$$

We now prove that assumption **(H)** implies (37). According to Lemma 3.4, there exists a standing shock U such that $u_0\in U+L^1_0(\mathbb{R})$. We now define functions a^+,a^- in $v(p^+)+L^1_0$ and $v(p^-)+L^1_0$ respectively, such that

$$a^-(y)\le u_0(y)\le a^+(y).$$

Let us explain for instance the construction of a^+. If $u_0(y)>v(y,p^+)$, we set

$$a^+(y)=u_0(y).$$

On the other hand, since $u\in U+L^1$ and U is asymptotic to $v(p^+),v(p^-)$, we have

$$\int_{\mathbb{R}}(v(y,p^+)-u_0(y))_+\,dy\ge\int(v(p^+)-U)-\|u_0-U\|_1=+\infty.$$

Hence there is enough room, between the graphs of $v(y,p^+)$ and $u_0(y)$ (restricted to the set where $u_0(y)\le v(y,p^+)$), to insert a function b^+ such that

$$u_0(y)\le v(y,p^+)\Rightarrow u_0(y)\le b^+(y)\le v(y,p^+),$$

$$\int_{\mathbb{R}}\mathbf{1}_{u_0\le v(y,p^+)}(v(y,p^+)-b^+(y))\,dy=\int_{\mathbb{R}}\mathbf{1}_{u_0>v(y,p^+)}(u_0(y)-v(y,p^+))\,dy.$$

On the set where $u_0(y)\le v(y,p^+)$, we define $a^+(y)=b^+(y)$. It is obvious that the function a^+ belongs to $v(p^+)+L^1_0$ and that $u_0\le a^+$. The function a^- is defined in a similar fashion. Thanks to the comparison principle, we have

$$S_ta^-\le S_tu_0\le S_ta^+\quad\forall t\ge0.$$

Consequently,

$$d(S_tu_0,\mathcal{A}_0)\le\left\|S_ta^+-v(p^+)\right\|_{L^1}+\left\|S_ta^--v(p^-)\right\|_{L^1}. \tag{38}$$

From the above inequality, it is clear that the stability of standing shocks follows from to the stability of solutions of equation (5) in L^1_0. Let us now prove that $\ell_1(u_0)=0$ if **(H)** is satisfied.

Let $\delta>0$. If $u_0\in\mathcal{A}$ is such that

$$\|(u_0-v(p^+))_+\|_1\le\delta,\quad\|(u_0-v(p^-))_-\|_1\le\delta,$$

then by construction

$$\|a^+-v(p^+)\|_1\le2\delta,\quad\|a^--v(p^-)\|_1\le2\delta.$$

And according to **(H)**, there exists $\delta_0 > 0$ such that if $\delta \le \delta_0$, then

$$\lim_{t\to\infty} \|S_t a^\pm - v(p^\pm)\|_1 = 0,$$

and thus the right-hand side of (38) vanishes as $t \to \infty$. Thus $\ell_1(u_0) = 0$.

Hence we now focus on the case where

$$\|(u_0 - v(p^+))_+\|_1 \ge \delta_0 \text{ or } \|(u_0 - v(p^-))_-\|_1 \ge \delta_0.$$

We then define the function

$$\bar u_0(y) := \begin{cases} v(y,p^+) + \alpha_+(u_0 - v(p^+)) & \text{if } u_0(y) > v(y,p^+),\\ u_0(y) & \text{if } v(y,p^-) \le u_0(y) \le v(y,p^+),\\ v(y,p^-) + \alpha_-(u_0 - v(p^-)) & \text{if } u_0(y) < v(y,p^-),\end{cases}$$

where

$$\alpha_\pm = \begin{cases} \frac{\delta_0}{\|(u_0 - v(p^\pm))_\pm\|_1} & \text{if } \|(u_0 - v(p^\pm))_\pm\|_1 > \delta_0,\\ 0 & \text{else.}\end{cases}$$

Since $\bar u_0 - u_0 \in L^1(\mathbb{R})$, $\bar u_0 \in \mathcal{A}$. Moreover,

$$\begin{aligned} \|\bar u_0 - u_0\|_1 &= (1-\alpha_+)\|(u_0 - v(p^+))_+\|_1 + (1-\alpha_-)\|(u_0 - v(p^-))_-\|_1\\ &\le d(u_0, \mathcal{A}_0) - \delta_0. \end{aligned}$$

Notice that $\ell_1(\bar u_0) = 0$. Since ℓ_1 is a contraction, we have

$$\ell_1(u_0) \le \ell_1(\bar u_0) + \|u_0 - \bar u_0\|_1 \le d(u_0, \mathcal{A}_0) - \delta_0. \tag{39}$$

We now argue by contradiction. Assume that for all $t \ge 0$,

$$\|(S_t u_0 - v(p^+))_+\|_1 \ge \delta_0 \text{ or } \|(S_t u_0 - v(p^-))_-\|_1 \ge \delta_0.$$

Then we may replace u_0 by $S_t u_0$, for $t \ge 0$ arbitrary, in inequality (39). We obtain

$$\ell_1(u_0) = \ell_1(S_t u_0) \le d(S_t u_0, \mathcal{A}_0) - \delta_0.$$

Passing to the limit as $t \to \infty$, we infer

$$\ell_1(u_0) \le \ell_1(u_0) - \delta_0,$$

which is absurd. Hence there exists $t_0 \ge 0$ such that

$$\|(S_{t_0} u_0 - v(p^+))_+\|_1 < \delta_0 \text{ and } \|(S_{t_0} u_0 - v(p^-))_-\|_1 < \delta_0.$$

We have already proved that $\ell_1(S_{t_0} u_0) = 0$. We deduce that $\ell_1(u_0) = 0$, and thus $\ell_0(u_0) = 0$.

Consequently, assumption **(H)** entails that $\ell_0(u) = 0$ for all $u \in \mathcal{A}$.

We conclude this article by a discussion of assumption **(H)**. In fact, it turns out that **(H)** is true for all fluxes $A \in W^{5,\infty}(\mathbb{T}\times\mathbb{R})$ satisfying (8). However, the proof of this result goes beyond the scope of this article. The L^1 stability of periodic solutions was first proved by A. Blanchet, J. Dolbeault and M. Kowalczyk in a linear context, see [4, 5]. The authors of [4] proved, under a technical assumption on the moments of order four of the function $S_t u_0 - v(\cdot, p)$, that $S_t u_0 - v(\cdot, p)$ converges towards zero in $L^1(\mathbb{R})$, with an algebraic rate of convergence. Their proof relies on a parabolic self-similar change of variables which transforms (1) into a Fokker-Planck equation with highly oscillating coefficients. The long time behavior of this rescaled equation is studied by means of entropy dissipation methods, together with homogenization techniques. These ideas were then used by the author in [8], and led to the proof of **(H)** for general fluxes and arbitrary initial data. Unlike in [4], however, the arguments of [8] do not use entropy dissipation methods, but rely rather on dynamical systems theory, with a scheme of proof similar to the one developed in Section 4. We also refer to [9, 10] for additional results and techniques concerning the asymptotic behavior of non linear viscous conservation laws in the homogeneous case.

Appendix A - Proof of Lemma 3.6

(i) Assume that the flux A is convex. Let $p_1, p_2 \in \mathbb{R}$ such that $p_1 \neq p_2$, and let $\lambda \in (0,1)$. In the following, we set

$$\begin{aligned} v_i(y) &= v(y, p_i), \quad i = 1, 2, \\ w = \lambda v_1 + (1-\lambda) v_2, &\quad p = \lambda p_1 + (1-\lambda) p_2, \\ u(y) &= v(y, \lambda p_1 + (1-\lambda) p_2). \end{aligned}$$

By definition of $v(\cdot, p)$ and of the homogenized flux $\bar{A}$, we have

$$\begin{aligned} -v_i' + A(y, v_i(y)) &= \bar{A}(p_i), \\ -u' + A(y, u(y)) &= \bar{A}(\lambda p_1 + (1-\lambda) p_2). \end{aligned}$$

Consequently, using the convexity of the flux A, we deduce that for all $y \in \mathbb{T}^N$,

$$\begin{aligned} -w'(y) + A(y, w(y)) &\leq -w'(y) + \lambda A(y, v_1(y)) + (1-\lambda) A(y, v_2(y)) \\ &= \lambda \bar{A}(p_1) + (1-\lambda) \bar{A}(p_2). \end{aligned} \tag{40}$$

Assume that $\bar{A}(\lambda p_1 + (1-\lambda) p_2) > \lambda \bar{A}(p_1) + (1-\lambda)\bar{A}(p_2)$, and write u, w as

$$u = p + f', \quad w = p + g',$$

with $f, g \in \mathcal{C}^2_{\text{per}}(\mathbb{T}^N)$. Since f and g are defined up to the addition of constants, we can assume that $f < g$ almost everywhere. Moreover, notice that

$$\sup_{y \in \mathbb{T}^N} \left(-g''(y) + A(y, p + g'(y))\right) < \inf_{y \in \mathbb{T}^N} \left(-f''(y) + A(y, p + f'(y))\right).$$

Thus there exists $\alpha > 0$ such that

$$-g'' + A(y, p + g'(y)) + \alpha g \leq -f'' + A(y, p + f'(y)) + \alpha f.$$

Hence, by the maximum principle, we infer that $g \leq f$, which is absurd. Thus

$$\bar{A}(\lambda p_1 + (1-\lambda) p_2) \leq \lambda \bar{A}(p_1) + (1-\lambda) \bar{A}(p_2).$$

If the flux A is strictly convex, then inequality (40) is strict for all $y \in \mathbb{T}^N$ (remember that the family $v(y,p)$ is strictly increasing with p for all $y \in \mathbb{T}^N$). Consequently, the same argument as above leads to

$$\bar{A}(\lambda p_1 + (1-\lambda) p_2) < \lambda \bar{A}(p_1) + (1-\lambda) \bar{A}(p_2).$$

(ii) Assume now that

$$\lim_{|p| \to \infty} \inf_{y \in \mathbb{T}} A(y, p) = +\infty.$$

For $p > 0$ arbitrary, let $y_p \in \mathbb{T}$ such that

$$v(y_p, p) = \max_{y \in \mathbb{T}} v(y, p).$$

Notice that since $\langle v(\cdot,p)\rangle = p$, we always have $v(y_p,p)\geq p$. Then

$$\begin{aligned}\bar{A}(p) &= \partial_y v(y,p) + A(y, v(y,p)) \quad \forall y\in\mathbb{T}\\ &= A(y_p, v(y_p,p))\\ &\geq \inf_{y\in\mathbb{T}} A\left(y, \max_{z\in\mathbb{T}} v(z,p)\right).\end{aligned}$$

The right-hand side goes to $+\infty$ as $p\to+\infty$. The limit as $p\to-\infty$ is treated in the same way.

Appendix B

Lemma 1. *Let $w\in L^1\cap L^\infty(\mathbb{R})$ such that $w'\in L^2(\mathbb{R})$ and $w''\in L^1_{loc}(\mathbb{R})$. Assume that w is such that*

$$\lim_{R\to\infty}\int_{\mathbb{R}} \operatorname{sgn}(w(y))w''(y)\phi\left(\frac{y}{R}\right)\,dy = 0$$

for all $\phi\in\mathcal{C}_0^\infty(\mathbb{R})$ such that $\phi\equiv 1$ in a neighbourhood of zero. Then

$$\frac{1}{\delta}|w'|^2\mathbf{1}_{|w|<\delta} \underset{\delta\to 0}{\rightharpoonup} 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}).$$

As a consequence,

$$\partial_{yy}|w| = \operatorname{sgn}(w)w'' \quad \textit{in } \mathcal{D}'(\mathbb{R}).$$

Proof. For $\delta>0$, let

$$\psi_\delta(x) := \begin{cases} \operatorname{sgn}(x) & \text{if } |x|\geq\delta,\\ \dfrac{x}{\delta} & \text{else.}\end{cases}$$

Then

$$\psi_\delta'(x) = \frac{1}{\delta}\mathbf{1}_{|x|<\delta},$$

and for all $R>0$, we have, using the chain rule

$$\int |w'|^2\psi_\delta'(w)\phi_R = -\int w''\psi_\delta(w)\phi_R - \int w'\psi_\delta(w)\phi_R',$$

where $\phi_R = \phi(\cdot/R)$.

Since $w'\in L^2$, we infer

$$\left|\int w'\psi_\delta(w)\phi_R'\right| \leq \int |w'|\ |\phi_R'| \leq R^{-1/2}\|w'\|_{L^2}\|\phi'\|_{L^2}.$$

Thus the above term vanishes as $R\to\infty$, uniformly in δ.

On the other hand,

$$\lim_{\delta\to0}\int w''\psi_\delta(w)\phi_R = \int w''\operatorname{sgn}(w)\phi_R,$$

and the right-hand side vanishes as $R\to\infty$ by assumption. We deduce that

$$\lim_{R\to\infty}\limsup_{\delta\to0}\int |w'|^2\psi_\delta'(w)\phi_R = 0.$$

Now, choose the function ϕ so that $\operatorname{sgn}(y)\phi'(y)\leq 0$ for all $y\in\mathbb{R}$. Then the integral $\int |w'|^2\psi_\delta'(w)\phi_R$ is non-negative and increasing with respect to R, and we deduce that

$$\lim_{\delta\to0}\int |w'|^2\psi_\delta'(w)\phi_R = 0 \quad \forall R.$$

Thus the first part of the lemma is proved, since for any function $\varphi_i n\mathcal{C}_0^\infty(\mathbb{R})$ there exists $R, C > 0$ such that

$$|\varphi| \leq C\phi_R.$$

Consider now $S_\delta \in W^{2,1}_{\text{loc}}(\mathbb{R})$ such that

$$S'_\delta = \psi_\delta \text{ and } S_\delta(0) = 0,$$

where the function ψ_δ was defined earlier. Then

$$S_\delta(w) \to |w| \quad \text{in } L^1_{\text{loc}}(\mathbb{R}),$$

and according to the chain rule,

$$\partial_{yy} S_\delta(w) = w''\psi_\delta(w) + |w'|^2 \frac{\mathbf{1}_{|w|\leq\delta}}{\delta}.$$

Passing to the limit in the sense of distributions in the above equality yields

$$\partial_{yy}|w| = w''\text{sgn}(w).$$

□

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